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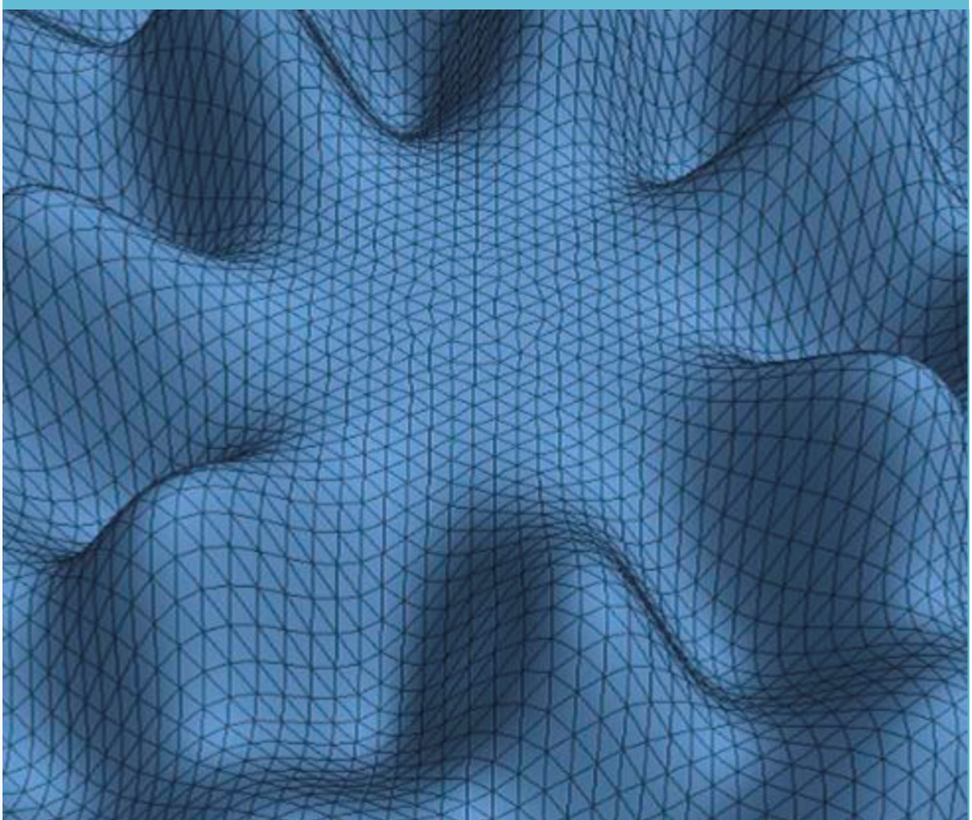

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Modulo and factor periodic Poisson stable functions

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Abstract. Functions as the basic concept of mathematics have to be permanently renewed to satisfy challenges, first of all, of modern industrial revolutions and science development. Oscillations and recurrence are mostly needed for the theoretical research and applications. If oscillations are preferable in engineering, the recurrence originates in celestial mechanics. The ultimate recurrence is the Poisson stability. Nowadays, needs for functions with irregular behavior are exceptionally strong in neuroscience and celestial dynamics, which is still in the developing mode. In the present research we have decided to combine periodic dynamics with the phenomenon of Poisson stability. That is, one of the simplest forms of oscillations is amalgamated with the most sophisticated recurrence type. The present products of the design are *modulo periodic Poisson stable functions* and *factor periodic Poisson stable functions*. The main results of the research are conditions for Poisson stability of the newly introduced functions. Numerical simulations, which confirm the contribution of periodicity and recurrence in the behavior of functions are provided.

Keywords. Poisson stability, modulo periodic Poisson stable functions, factor periodic Poisson stable functions.

Introduction

The theory of differential equations and dynamical systems is, mainly, a doctrine on oscillations and recurrence, which are basic in science and applications [1–5]. In literature, there is no clear difference for oscillations and recurrence. Nevertheless, if the line of oscillations contains periodic, quasi-periodic and almost periodic functions [6–10], the Poisson stable functions are unique with the recurrence property, since they can be unbounded. The functions, which in literature are called *recurrent functions* [4,5] belong to the both classes of functions. It is clear that the process of invention of new types functions is unstoppable, to response demands of the progress. In our research, we also have made contribution to the process. In paper [11], to strengthen the role of recurrence as a chaotic ingredient we have extended the

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Poisson stability to the unpredictability property. Thus, the Poincaré chaos has been determined, and one can say that the *unpredictability implies chaos* now. The unpredictable point in the functional space of the Bebutov dynamics is the unpredictable function [12–20]. Accordingly, we have provided a dynamical method, how to construct Poisson stable functions. Deterministic and stochastic dynamics have been utilized. Deterministically unpredictable functions have been constructed as solutions of hybrid systems, consisting of discrete and differential equations [19], and randomly they are results of the Bernoulli process inserted into a linear differential equation [18, 20, 21]. Unpredictable oscillations in neural networks have been researched in [19, 20, 22–24].

In the papers [16–18] and books [19,20] discussing existence of unpredictable solutions, we have developed a new method how to approve Poisson stable solutions, since unpredictable functions are a subset of Poisson stable functions, and to verify the unpredictability one has to check, if the Poisson stability is valid. The method is distinctly different than the *comparability method by character of recurrence* introduced in [25] and later has been realized in several articles [26–32].

Unlike the papers [12, 14, 16–24], the present research is busy with a new type of Poisson stable functions. In the papers [26–29] and others, quasilinear systems are with constant matrices of coefficients, and the newly introduced functions will allow to research systems with periodic and, even with Poisson stable coefficients [33]. Another significant novelty, which is achieved in the present paper as well as in our former studies [12, 17, 19, 20] is the numerical simulation of the Poisson stable functions and solutions. We believe that altogether, the present suggestions can shape a new interesting science direction, not only in the theoretical study of differential equations, but also about rich opportunities for applications in mechanics, electronics, artificial neural networks, neuroscience.

Preliminaries

In this part of the paper, we introduce definitions for *modulo periodic Poisson stable*, *factor periodic Poisson stable*, and *modulo almost Poisson stable functions* as well as for *compartmental Poisson stability*.

Let us start with the definition of the Poisson stable function.

Definition 1 [5]. A continuous and bounded function $\psi(t) : \mathbb{R} \to \mathbb{R}^n$ is called *Poisson stable*, if there exists a sequence t_k, which diverges to infinity such that the sequence $\psi(t+t_k)$ converges to $\psi(t)$ as $k \to \infty$ uniformly on bounded intervals of $\mathbb{R}$.

We shall call the sequence t_k, in Definition 1, *the Poisson sequence* for the function $\psi(t)$.

Definition 2. A function $f(t) = \phi(t) + \psi(t)$ is said to be the *modulo periodic Poisson stable (MPPS)* function, if $\phi(t)$ is an ω-periodic continuous function and $\psi(t)$ is a Poisson stable function.

Definition 3 [10]. A continuous function $\phi(t)$ is called *quasiperiodic* with periods $2\pi/\omega_1, 2\pi/\omega_2, \cdots, 2\pi/\omega_m$ if for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that each number ρ satisfying the system of inequalities $|\omega_k\rho| < \delta(mod\ 2\pi)$, $k = 1, 2, \cdots, m$, also satisfies the inequality $\sup_{t\in\mathbb{R}} \|\phi(t+\rho) - \phi(t)\| \le \epsilon$, that is, it is ϵ-almost period of $\phi(t)$.

Definition 4. A function $f(t) = \phi(t) + \psi(t)$ is said to be a *modulo quasiperiodic Poisson stable (MQPPS)* function if $\phi(t)$ is a quasiperiodic function, and $\psi(t)$ is a Poisson stable function.

Definition 5. A function $f(t) = \phi(t) + \psi(t)$ is said to be a *modulo almost periodic Poisson stable (MAPPS)* function if $\phi(t)$ is a continuous almost periodic function, and $\psi(t)$ is a Poisson stable function.

Definition 6. A product $\phi(t)\psi(t)$ is said to be a *factor periodic Poisson stable (FPPS) function*, if $\phi(t)$ is a continuous periodic and $\psi(t)$ is a Poisson stable functions.

Finally, we shall introduce definitions, which can also be useful in the future investigations.

Definition 7. A function $f(t)$ is said to be a *compartmental periodic Poisson stable (CPPS)* function if $f(t) = G(t, t)$, where $G(u, s)$ is a continuous bounded function, periodic in u, and Poisson stable in s.

Definition 8. A function $f(t)$ is said to be a *compartmental quasiperiodic Poisson stable (CQPPS)* function if $f(t) = G(t, t)$, where $G(u, s)$ is a continuous bounded function, quasiperiodic in u, and Poisson stable in s.

Definition 9. A function $f(t)$ is said to be a *compartmental almost periodic Poisson stable (CAPPS)* function if $f(t) = G(t, t)$, where $G(u, s)$ is a continuous bounded function, almost periodic in u, and Poisson stable in s.

In the present research, we will focus on MPPS and FPPS functions.

Main results

Theorem 1. *For arbitrary sequence of positive real numbers t_k, $k = 1, 2, ...,$ and a positive number ω there exists a subsequence t_{k_l}, $l = 1, 2, ...,$ and a number τ_ω, $0 \le \tau_\omega < \omega$, such that $t_{k_l} \to \tau_\omega (mod\ \omega)$ as $l \to \infty$.*

Proof. Consider the sequence τ_k such that $t_k \equiv \tau_k(mod\ \omega)$, and $0 \le \tau_k < \omega$ for all $k \ge 1$. The boundedness of the sequence τ_k implies that there exists a subsequence τ_{k_l}, which converges to a number τ_ω [34]. □

Consider a Poisson stable function $\psi(t)$, and the Poisson sequence t_k. By Lemma 1 for fixed $\omega > 0$ there exists a subsequence t_{k_l} and a number τ_ω such that $t_{k_l} \to \tau_\omega(mod\ \omega)$ as $l \to \infty$. In what follows, we shall call the number τ_ω as the *Poisson shift* for the function $\psi(t)$ with respect to the ω. The set of Poisson shifts $\mathcal{T}_\omega$ is not empty, in general case, it can consist of several or even an infinite number of elements. The number $\kappa_\omega = inf\ \mathcal{T}_\omega$, $0 \le \kappa_\omega < \omega$, is said to be a *Poisson number for the function $\phi(t)$ with respect to the number ω*. In what follows, we shall call κ_ω simply the *Poisson number.*

Lemma 1. $\kappa_\omega \in T_\omega$.

Proof. Assume on the contrary that κ_ω is not in T_ω. Then there exists a strictly decreasing sequence τ_m, $m \geq 1$, in T_ω, such that $\tau_m \to \kappa_\omega$. For each natural m, denote by t_i^m a subsequence of t_k such that $t_i^m \to \tau_m (mod\ \omega)$ as $i \to \infty$.

Fix a sequence of positive numbers ϵ_n, which converges to zero. One can find numbers i_n, $n = 1, 2, ...$, such that $|t_{i_n}^n - \tau_n| < \epsilon_n (mod\ \omega)$. It is clear that $t_{i_n}^n \to \kappa_\omega (mod\ \omega)$ as $n \to \infty$. □

Remark 1. *The last assertion implies that if* $\kappa_\omega = 0$, *then there exists a subsequence* t_{k_l} *such that* $t_{k_l} \to 0 (mod\ \omega)$ *as* $l \to \infty$.

Theorem 2. *If* $f(t) = \phi(t) + \psi(t)$ *is an MPPS function, and* $\kappa_\omega = 0$, *then the function* $f(t)$ *is Poisson stable.*

Proof. According to Lemma 1, there exists a subsequence t_{k_l}, which tends to zero in modulus ω as $l \to \infty$. Without loss of generality assume that $t_k \to 0 (mod\ \omega)$ as $k \to \infty$. Fix a positive number ϵ, and bounded interval $I \subset \mathbb{R}$. The periodic function $\phi(t)$ is uniformly continuous on $\mathbb{R}$. Consequently, there exists a number k_1 such that

$$\|\phi(t + t_k) - \phi(t)\| < \frac{\epsilon}{2}$$

for all $t \in \mathbb{R}$ and $k > k_1$. Moreover, there exists an integer k_2 such that

$$\|\psi(t + t_k) - \psi(t)\| < \frac{\epsilon}{2}$$

for $t \in I$, $k > k_2$. This is why,

$$\|f(t + t_k) - f(t)\| \leq \|\phi(t + t_k) - \phi(t)\| + \|\psi(t + t_k) - \psi(t)\| < \epsilon,$$

if $t \in I$ and $k > \max(k_1, k_2)$. That is, the function $f(t)$ is Poisson stable. □

Theorem 3. *Assume that* $\psi(t)$ *is a Poisson stable function. If* $\kappa_\omega = 0$, *for some positive number* ω, *then* $\psi(t)$ *is an MPPS function.*

Proof. Let us write $\psi(t) = g(t) + (\psi(t) - g(t))$, where $g(t)$ is a continuous $\omega-$periodic function. Since $\kappa_\omega = 0$, then the subtraction $\psi(t) - g(t)$ is Poisson stable by Theorem 2. □

Remark 2. *The last result is a source for the optimization problem how to choose the function* $g(t)$ *and the period* ω *to minimize the difference* $\psi(t) - g(t)$. *In other words, the problem of approximation of Poisson stable functions with periodic ones. It is of exceptional interest for celestial mechanics [2].*

Theorem 4. *If* $g(t) = \phi(t)\psi(t)$ *is a FPPS function, and* $\kappa_\omega = 0$, *then the function* $g(t)$ *is Poisson stable.*

Proof. Denote $n_\phi = \max\limits_{t\in\mathbb{R}} \|\phi(t)\|$ and $n_\psi = \sup\limits_{t\in\mathbb{R}} \|\psi(t)\|$. According to Lemma 1, there exists a subsequence t_{k_l}, which tends to zero in modulus ω as $l \to \infty$. Without loss of generality assume that $t_k \to 0(mod\,\omega)$ as $k \to \infty$. Fix a positive number ϵ, and bounded interval $I \subset \mathbb{R}$. The periodic function $\phi(t)$ is uniformly continuous on $\mathbb{R}$. Consequently, there exists a number k_1 such that

$$\|\phi(t+t_k) - \phi(t)\| \leq \frac{\epsilon}{2m_\psi}$$

for all $t \in \mathbb{R}$ and $k > k_1$. Moreover, there exists an integer k_2 such that

$$\|\psi(t+t_k) - \psi(t)\| \leq \frac{\epsilon}{2m_\phi}$$

for $t \in I$, $k > k_2$. This is why

$$\|g(t+t_k) - g(t)\| = \|\phi(t+t_k)\psi(t+t_k) - \phi(t)\psi(t)\| \leq$$
$$m_\psi\|\phi(t+t_k) - \phi(t)\| + m_\phi\|\psi(t+t_k) - \psi(t)\| < \epsilon,$$

if $t \in I$ and $k > \max(k_1, k_2)$. That is, the function $g(t)$ is Poisson stable. □

Numerical examples

Let us take into account the logistic discrete equation

$$\lambda_{i+1} = F_\mu(\lambda_i), \tag{1}$$

$i \in \mathbb{Z}$ and $F_\mu(s) = \mu s(1-s)$. The interval $[0,1]$ is invariant under the iterations of (1) for $\mu \in (0,4]$. It was shown in Theorem 4.1 [12] that the logistic map (1) possesses an unpredictable solution for each $\mu \in [3 + (2/3)^{1/2}, 4]$.

Define the following integral

$$\Theta(t) = \int_{-\infty}^{t} e^{-3(t-s)}\Omega(s)ds, \tag{2}$$

where $\Omega(t)$ is a piecewise constant function defined on the real axis through the equation $\Omega(t) = \psi_i$ for $t \in [i, i+1)$, $i \in \mathbb{Z}$. It is worth noting that $\Theta(t)$ is bounded on the whole real axis such that $\sup\limits_{t\in\mathbb{R}} |\Theta(t)| \leq 1/3$. Moreover, it was proved in [15] that the function $\Theta(t)$ is Poisson stable.

Next, we shall use the property of the function $\Theta(t)$ to construct MPPS and FPPS functions, which are Poisson stable by Theorems 2 and 4.

An example of the modulo periodic Poisson stable function. Consider the MPPS function

$$G(t) = 0.5sin(0.02\pi t) + 1.5\Theta^2(t). \tag{3}$$

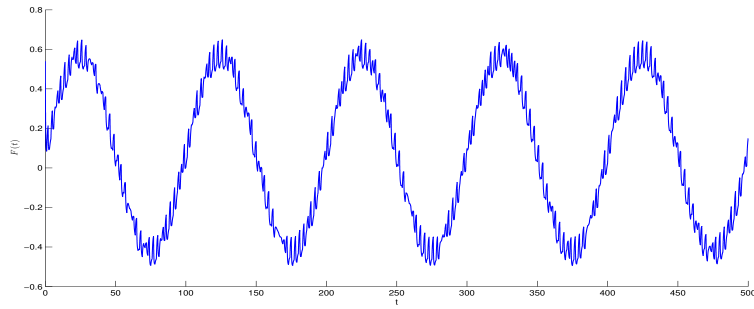

Figure 1 – The graph of the function $F(t)$.

One can easily verify that the conditions of Theorem 2 are true for the function. We do not reliably know the initial value of the Poisson stable function $\Theta(t)$, so we cannot visualize the MPPS function $G(t)$ precisely, but we can show a function $F(t)$, which approaches $G(t)$ as time increases.

In Figure 1 the function

$$F(t) = 0.5sin(0.02\pi t) + 1.5\eta^2(t), \tag{4}$$

with initial value $F(0) = 1.5\eta^2(0)$ is shown. The function $F(t)$ asymptotically converges to the MPPS function $G(t)$, and $\eta(t)$ is the solution of the differential equation $x' = -3x + \Omega(t)$ with the initial value $\eta(0) = 0.6$ [17, 22, 23].

An example of the factor periodic Poisson stable function. In Figure 2 the function $V(t)$ with initial value $V(0) = 0.6$ is illustrated, which approximates the following FPPS function

$$W(t) = cos(0.04t)\Theta(t). \tag{5}$$

The conditions of Theorem 4 for the function $W(t)$ are easily verifiable.

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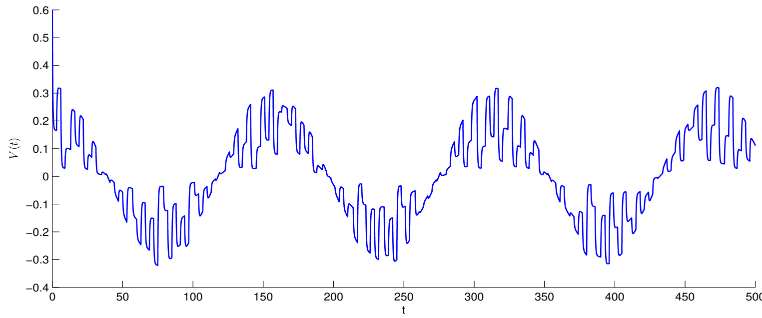

Figure 2 – The graph of the function $V(t)$.

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Ахмет М., Тлеубергенова М., Жаманшин А. ПЕРИОДТЫ ҚОСЫЛҒЫШТЫ ЖӘНЕ ПЕРИОДТЫ КОЭФФИЦИЕНТТІ ПУАССОН БОЙЫНША ОРНЫҚТЫ ФУНКЦИЯЛАР

Математиканың негізгі ұғымы ретінде функциялар, ең алдымен, қазіргі өнеркәсіп пен ғылымның дамуының міндеттеріне жауап беру үшін үнемі толықтырылып отыруы керек. Тербелістер мен рекуренттілік негізінен теориялық зерттеулер мен қолданулар үшін қажет. Техника саласында тербелістер қолайлы болса, рекурренттілік аспан механикасында пайда болды. Ең қиын рекурренттілік – бұл Пуассон бойынша орнықтылық болып табылады. Бүгінгі таңда нейробиология мен аспан механикасы сынды дамып келе жатқан салаларда реттелмеген функцияларға қажеттілік артуда. Бұл зерттеуде біз периодтылықты Пуассон бойынша орнықтылық құбылысымен біріктіруді ұсынамыз. Яғни, тербелістің қарапайым жағдайларының бірі рекуренттіліктің ең күрделі түрімен біріктірілген. Дәлірек айтқанда, зерттеудің объектілері *периодты қосылғышты Пуассон бойынша орнықты* және *периодты коэффициентті Пуассон бойынша орнықты* функциялар болып табылады. Мақалада анықталған функциялардың Пуассон бойынша орнықтылығының шарттары зерттеудің негізгі нәтижелері болып есептеледі. Жаңа функциялардың әрекетіндегі периодтылық пен рекурренттіліктің рөлін көрсету үшін сандық талдау жүргізілді.

Кілттік сөздер. Пуассон бойынша орнықтылық, периодты қосылғышты Пуассон бойынша орнықты функциялар, периодты коэффициентті Пуассон бойынша орнықты функциялар.

Ахмет М., Тлеубергенова М., Жаманшин А. ФУНКЦИИ УСТОЙЧИВЫЕ ПО ПУАССОНУ С ПЕРИОДИЧЕСКИМИ КОЭФФИЦИЕНТОМ И СЛАГАЕМЫМ

Функции как основная концепция математики должны постоянно пополнятся, чтобы отвечать на вызовы, современной промышленной революции и развитию науки. С этой целью в теоретических исследованиях и приложениях необходимы колебания и рекуррентность. Если колебания предпочтительнее в технике, то рекуррентность появилась в небесной механике. Наиболее сложная рекуррентность — это устойчивость по Пуассону. Сегодня потребность в функциях с нерегулярным поведением особенно высока в нейробиологии и небесной механике, которая все еще находится в стадии развития. В настоящем исследовании мы предлагаем совместить периодичность с устойчивостью по Пуассону. То есть одна из простейших форм колебаний сочетается с наиболее сложным типом рекуррентности. Более точно, объектами исследования являются *функции устойчивые по Пуассону с периодическим слагаемым* и *функции устойчивые по Пуассону с периодическим коэффициентом.* Основными результатами исследования являются условия устойчивости по Пуассону для функций, определенных в статье. Осуществлен численный анализ, иллюстрирующий роль периодичности и рекуррентности в поведении

новых функций.

Ключевые слова. Устойчивость по Пуассону, функции устойчивые по Пуассону с периодическим слагаемым, функции устойчивые по Пуассону с периодическим коэффициентом.

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Cauchy problem for the Jacobi fractional heat equation

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Abstract. In this work we study a Cauchy problem for the Jacobi fractional heat equation. The well-posedness results and a priori estimates are obtained in the Sobolev type spaces $W_e^{s,p}(\mathbb{R}^+, \nu_{\alpha,\beta})$.

Keywords. Jacobi operator, fractional heat equation, Fourier-Jacobi transform, inverse Fourier-Jacobi transform, Sobolev type space.

1 Introduction

In this paper we consider a Cauchy problem for the Heat equation associated with the Jacobi operator

$$\Delta_{\alpha,\beta} = A^{-1}_{\alpha,\beta}(t)\frac{d}{dt}\Big(A_{\alpha,\beta}(t)\frac{d}{dt}\Big), \quad t \in (0, +\infty), \tag{1}$$

here $A_{\alpha,\beta}(t) = 2^{2\rho}(\sinh(t))^{2\alpha+1}(\cosh(t))^{2\beta+1}, \rho = \alpha + \beta + 1$, with $\alpha \geq -1/2$ and $\beta \in \mathbb{R}$.

We can rewrite the expression (1) in the form

$$\Delta_{\alpha,\beta} = \frac{d^2}{dt^2} + g(t)\frac{d}{dt},$$

where $g(t) = (2\alpha + 1)\coth(t) + (2\beta + 1)\tanh(t)$.

The singular points for $\Delta_{\alpha,\beta}$ are 0 and $+\infty$. $\lim_{t\to+\infty} g(t) = 2\alpha + 2\beta + 2 = 2\rho$. The spectral decomposition of the Jacobi operator was considered by M. Flensted-Jensen in 1972 [1]. There were obtained a generalization of the classical Paley-Wiener Theorem and a generalized

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Fourier transform $\mathcal{F}_{\alpha,\beta}$, called Jacobi-Fourier transform. For more information about harmonic analysis associated with the Jacobi operator, we refer the readers to the papers [2–7].

Let $0 < \gamma < 1$. The aim of this paper is to study Cauchy problem for the non-homogeneous time-fractional heat equation associated with the Jacobi operator:

$$\mathcal{D}^{\gamma}_{0^+,t}u(t,x) - \Delta_{\alpha,\beta}u(t,x) + mu(t,x) = f(t,x), \quad x \in \mathbb{R}^+, \quad 0 < t < T < +\infty,$$

where m is a positive number and $\mathcal{D}^{\gamma}_{0^+,t}, 0 < \gamma < 1$, is the left-sided Caputo fractional derivative, under the condition

$$u(0,x) = \phi(x), \quad x \in \mathbb{R}^+.$$

The contents of this paper as follows. In Section 2, we collect some results about harmonic analysis associated with the Jacobi operator on $\mathbb{R}^+$ and here we introduce the Sobolev type space $W^{r,s}_e(\mathbb{R}^+, \nu_{\alpha,\beta})$, also give some necessary information about fractional derivative. In Section 3, we prove our main Theorem 3 about the solvability of Cauchy problems associated with the Jacobi operator on $\mathbb{R}^+$.

2 Preliminaries

2.1 The Jacobi operator. The eigenfunction of the operator $\Delta_{\alpha,\beta}$ is a unique solution of the equation [1]

$$\Delta_{\alpha,\beta}\varphi^{\alpha,\beta}_{\lambda}(t) + (\lambda^2 + \rho^2)\varphi^{\alpha,\beta}_{\lambda}(t) = 0, \quad \lambda \in \mathbb{C},$$

satisfying

$$\varphi^{\alpha,\beta}_{\lambda}(0) = 1, \quad \frac{d}{dt}\varphi^{\alpha,\beta}_{\lambda}(0) = 0$$

and given by the expression

$$\varphi^{\alpha,\beta}_{\lambda}(t) = F\Big(\frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda); \alpha + 1; -\sinh^2 t\Big), \tag{2}$$

where F is the Gauss hypergeometric function [8] and $\rho = \alpha+\beta+1$. The eigenfunction $\varphi^{\alpha,\beta}_{\lambda}(t)$ (2) is called the Jacobi function. The Jacobi function $\varphi^{\alpha,\beta}_{\lambda}(t)$ is analytic for $t \in [0, +\infty)$ and

$$\varphi^{\alpha,\beta}_{\lambda}(t) = \varphi^{\alpha,\beta}_{-\lambda}(t) \quad \text{and} \quad \overline{\varphi^{\alpha,\beta}_{\lambda}(t)} = \varphi^{\alpha,\beta}_{\overline{\lambda}}(t).$$

In particularly, we have

$$\varphi^{-\frac{1}{2},\frac{1}{2}}_{\lambda}(t) = \cos(\lambda t) \quad \text{and} \quad \varphi^{\frac{1}{2},\frac{1}{2}}_{\lambda}(t) = \frac{\sin(\lambda t)}{\lambda \sinh t}.$$

Remark 1 [1, Proposition 1, p. 144]. *For each fixed $t \in (0, +\infty)$, $\varphi^{\alpha,\beta}_{\lambda}(t)$ is an entire function as a function of λ,.*

Properties of the Jacobi functions $\varphi_\lambda^{\alpha,\beta}(t)$ are:
i) For all $\lambda \in \mathbb{C}$ and $t \in [0, +\infty)$ with $|Im\lambda| \leq \rho$, we have ([1, Lemma 11, p. 153])

$$|\varphi_\lambda^{\alpha,\beta}(t)| \leq 1.$$

ii) For all $n \in \mathbb{Z}^+$ there exists $K_n > 0$ such that ([1, Theorem 2, p. 145])

$$|\frac{d^n}{dt^n}\varphi_\lambda^{\alpha,\beta}(t)| \leq K_n(1+t)(1+|\lambda|)^n e^{(|Im\lambda|-\rho)t}$$

and

$$|\frac{d^n}{d\lambda^n}\varphi_\lambda^{\alpha,\beta}(t)| \leq K_n(1+t)^{n+1} e^{(|Im\lambda|-\rho)t}$$

for all $\lambda \in \mathbb{C}$, $t \in [0, +\infty)$.

Let us introduce the following function spaces ([1, p. 146-147], [5, Notations, p. 368]).

Let $\mathcal{S}_e(\mathbb{R})$ be the space of even, infinitely differentiable, and rapidly decreasing functions on $\mathbb{R}$, equipped with usual Schwartz topology, and $\mathcal{S}_e^r(\mathbb{R}) = \{(\cosh t)^{\frac{-2\rho}{r}}\mathcal{S}_e(\mathbb{R})\}$, $0 < r \leq 2$, be the space, equipped with the topology defined by the semi-norms

$$N_{n,k}(f) = \sup_{t\geq 0}(\cosh t)^{\frac{2\rho}{r}}(1+t)^n|\frac{d^k}{dt^k}f(t)|.$$

It is clear that $\mathcal{S}_e^r(\mathbb{R})$ is invariant under $\Delta_{\alpha,\beta}$ and the semi-norms defined by

$$N_{n,k}(f) = \sup_{t\geq 0}(\cosh t)^{\frac{2\rho}{r}}(1+t)^n|\Delta_{\alpha,\beta}^k f(t)|$$

are continuous on $\mathcal{S}_e^r(\mathbb{R})$.

Let $L^p(\mathbb{R}^+, \mu_{\alpha,\beta}), 1 \leq p < +\infty$, be the space of measurable functions f on $\mathbb{R}^+ = [0, +\infty)$ such that

$$\|f\|_{p,\mu}^p = \int_0^{+\infty} |f(t)|^p d\mu_{\alpha,\beta}(t) < +\infty,$$

where $d\mu_{\alpha,\beta}(t) = (2\pi)^{-\frac{1}{2}}2^{2\rho}(\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}dt$ or $d\mu_{\alpha,\beta}(t) = (2\pi)^{-\frac{1}{2}}A_{\alpha,\beta}(t)dt$.

Remark 2 [1, p. 146]. *Notice that* $\mathcal{S}_e^r(\mathbb{R}) \subset L^r(\mathbb{R}^+, \mu_{\alpha,\beta})$ *for all* $0 < r \leq 2$.

Let $L^p(\mathbb{R}^+, \nu_{\alpha,\beta}), 1 \leq p < \infty$ *be the space of measurable functions* f *on* $\mathbb{R}^+$ *such that*

$$\|f\|_{p,\nu}^p = \int_0^{+\infty} |f(\lambda)|^p d\nu_{\alpha,\beta}(\lambda) < +\infty,$$

where $d\nu_{\alpha,\beta}(\lambda) = (2\pi)^{-\frac{1}{2}}|c_{\alpha,\beta}(\lambda)|^{-2}d\lambda$. *Here,* $c_{\alpha,\beta}(\lambda)$ *is the Harish-Chandra's function, given by*

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(i\lambda)\Gamma(\alpha+1)}{\Gamma(\frac{\rho+i\lambda}{2})\Gamma(\frac{\alpha-\beta+1+i\lambda}{2})}.$$

Note that for real λ, α, β, we have $\overline{c_{\alpha,\beta}(\lambda)} = c_{\alpha,\beta}(-\lambda)$.

We will use $L^p(\mu)$ and $L^p(\nu)$ instead of $L^p(\mathbb{R}^+, \mu_{\alpha,\beta})$ and $L^p(\mathbb{R}^+, \nu_{\alpha,\beta})$, respectively for our convenience.

For $f \in L^1(\mu)$ the Fourier-Jacobi transform $\mathcal{F}_{\alpha,\beta}$ of f is defined by ([1, Proposition 3, p. 146], [5, Definition 1.1, p. 369])

$$\widehat{f}(\lambda) = (\mathcal{F}_{\alpha,\beta} f)(\lambda) = \int_0^{+\infty} f(t)\varphi_\lambda^{\alpha,\beta}(t)d\mu_{\alpha,\beta}(t) \tag{3}$$

and for $f \in L^1(\nu)$ the inverse Fourier-Jacobi transform $\mathcal{F}^{-1}_{\alpha,\beta}$ is given by

$$f(t) = \left(\mathcal{F}^{-1}_{\alpha,\beta} \hat{f}\right)(t) = \int_0^{+\infty} \widehat{f}(\lambda)\varphi_\lambda^{\alpha,\beta}(t)d\nu_{\alpha,\beta}(\lambda), \tag{4}$$

where $\varphi_\lambda^{\alpha,\beta}(t)$ is the Jacobi functions (2).

Proposition 1 ([1, Proposition 3, p. 146]). *Fourier-Jacobi Transform $\mathcal{F}_{\alpha,\beta}$ is a linear, norm-preserving map of $L^2(\mu)$ onto $L^2(\nu)$.*

In particularly, we have the Plancherel's identity

$$\|\widehat{f}\|_{2,\nu} = \|f\|_{2,\mu}. \tag{5}$$

Remark 3. *For $\alpha = \beta = -\frac{1}{2}$, we have the Fourier-cosine transform*

$$\widehat{f_c}(\lambda) = (\mathcal{F}_c f)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \cos(\lambda t) f(t) dt,$$

and the inverse Fourier-cosine transform is defined by

$$f(t) = \left(\mathcal{F}_c^{-1} \widehat{f_c}\right)(t) = \frac{4}{\sqrt{2\pi}} \int_0^{+\infty} \cos(\lambda t) \widehat{f_c}(\lambda) d\lambda.$$

Remark 4 [5, Remark, p. 370]. *It is clear that for all $f \in \mathcal{S}_e^r(\mathbb{R})$, we have*

$$\mathcal{F}_{\alpha,\beta}(\Delta_{\alpha,\beta}(f)) = -(\lambda^2 + \rho^2)\mathcal{F}_{\alpha,\beta}(f).$$

Notation [6, Definition 3.2, p. 175]. *For $s \in \mathbb{R}$, we denote by $W_e^{s,p}(\mathbb{R}^+, \nu_{\alpha,\beta})$ the space of functions satisfying*

$$\int_0^{+\infty} (\lambda^2 + \rho^2)^{ps} \left|\mathcal{F}_{\alpha,\beta}(u)\right|^p d\nu_{\alpha,\beta}(\lambda) < +\infty \quad \textit{for all} \quad u \in \mathcal{S}_e^2(\mathbb{R}).$$

The norm of $W_e^{s,p}(\mathbb{R}^+,\nu_{\alpha,\beta})$ can be taken by

$$\|u\|^p_{W_e^{s,p}(\mathbb{R}^+,\nu_{\alpha,\beta})} = \int_0^{+\infty} (\lambda^2+\rho^2)^{ps} \left|\mathcal{F}_{\alpha,\beta}(u)\right|^p d\nu_{\alpha,\beta}(\lambda). \tag{6}$$

This is the Sobolev type space on $\mathbb{R}^+$. We will use $W_e^{s,p}(\nu)$ instead of $W_e^{s,p}(\mathbb{R}^+,\nu_{\alpha,\beta})$ for our convenience.

Theorem 1. [6, Theorem 3.3, p. 176]. *For $s\in\mathbb{R}$ and $1\le p<+\infty$ the space $\mathcal{S}_e^2(\mathbb{R})$ is dense in $W_e^{s,p}(\nu)$.*

Theorem 2 [6, Theorem 3.4, p. 177]. *For $s,t\in\mathbb{R}, t<s$ and $1\le p<+\infty$ the space $W_e^{s,p}(\nu)$ is continuously included in the space $W_e^{t,p}(\nu)$.*

Also, we deal with the spaces $C([0,T],W_e^{1,2}(\nu))$ and $C([0,T],L^2(\mu))$ with the norms

$$\|u\|^2_{C([0,T],W_e^{1,2}(\nu))} := \max_{0<t<T} \|u(t,\cdot)\|^2_{W_e^{1,2}(\nu)}$$

and

$$\|u\|^2_{C([0,T],L^2(\mu))} := \max_{0<t<T} \|f(t,\cdot)\|^2_{2,\mu},$$

respectively.

2.2 Fractional differentiation operators. In this subsection, we introduce fractional differentiation operators and other conceptions. We refer the readers to the papers [9–12] to get acquainted with some new results for diffusion equations with Caputo fractional derivative.

Definition 1 [13, p. 69]. *Let $[a,b]$ $(-\infty<a<b<\infty)$ be a finite interval on the real axis $\mathbb{R}$. The left and right Riemann-Liouville fractional integrals I^γ_{a+} and I^γ_{b-} of order $\gamma\in\mathbb{R}$ $(\gamma>0)$ are defined by*

$$I^\gamma_{a+}[f](t) := \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1} f(s)ds, \quad t\in(a,b],$$

and

$$I^\gamma_{b-}[f](t) := \frac{1}{\Gamma(\gamma)}\int_t^b (t-s)^{\gamma-1} f(s)ds, \quad t\in[a,b),$$

respectively. Here Γ denotes the Euler gamma function.

Definition 2 [13, p. 70]. *The left and right Riemann-Liouville fractional derivatives D^γ_{a+} and D^γ_{b-} of order $\gamma\in\mathbb{R}$ $(0<\gamma<1)$ are given by*

$$D^\gamma_{a+}[f](t) := \frac{d}{dt} I^{1-\gamma}_{a^+}[f](t), \quad \forall t\in(a,b],$$

and

$$D^\gamma_{b-}[f](t) := -\frac{d}{dt} I^{1-\gamma}_{b^-}[f](t), \quad \forall t\in[a,b),$$

respectively.

Definition 3 [13, p. 91]. *The left and right Caputo fractional derivatives* D^{γ}_{a+} *and* D^{γ}_{b-} *of order* $\gamma \in \mathbb{R}$ $(0 < \gamma < 1)$ *are defined by*

$$\mathcal{D}^{\gamma}_{a+}[f](t) := D^{\gamma}_{a+}[f(t) - f(a)], \quad t \in (a, b],$$

and

$$\mathcal{D}^{\gamma}_{b-}[f](t) := D^{\gamma}_{b-}[f(t) - f(b)], \quad t \in [a, b),$$

respectively.

Definition 4 [14]. *Let* X *be a Banach space. We say that* $u \in C^{\gamma}([0, T], X)$ *if* $u \in C([0, T], X)$ *and* $\mathcal{D}^{\gamma}_{t} u \in C([0, T], X)$.

The classical Mittag-Leffler function $\mathbb{E}_{\gamma,1}(t)$ and the Mittag-Leffler type function $\mathbb{E}_{\gamma,\gamma}(t)$ are given by the expressions

$$\mathbb{E}_{\gamma,1}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k + 1)} \quad \mathbb{E}_{\gamma,\gamma}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k + \gamma)}.$$

In the case $\gamma = 1$, we obtain $\mathbb{E}_{1,1}(t) = e^t$. For more information about the classical Mittag-Leffler function $\mathbb{E}_{\gamma,1}(t)$ and the Mittag-Leffler type function $\mathbb{E}_{\gamma,\gamma}(t)$ see e.g. [9, p. 40 and p. 42].

In [15] the following estimate for the Mittag-Leffler function is proved, when $0 < \gamma < 1$ (not true for $\gamma \geq 1$)

$$\frac{1}{1 + \Gamma(1-\gamma)t} \leq \mathbb{E}_{\gamma,1}(-t) \leq \frac{1}{1 + \Gamma(1+\gamma)^{-1} t}, \quad t > 0.$$

Then it follows

$$0 < \mathbb{E}_{\gamma,1}(-t) < 1, \quad t > 0.$$

If $\gamma = 1$, we know that $0 < e^{-t} < 1$, when $t > 0$.

3 Main problem

The section deals with a Cauchy problem for the time-fractional heat equation generated by the Jacobi operator $\Delta_{\alpha,\beta}$ (1).

Let $0 < \gamma < 1$. We consider the non-homogeneous time-fractional heat equation

$$\mathcal{D}^{\gamma}_{0+,t} u(t, x) - \Delta_{\alpha,\beta} u(t, x) + m u(t, x) = f(t, x), \quad x \in \mathbb{R}^+, \quad 0 < t < T, \tag{7}$$

with initial condition

$$u(0, x) = \phi(x), \quad x \in \mathbb{R}^+, \tag{8}$$

where the functions f and ϕ are given functions. Our aim is to find a unique solution u of the problem (7)-(8).

Theorem 3. *Let $0 < \gamma < 1$. Suppose that $f \in C^1([0,T], L^2(\mu))$ and $\phi \in W_e^{1,2}(\nu)$. Then the problem (7)-(8) has a unique solution $u \in C^\gamma([0,T], L^2(\mu)) \cap C([0,T], W_e^{1,2}(\nu))$ and can be represented by the formula*

$$u(t,x) = \int_0^{+\infty} \int_0^{+\infty} \int_0^t (t-\tau)^{\gamma-1} \mathbb{E}_{\gamma,\gamma}\left(-(\lambda^2+\rho^2)(t-\tau)^\gamma\right) f(\tau,y)$$

$$\times \varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x) d\tau d\mu_{\alpha,\beta}(y) d\nu_{\alpha,\beta}(\lambda)$$

$$+ \int_0^{+\infty} \int_0^{+\infty} \mathbb{E}_{\gamma,1}\left(-(\lambda^2+\rho^2)t^\gamma\right) \phi(y)\varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x) d\mu_{\alpha,\beta}(y) d\nu_{\alpha,\beta}(\lambda).$$

Proof. Let $0 < \gamma < 1$. We first prove that the problem (7)-(8) has only one solution, if the later exists. Suppose the proposition is false. Assume that there exist two different solutions $u_1(t,x)$ and $u_2(t,x)$. Denote $u_0(t,x) = u_1(t,x) - u_2(t,x)$. Then $u_0(t,x)$ satisfies the following equation

$$\mathcal{D}_{0+,t}^\gamma u_0(t,x) - \Delta_{\alpha,\beta} u_0(t,x) + m u_0(t,x) = 0, \quad x \in \mathbb{R}^+, \quad 0 < t < T, \tag{9}$$

$$u_0(0,x) = 0, \quad x \in \mathbb{R}^+. \tag{10}$$

The problem (9)-(10) has the only trivial solution. This implies uniqueness of the solution.

Let us prove the existence of the solutions. Using the Fourier-Jacobi transform $\mathcal{F}_{\alpha,\beta}$ (3) on both sides of (7)-(8) we have

$$\mathcal{D}_{0+,t}^\gamma \widehat{u}(t,\lambda) + (\lambda^2+\rho^2+m)\widehat{u}(t,\lambda) = \widehat{f}(t,\lambda), \tag{11}$$

$$\widehat{u}(0,\lambda) = \widehat{\phi}(\lambda), \tag{12}$$

for all $\lambda \in \mathbb{R}$ and $0 < t < T$. The solution (see [13, ex. 4.9, p. 231]) of the problem (11)-(12) is given by

$$\widehat{u}(t,\lambda) = \int_0^t (t-\tau)^{\gamma-1} \mathbb{E}_{\gamma,\gamma}\left(-(\lambda^2+\rho^2+m)(t-\tau)^\gamma\right) \widehat{f}(\tau,\lambda) d\tau + \widehat{\phi}(\lambda) \mathbb{E}_{\gamma,1}\left(-(\lambda^2+\rho^2+m)t^\gamma\right), \tag{13}$$

where $\mathbb{E}_{\gamma,1}(z)$ is the classical Mittag-Leffler function and $\mathbb{E}_{\gamma,\gamma}(z)$ is the Mittag-Leffler type function. Now using the inverse Fourier-Jacobi transform $\mathcal{F}_{\alpha,\beta}^{-1}$ (4) to (13), we obtain the formula for the solution of the problem (7)-(8), given by

$$u(t,x) = \int_0^{+\infty} \int_0^{+\infty} \int_0^t (t-\tau)^{\gamma-1} \mathbb{E}_{\gamma,\gamma}\left(-(\lambda^2+\rho^2+m)(t-\tau)^\gamma\right) f(\tau,y)$$

$$\times\varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x)d\tau d\mu_{\alpha,\beta}(y)d\nu_{\alpha,\beta}(\lambda)$$

$$+\int_0^{+\infty}\int_0^{+\infty}\mathbb{E}_{\gamma,1}\left(-(\lambda^2+\rho^2+m)t^\gamma\right)\phi(y)\varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x)d\mu_{\alpha,\beta}(y)d\nu_{\alpha,\beta}(\lambda).$$

Using the property

$$\frac{d}{d\tau}\left(\mathbb{E}_{\gamma,1}(c\tau^\gamma)\right)=c\tau^{\gamma-1}\mathbb{E}_{\gamma,\gamma}(c\tau^\gamma),\quad c=\text{constant},$$

of the Mittag-Leffler function, we obtain

$$\frac{\partial}{\partial\tau}\left(\mathbb{E}_{\gamma,1}\left(-(\lambda^2+\rho^2+m)(t-\tau)^\gamma\right)\right)=(\lambda^2+\rho^2+m)(t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-(\lambda^2+\rho^2+m)(t-\tau)^\gamma\right)$$

and we can write (13) in the form

$$\widehat{u}(t,\lambda)=\int_0^t(t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-(\lambda^2+\rho^2+m)(t-\tau)^\gamma\right)\widehat{f}(\tau,\lambda)d\tau+\widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}\left(-(\lambda^2+\rho^2+m)t^\gamma\right)$$

$$=\frac{1}{\lambda^2+\rho^2+m}\int_0^t\frac{\partial}{\partial\tau}\left(\mathbb{E}_{\gamma,1}\left(-(\lambda^2+\rho^2+m)(t-\tau)^\gamma\right)\right)\widehat{f}(\tau,\lambda)d\tau+\widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}\left(-(\lambda^2+\rho^2+m)t^\gamma\right)$$

$$=\frac{\widehat{f}(t,\lambda)}{\lambda^2+\rho^2+m}-\frac{\widehat{f}(0,\lambda)\mathbb{E}_{\gamma,1}\left(-(\lambda^2+\rho^2+m)t^\gamma\right)}{\lambda^2+\rho^2+m}$$

$$-\frac{1}{\lambda^2+\rho^2+m}\int_0^t\mathbb{E}_{\gamma,1}\left(-(\lambda^2+\rho^2+m)(t-\tau)^\gamma\right)\frac{\partial}{\partial\tau}\widehat{f}(\tau,\lambda)d\tau+\widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}\left(-(\lambda^2+\rho^2+m)t^\gamma\right)$$

using the rule of integration by parts and $\mathbb{E}_{\gamma,1}(0)=1$.

Let $f\in C^1([0,T],L^2(\mu))$, $\phi\in W_e^{1,2}(\nu)$, then we can estimate u as follows

$$\|u(t,\cdot)\|^2_{W_e^{1,2}(\nu)}=\int_0^{+\infty}\left|(\lambda^2+\rho^2)\widehat{u}(t,\lambda)\right|^2d\nu_{\alpha,\beta}(\lambda)\lesssim\int_0^{+\infty}\left|\frac{(\lambda^2+\rho^2)\widehat{f}(t,\lambda)}{\lambda^2+\rho^2+m}\right|^2d\nu_{\alpha,\beta}(\lambda)$$

$$+\int_0^{+\infty}\left|\frac{(\lambda^2+\rho^2)\widehat{f}(0,\lambda)\mathbb{E}_{\gamma,1}\left(-(\lambda^2+\rho^2+m)t^\gamma\right)}{\lambda^2+\rho^2+m}\right|^2d\nu_{\alpha,\beta}(\lambda)$$

$$+\int_0^{+\infty}\left|\int_0^t\frac{(\lambda^2+\rho^2)\mathbb{E}_{\gamma,1}\left(-(\lambda^2+\rho^2+m)(t-\tau)^\gamma\right)}{\lambda^2+\rho^2+m}\frac{\partial}{\partial\tau}\widehat{f}(\tau,\lambda)d\tau\right|^2d\nu_{\alpha,\beta}(\lambda)$$

$$+\int_0^{+\infty}\left|(\lambda^2+\rho^2)\widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}\left(-(\lambda^2+\rho^2+m)t^\gamma\right)\right|^2d\nu_{\alpha,\beta}(\lambda)$$

$$\lesssim\int_0^{+\infty}\left|\widehat{f}(t,\lambda)\right|^2d\nu_{\alpha,\beta}(\lambda)+\int_0^{+\infty}\left|\widehat{f}(0,\lambda)\right|^2d\nu_{\alpha,\beta}(\lambda)$$

$$+\int_0^{+\infty}\left(\int_0^t\left|\frac{\partial}{\partial\tau}\widehat{f}(\tau,\lambda)\right|d\tau\right)^2 d\nu_{\alpha,\beta}(\lambda)+\int_0^{+\infty}\left|(\lambda^2+\rho^2)\widehat{\phi}(\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda)$$

$$\lesssim \|\widehat{f}(t,\cdot)\|_{2,\nu}^2+\|\widehat{f}(0,\cdot)\|_{2,\nu}^2+\int_0^T\|\frac{\partial}{\partial t}\widehat{f}(t,\cdot)\|_{2,\nu}^2dt+\|\phi\|^2_{W_e^{1,2}(\nu)}$$

$$=\|f(t,\cdot)\|_{2,\mu}^2+\|f(0,\cdot)\|_{2,\mu}^2+\int_0^T\|\frac{\partial}{\partial t}f(t,\cdot)\|_{2,\mu}^2dt+\|\phi\|^2_{W_e^{1,2}(\nu)},$$

here we have used the Cauchy-Schwarz inequality, Fubini's theorem and $a\lesssim b$ denotes $a\leq cb$ for some positive constant c independent of a and b. Thus

$$\|u(t,\cdot)\|^2_{W_e^{1,2}(\nu)}\lesssim\|f(t,\cdot)\|_{2,\mu}^2+\|f(0,\cdot)\|_{2,\mu}^2+\int_0^T\|\frac{\partial}{\partial t}f(t,\cdot)\|_{2,\mu}^2dt+\|\phi\|^2_{W_e^{1,2}(\nu)}.$$

Then we obtain

$$\|u\|^2_{C([0,T],W_e^{1,2}(\nu))}\lesssim\|f\|^2_{C^1([0,T],L^2(\mu))}+\|\phi\|^2_{W_e^{1,2}(\nu)}<+\infty.$$

Let us estimate the function $\mathcal{D}^{\gamma}_{0+,t}u$:

$$\|\mathcal{D}^{\gamma}_{0+,t}u(t,\cdot)\|_{2,\mu}^2=\|\mathcal{D}^{\gamma}_{0+,t}\widehat{u}(t,\cdot)\|_{2,\nu}^2=\int_0^{+\infty}\left|\mathcal{D}^{\gamma}_{0+,t}\widehat{u}(t,\cdot)\right|^2d\nu_{\alpha,\beta}(\lambda)$$

$$=\int_0^{+\infty}\left|\widehat{f}(t,\lambda)-(\lambda^2+\rho^2+m)\widehat{u}(t,\lambda)\right|^2d\nu_{\alpha,\beta}(\lambda)$$

$$\lesssim\int_0^{+\infty}\left|\widehat{f}(t,\lambda)\right|^2d\nu_{\alpha,\beta}(\lambda)+\int_0^{+\infty}\left|(\lambda^2+\rho^2+m)\widehat{u}(t,\lambda)\right|^2d\nu_{\alpha,\beta}(\lambda)$$

$$\lesssim\|\widehat{f}(t,\cdot)\|_{2,\nu}^2+\int_0^{+\infty}\left|(\lambda^2+\rho^2)\widehat{u}(t,\lambda)\right|^2d\nu_{\alpha,\beta}(\lambda).$$

Thus we have

$$\|\mathcal{D}^{\gamma}_{0+,t}u(t,\cdot)\|_{2,\mu}^2\lesssim\|f(t,\cdot)\|_{2,\mu}^2+\|u(t,\cdot)\|^2_{W_e^{1,2}(\nu)}$$

and

$$\|\mathcal{D}^{\gamma}_{0+,t}u\|^2_{C([0,T],L^2(\mu))}$$
$$\lesssim\|f\|^2_{C([0,T],L^2(\mu))}+\|u\|^2_{C([0,T],W_e^{1,2}(\nu))}\lesssim\|f\|^2_{C^1([0,T],L^2(\mu))}+\|\phi\|^2_{W_e^{1,2}(\nu)}<+\infty.$$

It is obvious that $\|u\|^2_{C([0,T],L^2(\mu))}<+\infty$ (Plancherel's identity and Theorem 2). Consequently we get

$$\|u\|^2_{C^{\gamma}([0,T],L^2(\mu))}<+\infty.$$

This ends the proof.

Remark 5. *Now, we show that in the limit case, i.e.* $\gamma = 1$*, Theorem 3 holds. Let* $\gamma = 1$*. Then instead of the problem (7)-(8), we consider a problem*

$$u_t(t,x) - \Delta_{\alpha,\beta}u(t,x) + mu(t,x) = f(t,x), \quad x \in \mathbb{R}^+, \quad 0 < t < T, \tag{14}$$

$$u(0,x) = \phi(x), \quad x \in \mathbb{R}^+, \tag{15}$$

where the functions f *and* ϕ *are given and sufficiently smooth functions. Using the Fourier-Jacobi transform* $\mathcal{F}_{\alpha,\beta}$ *(3) on both sides of the problem (14)-(15), we obtain*

$$\widehat{u}_t(t,\lambda) + (\lambda^2 + \rho^2 + m)\widehat{u}(t,\lambda) = \widehat{f}(t,\lambda), \tag{16}$$

$$\widehat{u}(0,\lambda) = \widehat{\phi}(\lambda), \tag{17}$$

for all $\lambda \in \mathbb{R}$ *and* $0 < t < T$*. If we solve the problem (16)-(17) relative to the variable* t *for every* λ*, we obtain a unique solution given by the expression*

$$\widehat{u}(t,\lambda) = \int_0^t \widehat{f}(\tau,\lambda)e^{-(\lambda^2+\rho^2+m)(t-\tau)}d\tau + \widehat{\phi}(\lambda)e^{-(\lambda^2+\rho^2+m)t}, \tag{18}$$

which can be obtained from (13), when $\gamma = 1$*, taking into account* $\mathbb{E}_{1,1}(t) = e^t$*. Then for this solution (18) all the above inequalities hold. And hence Theorem 3 based on these inequalities holds.*

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Бекболат Б., Тоқмағамбетов Н.Е. ЯКОБИ БӨЛШЕК РЕТТІ ЖЫЛУӨТКІЗГІШТІК ТЕҢДЕУІ ҮШІН КОШИ ЕСЕБІ

Бұл жұмыста біз Якоби бөлшек ретті жылуөткізгіштік теңдеуі үшін Коши есебін қарастырдық. Шешімнің тұрақтылық нәтижелерін және алдын ала бағалауларды Соболев типтес $W_e^{s,p}(\mathbb{R}^+, \nu_{\alpha,\beta})$ кеңістіктерінде алдық.

Кілттік сөздер. Якоби операторы, бөлшек ретті жылуөткізгіштік теңдеуі, Фурье-Якоби түрлендіруі, кері Фурье-Якоби түрлендіруі, Соболев типтес кеңістіктер.

Бекболат Б., Тоқмағамбетов Н.Е. ЗАДАЧА КОШИ ДЛЯ ДРОБНОГО УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ ЯКОБИ

В этой работе мы изучаем задачу Коши для дробного уравнения теплопроводности Якоби. Результаты корректности и априорные оценки получены в пространствах типа Соболева $W_e^{s,p}(\mathbb{R}^+, \nu_{\alpha,\beta})$.

Ключевые слова. Оператор Якоби, дробное уравнение теплопроводности, преобразование Фурье-Якоби, обратное преобразование Фурье-Якоби, пространство типа Соболева.

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