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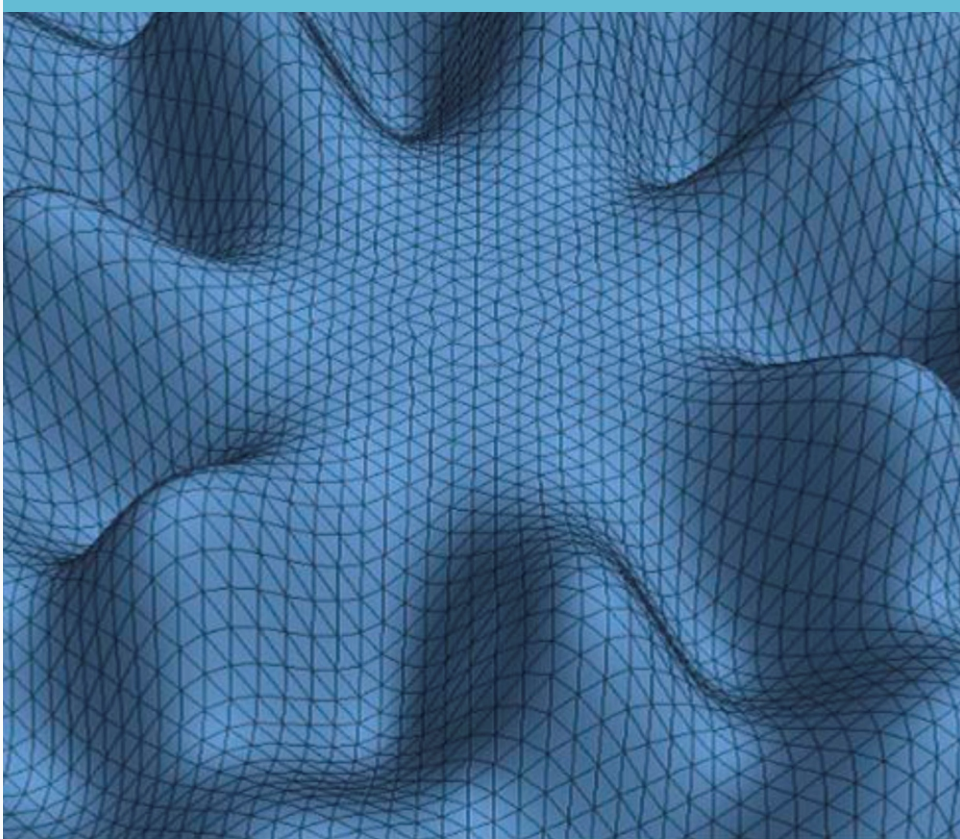

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Propagation of waves along the star-graph

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Abstract. In this paper, we study a mixed problem for a wave equation on a star-graph with finite arc lengths. The initial data is considered to be sufficiently smooth. In this case, the solution is determined by the d'Alembert formula derived in this article. When the star-graph consists of only two arcs, the d'Alembert formula given in the article coincides with the well-known d'Alembert formula for the mixed problem for the wave equation on a finite interval. In the case of multipoint mixed problems for the wave equation on an interval, similar formulas are given in the recent work of N.E. Tokmaganbetov, B.E. Kanguzhin, B. Bekbolat.

Keywords. Eigenvalues, Kirchhoff conditions, star-graph, wave equation, d'Alembert formula.

1 Introduction

It is known [1] that the solution of the Cauchy problem for the wave equation is given by the d'Alembert formula. The physical meaning of the d'Alembert formula corresponds to wave propagation. It is important that solutions to the wave equation can have discontinuities that propagate along the characteristics. Discontinuous solutions of the wave equation for a string and a rod have no physical meaning. However, the same equation is satisfied by the gas pressure in a long narrow pipe. The pressure can be discontinuous. Discontinuous solutions of the wave equation in gas dynamics are called the shock waves.

The d'Alembert method or the method of incident and reflected waves allows solving not only the Cauchy problem for the wave equation, but also finding solutions to mixed problems. In the case of a semi-bounded string, the effect of reflected wave that depends on the form of the boundary condition is observed. In the case of bounded strings, waves are also reflected, but this effect occurs in a more complex scenario. Details of highlighted effects can be found in the book of A.I. Komech [2].

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In [3, 4], the d'Alembert formula was modified for the mixed multipoint problem for the wave equation. In this case, the solution to the mixed multipoint problem is assumed to be sufficiently smooth. An analogue of the d'Alembert formula for the mixed multipoint problem for the wave equation with initial data of discontinuous first derivatives was derived in [5]. In this paper, we state and prove the d'Alembert formula for strings representing a star-graph.

2 Basic concepts and notation

Let m be a fixed natural number. We consider the following mixed problem for the system of wave equations

$$\begin{gathered}
\frac{\partial^2 u_{m+1}(x_{m+1},t)}{\partial t^2}-\frac{\partial^2 u_{m+1}(x_{m+1},t)}{\partial x_{m+1}^2}=0,\ \ 0<x_{m+1}<b_{m+1},\ \ t>0,\\
\frac{\partial^2 u_{m}(x_{m},t)}{\partial t^2}-\frac{\partial^2 u_{m}(x_{m},t)}{\partial x_{m}^2}=0,\ \ 0<x_{m}<b_{m},\ \ t>0,\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\frac{\partial^2 u_{1}(x_{1},t)}{\partial t^2}-\frac{\partial^2 u_{1}(x_{1},t)}{\partial x_{1}^2}=0,\ \ 0<x_{1}<b_{1},\ \ t>0,
\end{gathered}\tag{1}$$

with conditions of the form (a)

$$u_{m+1}(b_{m+1},t)=u_1(0,t)=\ldots=u_m(0,t),\ \ t>0,$$

$$\frac{\partial u_{m+1}(b_{m+1},t)}{\partial x_{m+1}}=\frac{\partial u_1(0,t)}{\partial x_1}+\ldots+\frac{\partial u_m(0,t)}{\partial x_m},\quad t>0,\tag{2}$$

and conditions of the form (b)

$$u_{m+1}(0,t)=0,\ u_1(b_1,t)=0,\ \ldots,\ u_m(b_m,t)=0,\ t>0,\tag{3}$$

and also the initial conditions

$$\begin{gathered}
u_{m+1}(x_{m+1},0)=\varphi_{m+1}(x_{m+1}),\ \ 0<x_{m+1}<b_{m+1},\\
\frac{\partial}{\partial t}u_{m+1}(x_{m+1},0)=\psi_{m+1}(x_{m+1}),\ \ 0<x_{m+1}<b_{m+1},\\
u_m(x_m,0)=\varphi_m(x_m),\quad 0<x_m<b_m,\\
\frac{\partial}{\partial t}u_m(x_m,0)=\psi_m(x_m),\ \ 0<x_m<b_m,\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
u_1(x_1,0)=\varphi_1(x_1),\ \ 0<x_1<b_1,
\end{gathered}\tag{4}$$

$$\frac{\partial}{\partial t}u_1(x_1,0)=\psi_1(x_1),\ \ 0<x_1<b_1.$$

By the results of the works [6–8], the problem (1)–(4) can be interpreted as a mixed problem for the wave equation on a star-graph $\Gamma=\{V,E\}$. Here V represents the set of vertices, numbered from 0 to $(m+1)$, and E is the set of arcs $e_1,\ldots,e_{m+1}$ [7], [8]. Each one of the wave equations (1) holds on each arc. The vertex $(m+1)\in V$ is called the inner vertex of the star-graph. The conditions of the form (a) means that the Kirchhoff laws hold at the inner vertex [9]. The vertices $0,1,\ldots,m$ are called the boundary vertices of the star-graph (Fig. 1). Conditions of type (b) represent a set of boundary conditions. For $m=1$ the problem (1)–(4) coincides with the mixed problem for the wave equation

$$\frac{\partial^2 w}{\partial t^2}-\frac{\partial w}{\partial x^2}=0,\ \ 0<x<b_1+b_2,\ \ t>0, \tag{5}$$

$$w(0,t)=0,\ \ w(b_1+b_2,t)=0,\ \ t>0, \tag{6}$$

$$w(x,0)=\varphi(x),\ \ \frac{\partial w(x,0)}{\partial t}=\psi(x),\ \ 0<x<b_1+b_2. \tag{7}$$

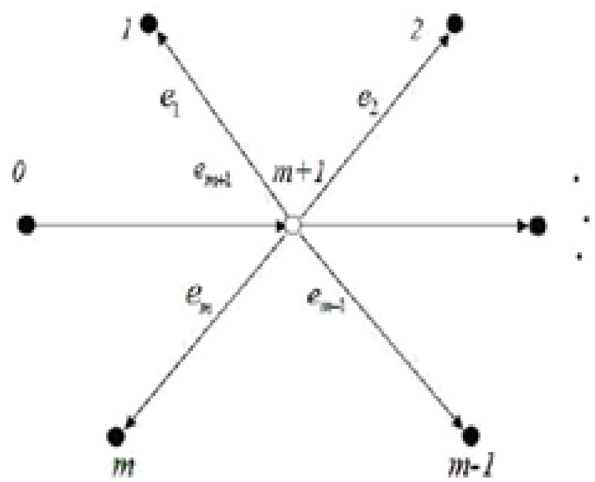

Fig. 1: Star-graph

In this case, the d'Alembert formula

$$w(x,t)=\frac{\widetilde{\varphi}(x+t)+\widetilde{\varphi}(x-t)}{2}+\frac{1}{2}\int\limits_{x-t}^{x+t}\widetilde{\psi}(\xi)d\xi,\ \ 0<x<b_1+b_2,\ t>0, \tag{8}$$

holds, here $\widetilde{\varphi}(x)$ and $\widetilde{\psi}(x)$ are the continuations of the functions $\varphi(x)$ and $\psi(x)$ from the interval $[0,\, b_1+b_2]$ to the entire real axis, which are obtained by the following algorithm:

1. first, we continue in an odd way from the interval $[0,\, b_1+b_2]$ to the interval $[-b_1-b_2,0]$;
2. then we continue periodically from the interval $[-b_1-b_2,b_1+b_2]$ to the entire real axis.

In this paper, an analog of formula (8) is obtained for the mixed problem (1)–(4) on a star-graph.

To do this, we need the dependence of the solution to the system of differential equations on the spectral parameter

$$-y_j''(x_j)=\rho^2 y_j(x_j),\quad 0<x_j<b_j,\quad j=1,2,\ldots,m+1, \tag{9}$$

with conditions of the form (a)

$$y_{m+1}(b_{m+1})=y_1(0)=\ldots=y_m(0),\quad y_{m+1}'(b_{m+1})=y_1'(0)+\ldots+y_m'(0) \tag{10}$$

and conditions of the form (b)

$$y_{m+1}(0)=0,\ y_m(b_m)=0,\ y_{m-1}(b_{m-1})=0,\ \ldots\ y_1(b_1)=0. \tag{11}$$

In the works [6], [7], the required dependence is presented. To formulate the result of [7], we need the following notation. We denote by $C_j(x_j,\lambda)$ and $S_j(x_j,\lambda)$ the solutions to equation (9) with the conditions

$$S_j(0,\lambda)=C_j'(0,\lambda)=0,\quad C_j(0,\lambda)=S_j'(0,\lambda)=1$$

for a fixed j from the set $\{1,2,\ldots,m+1\}$. Really, these functions have explicit representations

$$C_j(x_j,\lambda)=\cos\rho x_j,\ \ S_j(x_j,\lambda)=\frac{\sin\rho x_j}{\rho},\ \ \rho^2=\lambda.$$

The following statement is proved in [7].

Statement 1. *Let $y_{m+1}(x_{m+1},\lambda)=S_{m+1}(x_{m+1},\lambda)$, $x_{m+1}\in e_{m+1}$. Then the solution to system (9), (10), (11) has the following form*

$$y_j(x_j,\lambda)=S_{m+1}(b_{m+1},\lambda)C_j(x_j,\lambda)+B_jS_{m+1}'(b_{m+1},\lambda)S_j(x_j,\lambda),\quad j=1,2,\ldots,m. \tag{12}$$

Moreover, the constants $B_1,B_2,\ldots,B_m$ satisfy the relation

$$B_1+B_2+\ldots+B_m=1. \tag{13}$$

Let us denote by $\{\lambda_n, \quad n \geq 1\}$ the eigenvalues of problem (9),(10),(11). Then, relations (11) imply the relations

$$S_{m+1}(b_{m+1}, \lambda)C_j(b_j, \lambda) + B_j S'_{m+1}(b_{m+1}, \lambda)S_j(b_j, \lambda) = 0, \quad j = 1, 2, \ldots, m, \tag{14}$$

for $\lambda = \lambda_n$. Hence

$$B_j = -\frac{S_{m+1}(b_{m+1}, \lambda_n)C_j(b_j, \lambda_n)}{S'_{m+1}(b_{m+1}, \lambda_n)S_j(b_j, \lambda_n)}, \ j = 1, 2, \ldots, m.$$

Then relation (13) implies the dispersion relation

$$\frac{S'_{m+1}(b_{m+1}, \lambda_n)}{S_{m+1}(b_{m+1}, \lambda_n)} + \sum_{j=1}^{m} \frac{C_j(b_j, \lambda_n)}{S_j(b_j, \lambda_n)} = 0. \tag{15}$$

We denote the left-hand side of (15) by $\Delta(\lambda_n)$. We note that

$$\Delta(\lambda) = \sqrt{\lambda} \sum_{j=1}^{m+1} \cot(b_j \sqrt{\lambda}).$$

Lemma 1. *Zeros of an entire function $\Delta(\lambda)$ are real and simple.*
Proof. The simplicity of the eigenvalues follows from the inequality

$$\Delta'(\lambda_n) = -\frac{1}{2} \sum_{j=1}^{m+1} \frac{1}{\sin^2(b_j \sqrt{\lambda_n})} < 0.$$

Example 1. We calculate the zeros of the function $\Delta(\lambda)$ for $m = 2$, $b_1 = \dfrac{5}{\pi}$, $b_2 = \dfrac{4}{\pi}$, $b_3 = \dfrac{3}{\pi}$ by the graphical method.

We find the first six zeros of the function $\Delta(\lambda)$:

$$\rho_1 = A = 1.23, \ \rho_2 = B = 2.17, \ \rho_3 = C = 3.02, \ \rho_4 = D = 3.7, \ \rho_5 = E = 4.46, \ \rho_6 = F = 5.41$$

from Figure 2.

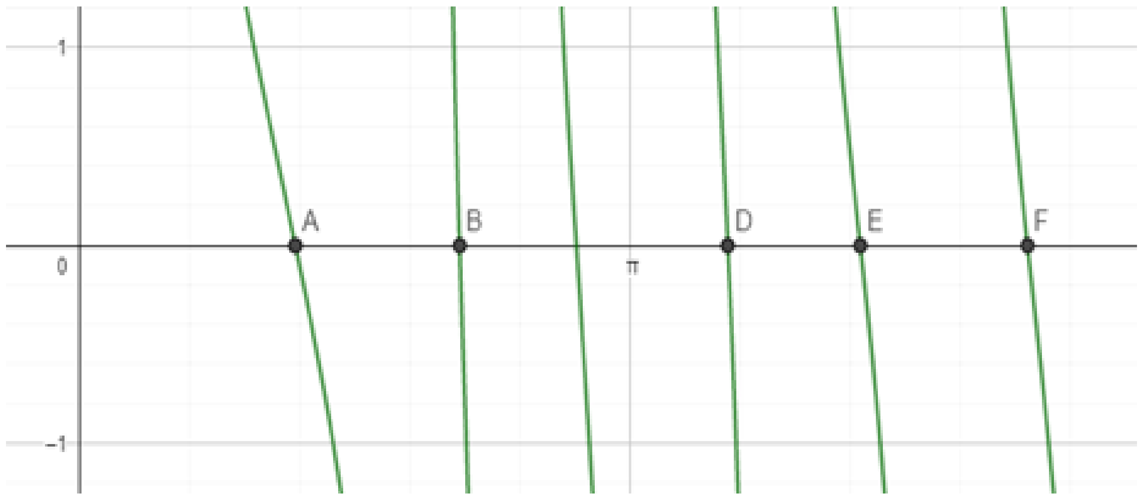

Fig. 2: Eigenvalues of the function $\Delta(\lambda)$.

In this case, the corresponding system of eigenfunctions has the following form

$$\begin{cases} w_1(x_1,\lambda_n) = \dfrac{\sin\sqrt{\lambda_n}(b_1-x_1)}{\sqrt{\lambda_n}} S_2(b_2,\lambda_n)\dots S_m(b_m,\lambda_n)S_{m+1}(b_{m+1},\lambda_n), \\ w_2(x_1,\lambda_n) = S_1(b_1,\lambda_n)\dfrac{\sin\sqrt{\lambda_n}(b_2-x_2)}{\sqrt{\lambda_n}} S_3(b_3,\lambda_n)\dots S_m(b_m,\lambda_n)S_{m+1}(b_{m+1},\lambda_n), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ w_m(x_m,\lambda_n) = S_1(b_1,\lambda_n)S_2(b_2,\lambda_n)S_3(b_3,\lambda_n)\dots \\ \qquad\qquad \dots S_{m-1}(b_{m-1},\lambda_n)\dfrac{\sin\sqrt{\lambda_n}(b_m-x_m)}{\sqrt{\lambda_n}} S_{m+1}(b_{m+1},\lambda_n), \\ w_{m+1}(x_{m+1},\lambda_n) = S_1(b_1,\lambda_n)S_2(b_2,\lambda_n)S_3(b_3,\lambda_n)\dots S_m(b_m,\lambda_n)\dfrac{\sin\sqrt{\lambda_n}\,x_{m+1}}{\sqrt{\lambda_n}}. \end{cases} \tag{16}$$

The work of N.P Bondarenko [10] implies that problem (9), (10), (11) is self-adjoint in $L_2(\Gamma)$. Therefore, system (16) represents an orthogonal basis in space $L_2(\Gamma)$. Recall that in $L_2(\Gamma)$ the inner product is introduced by the rule

$$(y,z) = \sum_{j=1}^{m+1}(y_j,z_j) = \sum_{j=1}^{m+1}\int_0^{b_j} y_j(x_j)\overline{z_j(x_j)}dx_j,\ y,\,z\in L_2(\Gamma).$$

We denote by $\Phi = (\varphi_1(x_1),\varphi_2(x_2),\dots,\varphi_{m+1}(x_{m+1}))$ and

$\Psi = (\psi_1(x_1),\, \psi_2(x_2), \ldots, \psi_{m+1}(x_{m+1}))$ the initial data (4). We denote by

$$W_n = (w_1(x_1, \lambda_n), w_2(x_2, \lambda_n), \ldots, w_{m+1}(x_{m+1}, \lambda_n))$$

the eigenfunction corresponding to the eigenvalue λ_n. Further, we assume that the initial data Φ, Ψ are subject to the matching conditions (2) and (3). Then the functions Φ, Ψ can be expanded into the series with respect to the eigenfunctions

$$\Phi = \sum_n D_n(\Phi) W_n, \;\; \Psi = \sum_n D_n(\Psi) W_n, \tag{17}$$

which converge uniformly in $L_2(\Gamma)$. A similar theorem for differential operators on an interval is proved in the monograph of M.A Naymark [11]. This theorem holds for differential operators defined on graphs. We note that the Fourier coefficients $\{D_n(\Phi)\}$, $\{D_n(\Psi)\}$ are determined by the standard formulas

$$D_n(\Phi) = \frac{(\Phi, W_n)}{(W_n, W_n)}, \qquad D_n(\Psi) = \frac{(\Psi, W_n)}{(W_n, W_n)}.$$

Relations (17) can be rewritten in the coordinate-wise form

$$\begin{cases} \varphi_j(x_j) = \sum\limits_n D_n(\Phi) w_j(x_j, \lambda_n), \\ \psi_j(x_j) = \sum\limits_n D_n(\Psi) w_j(x_j, \lambda_n), \;\; j = 1, 2, \ldots, m+1. \end{cases} \tag{18}$$

Without loss of generality, we derive an analogue of the d'Alembert formula for $\Psi \neq 0$. Since there is a standard technique for obtaining the d'Alembert formula for case $\Psi \neq 0$, if the formula is known at $\Phi \neq 0$ $\Psi \equiv 0$. Let $\Psi \equiv 0$. The solution of the mixed problem on the graph (1)–(4) is sought in the form

$$u_j(x_j t) = \sum_n d_n(t) w_j(x_j, \lambda_n), \;\; 0 < x_j < b_j, \;\; j = 1, \ldots, m+1.$$

Then it is easy to understand that

$$d_n(t) = D_n(\Phi) \cos\sqrt{\lambda_n} \cdot t, \;\; n \geq 1.$$

Thus, the solution can be represented as follows

$$u_j(x_j, t) = \sum_n D_n(\Phi) \cos\sqrt{\lambda_n} t \cdot w_j(x_j, \lambda_n), \;\;\; 0 < x_j < b_j, \;\; t > 0, \;\; j = 1, \ldots, m+1. \tag{19}$$

The next lemma contains one useful property of the product $\cos\sqrt{\lambda_n} t \cdot w_j(x_j, \lambda_n)$ with fixed n and j.

Lemma 2. *The following identity*

$$\cos\sqrt{\lambda_n} t \cdot w_j(x_j, \lambda_n) = \frac{1}{2} w_j(x_j - t, \lambda_n) + \frac{1}{2} w_j(x_j + t, \lambda_n)$$

holds for fixed n and j.

Proof. The identity

$$\cos\sqrt{\lambda_n}t\cdot\frac{\sin\sqrt{\lambda_n}(b_j-x_j)}{\sqrt{\lambda_n}}=\frac{1}{2}\frac{\sin\sqrt{\lambda_n}(b_j-x_j+t)}{\sqrt{\lambda_n}}+\frac{1}{2}\frac{\sin\sqrt{\lambda_n}(b_j-x_j-t)}{\sqrt{\lambda_n}}$$

implies the proof of Lemma 2. The series (18) converge uniformly on the arc e_j for each fixed $j=1,2,\dots,m+1$.

Continuation by rule A: we continue the functions $\varphi_j(x_j)$ and $\psi_j(x_j)$ from the arc e_j for the entire real axis $x_j\in R$, we will define their values by the right-hand sides of relations (18). We denote the obtained corresponding extensions by $\widetilde{\varphi}(x_j)$ and $\widetilde{\psi}_j(x_j)$ for $x_j\in R$.

Then Lemma 2 and formula (19) imply the main statement of this work.

Theorem 1. *Let the initial data* $\Phi=(\varphi_1(x_1),\varphi_2(x_2),\dots,\varphi_{m+1}(x_{m+1}))$ *and* $\Psi=(\psi_1(x_1),\psi_2(x_2),\dots,\ \psi_{m+1}(x_{m+1}))$ *be twice continuously differentiable functions on graph* Γ *and satisfy conditions* (2) *and* (3). *Then the mixed problem* (1)–(4) *on the graph has a unique solution, which can be represented in the form*

$$u_j(x_j,t)=\frac{1}{2}\widetilde{\varphi}_j(x_j+t)+\frac{1}{2}\widetilde{\varphi}_j(x_j+t)+\frac{1}{2}\int\limits_{x_j-t}^{x_j+t}\widetilde{\psi}(\xi)d\xi.\quad j=1,2,\dots,m+1,$$

where $\widetilde{\varphi}_j$ *and* $\widetilde{\psi}_j$ *are continuations of the functions* φ_j *and* ψ_j *from the arc* e_j *to the entire real axis by the rule A.*

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References

[1] Tikhonov A.N., Samarskii A.A. *Equation of mathematical physics,* .: Nauka, 1977 (in Russian).

[2] Komech A.I. *Practical Solution of Equations of Mathematical Physics*, Publishing House of Moscow State University, 1986 (in Russian).

[3] Bekbolat B., Kanguzhin B., Tokmagambetov N. *To the question of a multipoint mixed boundary value problem for a wave equation*, News of the national academy of sciences of the Republic of Kazakhstan. Series Physico-Mathematical, 4: 326 (2019).

[4] Kanguzhin B., Kozhataeva M.Zh. *The d'Alembert formula in the case of point problem*, Doklady NAN RK, 5 (1998), 29-33 (in Russian).

[5] Kanguzhin B., *Propagation of nonsmooth waves under singular perturbations of the wave equation,* Eurasian Math. J., [in print].

[6] Zhapsarbaeva L.K., Kanguzhin B., Koshkarbaev N. *On the asymptotics with respect to the spectral parameter of solutions of differential equations on a tree with Kirchhoff conditions at its interior vertices,* Math. J. 17:4 (2017).

[7] Zhapsarbaeva L.K., Kanguzhin B., Seitova A.A. *Asymptotics of the eigenvalues of the double differentiation operator with Birkhoff-regular boundary conditions on a star graph,* Math. J., 18:2 (2018).

[8] Kanguzhin B., Zhapsarbaeva L., Madibaiuly Zh. *Lagrange formula for differential operators and self-adjoint restrictions of the maximal operator on a tree,* Eurasian Math. J., 2 (2019), 16-19.

[9] Afanaseva N.A., Bulot L.P. *Electrical Engineering and Electronics.* SPb.: SPbGUN I P.T. Uchebnoe posobie, 2010 (in Russian).

[10] Bondarenko N.P. *Discrete mathematical models.* Uchebnoe posobie, Saratov, 2015 (in Russian).

[11] Naymark M.A. *Linear Differential Operators,* M.: Nauka, 1969 (in Russian).

Аймал Раса Г.Х., Кангужин Б.Е., Қайырбек Ж.А. ЖҰЛДЫЗ-ГРАФ БОЙЫНДАҒЫ ТОЛҚЫННЫҢ ТАРАЛУЫ

Бұл мақалада доға ұзындығы шектелген жұлдыз-графтағы аралас толқын теңдеуінің есебін зерттейміз. Бастапқы шарттар жеткілікті жатық болып саналады. Бұл жағдайда шешім мақалада келтірілген Даламбер формуласына сәйкес анықталады. Жұлдыз-граф тек екі доғадан тұрғанда, мақалада келтірілген Даламбер формуласы ақырлы кесіндідегі толқын теңдеуі үшін белгілі Даламбер формуласымен сәйкес келеді. Кесіндідегі толқын теңдеуіне арналған көп нүктелі аралас есептер жағдайына ұқсас формулалар Н.Е. Тоқмағанбетов, Б.Е.Кангужин, Б. Бекболаттың жұмысында келтірілген.

Кілттік сөздер. Меншікті мәндер, Кирхгоф шарттары, жұлдыз-граф, толқын теңдеуі, Даламбер формуласы.

Аймал Раса Г.Х., Кангужин Б.Е., Қайырбек Ж.А. РАСПРОСТРАНЕНИЕ ВОЛН ВДОЛЬ ГРАФА-ЗВЕЗДЫ

В данной статье изучается смешанная задача для волнового уравнения на графе-звезде с конечными длинами дуг. Начальные данные считаются достаточна гладкими. В таком случае решение определяется согласно выведенной в статье формуле Даламбера. Когда граф-звезда состоит только из двух дуг, приведенная в статье формула Даламбера совпадает с известной формулой Даламбера для смешанной задачи для волнового уравнения на конечном отрезке. В случае многоточечных смешанных задач для волнового уравнения на отрезке подобные формулы приведены в недавней работе Н.Е. Токмаганбетова, Б.Е. Кангужина, Б. Бекболата.

Ключевые слова. Собственные значения, условия Кирхгофа, граф-звезда, волновое уравнение, формула Даламбера.

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On Burgers equation with dynamic boundary conditions in angular domain

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Abstract. Earlier, in the works [1] and [2] the correctness of the Dirichlet boundary value problem for the Burgers equation was established. In contrast to these works, in Sobolev spaces and in an angular domain, we show the correctness of the boundary value problem for the Burgers equation with dynamic boundary conditions. The methods of functional analysis, a priori estimates, and Faedo-Galerkin are used.

Keywords. Burgers equation, Sobolev class, degenerating domain, dynamic boundary condition, a priori estimate.

1 Introduction

The study of the Burgers equation has a long history, some of which is given in [1] and [2], as well as in the monographs [3] and [4].

In the works [1] and [2] in Sobolev spaces, the correctness of the boundary value problem for the Burgers equation was established. In this case, the domain of independent variables degenerated according to a nonlinear law, and homogeneous Dirichlet conditions were set on the boundary.

In angular domains, problems of linear thermal conductivity with time derivatives in boundary conditions were studied in [5]. The correctness of the problems under consideration was proved in weighted Hölder classes. Further, these results were developed in [6]–[8].

The infiltration of the wetting front into a porous medium is a classical problem with a free boundary. Historically, the first and best known example is the Green-Ampt model for water flow in soils [9]. There is a huge variety of situations (chemically reacting media,

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deformable media, capillarity effects, mass transfer, mixture flows, media with a complex structure, pollution, reclamation, soil freezing, production of composite materials, brewing, etc.).

As is known, nonlinear Burgers equations and their modifications are also suitable models of fluid motion in porous media [10]–[15].

The range of application of boundary value problems for parabolic equations in a domain with a boundary that changes over time is quite wide. Such problems arise in the study of thermal processes in electrical contacts [16], the processes of ecology and medicine [17], in solving some problems of hydromechanics [18], thermomechanics in thermal shock [19] and so on.

Extensive literature is devoted to the study of the solvability of linear and nonlinear parabolic equations in cylindrical domains. However, as for nonlinear boundary value problems in degenerating non-cylindrical domains, they have been studied relatively little.

For angular domains in Lebesgue classes, there were studied boundary value problems of heat conduction with the homogeneous Dirichlet boundary conditions and established theorems on their solvability by reducing them to the Volterra singular integral equations of the second kind [20], [21].

In [22] there were studied various cases of the nonhomogeneous Dirichlet type boundary conditions. In these cases, it is shown that both unique solvability and non-unique solvability for the corresponding boundary value problems takes place.

In this paper, in Sobolev classes, we study the solvability of the boundary value problem for the Burgers equation in an angular domain with time derivatives in boundary conditions (in a sense, an analogue of Solonnikov-Fasano problem [5] for the Burgers equation). We are considering the case: when one part of the boundary is motionless, and the other part is movable.

In Section 2, we give a statement of the boundary value problem under study. Here also, this problem is reduced to the study of the solvability of two subproblems, and we formulate the main results of the work. We study the questions of unique solvability of two auxiliary boundary value problems for the Burgers equation in rectangular domains, which are used in the proof of the main results of the work. Sections 3–7 are devoted to the first auxiliary problem, in which its correctness in the Sobolev classes is established by the methods of a priori estimates and Faedo-Galerkin. The correctness of the second auxiliary boundary value problem is shown in Section 8. In Sections 9–11, we prove Theorem 1 on the unique solvability of the problem posed in Section 2. A brief conclusion concludes the work.

2 Problem statement and main result

Let $Q_{xt} = \{x, t \,|\, 0 < x < kt,\ 0 < t < T < \infty,\ k > 0\}$ be a domain that degenerates at $t = 0$, and let Ω_t be a section of the domain Q_{xt} for a fixed value of the variable $t \in (0, T)$. In the domain Q_{xt} we consider the following boundary value problem for the Burgers equation:

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = f, \tag{1}$$

$$\frac{d}{dt}u(0,t) + b_0\left[\frac{1}{3}(u)^2 - \nu\partial_x u\right]|_{x=0} = 0, \tag{2}$$

$$\frac{d}{dt}u(kt,t) - b_1\left[\frac{1}{3}(u)^2 - \nu\partial_x u\right]|_{x=kt} = 0, \tag{3}$$

where $\frac{d}{dt}u(\varphi(t),t) = [\partial_t u(x,t) + \varphi'(t)\partial_x u(x,t)]_{x=\varphi(t)}$,

$$f \in L_2(Q_{xt}),\ \nu = \text{const} > 0,\ b_0 = \text{const} > 0,\ b_1 = \text{const} > 0. \tag{4}$$

Remark 1. We believe that the presence of the nonlinear term $u^2/3$ in the boundary conditions (2)–(3) is dictated only by the presence of the convective component in the Burgers equation, which provides a nonlinear "mass" transfer and exchange at the boundary. We proceeded from the fact that in equation (1) the convective and diffusion terms can be written in the form $\partial_x\left(u^2/2 - \nu\partial_x u\right)$.

Problem 1. Under conditions (4), establish the solvability of boundary value problem (1)–(3).

Theorem 1 (Main result). *Let $f \in L_2(Q_{xt})$ (4). Then boundary value problem (1)–(3) has a unique solution*

$$u \in H^{2,1}(Q_{xt}) \equiv \left\{L_2(0,T;H^2(0,kt)) \cap H^1(0,T;L_2(0,kt))\right\},$$
$$u(kt,t),\ u(0,t) \in H^1(0,T).$$

The proof of Theorem 1 is given below.

3 The first auxiliary initial boundary value problem

In the domain $Q_{yt} = \{y,t|\ y \in (0,1),\ t \in (0,T)\}$ we consider the following auxiliary initial boundary value problem:

$$\partial_t w + \alpha(t)w\partial_y w - \beta(t)\partial_y^2 w + \gamma(y,t)\partial_y w = g, \tag{5}$$

$$\frac{d}{dt}w(0,t) + \frac{b_0}{\alpha(t)}\left[\frac{\alpha(t)}{3}w^2 - \beta(t)\partial_y w\right]|_{y=0} = 0, \tag{6}$$

$$\frac{d}{dt}w(1,t) - \frac{b_1}{\alpha(t)}\left[\frac{\alpha(t)}{3}w^2 - \beta(t)\partial_y w\right]|_{y=1} = 0,\ \ 0 < t < T, \tag{7}$$

$$w(y,0)=0,\ \ 0<y<1, \tag{8}$$

where b_0, b_1 are given positive constants, and the given continuous functions $\alpha(t)$, $\beta(t)$, $\gamma(y,t)$ satisfy conditions

$$\alpha'(t)\le 0,\ \alpha_1\le\alpha(t)\le\alpha_2,\ \beta_1\le\beta(t)\le\beta_2,\ |\gamma(y,t)|\le\gamma_1,\ |\partial_y\gamma(y,t)|\le\gamma_1,\ \forall\, t\in[0,T], \tag{9}$$

with the given positive constants $\alpha_i,\ \beta_i,\ i=1,2,\ \gamma_1,\ \alpha(t)\in C^1([0,T]),\ \partial_y\gamma(y,t)\in C(\bar{Q}_{yt})$.

Theorem 2. *Let $g\in L_2(Q_{yt})$ and conditions (9) be satisfied. Then boundary value problem (5)–(8) has a unique solution*

$$w\in H^{2,1}(Q_{yt})\equiv L_2(0,T;H^2(0,1))\cap H^1(0,T;L_2(0,1)),\ \ w(1,t),\ w(0,T)\in H^1(0,T).$$

To apply the Faedo-Galerkin method, we need to solve the following spectral problem:

$$-Y''(y)=\lambda^2Y(y),\ y\in(0,1), \tag{10}$$

$$Y'(0)+\lambda^2Y(0)=0, \tag{11}$$

$$Y'(1)-\lambda^2Y(1)=0, \tag{12}$$

obtained by applying the variable separation method ($u(y,t)=F(t)Y(y)$) from the following problem

$$\partial_t u-\partial_y^2u=0,\ y\in(0,1),\ t\in(0,T),$$
$$\partial_t u-\partial_x u\big|_{y=0}=0,\ \ \partial_t u+\partial_x u\big|_{y=1}=0,$$
$$u(y,0)=u_0(y).$$

4 Solving spectral problem (10)–(12)

We seek the general solution to equation (10) in the form

$$Y(y)=C_1\exp\{i\lambda y\}+C_2\exp\{-i\lambda y\},\ i=\sqrt{-1}. \tag{13}$$

Satisfying (13) to boundary conditions (11)–(12), we obtain

$$Y_{01}(y)=1,\ \lambda_{01}=0,\ \tan\frac{\lambda_{01}}{2}=-\lambda_{01}, \tag{14}$$

$$Y_{2n-1}(y)=\cos\frac{\lambda_{2n-1}(1-2y)}{2},\ \lambda_{2n-1}=(2n-1)\pi+\varepsilon_{2n-1},\ \tan\frac{\lambda_{2n-1}}{2}=-\lambda_{2n-1},\ n\in\mathbb{N}, \tag{15}$$

$$Y_{02}(y) = \sin\frac{\lambda_{02}(1-2y)}{2},\ \lambda_{02} \approx \frac{2\pi}{5},\ \cot\frac{\lambda_{02}}{2} = \lambda_{02}, \tag{16}$$

$$Y_{2n}(y) = \sin\frac{\lambda_{2n}(1-2y)}{2},\ \lambda_{2n} = 2n\pi + \varepsilon_{2n},\ \cot\frac{\lambda_{2n}}{2} = \lambda_{2n},\ n \in \mathbb{N}. \tag{17}$$

It is easy to see that the solutions of equations

$$\tan\frac{\lambda_{2n-1}}{2} = -\lambda_{2n-1},\ n \in \mathbb{N}, \tag{18}$$

and

$$\cot\frac{\lambda_{2n}}{2} = \lambda_{2n},\ n \in \mathbb{N}, \tag{19}$$

are, respectively, close to points $(2n-1)\pi$ and $2n\pi,\ n \in \mathbb{N}$, and with the growth of n they approach arbitrarily close from the right to the corresponding specified points $(2n-1)\pi$ and $2n\pi,\ n \in \mathbb{N}$, i.e. $\varepsilon_n \to 0+$ at $n \to \infty$. If we introduce the notation $2x = (1-2y)\pi$, then we get: $x \in (-\pi/2, \pi/2)$.

By the Paley-Wiener theorem ([24], chapter V, 86, example), the system of functions (15) and (17) is complete in $L_2(0,1)$, since the system of functions:

$$\frac{\sqrt{2}\cos x}{\sqrt{\pi}},\ \frac{\sqrt{2}\sin 2x}{\sqrt{\pi}},\ \frac{\sqrt{2}\cos 3x}{\sqrt{\pi}},\ \frac{\sqrt{2}\sin 4x}{\sqrt{\pi}}, ..., \tag{20}$$

which is complete in $L_2(-\pi/2, \pi/2)$, will differ little from it. For the latter system, it is sufficient to make the replacement $x_1 = x + \pi/2$. We get the system of sines:

$$\frac{\sqrt{2}\sin x_1}{\sqrt{\pi}},\ \frac{\sqrt{2}\sin 2x_1}{\sqrt{\pi}},\ \frac{\sqrt{2}\sin 3x_1}{\sqrt{\pi}},\ \frac{\sqrt{2}\sin 4x_1}{\sqrt{\pi}}, ...,$$

which is complete in $L_2(0,\pi)$.

Note that the system of functions (15) and (17) is not orthogonal in $L_2(0,1)$.

Remark 2. The applicability of the Paley-Wiener theorem ([24], chapter V, 86, example) follows from the relations:

$$\lambda_1 \approx 3.673,\ \ \lambda_1 - \pi \approx 0.533,\ \ M\pi = |\lambda_1 - \pi| < 0.54 < \ln 2 \approx 0.693,\ \ \theta = \exp\{M\pi\} - 1 < 1.$$

5 Setting and solving the approximate problem

We multiply equation (5) scalarly in $L_2(0,1)$ by function $v \in H^1(0,1)$. As a result, taking into account initial (8) and boundary conditions (6)–(7) we will have a weak statement of problem (5)–(8):

$$\int\limits_0^1 \partial_t w v dy + \alpha(t)\int\limits_0^1 w\partial_y w v dy + \beta(t)\int\limits_0^1 \partial_y w \partial_y v dy \int\limits_0^1 \gamma(y,t)\partial_y w v dy +$$

$$+\frac{\alpha(t)}{b_1}\frac{d}{dt}w(1,t)v(1,t)-\frac{\alpha(t)}{3}w^2(1,t)v(1,t)+\frac{\alpha(t)}{b_0}\frac{d}{dt}w(0,t)v(0,t)$$

$$+\frac{\alpha(t)}{3}w^2(0,t)v(0,t)=\int\limits_0^1 gvdy,\ \ \forall\, v\in H^1(0,1), \tag{21}$$

$$w(y,0)=0,\ \ y\in(0,1). \tag{22}$$

We introduce the following approximate solution

$$w_n(y,t)=\sum_{j=1}^n c_j(t)Y_j(y),\ \ w_n(y,0)=\sum_{j=1}^n c_j(0)Y_j(y). \tag{23}$$

Next, we will satisfy this solution to an approximate version of problem (21)–(22):

$$\int\limits_0^1 \partial_t w_n Y_j dy+\alpha(t)\int\limits_0^1 w_n\partial_y w_n Y_j dy+\beta(t)\int\limits_0^1 \partial_y w_n\partial_y Y_j dy+\int\limits_0^1 \gamma(y,t)\partial_y w_n Y_j dy$$

$$+\frac{\alpha(t)}{b_1}\frac{d}{dt}w_n(1,t)Y_j(1)-\frac{\alpha(t)}{3}w_n^2(1,t)Y_j(1)+\frac{\alpha(t)}{b_0}\frac{d}{dt}w_n(0,t)Y_j(0)$$

$$+\frac{\alpha(t)}{3}w_n^2(0,t)Y_j(0)=\int\limits_0^1 gY_j dy, \tag{24}$$

$$w_n(y,0)=0,\ \ y\in(0,1), \tag{25}$$

for all $j=1,...,n$, and $t\in[0,T]$.

Lemma 1. *Problem (24)–(25) has a unique solution $w_n(y,t)$.*

Proof. Since the system of functions $Y_1(y),Y_2(y),...$ is a basis in $L_2(0,1)$, we have

$$\det\{W_n\}=\left\|\left(\{Y_k(y),\bar Y_k(0),\bar Y_k(1)\},\{Y_j(y),\bar Y_j(0),\bar Y_j(1)\}\right)\right\|_{k,j=1}^n\neq 0,\ \forall\ \text{finite}\ n;$$

W_n is a Gram matrix, $\bar Y_k(0)=\frac{\sqrt{\alpha(t)}}{\sqrt{b_0}}Y_k(0),\ \ \bar Y_k(1)=\frac{\sqrt{\alpha(t)}}{\sqrt{b_1}}Y_k(1),\ k=1,...,n,$

$$\left(\{Y_k(y),\bar Y_k(0),\bar Y_k(1)\},\{Y_j(y),\bar Y_k(0),\bar Y_k(1)\}\right)=(Y_k(y),Y_j(y))+\left(\bar Y_k(0)\bar Y_j(0)\right)+\left(\bar Y_k(1)\bar Y_j(1)\right),$$

$(\cdot,\cdot)$ is the scalar product in $L_2(0,1)$, $A_n=(\partial_y Y_k(y),\partial_y Y_j(y))_{k,j=1}^n$,

$$w_n^2(1,t)Y_j(1,t)-w_n^2(0,t)Y_j(0,t)=[\sum_{k=1}^n c_k(t)Y_k(1)]^2Y_j(1)-[\sum_{k=1}^n c_k(t)Y_k(0)]^2Y_j(0).$$

Further, if we introduce the notation

$$G_n(t) = \{g_1(t), ..., g_n(t)\},\ P_n(t) = \{p_1(t), ..., p_n(t)\},\ H_n(t) = \{h_1(t), ..., h_n(t)\},$$

$$C_n(t) = \{c_1(t), ..., c_n(t)\},$$

where

$$g_j(t) = \int_0^1 gY_j(y)dy,\ \ p_j(t) = -\alpha(t)\int_0^1 w_n\partial_y w_n Y_j(y)dy - \int_0^1 \gamma(y,t)\partial_y w_n(y,t)Y_j(y)dy,$$

$$h_j(t) = \frac{\alpha(t)}{3}[\sum_{k=1}^n c_k(t)Y_k(1)]^2Y_j(1) - [\sum_{k=1}^n c_k(t)Y_k(0)]^2Y_j(0),$$

for all $j = 1, ..., n$, then problem (24)–(25) is equivalent to the following Cauchy problem for a finite system of nonlinear ordinary differential equations

$$C_n'(t) = W_n^{-1}\left[-\beta(t)A_nC_n(t) + P_n(t) + H_n(t) + G_n(t)\right],\ \ C_n(0) = 0. \tag{26}$$

Note that the functions $p_j(t),\ h_j(t)$ are well defined, and the function $g_j(t)$ is square integrable (by virtue of $g \in L_2(Q_{yt})$). Therefore, the Cauchy problem (26) is uniquely solvable on some interval $[0, T']$, where $T' \leq T$. However, according to the a priori estimates established below, we find that this solution $C_n(t)$ continues to a finite time T.

Thus, we find functions $C_n(t) = \{c_j(t),\ j = 1, ..., n\}$ as a solution to the Cauchy problem (26) for each fixed finite n, and together with them the only approximate solution $w_n(y,t)$ to problem (24)–(25). Lemma 1 is completely proved.

6 A priori estimates

Lemma 2. *There exists a positive constant K_1 independent of n, such that for all $t \in [0, T]$ the following estimate takes place*

$$\|w_n(y,t)\|^2_{L_2(0,1)} + \frac{\alpha_1}{b_1}|w_n(1,t)|^2 + \frac{\alpha_1}{b_0}|w_n(0,t)|^2 + \beta_1\int_0^t \|\partial_y w_n(y,\tau)\|^2_{L_2(0,1)}d\tau \leq K_1. \tag{27}$$

Proof. Multiplying (24) by $c_j(t)$, summing the result over j from 1 to n and using the equality

$$\int_0^1 w_n(y,t)\partial_y w_n(y,t)w_n(y,t)dy = \frac{1}{3}w_n^3(1,t) - \frac{1}{3}w_n^3(0,t),$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\int\limits_0^1 |w_n(y,t)|^2 dy + \beta(t)\int\limits_0^1 |\partial_y w_n(y,t)|^2 dy + \frac{\alpha(t)}{2b_1}\frac{d}{dt}|w_n(1,t)|^2 + \frac{\alpha(t)}{2b_0}\frac{d}{dt}|w_n(0,t)|^2$$

$$= -\int\limits_0^1 \gamma(y,t)\partial_y w_n(y,t) w_n(y,t) dy + \int\limits_0^1 g(y,t) w_n(y,t) dy. \tag{28}$$

First, note that due to property $\alpha'(t) \leq 0$ inequalities

$$\int\limits_0^t \frac{\alpha(t)}{2b_1}\frac{d}{dt}|w_n(1,t)|^2\, dt \geq \frac{\alpha_1}{2b_1}|w_n(1,t)|^2, \quad \int\limits_0^t \frac{\alpha(t)}{2b_0}\frac{d}{dt}|w_n(0,t)|^2\, dt \geq \frac{\alpha_1}{2b_0}|w_n(0,t)|^2$$

hold, which are obtained by integrating the left sides of the inequalities by parts.

Now, by integrating (28) with respect to t from 0 to t and using Cauchy inequality

$$-\int\limits_0^1 \gamma(y,t)\partial_y w_n(y,t) w_n(y,t) dy \leq \frac{\beta_1}{2}\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + \frac{\gamma_1^2}{2\beta_1}\|w_n(y,t)\|^2_{L_2(0,1)},$$

$$\int\limits_0^1 g(y,t) w_n(y,t) dy \leq \frac{1}{2}\|g(y,t)\|^2_{L_2(0,1)} + \frac{1}{2}\|w_n(y,t)\|^2_{L_2(0,1)},$$

we get

$$\|w_n(y,t)\|^2_{L_2(0,1)} + \frac{\alpha_1}{b_1}|w_n(1,t)|^2 + \frac{\alpha_1}{b_0}|w_n(0,t)|^2 + \beta_1\int\limits_0^t \|\partial_y w_n(y,\tau)\|^2_{L_1(0,1)} d\tau$$

$$\leq \left(\frac{\gamma_1^2}{\beta_1} + 1\right)\int\limits_0^t \|w_n(y,\tau)\|^2_{L_2(0,1)} d\tau + \int\limits_0^T \|g(y,\tau)\|^2_{L_2(0,1)} d\tau. \tag{29}$$

From (29) follows

$$\|w_n(y,t)\|^2_{L_2(0,1)} \leq \left(\frac{\gamma_1^2}{\beta_1} + 1\right)\int\limits_0^t \|w_n(y,\tau)\|^2_{L_2(0,1)} d\tau + \int\limits_0^T \|g(y,\tau)\|^2_{L_2(0,1)} d\tau.$$

By applying the Gronwall's inequality, we obtain the estimate for $\|w_n(y,t)\|^2_{L_2(0,1)}$. By using this estimate in (29), we establish the required estimate for Lemma 2.

Lemma 3. *For a positive constant K_2 independent of n, for all $t \in (0,T]$ the following inequality takes place:*

$$\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + \frac{\alpha_1}{b_1\beta_1}\int\limits_0^t |\frac{d}{d\tau}w_n(1,\tau)|^2 d\tau + \frac{\alpha_1}{b_0\beta_1}\int\limits_0^t |\frac{d}{d\tau}w_n(0,\tau)|^2 d\tau$$

$$+\beta_1 \int\limits_0^t \|\partial_y^2 w_n(y,\tau)\|^2_{L_2(0,1)} d\tau \le K_2. \tag{30}$$

Proof. Taking into account equality

$$\sum_{j=1}^n c_j\lambda_j^2 Y_j(y) = -\sum_{j=1}^n c_j \partial_y^2 Y_j(y) = -\partial_y^2 w_n(y,t),$$

which follows from (10) and (23), and multiplying equality (24) by $c_j\lambda_j^2$ and summing over j from 1 to n, we obtain

$$\frac{1}{2}\frac{d}{d\,t}\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + \beta(t)\|\partial_y^2 w_n(y,t)\|^2_{L_2(0,1)}$$

$$= \alpha(t)\Big(w_n(y,t)\partial_y w_n(y,t), \partial_y^2 w_n(y,t)\Big) + \Big(\gamma(y,t)\partial_y w_n(y,t), \partial_y^2 w_n(y,t)\Big)$$

$$-\big(g(y,t), \partial_y^2 w_n(y,t)\big) + \frac{d}{dt}w_n(y,t)\partial_y w_n(y,t)\Big|_{y=0}^{y=1}$$

$$\le \alpha_2\Big|\Big(w_n(y,t)\partial_y w_n(y,t), \partial_y^2 w_n(y,t)\Big)\Big| + \gamma_1\big|\big(\partial_y w_n(y,t), \partial_y^2 w_n(y,t)\big)\big|$$

$$+\big|\big(g(y,t), \partial_y^2 w_n(y,t)\big)\big| - \frac{\alpha_1}{b_1\beta(t)}|\frac{d}{dt}w_n(1,t)|^2 + \frac{\alpha(t)}{3\beta(t)}|w_n(1,t)|^2|\frac{d}{dt}w_n(1,t)|$$

$$-\frac{\alpha_1}{b_0\beta(t)}|\frac{d}{dt}w_n(0,t)|^2 + \frac{\alpha(t)}{3\beta(t)}|w_n(0,t)|^2|\frac{d}{dt}w_n(0,t)|,$$

or

$$\frac{1}{2}\frac{d}{d\,t}\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + \frac{\alpha_1}{b_1\beta_2}|\frac{d}{dt}w_n(1,t)|^2 + \frac{\alpha_1}{b_0\beta_2}|\frac{d}{dt}w_n(0,t)|^2$$

$$+\beta_1\|\partial_y^2 w_n(y,t)\|^2_{L_2(0,1)} \le \alpha_2\Big|\Big(w_n(y,t)\partial_y w_n(y,t), \partial_y^2 w_n(y,t)\Big)\Big|$$

$$+\frac{\alpha_2}{3\beta_1}|w_n(1,t)|^2|\frac{d}{dt}w_n(1,t)| + \frac{\alpha_2}{3\beta_1}|w_n(0,t)|^2|\frac{d}{dt}w_n(0,t)|$$

$$+\gamma_1\big|\big(\partial_y w_n(y,t), \partial_y^2 w_n(y,t)\big)\big| + \big|\big(g(y,t), \partial_y^2 w_n(y,t)\big)\big|. \tag{31}$$

First, we consider the estimates of the nonlinear terms from (31). First of all, we have

$$\Big|\Big(w_n(y,t)\partial_y w_n(y,t), \partial_y^2 w_n(y,t)\Big)\Big| \le \|w_n(y,t)\|_{L_4(0,1)}\|\partial_y w_n(y,t)\|_{H^1(0,1)}\|\partial_y w_n(y,t)\|_{L_4(0,1)}$$

$$\le \|w_n(y,t)\|_{L_4(0,1)}\|\partial_y w_n(y,t)\|_{H^1(0,1)}\|\partial_y w_n(y,t)\|_{L_\infty(0,1)}. \tag{32}$$

Further, taking into account the interpolation inequality from ([25], Theorems 5.8–5.9, p.140–141)

$$\alpha_2\|\partial_y w_n(y,t)\|_{L_4(0,1)} \le C\|\partial_y w_n(y,t)\|_{H^1(0,1)}^{1/2}\|\partial_y w_n(y,t)\|_{L_2(0,1)}^{1/2}, \ \forall\, \partial_y w_n(y,t) \in H^1(0,1),$$

from (32) we obtain

$$\alpha_2\Big|\Big(w_n(y,t)\partial_y w_n(y,t), \partial_y^2 w_n(y,t)\Big)\Big|$$

$$\le C\|w_n(y,t)\|_{L_4(0,1)}\|\partial_y w_n(y,t)\|_{H^1(0,1)}^{3/2}\|\partial_y w_n(y,t)\|_{L_2(0,1)}^{1/2}$$

$$\le \frac{\beta_1}{8}\|\partial_y^2 w_n(y,t)\|_{L_2(0,1)}^2 + \left[\frac{\beta_1}{8} + C_2\|w_n(y,t)\|_{L_4(0,1)}^4\right]\|\partial_y w_n(y,t)\|_{L_2(0,1)}^2. \tag{33}$$

Here we have used Young's inequality $(p^{-1}+q^{-1}=1):$

$$|AB| = \left|\left(a^{1/p}A\right)\left(a^{1/q}\frac{B}{a}\right)\right| \le \frac{a}{p}\,|A|^p + \frac{a}{qa^q}\,|B|^q\,, \tag{34}$$

where

$$A = \|\partial_y w_n(y,t)\|_{H^1(0,1)}^{3/2}\,,\ \ B = C\,\|w_n(y,t)\|_{L_4(0,1)}\,\|\partial_y w_n(y,t)\|_{L_2(0,1)}^{1/2}\,,\ \ a = \frac{\beta_1}{6},\ \ p = \frac{4}{3},\ \ q = 4.$$

Note that for two nonlinear terms on the right-hand side of (31) the following estimates hold:

$$\frac{\alpha_2}{3\beta_1}|w_n(1,t)|^2|\frac{d}{dt}w_n(1,t)| \le \frac{b_1\alpha_2^2}{18\beta_1\alpha_1}|w_n(1,t)|^4 + \frac{\alpha_1}{2b_1\beta_1}|\frac{d}{dt}w_n(1,t)|^2$$

$$\le K_1^2\frac{b_1\alpha_2^2}{18\beta_1\alpha_1} + \frac{\alpha_1}{2b_1\beta_1}|\frac{d}{dt}w_n(1,t)|^2, \tag{35}$$

$$\frac{\alpha_2}{3\beta_1}|w_n(0,t)|^2|\frac{d}{dt}w_n(0,t)| \le \frac{b_0\alpha_2^2}{18\beta_1\alpha_1}|w_n(0,t)|^4 + \frac{\alpha_1}{2b_0\beta_1}|\frac{d}{dt}w_n(0,t)|^2$$

$$\le K_1^2\frac{b_0\alpha_2^2}{18\beta_1\alpha_1} + \frac{\alpha_1}{2b_0\beta_1}|\frac{d}{dt}w_n(0,t)|^2. \tag{36}$$

In inequalities (35)–(36) estimate (27) from Lemma 2 is used.

Further, for the last two terms from (31) we will have:

$$\gamma_1 \left| \Big(\partial_y w_n(y,t), \partial_y^2 w_n(y,t) \Big) \right| \le \frac{\beta_1}{8} \|\partial_y^2 w_n(y,t)\|^2_{L_2(0,1)} + C_3 \|\partial_y w_n(y,t)\|^2_{L_2(0,1)}, \qquad (37)$$

$$\left| \big(g(y,t), \partial_y^2 w_n(y,t) \big) \right| \le \frac{\beta_1}{4} \|\partial_y^2 w_n(y,t)\|^2_{L_2(0,1)} + C_4 \|g(y,t)\|^2_{L_2(0,1)}. \qquad (38)$$

From (31), (33)–(38) we obtain

$$\frac{d}{dt} \|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + \frac{\alpha_1}{b_1\beta_1} \Big|\frac{d}{dt} w_n(1,t)\Big|^2 + \frac{\alpha_1}{b_0\beta_1} \Big|\frac{d}{dt} w_n(0,t)\Big|^2 + \beta_1 \|\partial_y^2 w_n(y,t)\|^2_{L_2(0,1)}$$

$$\le 2C_4 \|g(y,t)\|^2_{L_2(0,1)} + \left[\frac{\beta_1}{4} + 2C_2 \|w_n(y,t)\|^4_{L_4(0,1)} + 2C_3 \right] \|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + K_0, \qquad (39)$$

where

$$K_0 = K_1^2 \frac{b_1 \alpha_2^2}{9\beta_1\alpha_1} + K_1^2 \frac{b_1 \alpha_2^2}{9\beta_1\alpha_1}, \ \ K_1 \text{ is the constant from Lemma 2,}$$

or, by integrating (39) with respect to t from 0 to t, we will have

$$\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + \frac{\alpha_1}{b_1\beta_1} \int_0^t \Big|\frac{d}{d\tau} w_n(1,\tau)\Big|^2 d\tau + \frac{\alpha_1}{b_0\beta_1} \int_0^t \Big|\frac{d}{d\tau} w_n(0,\tau)\Big|^2 d\tau$$

$$+\beta_1 \int_0^t \|\partial_y^2 w_n(y,\tau)\|^2_{L_2(0,1)} d\tau \le A_4 \|g(y,t)\|^2_{L_2(Q)} + \int_0^t A_5(\tau) \|\partial_y w_n(y,\tau)\|^2_{L_2(0,1)} d\tau + K_0 T, \qquad (40)$$

where

$$A_4 = 2C_4, \ \ A_5(t) = \frac{\beta_1}{4} + 2C_2 \|w_n(y,t)\|^4_{L_4(0,1)} + 2C_3.$$

From inequality (40) in the same way as in the proof of Lemma 2 we obtain the desired estimate (30). Lemma 3 is completely proved.

Lemma 4. *For positive constants K_3, K_4 and K_5 independent of n, for all $t \in (0,T]$ the following inequalities hold:*

$$\|\partial_t w_n(y,t)\|^2_{L_2(Q_{yt})} \le K_3, \qquad (41)$$

$$\Big\|\frac{d}{dt} w_n(1,t)\Big\|^2_{L_2(0,T)} \le K_4, \qquad (42)$$

$$\Big\|\frac{d}{dt} w_n(0,t)\Big\|^2_{L_2(0,T)} \le K_5. \qquad (43)$$

Proof. Let us write down initial boundary value problem (5)–(8) for the approximate solution $w_n(y,t)$:

$$\partial_t w_n + \alpha(t) w_n \partial_y w - \beta(t)\partial_y^2 w_n + \gamma(y,t)\partial_y w_n = g, \tag{44}$$

$$\frac{d}{dt}w_n(0,t) + \frac{b_0}{\alpha(t)}\left[\frac{\alpha(t)}{3}w_n^2 - \beta(t)\partial_y w_n\right]\Big|_{y=0} = 0, \tag{45}$$

$$\frac{d}{dt}w_n(1,t) - \frac{b_1}{\alpha(t)}\left[\frac{\alpha(t)}{3}w_n^2 - \beta(t)\partial_y w_n\right]\Big|_{y=1} = 0,\ \ 0<t<T, \tag{46}$$

$$w_n(y,0) = 0,\ \ 0<y<1. \tag{47}$$

From the equation and boundary conditions (44)–(47) respectively, we obtain

$$\|\partial_t w_n\|_{L_2(Q_{yt})} \le \alpha_2^2 \|w_n\partial_y w_n\|_{L_2(Q_{yt})} + \beta_2\|\partial_y^2 w_n\|_{L_2(Q_{yt})} + \gamma_1\|\partial_y w_n\|_{L_2(Q_{yt})} + \|g\|_{L_2(Q_{yt})}, \tag{48}$$

$$\|\frac{d}{dt}w_n\|_{L_2(0,T)} \le \frac{b_0\alpha_2}{3\alpha_1}\|w_n(0,t)\|^{1/2}_{L_4(0,T)} + \frac{b_0\beta_2}{\alpha_1}\|\partial_y w_n(0,t)\|_{L_2(0,T)}, \tag{49}$$

$$\|\frac{d}{dt}w_n\|_{L_2(0,T)} \le \frac{b_1\alpha_2}{3\alpha_1}\|w_n(1,t)\|^{1/2}_{L_4(0,T)} + \frac{b_1\beta_2}{\alpha_1}\|\partial_y w_n(1,t)\|_{L_2(0,T)}. \tag{50}$$

According to embedding $H^1(0,1) \hookrightarrow L_\infty(0,1)$ inequality $\|w_n\|_{L_\infty(0,1)} \le C\|w_n\|_{H^1(0,1)}$ holds. Hence, taking into account Lemmas 2 and 3, we obtain

$$\|w_n\partial_y w_n\|_{L_2(Q_{yt})} \le C\|w_n\|_{L_\infty(0,T;H^1(0,1))}\|\partial_y w_n\|_{L_2(Q_{yt})}. \tag{51}$$

Estimate (41) follows from (48), (51) and from the statements of Lemmas 2 and 3. Estimates (42)–(43) follow, respectively, from (49)–(50) and the statements of Lemmas 2 and 3. Lemma 4 is completely proved.

7 Unique solvability of the first auxiliary problem (5)–(8)

Lemmas 2–4 show that the sequences of Galerkin approximations

$$\{w_n(y,t),\ w_n(1,t),\ w_n(0,t), n = 1,2,3,...\}$$

are bounded in the direct product of spaces

$$L_\infty(0,T;H^1(0,1)) \cap L_2(0,T;H^2(0,1)) \times L_\infty(0,T) \times L_\infty(0,T),$$

and the sequences

$$\{\partial_t w_n(y,t),\ \frac{d}{dt}w_n(1,t),\ \frac{d}{dt}w_n(0,y),\ n = 1,2,3,...\}$$

are bounded in

$$L_2(0,T;L_2(0,1)) \times L_2(0,T) \times L_2(0,T),$$

respectively.

Thus, we can extract weakly convergent subsequences (we preserve the notation of the index n for the subsequences):

$$w_n(y,t) \to w(y,t) \;\text{ weakly in }\; L_2(0,T;H^2(0,1)) \cap H^1(0,T;L_2(0,1)), \tag{52}$$

$$w_n(y,t) \to w(y,t) \;\text{ strongly in }\; L_2(0,T;L_2(0,1)) \;\text{ and almost everywhere in }\; Q_{yt}, \tag{53}$$

$$\{w_n(1,t),\, w_n(0,t)\} \to \{w(1,t),\, w(0,t)\} \;\text{ weakly }\; H^1(0,T) \times H^1(0,T), \tag{54}$$

$$\{w_n(1,t),\, w_n(0,t)\} \to \{w(1,t),\, w(0,t)\} \;\text{ strongly in }\; L_2(0,T) \times L_2(0,T). \tag{55}$$

Lemma 5. *Let conditions (9) be satisfied and $g \in L_2(Q_{yt})$. Then initial boundary value problem (5)–(8) has a weak solution in space $H^{2,1}(Q_{yt})$.*

Proof. Let $\varphi(t) \in \mathcal{D}((0,T))$, i.e. from the class of infinitely differentiable finite functions. We introduce the notation $v_j(y,t) = \varphi(t)Y_j(y)$, where $Y_j(y) \in H^1(0,1)$. Now, multiplying integral identity (24) by the function $\varphi(t) \in \mathcal{D}((0,T))$ and integrating the result obtained with respect to t from 0 to T, we obtain

$$\int_0^T\int_0^1 \left[\partial_t w_n + \alpha(t) w_n \partial_y w_n - \beta(t)\partial_y^2 w_n + \gamma(y,t)\partial_y w_n\right] v_j \, dy\, dt$$

$$+ \int_0^T \left[\beta(t)\partial_y w_n(1,t) + \frac{\alpha(t)}{b_1}\frac{d}{dt} w_n(1,t) - \frac{\alpha(t)}{3} w_n^2(1,t)\right] v_j(1,t)\, dt$$

$$+ \int_0^T \left[-\beta(t)\partial_y w_n(0,t) + \frac{\alpha(t)}{b_0}\frac{d}{dt} w_n(0,t) + \frac{\alpha(t)}{3} w_n^2(0,t)\right] v_j(0,t)\, dt$$

$$= \int_0^T\int_0^1 g v_j \, dy\, dt, \;\; \forall\, \varphi(t) \in \mathcal{D}((0,T)), \;\; \forall\, j = 1, ..., n. \tag{56}$$

Since $\mathcal{D}((0,T);H^1(0,1))$ is dense in $L_2(0,T;H^1(0,1))$, then integral identity (56) can be rewritten as

$$\int_0^T\int_0^1 \left[\partial_t w_n + \alpha(t) w_n \partial_y w_n - \beta(t)\partial_y^2 w_n + \gamma(y,t)\partial_y w_n\right] v \, dy\, dt$$

$$+\int_0^T \left[\beta(t)\partial_y w_n(1,t) + \frac{\alpha(t)}{b_1}\frac{d}{dt}w_n(1,t) - \frac{\alpha(t)}{3}w_n^2(1,t)\right] v(1,t)\,dt$$

$$+\int_0^T \left[-\beta(t)\partial_y w_n(0,t) + \frac{\alpha(t)}{b_0}\frac{d}{dt}w_n(0,t) + \frac{\alpha(t)}{3}w_n^2(0,t)\right] v(0,t)\,dt$$

$$= \int_0^T \int_0^1 gv\,dy\,dt, \ \ \forall\, v(y,t) \in L_2(0,T;H^1(0,1)). \tag{57}$$

In integral identity (57) we pass to the limit as $n \to \infty$. In the expressions corresponding to the linear terms of equation (5) and boundary conditions (6)–(7), passing to the limit is carried out according to relations (52) and (54). As for the nonlinear terms, here we have the following:

$$\int_0^T \int_0^1 \alpha(t) w_n(y,t)\partial_y w_n(y,t) v(y,t)\,dy\,dt = \int_0^T \alpha(t) \int_0^1 [w_n(y,t) - w(y,t)]\partial_y w_n(y,t) v(y,t)\,dy\,dt$$

$$+\int_0^T \alpha(t) \int_0^1 w(y,t)\partial_y w_n(y,t) v(y,t)\,dy\,dt \to \int_0^T \alpha(t) \int_0^1 w(y,t)\partial_y w(y,t) v(y,t)\,dy\,dt, \tag{58}$$

since according to (53) and (52) the following limit relation holds

$$\int_0^T \alpha(t) \int_0^1 [w_n(y,t) - w(y,t)]\partial_y w_n(y,t) v(y,t)\,dy\,dt \to 0.$$

Further, according to (55) and (54), similarly to the previous one, we will have

$$\int_0^T w_n(1,t) w_n(1,t) v(1,t)\,dt = \int_0^T [w_n(1,t) - w(1,t)] w_n(1,t) v(1,t)\,dt$$

$$+\int_0^T w(1,t) w_n(1,t) v(1,t) dt \to \int_0^T w^2(1,t) v(1,t)\,dt, \tag{59}$$

$$\int_0^T w_n(0,t) w_n(0,t) v(0,t) dt = \int_0^T [w_n(0,t) - w(0,t)] w_n(0,t) v(0,t)\,dt$$

$$+\int_0^T w(0,t)w_n(0,t)v(0,t)dt \to \int_0^T w^2(0,t)v(0,t)\,dt. \tag{60}$$

So, passing to the limit at $n\to\infty$ in integral identity (57), taking into account limit relations (58)–(60), as well as in initial condition (25), we get

$$\int_0^T\int_0^1 \left[\partial_t w + \alpha(t)w\partial_y w - \beta(t)\partial_y^2 w + \gamma(y,t)\partial_y w\right] v\,dy\,dt$$

$$+\int_0^T \left[\beta(t)\partial_y w(1,t) + \frac{\alpha(t)}{b_1}\frac{d}{dt}w(1,t) - \frac{\alpha(t)}{3}w^2(1,t)\right] v(1,t)\,dt$$

$$+\int_0^T \left[-\beta(t)\partial_y w(0,t) + \frac{\alpha(t)}{b_0}\frac{d}{dt}w(0,t) + \frac{\alpha(t)}{3}w^2(0,t)\right] v(0,t)\,dt$$

$$= \int_0^T\int_0^1 gv\,dy\,dt,\ \ \forall\, v(y,t)\in L_2(0,T;H^1(0,1)). \tag{61}$$

$$\int_0^1 w(y,0)\psi(y)\,dy = 0,\ \ \forall\,\psi\in L_2(0,1). \tag{62}$$

Note that integral identity (61) is also valid for any test function $v(y,t) \in L_2(0,T;H_0^1(0,1)) \subset L_2(0,T;H^1(0,1))$.

Further, returning to (61) and taking into account that traces $v(1,t)$ and $v(0,t)$ from $L_2(0,T)$ of test function $v\in L_2(0,T;H^1(0,1))$ are independent of each other and are arbitrary, in this case identities

$$\int_0^T\int_0^1 \left[\partial_t w + \alpha(t)w\partial_y w - \beta(t)\partial_y^2 w + \gamma(y,t)\partial_y w - g\right] v\,dy\,dt = 0,\ \ \forall\, v(y,t)\in L_2(0,T;H_0^1(0,1)), \tag{63}$$

$$\int_0^T \left[\beta(t)\partial_y w(1,t) + \frac{\alpha(t)}{b_1}\frac{d}{dt}w(1,t) - \frac{\alpha(t)}{3}w^2(1,t)\right] \psi_1(t)\,dt = 0,\ \ \forall\,\psi_1(t)\in L_2(0,T), \tag{64}$$

$$\int_0^T \left[-\beta(t)\partial_y w(0,t) + \frac{\alpha(t)}{b_0}\frac{d}{dt}w(0,t) + \frac{\alpha(t)}{3}w^2(0,t)\right] \psi_0(t)\,dt = 0,\ \ \forall\,\psi_0(t)\in L_2(0,T), \tag{65}$$

follow from (57), that is, the integrands in square brackets from (63)–(65) define zero functionals over spaces $L_2(0,T;H_0^1(0,1))$ and $L_2(0,T)$, and belong to spaces $0\in L_2(0,T;H^{-1}(0,1))\subset\mathcal{D}'(Q_{yt})$ and $0\in L_2(0,T)\subset\mathcal{D}'((0,T))$. Thus, from (63)–(65) we obtain that the weak limit function $w(y,t)$ satisfies equation (5) and boundary conditions (6)–(7), and from (62) it follows that it satisfies initial condition (8). This completes the proof of Lemma 5.

Lemma 6. *Under the conditions of Lemma 5 the solution $w\in H^{2,1}(Q_{yt})$ of initial boundary value problem (5)–(8) is unique.*

Proof. Let boundary value problem (5)–(8) have two different solutions $w^{(1)}(y,t)$ and $w^{(2)}(y,t)$. Then their difference $w(y,t)=w^{(1)}(y,t)-w^{(2)}(y,t)$ will satisfy the following homogeneous problem:

$$\partial_t w+\alpha(t)w\partial_y w^{(1)}+\alpha(t)w^{(2)}\partial_y w-\beta(t)\partial_y^2 w=0, \tag{66}$$

$$\frac{d}{dt}w(0,t)+\frac{b_0}{\alpha(t)}\left[\frac{\alpha(t)}{3}w\left(w^{(1)}+w^{(2)}\right)-\beta(t)\partial_y w\right]\Big|_{y=0}=0, \tag{67}$$

$$\frac{d}{dt}w(1,t)-\frac{b_1}{\alpha(t)}\left[\frac{\alpha(t)}{3}w\left(w^{(1)}+w^{(2)}\right)-\beta(t)\partial_y w\right]\Big|_{y=1}=0. \tag{68}$$

According to Lemmas 2 and 3 we have

$$\begin{gathered}w^{(i)}(y,t)\in L_\infty(0,T;H^1(0,1))\cap L_2(0,T;H^2(0,1)),\\ w^{(i)}(1,t)\quad\text{and}\quad w^{(i)}(0,t)\in L_\infty(0,T),\qquad i=1,2.\end{gathered} \tag{69}$$

Multiplying equation (66) by function $w(y,t)$ scalarly in $L_2(0,1)$ and taking into account (67)–(69), we obtain

$$\begin{gathered}\frac{1}{2}\frac{d}{d\,t}\|w(y,t)\|^2_{L_2(0,1)}+\frac{\alpha_1}{2b_1}\frac{d}{dt}|w(1,t)|^2+\frac{\alpha_1}{2b_0}\frac{d}{dt}|w(0,t)|^2\\ +\beta_1\|\partial_y w(y,t)\|^2_{L_2(0,1)}\le\frac{\alpha(t)}{3}|w(1,t)|^2\left[w^{(1)}(1,t)+w^{(2)}(1,t)\right]\\ +\frac{\alpha(t)}{3}|w(0,t)|^2\left[w^{(1)}(0,t)+w^{(2)}(0,t)\right]-\alpha(t)\int_0^1\left[w^2\partial_y w^{(1)}+w^{(2)}w\partial_y w\right]dy.\end{gathered} \tag{70}$$

Now we estimate the right-hand side of (70). According to (69) and by Lemma 2 we have:

$$\begin{gathered}\frac{\alpha(t)}{3}\left[w^{(1)}(1,t)+w^{(2)}(1,t)\right]|w(1,t)|^2\\ \le\frac{\alpha_2}{3}\left[\|w^{(1)}(1,t)\|_{L_\infty(0,T)}+\|w^{(2)}(1,t)\|_{L_\infty(0,T)}\right]|w(1,t)|^2\le C_1|w(1,t)|^2,\end{gathered} \tag{71}$$

$$\frac{\alpha(t)}{3}\left[w^{(1)}(0,t)+w^{(2)}(0,t)\right]|w(0,t)|^2$$

$$\leq \frac{\alpha_2}{3}\left[\|w^{(1)}(0,t)\|_{L_\infty(0,T)}+\|w^{(2)}(0,t)\|_{L_\infty(0,T)}\right]|w(0,t)|^2 \leq C_2|w(0,t)|^2, \tag{72}$$

$$\alpha(t)\int\limits_0^1 \left[w^2\partial_y w^{(1)}+w^{(2)}w\partial_y w\right]dy = \alpha(t)\left[|w(1,t)|^2 w^{(1)}(1,t)-|w(0,t)|^2 w^{(1)}(0,t)\right]$$

$$+\alpha(t)\int\limits_0^1\left[-2w^{(1)}w\partial_y w+w^{(2)}w\partial_y w\right]dy \leq C_3|w(1,t)|^2+C_4|w(0,t)|^2$$

$$+\frac{\alpha_2^2}{\beta_1}\left[2\|w^{(1)}\|_{L_\infty(Q_{yt})}+\|w^{(2)}\|_{L_\infty(Q_{yt})}\right]^2\|w\|^2_{L_2(0,1)}+\frac{\beta_1}{2}\|\partial_y w\|^2_{L_2(0,1)}$$

$$\leq C_3|w(1,t)|^2+C_4|w(0,t)|^2+C_5\|w(y,t)\|^2_{L_2(0,1)}+\frac{\beta_1}{2}\|\partial_y w\|^2_{L_2(0,1)}. \tag{73}$$

Based on relations (70)–(73) we establish

$$\frac{d}{dt}\|w(y,t)\|^2_{L_2(0,1)}+\frac{\alpha_1}{b_1}\frac{d}{dt}|w(1,t)|^2+\frac{\alpha_1}{b_0}\frac{d}{dt}|w(0,t)|^2+\beta_1\|\partial_y w(y,t)\|^2_{L_2(0,1)}$$

$$\leq 2(C_1+C_3)|w(1,t)|^2+2(C_2+C_4)|w(0,t)|^2+2C_5\|w(y,t)\|^2_{L_2(0,1)},\ \ \forall\, t\in(0,T].$$

Hence, applying Gronwall's inequality, we obtain:

$$\|w(y,t)\|^2_{L_2(0,1)}+|w(1,t)|^2+|w(0,t)|^2\equiv 0,\ \forall\, t\in(0,T].$$

This means that $w^{(1)}(y,t)\equiv w^{(2)}(y,t)$ in $L_2(Q_{yt})$, $w^{(1)}(1,t)\equiv w^{(2)}(1,t)$ and $w^{(1)}(0,t)\equiv w^{(2)}(0,t)$ in $L_2(0,T)$, i.e. the solution to initial boundary value problem (5)–(8) can be only one. Lemma 6 is completely proved.

Thus, the statement of Lemmas 5 and 6 implies the validity of Theorem 2. Theorem 2 is completely proved. Theorem 2 will also be used in the following sections to solve Problem 1, i.e. in the proof of Theorem 1.

8 The second auxiliary initial boundary value problem

In the domain $Q_{xt}=\{x,t\,|\,0<x<t_0+kt,\ 0<t<T,\ t_0>0\}$, we consider the following initial boundary value problem

$$\partial_t u+u\partial_x u-\nu\partial_x^2 u=f, \tag{74}$$

$$\frac{d}{dt}u(0,t)+b_0\left[\frac{1}{3}(u)^2-\nu\partial_x u\right]\Big|_{x=0}=0, \tag{75}$$

$$\frac{d}{dt}u(t_0+kt,t)-b_1\left[\frac{1}{3}(u)^2-\nu\partial_x u\right]\Big|_{x=t_0+kt}=0, \tag{76}$$

with initial condition

$$u(x,0)=0,\ \ x\in(0,t_0), \tag{77}$$

where $\nu,\ b_0,\ b_1$ are given positive constants, and the function $f(x,t)$ satisfy condition

$$f\in L_2(Q_{xt}). \tag{78}$$

Problem 2. Prove the unique solvability of initial boundary value problem (74)–(77) under condition (78).

Using the reversible transformation of independent variables

$$y=y(x,t)=\frac{x}{t_0+kt},\ \ t=t;\ \ x=x(y,t)=y(t_0+kt),\ t=t;$$

we move from $\{x,t\}$ to $\{y,t\}$. In this case, the domain Q_{xt} is transformed into a rectangular domain $Q_{yt}=\{y,t:\ 0<y<1,\ 0<t<T\}$. Problem 2 takes the following form:

$$\partial_t w+\frac{1}{t_0+kt}w\partial_y w-\frac{\nu}{(t_0+kt)^2}\partial_y^2 w-\frac{ky}{t_0+kt}\partial_y w=g(y,t), \tag{79}$$

$$\frac{d}{dt}w(0,t)+b_0\left[\frac{1}{3}(w)^2-\frac{\nu}{t_0+kt}\partial_y w\right]\Big|_{y=0}=0, \tag{80}$$

$$\frac{d}{dt}w(1,t)-b_1\left[\frac{1}{3}(w)^2-\frac{\nu}{t_0+kt}\partial_y w\right]\Big|_{y=1}=0, \tag{81}$$

with the initial condition

$$w(y,0)=0,\ \ y\in(0,1), \tag{82}$$

where $w(y,t)=u(x(y,t),t),\ g(y,t)=f(x(y,t),t)$.

Initial boundary value problem (79)–(82) is a particular case of the first auxiliary problem (5)–(8), where

$$\alpha(t)=\frac{1}{t_0+kt},\ \beta(t)=\frac{\nu}{(t_0+kt)^2},\ \gamma(y,t)=\frac{ky}{t_0+kt},$$

and conditions (9) are provided. Therefore, as a consequence of Theorem 2 we obtain

Theorem 3. *Let condition (78) be satisfied and $g\in L_2(Q_{yt})$. Then initial boundary value problem (79)–(82) is uniquely solvable in space*

$$\{w(y,t),\ w(1,t),\ w(0,t)\}\in H^{2,1}(Q_{yt})\times H^1(0,T)\times H^1(0,T).$$

Further, taking into account the correspondence of spaces in domains Q_{xt} and Q_{yt}:

$$g \in L_2(Q_{yt}) \Longleftrightarrow f \in L_2(Q_{xt}),$$

$$w \in H^{2,1}(Q_{yt}) = L_2(0,T;H^2(0,1)) \cap H^1(0,T;L_2(0,1)) \Longleftrightarrow u \in H^{2,1}(Q_{xt})$$
$$= L_2(0,T;H^2(0,t_0+kt)) \cap H^1(0,T;L_2(0,t_0+kt)),$$
$$w(1,t) \in H^1(0,T) \Longleftrightarrow u(t_0+kt,t) \in H^1(0,T),$$
$$w(0,t) \in H^1(0,T) \Longleftrightarrow u(0,t) \in H^1(0,T),$$

we establish the following statement

Theorem 4. *Let condition (78) be satisfied and $f \in L_2(Q_{xt})$. Then initial boundary value problem (74)–(77) is uniquely solvable in space*

$$\{u(x,t),\ u(t_0+kt,t),\ u(0,t)\} \in H^{2,1}(Q_{xt}) \times H^1(0,T) \times H^1(0,T).$$

9 To solving Problem 1

To the domain $Q_{xt} = \{x,t \,|\, 0 < x < kt,\ 0 < t < T\}$ from Section 2 we will put a family of domains $Q_{xt}^n = \{x,t \,|\, 0 < x < kt,\ 1/n < t < T\}$, $n \in \mathbb{N}^* \equiv \{n \in \mathbb{N} \,|\, n \geq n_1, 1/n_1 < T\}$, representing the trapezoids, and Ω_t is a section of Q_{xt}^n for a given value of the variable $t \in (1/n,T)$. Note that at the point $t = 1/n$ the domain Q_{xt}^n no longer degenerates to a point, in addition, between the original domain Q_{xt} and the domains Q_{xt}^n the strict inclusions $Q_{xt}^n \subset Q_{xt}^{n+1} \subset ... \subset Q_{xt}$ take place and, it is obvious, that $\lim\limits_{n\to\infty} Q_{xt}^n = Q_{xt}$.

On the trapezoid Q_{xt}^n we consider the following boundary value problems for the Burgers equation with respect to the functions $u_n(x,t)$:

$$\partial_t u_n + u_n \partial_x u_n - \nu \partial_x^2 u_n = f_n, \tag{83}$$

$$\frac{d}{dt} u_n(0,t) + b_0 \left[\frac{1}{3}(u_n)^2 - \nu \partial_x u_n \right] |_{x=0} = 0, \tag{84}$$

$$\frac{d}{dt} u_n(kt,t) - b_1 \left[\frac{1}{3}(u_{1n})^2 - \nu \partial_x u_n \right] |_{x=kt} = 0, \tag{85}$$

with initial conditions

$$u_n(x,1/n) = 0,\ \ x \in (0,k/n). \tag{86}$$

For each fixed $n \in \mathbb{N}^*$, initial boundary value problems (83)–(86) are problems of the form (74)–(77) under condition (78), for which Theorem 4 is valid. From Theorem 4 we obtain

Theorem 5. *Let $f_n \in L_2(Q_{xt}^n)$. Then, for each fixed $n \in \mathbb{N}^*$ initial boundary value problems (83)–(86) are uniquely solvable in the space*

$$\{u_n(x,t),\ u_n(kt,t),\ u_n(0,t)\} \in H^{2,1}(Q_{xt}^n) \times H^1(1/n,T) \times H^1(1/n,T).$$

To continue the proof of Theorem 1 we need the following statement:

Theorem 6. *Under the conditions of Theorems 1 and 5 the following estimate holds*

$$\|u_n(x,t)\|^2_{H^{2,1}(Q_{xt}^n)} + \|u_n(kt,t)\|^2_{H^1(1/n,T)} + \|u_n(0,t)\|^2_{H^1(1/n,T)} \le C\|f_n(x,t)\|^2_{L_2(Q_{xt}^n)}. \tag{87}$$

To prove Theorem 6 we will establish a number of lemmas.

Lemma 7. *There exists a positive constant K_1 independent of n, such that for all $t \in [1/n,T]$ the following estimate takes place*

$$\|u_n(x,t)\|^2_{L_2(0,kt)} + |u_n(kt,t)|^2 + |u_n(0,t)|^2 + \int\limits_{1/n}^{t} \|\partial_x u_n(x,\tau)\|^2_{L_2(0,kt)} d\tau \le K_1\|f_n(x,t)\|^2_{L_2(Q_{xt}^n)}. \tag{88}$$

Proof. Multiplying equation (83) by $u_n(x,t)$ scalarly in $L_2(0,kt)$ and using the following equalities

$$\int\limits_0^{kt} u_n(x,t)\partial_x u_n(x,t) u_n(x,t)dx = \frac{1}{3}u_n^3(kt,t) - \frac{1}{3}u_n^3(0,t),$$

$$\frac{d}{d\,t}\|u_n(x,t)\|^2_{L_2(0,kt)} = 2\int\limits_0^{kt} \partial_t u_n(x,t)\, u_n(x,t)\, dx + k|u_n(kt,t)|^2,$$

we get

$$\frac{1}{2}\frac{d}{dt}\int\limits_0^{kt} |u_n(x,t)|^2 dx + \frac{1}{2b_1}\frac{d}{dt}|u_n(kt,t)|^2 + \frac{1}{2b_0}\frac{d}{dt}|u_n(0,t)|^2$$

$$+\nu\int\limits_0^{kt} |\partial_x u_n(x,t)|^2 dx = \int\limits_0^{kt} f_n(x,t)u_n(x,t)dx + \frac{k}{2}|u_n(kt,t)|^2,$$

or

$$\frac{d\theta(t)}{dt} + 2\nu\|\partial_x u_n(x,t)\|^2_{L_2(0,kt)} \le K_0\left[\|f_n(x,t)\|^2_{L_2(0,kt)} + \theta(t)\right],$$

where

$$\theta(t) = \|u_n(x,t)\|^2_{L_2(0,kt)} + \frac{1}{b_1}|u_n(kt,t)|^2 + \frac{1}{b_0}|u_n(0,t)|^2, \ K_0 = \text{const} > 0.$$

Note that K_0 does not depend on n.

From this we obtain two inequalities

$$\frac{d\,\theta(t)}{dt} \le K_0\left[\|f_n(x,t)\|^2_{L_2(0,kt)} + \theta(t)\right], \ \ \theta(0) = 0, \tag{89}$$

$$2\nu\|\partial_x u_n(x,t)\|^2_{L_2(0,kt)} \le K_0\left[\|f_n(x,t)\|^2_{L_2(0,kt)} + \|u_n(x,t)\|^2_{L_2(0,kt)}\right]. \tag{90}$$

Finally, applying Gronwall's inequality from (89)–(90), we obtain estimate (88). This completes the proof of Lemma 7.

Lemma 8. *There exists a positive constant K_2 independent of n, such that for all $t \in [1/n, T]$ the following estimate takes place*

$$\|\partial_x u_n(x,t)\|^2_{L_2(0,kt)} + \int\limits_{1/n}^{t}\left[|\frac{d}{d\tau}u_n(k\tau,\tau)|^2 + |\frac{d}{d\tau}u_n(0,\tau)|^2\right]d\tau$$

$$+|\partial_x u_n(kt,t)|^2 + \int\limits_{1/n}^{t}\|\partial_x^2 u_n(x,\tau)\|^2_{L_2(0,kt)}d\tau \le K_2\|f_n(x,t)\|^2_{L_2(Q^n_{xt})}. \tag{91}$$

Proof. Multiplying equation (83) by $-\partial_x^2 u_n(x,t)$ scalarly in $L_2(0,kt)$ and using the following equality:

$$\frac{d}{d\,t}\|\partial_x u_n(x,t)\|^2_{L_2(0,kt)} = 2\int\limits_0^{kt}\partial_t\partial_x u_n(x,t)\,\partial_x u_n(x,t)\,dx + k|\partial_x u_n(kt,t)|^2,$$

we obtain

$$\frac{1}{2}\frac{d}{d\,t}\|\partial_x u_n(x,t)\|^2_{L_2(0,kt)} + \nu\|\partial_x^2 u_n(x,t)\|^2_{L_2(0,kt)}$$

$$= \big(u_n(x,t)\partial_x u_n(x,t), \partial_x^2 u_n(x,t)\big) -$$

$$\big(f_n(x,t), \partial_x^2 u_n(x,t)\big) + \partial_t u_n(x,t)\partial_x u_n(x,t)\Big|_{x=0}^{x=kt} + \frac{k}{2}|\partial_x u_n(kt,t)|^2$$

$$\le \big|\big(u_n(x,t)\partial_x u_n(x,t), \partial_x^2 u_n(x,t)\big)\big| + \big|\big(f_n(x,t), \partial_x^2 u_n(x,t)\big)\big|$$

$$-\frac{1}{b_1\nu}\left|\frac{d}{dt}u_n(kt,t)\right|^2+\frac{1}{3\nu}|u_n(kt,t)|^2|\partial_t u_n(kt,t)|-k|\partial_x u_n(kt,t)|^2$$

$$-\frac{1}{b_0\nu}\left|\frac{d}{dt}u_n(0,t)\right|^2+\frac{1}{3\nu}|u_n(0,t)|^2|\frac{d}{dt}u_n(0,t)|+\frac{k}{2}|\partial_x u_n(kt,t)|^2,$$

or

$$\frac{1}{2}\frac{d}{d\,t}\|\partial_x u_n(x,t)\|^2_{L_2(0,kt)}+\frac{1}{b_1\nu}\left|\frac{d}{dt}u_n(kt,t)\right|^2+\frac{1}{b_0\nu}\left|\frac{d}{dt}u_n(0,t)\right|^2$$

$$+\frac{k}{2}|\partial_x u_n(kt,t)|^2+\nu\|\partial_x^2 u_n(x,t)\|^2_{L_2(0,kt)}\le\left|\big(u_n(x,t)\partial_x u_n(x,t),\partial_x^2 u_n(x,t)\big)\right|$$

$$+\frac{1}{3\nu}|u_n(kt,t)|^2|\partial_t u_n(kt,t)|+\frac{1}{3\nu}|u_n(0,t)|^2|\partial_t u_n(0,t)|+\left|\big(f_n(x,t),\partial_x^2 u_n(x,t)\big)\right|. \tag{92}$$

First, we consider the estimates of the nonlinear terms from (92). First of all, we have

$$\left|\big(u_n(x,t)\partial_x u_n(x,t),\partial_x^2 u_n(x,t)\big)\right|$$

$$\le\|u_n(x,t)\|_{L_4(0,kt)}\|\partial_x u_n(x,t)\|_{H^1(0,kt)}\|\partial_x u_n(x,t)\|_{L_4(0,kt)}$$

$$\le\|u_n(x,t)\|_{L_4(0,kt)}\|\partial_x u_n(x,t)\|_{H^1(0,kt)}\|\partial_x u_n(x,t)\|_{L_\infty(0,kt)}. \tag{93}$$

Further, taking into account the interpolation inequality from ([25], Theorems 5.8–5.9, p.140–141)

$$\|\partial_x u_n(x,t)\|_{L_4(0,kt)}\le C\|\partial_x u_n(x,t)\|^{1/2}_{H^1(0,kt)}\|\partial_x u_n(x,t)\|^{1/2}_{L_2(0,kt)},\ \forall\,\partial_y u_n(x,t)\in H^1(0,kt),$$

from (93) we obtain

$$\left|\big(u_n(x,t)\partial_x u_n(x,t),\partial_x^2 u_n(x,t)\big)\right|$$

$$\le C\|u_n(x,t)\|_{L_4(0,kt)}\|\partial_x u_n(x,t)\|^{3/2}_{H^1(0,kt)}\|\partial_x u_n(x,t)\|^{1/2}_{L_2(0,kt)}$$

$$\le\frac{\nu}{8}\|\partial_x^2 u_n(x,t)\|^2_{L_2(0,kt)}+\left[\frac{\nu}{8}+C_2\|u_n(x,t)\|^4_{L_4(0,kt)}\right]\|\partial_x u_n(x,t)\|^2_{L_2(0,kt)}. \tag{94}$$

Here we have used Young's inequality (34), where $a=\nu/6,\ p=4/3,\ q=4,$

$$A=\|\partial_x u_n(x,t)\|^{3/2}_{H^1(0,kt)},\ \ B=C\,\|u_n(x,t)\|_{L_4(0,kt)}\,\|\partial_x u_n(x,t)\|^{1/2}_{L_2(0,kt)}.$$

Note that for two nonlinear terms on the right-hand side of (92) the following estimates hold:

$$\frac{1}{3\nu}|u_n(kt,t)|^2|\partial_t u_n(kt,t)|\le\frac{b_1}{18\nu}|u_n(kt,t)|^4+\frac{1}{2b_1\nu}|\partial_t u_n(kt,t)|^2$$

$$\le K_1^2\frac{b_1}{18\nu}+\frac{1}{2b_1\nu}|\partial_t u_n(kt,t)|^2\le K_1^2\frac{b_1}{18\nu}+\frac{1}{b_1\nu}|\frac{d}{dt}u_n(kt,t)|^2+\frac{k^2}{b_1\nu}|\partial_x u_n(kt,t)|^2, \tag{95}$$

$$\frac{1}{3\nu}|u_n(0,t)|^2|\partial_t u_n(0,t)| \le \frac{b_0}{18\nu}|u_n(0,t)|^4 + \frac{1}{2b_0\nu}|\partial_t u_n(0,t)|^2$$

$$\le K_1^2\frac{b_0}{18\nu} + \frac{1}{2b_0\nu}|\partial_t u_n(0,t)|^2 = K_1^2\frac{b_0}{18\nu} + \frac{1}{2b_0\nu}|\frac{d}{dt}u_n(0,t)|^2. \tag{96}$$

In inequalities (95)–(96), estimate (88) from Lemma 7 is used.

In (95), it is necessary to estimate the last term on the right. Taking into account the interpolation inequality from ([25], Theorem 5.9, p.140–141), we get

$$\frac{k^2}{b_1\nu}|\partial_x u_n(kt,t)|^2 \le \frac{k^2}{b_1\nu}\|\partial_x u_n(x,t)\|^2_{L_\infty(0,kt)} \le \frac{k^2K^2}{b_1\nu}\|\partial_x u_n\|_{H^1(0,kt)}\|\partial_x u_n\|_{L_2(0,kt)}$$

$$\le \frac{2k^2K^2}{b_1\nu}\left[\|\partial_x u_n\|_{L_2(0,kt)} + \|\partial_x^2 u_n\|_{L_2(0,kt)}\right]\|\partial_x u_n\|_{L_2(0,kt)}$$

$$= \frac{2k^2K^2}{b_1\nu}\|\partial_x u_n\|^2_{L_2(0,kt)} + \frac{2k^2K^2}{b_1\nu}\|\partial_x^2 u_n\|_{L_2(0,kt)}\|\partial_x u_n\|_{L_2(0,kt)}$$

$$\le \frac{\nu}{8}\|\partial_x^2 u_n\|^2_{L_2(0,kt)} + \left[\frac{2k^2K^2}{b_1\nu} + \frac{32k^4K^4}{b_1^2\nu^3}\right]\|\partial_x u_n\|^2_{L_2(0,kt)}, \tag{97}$$

where K is a constant from Theorem 5.9, p.140–141 [25].

Further, for the last term from (92) we will have:

$$\left|\left(f_n(x,t), \partial_x^2 u_n(x,t)\right)\right| \le \frac{\nu}{4}\|\partial_x^2 u_n(x,t)\|^2_{L_2(0,kt)} + C_4\|f_n(x,t)\|^2_{L_2(0,kt)}. \tag{98}$$

From (92), (94)–(98) we obtain

$$\frac{d}{d\,t}\|\partial_x u_n(x,t)\|^2_{L_2(0,kt)} + \frac{1}{b_1\nu}\left|\frac{d}{dt}u_n(kt,t)\right|^2 + \frac{1}{b_0\nu}\left|\frac{d}{dt}u_n(0,t)\right|^2 + \nu\|\partial_x^2 u_n(x,t)\|^2_{L_2(0,kt)}$$

$$\le 2C_4\|f_n(x,t)\|^2_{L_2(0,kt)} + C_5(t)\|\partial_x u_n(x,t)\|^2_{L_2(0,kt)} + K_0, \tag{99}$$

where

$$C_5(t) = \frac{\nu}{4} + 2C_2\|u_n(x,t)\|^4_{L_4(0,kt)} + \frac{4k^2K^2}{b_1\nu} + \frac{64k^4K^4}{b_1^2\nu^3},\ \ K_0 = K_1^2\frac{b_0}{9\nu} + K_1^2\frac{b_1}{9\nu},$$

or, by integrating (99) with respect to t from $1/n$ to t, we will have

$$\|\partial_x u_n(x,t)\|^2_{L_2(0,kt)} + \frac{1}{b_1\nu}\int\limits_{1/n}^{t}\left|\frac{d}{d\tau}u_n(kt,\tau)\right|^2 d\tau + \frac{1}{b_0\nu}\int\limits_{1/n}^{t}\left|\frac{d}{d\tau}u_n(0,\tau)\right|^2 d\tau$$

$$+\nu \int\limits_{1/n}^{t} \|\partial_x^2 u_n(x,\tau)\|^2_{L_2(0,kt)} d\tau \leq A_4 \|f_n(x,t)\|^2_{L_2(Q^n_{xt})} + \int\limits_{1/n}^{t} A_5(\tau) \|\partial_x u_n(x,\tau)\|^2_{L_2(0,kt)} d\tau + K_0 T, \tag{100}$$

where $A_4 = 2C_4$.

From inequality (100) in the same way as in the proof of Lemma 7 we obtain desired estimate (91). Lemma 8 is completely proved.

Lemma 9. *There exists a positive constant K_3 independent of n, such that the following estimate takes place*

$$\|\partial_t u_n(x,t)\|^2_{L_2(Q^n_{xt})} + \nu^2 \|\partial_x^2 u_n(x,t)\|^2_{L_2(Q^n_{xt})} \leq K_3 \|f_n(x,t)\|^2_{L_2(Q^n_{xt})}. \tag{101}$$

Proof. From equation (83) we will have

$$\|f_n\|^2_{L_2(Q^n_{xt})} = \left(\partial_t u_n + u_n \partial_x u_n - \nu \partial_x^2 u_n,\, \partial_t u_n + u_n \partial_x u_n - \nu \partial_x^2 u_n\right)_{L_2(Q^n_{xt})}$$

$$= \|\partial_t u_n\|^2_{L_2(Q^n_{xt})} + \nu^2 \|\partial_x^2 u_n\|^2_{L_2(Q^n_{xt})} + \|u_n \partial_x u_n\|^2_{L_2(Q^n_{xt})}$$

$$-2\nu \left(\partial_t u_n,\, \partial_x^2 u_n\right)_{L_2(Q^n_{xt})} + 2\nu \left(\partial_t u_n,\, u_n \partial_x u_n\right)_{L_2(Q^n_{xt})} - 2\nu \left(u_n \partial_x u_n,\, \partial_x^2 u_n\right)_{L_2(Q^n_{xt})},$$

or

$$\|\partial_t u_n\|^2_{L_2(Q^n_{xt})} + \nu^2 \|\partial_x^2 u_n\|^2_{L_2(Q^n_{xt})} = \|f_n\|^2_{L_2(Q^n_{xt})} - \|u_n \partial_x u_n\|^2_{L_2(Q^n_{xt})}$$

$$+2\nu \left(u_n \partial_x u_n,\, \partial_x^2 u_n\right)_{L_2(Q^n_{xt})} - 2\left(\partial_t u_n,\, u_n \partial_x u_n\right)_{L_2(Q^n_{xt})} + 2\nu \left(\partial_t u_n,\, \partial_x^2 u_n\right)_{L_2(Q^n_{xt})}. \tag{102}$$

We estimate two terms from (102), by using the Cauchy's inequality:

$$\left| -2\left(\partial_t u_n,\, u_n \partial_x u_n\right)_{L_2(Q^n_{xt})} \right| \leq \frac{1}{2} \|\partial_t u_n\|^2_{L_2(Q^n_{xt})} + 2 \|u_n \partial_x u_n\|^2_{L_2(Q^n_{xt})}, \tag{103}$$

$$\left| 2\nu \left(u_n \partial_x u_n,\, \partial_x^2 u_n\right)_{L_2(Q^n_{xt})} \right| \leq 4 \|u_n \partial_x u_n\|^2_{L_2(Q^n_{xt})} + \frac{\nu^2}{4} \|\partial_x^2 u_n\|^2_{L_2(Q^n_{xt})}. \tag{104}$$

Now, for the last term from (102) we have:

$$\left(\partial_t u_n,\, \partial_x^2 u_n\right)_{L_2(Q^n_{xt})} = -\int\limits_{1/n}^{T} \int\limits_{0}^{kt} \partial_t \left(\partial_x u_n\right) \partial_x u_n \, dx \, dt + \int\limits_{1/n}^{T} \left[\partial_t u_n \partial_x u_n\right] \Big|_{x=0}^{x=kt} dt$$

$$= -\frac{1}{2} \int\limits_{0}^{kt} \left(\partial_x u_n(x,T)\right)^2 \, dx + \int\limits_{1/n}^{T} \partial_t u_n(kt,t) \partial_x u_n(kt,t) \, dt - \int\limits_{1/n}^{T} \partial_t u_n(0,t) \partial_x u_n(0,t) \, dt.$$

Since from boundary conditions (84)–(85) we have

$$\partial_t u_n(kt,t) = b_1\left[\frac{1}{3}\left(u_n(kt,t)\right)^2 - \nu\partial_x u_n(kt,t)\right] - k\partial_x u_n(kt,t),$$

$$\frac{d}{dt}u_n(0,t) = \partial_t u_n(0,t) = -b_0\left[\frac{1}{3}\left(u_n(0,t)\right)^2 - \nu\partial_x u_n(0,t)\right],$$

then

$$\left(\partial_t u_n,\, \partial_x^2 u_n\right)_{L_2(Q_{xt}^n)} = -\frac{1}{2}\int\limits_0^{kt} |\partial_x u_n(x,T)|^2\,dx$$

$$+\frac{b_1}{3}\int\limits_{1/n}^{T} |u_n(kt,t)|^2\partial_x u_n(kt,t)\,dt - (b_1\nu + k)\int\limits_{1/n}^{T} |\partial_x u_n(kt,t)|^2\,dt$$

$$+\frac{b_0}{3}\int\limits_{1/n}^{T} |u_n(0,t)|^2\partial_x u_n(0,t)\,dt - b_0\nu\int\limits_{1/n}^{T} |\partial_x u_n(0,t)|^2\,dt,$$

and the following inequality holds

$$2\nu\left(\partial_t u_n,\, \partial_x^2 u_n\right)_{L_2(Q_{xt}^n)} \le \frac{2b_1\nu}{3}\int\limits_{1/n}^{T} u_n(kt,t)\partial_x u_n(kt,t)\,dt + \frac{2b_0\nu}{3}\int\limits_{1/n}^{T} u_n(0,t)\partial_x u_n(0,t)\,dt. \tag{105}$$

We need the norm of the operator of the following embedding of the Sobolev space in the space of continuous functions: $H^1(0,kt) \hookrightarrow C([0,kt])$, i.e. there exists a number B independent of $v(x)$, such that

$$\|v(x)\|_{C([0,kt])} \le B\|v(x)\|_{H^1(0,kt)},\ \forall\, v(x) \in H^1(0,kt),\ \forall\, t \in [1/n,T]. \tag{106}$$

Let us estimate the terms on the right-hand side of inequality (105). We have

$$\int\limits_{1/n}^{T} u_n(kt,t)\partial_x u_n(kt,t)\,dt \le C_1(\varepsilon_1)\|u_n(kt,t)\|^2_{L_2(1/n,T)} + \varepsilon_1\|\partial_x u_n(kt,t)\|^2_{L_2(1/n,T)}, \tag{107}$$

$$\int\limits_{1/n}^{T} u_n(0,t)\partial_x u_n(0,t)\,dt \le C_0(\varepsilon_0)\|u_n(0,t)\|^2_{L_2(1/n,T)} + \varepsilon_0\|\partial_x u_n(0,t)\|^2_{L_2(1/n,T)}, \tag{108}$$

where $2\sqrt{\varepsilon_0 C_0(\varepsilon_0)} = 1,\ 2\sqrt{\varepsilon_1 C_1(\varepsilon_1)} = 1$.

Now, taking into account (106) and choosing $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ so that equality $\frac{8B^2}{3}(b_0\varepsilon_0 + b_1\varepsilon_1) = \nu$ holds, from (102)–(105) we obtain the estimate

$$\|\partial_t u_n\|^2_{L_2(Q^n_{xt})} + \nu^2\|\partial_x^2 u_n\|^2_{L_2(Q^n_{xt})} \le 2\|f_n\|^2_{L_2(Q^n_{xt})} + 12\|u_n\partial_x u_n\|^2_{L_2(Q^n_{xt})}$$

$$+\frac{4b_1\nu C_1(\varepsilon_1)}{3}\|u_n(kt,t)\|^2_{L_2(1/n,T)} + \frac{4b_0\nu C_0(\varepsilon_0)}{3}\|u_n(0,t)\|^2_{L_2(1/n,T)}. \tag{109}$$

According to (88) from Lemma 7 we have the estimates

$$\|u_n(kt,t)\|^2_{L_2(1/n,T)} \le K_1 T\|f_n\|^2_{L_2(Q^n_{xt})},\ \ \|u_n(0,t)\|^2_{L_2(1/n,T)} \le K_1 T\|f_n\|^2_{L_2(Q^n_{xt})}, \tag{110}$$

where K_1 is constant from Lemma 7.

It remains to estimate the term $12\|u_n\partial_x u_n\|^2_{L_2(Q^n_{xt})}$ in (109). Using embedding $H^1(0,kt) \hookrightarrow L_\infty(0,kt)$ with embedding constant C_0 and estimate (91) from Lemma 8, we obtain

$$\|u_n\partial_x u_n\|^2_{L_2(Q^n_{xt})} \le \int\limits_{1/n}^{T} \|u_n\|^2_{L_\infty(0,kt)}\|\partial_x u_n\|^2_{L_2(0,kt)}\,dt$$

$$\le C_0\int\limits_{1/n}^{T} \|u_n\|^2_{H^1(0,kt)}\|\partial_x u_n\|^2_{L_2(0,kt)}\,dt$$

$$\le C_0\|u_n\|^2_{L_\infty(1/n,T;H^1(0,kt))}\|\partial_x u_n\|^2_{L_2(Q^n_{xt})} \le C_0K_2\|f\|^2_{L_2(Q_{xt})}\|\partial_x u_n\|^2_{L_2(0,kt)}\,dt, \tag{111}$$

since $\|f_n\|_{L_2(Q^n_{xt})} \le \|f\|_{L_2(Q_{xt})}$ by definition (where K_2 is the constant from Lemma 8).

Based on inequalities (109)–(111) we establish estimate (101) of Lemma 9. This completes the proof of Lemma 9.

Lemma 10. *For positive constants K_3, K_4 and K_5 independent of n, for all $t \in (1/n, T]$ the following inequalities take place:*

$$\|\partial_t u_n(x,t)\|^2_{L_2(Q^n_{xt})} \le K_3\|f_n\|^2_{L_2(Q^n_{xt})}, \tag{112}$$

$$\|\frac{d}{dt}u_n(kt,t)\|^2_{L_2(1/n,T)} \le K_4\|f_n\|^2_{L_2(Q^n_{xt})}, \tag{113}$$

$$\|\frac{d}{dt}u_n(0,t)\|^2_{L_2(1/n,T)} \le K_5\|f_n\|^2_{L_2(Q^n_{xt})}. \tag{114}$$

Proof. Estimate (112) follows from (101) in Lemma 9. It remains to prove the validity of estimates (113)–(114). From boundary conditions (84)–(85) we obtain

$$\|\frac{d}{dt}u_n(0,t)\|_{L_2(1/n,T)} \leq \frac{b_0}{3}\|u_n(0,t)\|^{1/2}_{L_4(1/n,T)} + b_0\nu\|\partial_x u_n(0,t)\|_{L_2(1/n,T)}, \tag{115}$$

$$\|\frac{d}{dt}u_n(kt,t)\|_{L_2(1/n,T)} \leq \frac{b_1}{3}\|u_n(kt,t)\|^{1/2}_{L_4(1/n,T)} + b_1\nu\|\partial_x u_n(kt,t)\|_{L_2(1/n,T)}. \tag{116}$$

Estimates (113)–(114) follow, respectively, from (115)–(116) and from the statements of Lemmas 7 and 8. Lemma 10 is completely proved.

Taking into account the obvious inequality

$$\|f_n\|_{L_2(Q^n_{xt})} \leq \|f\|_{L_2(Q_{xt})} \ \ \forall\, n \in \mathbb{N}^*,$$

from Lemmas 7–10 we directly obtain the validity of estimate (87) from Theorem 6. Thus, we have proved the validity of Theorem 6.

10 Proof of Theorem 1. Existence

Proof of Theorem 1 is based on Theorem 6. In boundary value problem (83)–(86) we continue with zeros each element of the sequence $\{u_n(x,t),\, f_n(x,t),\, \{x,t\} \in Q^n_{xt};\, u_n(kt,t),\, u_n(0,t),\, t \in (1/n,T);\, n \in \mathbb{N}^*\}$, respectively, over the entire domain Q_{xt} and for the entire interval $(0,T)$. As a result, we obtain a sequence of functions that we denote by

$$\left\{\widetilde{u_n(x,t)},\ \widetilde{f_n(x,t)},\ \widetilde{u_n(kt,t)},\ \widetilde{u_n(0,t)},\ n \in \mathbb{N}^*\right\}. \tag{117}$$

Obviously, each four functions from sequence (117) in the domain Q_{xt} satisfy boundary value problem (1)–(3) according to the statement of Theorems 5–6. In addition, note that estimate (87) will be strengthened if we replace $\|\widetilde{f_n(x,t)}\|_{L_2(Q_{xt})}$ on its right-hand side by expression $\|f(x,t)\|_{L_2(Q_{xt})}$, since

$$\|\widetilde{f_n(x,t)}\|_{L_2(Q_{xt})} \leq \|f(x,t)\|_{L_2(Q_{xt})}, \quad \text{for } \ \forall\, n \in \mathbb{N}^*.$$

Therefore, we obtain a bounded sequence of functions (117), from which we can extract a weakly convergent subsequence (we preserve the notation of the index n for the subsequence), i.e. we will have

$$\widetilde{u_n(x,t)} \to u(x,t) \ \text{ weakly in } \ H^{2,1}(Q_{xt}), \tag{118}$$

$$\widetilde{u_n(kt,t)} \to z_1(t) \equiv u(kt,t) \ \text{ weakly in } \ H^1(0,T), \tag{119}$$

$$\widetilde{u_n(0,t)} \to z_0(t) \equiv u(0,t) \ \text{ weakly in } \ H^1(0,T). \tag{120}$$

From (118)–(120), respectively, it follows that

$$\widetilde{u_n(x,t)} \to u(x,t) \ \text{ strongly in } \ L_2(Q_{xt}), \tag{121}$$

$$\widetilde{u_n(kt,t)} \to z_1(t) \equiv u(kt,t) \ \text{ strongly in } \ L_2(0,T), \tag{122}$$

$$\widetilde{u_n(0,t)} \to z_0(t) \equiv u(0,t) \ \text{ strongly in } \ L_2(0,T). \tag{123}$$

Then, according to (118)–(123) in the following integral identities we can pass to the limit as $n \to \infty$:

$$\int\limits_{Q_{xt}} \left[\partial_t \widetilde{u_n(x,t)} + \widetilde{u_n(x,t)}\partial_x \widetilde{u_n(x,t)} - \nu\partial_x^2 \widetilde{u_n(x,t)} - \widetilde{f_n(x,t)}\right] \psi(x,t)\,dx\,dt$$

$$\to \int\limits_{Q_{xt}} \left[\partial_t u(x,t) + u(x,t)\partial_x u(x,t) - \nu\partial_x^2 u(x,t) - f(x,t)\right] \psi(x,t)\,dx\,dt = 0, \ \ \forall\,\psi \in L_2(Q_{xt}), \tag{124}$$

$$\int\limits_0^T \left[\partial_t \widetilde{u_n(x,t)} + \frac{b_0}{3}(\widetilde{u_n(x,t)})^2 - b_0\nu\partial_x \widetilde{u_n(x,t)}\right]\Bigg|_{x=0} \psi_0(t)\,dt$$

$$\to \int\limits_0^T \left[\partial_t u(x,t) + \frac{b_0}{3}(u(x,t))^2 - b_0\nu\partial_x u(x,t)\right]\Bigg|_{x=0} \psi_0(t)\,dt = 0 \ \ \forall\,\psi_0 \in L_2(0,T), \tag{125}$$

$$\int\limits_0^T \left[\partial_t \widetilde{u_n(x,t)} - \frac{b_1}{3}(\widetilde{u_n(x,t)})^2 + b_1\nu \widetilde{u_n(x,t)}\right]\Bigg|_{x=kt} \psi_1(t)\,dt$$

$$\to \int\limits_0^T \left[\partial_t u(x,t) - \frac{b_1}{3}(u(x,t))^2 + b_1\nu u(x,t)\right]\Bigg|_{x=kt} \psi_1(t)\,dt = 0 \ \ \forall\,\psi_1 \in L_2(0,T). \tag{126}$$

So, we have established that boundary value problem (1)–(3) has the solution $u_1(x,t) \in H^{2,1}(Q_{xt})$ in the sense of integral identities (124)–(126). The existence part of Theorem 1 has been proved.

11 Proof of Theorem 1. Uniqueness

Let boundary value problem (1)–(3) have two different solutions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$. Then their difference $u(x,t) = u^{(1)}(x,t) - u^{(2)}(x,t)$ will satisfy the following homogeneous problem:

$$\partial_t u + u\partial_x u^{(1)} + u^{(2)}\partial_x u - \nu\partial_x^2 u = 0, \tag{127}$$

$$\frac{d}{dt}u(0,t)+b_0\left[\frac{1}{3}u\left(u^{(1)}+u^{(2)}\right)-\nu\partial_x u\right]\big|_{x=0}=0, \tag{128}$$

$$\frac{d}{dt}u(kt,t)-b_1\left[\frac{1}{3}u\left(u^{(1)}+u^{(2)}\right)-\nu\partial_x u\right]\big|_{x=kt}=0. \tag{129}$$

According to Lemmas 7 and 8 we have

$$u^{(i)}(x,t)\in L_\infty(0,T;H^1(0,kt))\cap L_2(0,T;H^2(0,kt)), \tag{130}$$

$$u^{(i)}(kt,t)\ \text{ and }\ u^{(i)}(0,t)\in L_\infty(0,T),\ i=1,2. \tag{131}$$

Multiplying equation (127) by function $u(x,t)$ scalarly in $L_2(0,kt)$ and taking into account (128)–(131), we obtain

$$\frac{1}{2}\frac{d}{d\,t}\|u(x,t)\|^2_{L_2(0,kt)}+\frac{1}{2b_1}\frac{d}{d\,t}|u(kt,t)|^2+\frac{1}{2b_0}\frac{d}{d\,t}|u(0,t)|^2$$

$$+\nu\left\|\partial_x u(x,t)\right\|^2_{L_2(0,kt)}\le\frac{1}{3}|u(kt,t)|^2\left[u^{(1)}(kt,t)+u^{(2)}(kt,t)\right]$$

$$+\frac{1}{3}|u(0,t)|^2\left[u^{(1)}(0,t)+u^{(2)}(0,t)\right]-\int\limits_0^{kt}\left[u^2\partial_x u^{(1)}+u^{(2)}u\partial_x u\right]\,dx+k|u(kt,t)|^2. \tag{132}$$

Here we have used the following equality

$$\frac{d}{d\,t}\|u(x,t)\|^2_{L_2(0,kt)}=2\int\limits_0^{kt}\partial_t u(x,t)\,u(x,t)\,dx+k|u(kt,t)|^2.$$

Let us estimate the right-hand side of (132). According to (130)–(131) and Lemma 7, we have:

$$\frac{1}{3}\left[u^{(1)}(kt,t)+u^{(2)}(kt,t)\right]|u(kt,t)|^2$$

$$\le\frac{1}{3}\left[\|u^{(1)}(kt,t)\|_{L_\infty(0,T)}+\|u^{(2)}(kt,t)\|_{L_\infty(0,T)}\right]|u(kt,t)|^2\le C_1|u(kt,t)|^2, \tag{133}$$

$$\frac{1}{3}\left[u^{(1)}(0,t)+u^{(2)}(0,t)\right]|u(0,t)|^2$$

$$\le\frac{1}{3}\left[\|u^{(1)}(0,t)\|_{L_\infty(0,T)}+\|u^{(2)}(0,t)\|_{L_\infty(0,T)}\right]|u(0,t)|^2\le C_2|u(0,t)|^2, \tag{134}$$

$$\int\limits_0^{kt}\left[u^2\partial_x u^{(1)}+u^{(2)}u\partial_x u\right]\,dx=\left[|u(kt,t)|^2u^{(1)}(kt,t)-|u(0,t)|^2u^{(1)}(0,t)\right]$$

$$+\int\limits_0^{kt}\left[-2u^{(1)}u\partial_x u+u^{(2)}u\partial_x u\right]\,dx\le C_3|u(kt,t)|^2+C_4|u(0,t)|^2$$

$$+\frac{1}{2\nu}\left[2\|u^{(1)}\|_{L_\infty(Q_{xt})}+\|u^{(2)}\|_{L_\infty(Q_{xt})}\right]^2\|u\|^2_{L_2(0,kt)}+\frac{\nu}{2}\|\partial_x u\|^2_{L_2(0,kt)}$$

$$\le C_3|u(kt,t)|^2+C_4|u(0,t)|^2+C_5\|u(x,t)\|^2_{L_2(0,kt)}+\frac{\nu}{2}\|\partial_x u\|^2_{L_2(0,kt)}. \tag{135}$$

Based on relations (132)–(135) we establish

$$\frac{d}{dt}\|u(x,t)\|^2_{L_2(0,kt)}+\frac{1}{b_1}\frac{d}{dt}|u(kt,t)|^2+\frac{1}{b_0}\frac{d}{dt}|u(0,t)|^2+\nu\,\|\partial_x u(x,t)\|^2_{L_2(0,kt)}$$

$$\le 2(C_1+C_3)|u(kt,t)|^2+2(C_2+C_4)|u(0,t)|^2+2C_5\|u(x,t)\|^2_{L_2(0,kt)},\ \ \forall\,t\in(0,T]. \tag{136}$$

Now we estimate the penultimate term from (136). Taking into account the interpolation inequality form ([25], Theorem 5.9, p.140–141), we will have

$$2C_5\|u(x,t)\|^2_{L_2(0,kt)}\le 2C_5K^2\|u\|_{H^1(0,kt)}\|u\|_{L_2(0,kt)}$$

$$\le 4C_5K^2\left[\|u\|_{L_2(0,kt)}+\|\partial_x u\|_{L_2(0,kt)}\right]\|u\|_{L_2(0,kt)}$$

$$=4C_5K^2\|u\|^2_{L_2(0,kt)}+4C_5K^2\|\partial_x u\|_{L_2(0,kt)}\|u\|_{L_2(0,kt)}$$

$$\le\frac{\nu}{2}\|\partial_x u\|^2_{L_2(0,kt)}+\left[4C_5K^2+\frac{8C_5^2K^4}{\nu}\right]\|u\|^2_{L_2(0,kt)}, \tag{137}$$

where K is the constant from Theorem 5.9, p.140–141 [25].

From (136)–(137), applying Gronwall's inequality, we obtain:

$$\|u_1(x,t)\|^2_{L_2(0,kt)}+|u_1(kt,t)|^2+|u_1(0,t)|^2\equiv 0,\ \forall\,t\in(0,t^*].$$

This means that $u^{(1)}(x,t)\equiv u^{(2)}(x,t)$ in $L_2(Q_{xt})$, $u^{(1)}(kt,t)\equiv u^{(2)}(kt,t)$ and $u^{(1)}(0,t)\equiv u^{(2)}(0,t)$ in $L_2(0,T)$, i.e. the solution to initial boundary value problem 1 (1)–(3) can be only one. The uniqueness is proved.

Thus, we have proved the main result of our work, Theorem 1.

12 Conclusion

In this work, in the Sobolev classes, we have established the solvability theorems for boundary value problem for the Burgers equation in a degenerating domain, the degeneration point of which is at the origin. Moveover, the moving part of the boundary obeys a linear law. The established results can be useful in the problems of modeling (a) nonlinear thermal fields in high voltage contact devices, (b) nonlinear processes of diffusion and propagation of foreign inclusions in the flows of water and atmospheric areas, etc.

References

[1] Benia Y., and Sadallah B.-K. *Existence of solutions to Burgers equations in domains that can be transformed into rectangles*, Electron. J. Diff. Equ., 157 (2016), 1-13.

[2] Benia Y., and Sadallah B.-K. *Existence of solutions to Burgers equations in a non-parabolic domain*, Electron. J. Diff. Equ., 20 (2018), 1-13.

[3] Burgers J.M. *The nonlinear diffusion equation. Asymptotic solutions and statistical problems*, D. Reidel Publishing Company, Dordrecht-Holland / Boston, USA, 1974.

[4] Vishik M.I., Fursikov A.V. *Mathematical problems of statistical hydromechanics*, M.: Nauka, 1980.

[5] Solonnikov V.A., Fasano A. *One-dimensional parabolic problem arising in the study of some free boundary problems*, Zap. Nauchn. Sem. POMI, 269 (2000), 322-338 (in Russian).

[6] Fasano A. *A One-dimensional Flow Problem in Porous Media with Hydrophile Grains*, Math. Meth. Appl. Sci., 22 (1999), 605-617.

[7] Fasano A., Solonnikov V. *Estimates of weighted Hölder norms of the solutions to a parabolic boundary value problem in an initially degenerate domain*, Rend. Mat. Acc. Lincei., S. 9, 13:1 (2002), 23–41.

[8] Fasano A., Solonnikov V. *Unsaturated incompressible flows in adsorbing porous media*, Math. Meth. Appl. Sci., 26 (2003), 1391–1419.

[9] Farina A., Preziosi L. *Non-isothermal injection moulding with resin cure and perform deformability*, Composites/A, 31 (2000), 1355–1372.

[10] Nouri Z., Bendaas S., Kadem H.E. *N wave and periodic wave solutions for Burgers equations*, International Journal of Analysis and Applications, 18:2 (2020), 304-318.

[11] Yang X.-J., Tenreiro Machado J.A. *A new fractal nonlinear Burgers' equation arising in the acoustic signals propagation*, Math. Meth. Appl. Sci., 42:18 (2019), 7539-7544.

[12] Selmi R., Chaabani A. *Well-posedness to 3D Burgers' equation in critical Gevrey Sobolev spaces*, Archiv der Mathematik, 112:6 (2019), 661-672.

[13] Molinet L., Pilod D., Vento S. *On well-posedness for some dispersive perturbations of Burgers' equation*, Annales De L Institute Henri Poincare – Analyse Non Lineaire, 35:7 (2018), 1719-1756.

[14] Rottmann-Matthes J. *Freezing similarity solutions in the multidimensional Burgers equation*, Nonlinearity, 30:12 (2017), 4558-4586.

[15] Zhu N., Liu Zh., Zhao K. *On the Boussinesq-Burgers equations driven by dynamic boundary conditions*, Journal of Differential Equations, 264:3 (2018), 2287-2309.

[16] Kim E.I., Omelchenko V.T., Kharin S.N. *Mathematical models of thermal processes in electrical contacts*, Academy of Sciences of the Kazakh SSR, Alma-Ata, 1977 (in Russian).

[17] Mitropol'ski Yu.A., Berezovskii A.A., Plotnizkii T.A. *Problems with free boundaries for a nonlinear evolutioonary equation in problems of metallurgy, medicine, ecology*, Ukr. Math. J., 44:1 (1992), 67-75 (in Russian).

[18] Verigin N.N. *On a class of hydromechanical problems for domains with movable boundaries*, Fluid dynamics with free boundaries, 46 (1980), 23-32 (in Russian).

[19] Kartashov E.M. *The problem of heat stroke in a domain with moving boundary based on new integral relations*, News of the Russian Academy of Sciences. Energetics, 4 (1997), 122-137 (in Russian).

[20] Amangaliyeva M.M., Jenaliyev M.T., Kosmakova M.T., Ramazanov M.I. *About Dirichlet boundary value problem for the heat equation in the infinite angular domain*, Boundary Value Problems, 2014:213 (2014), 1-21.

[21] Amangaliyeva M.M., Dzhenaliev M.T., Kosmakova M.T., Ramazanov M.I. *On one homogeneous problem for the heat equation in an infinite angular domain*, Sib. Math. J., 56:6 (2015), 982-995.

[22] Amangaliyeva M.M., Jenaliyev M.T., Kosmakova M.T., Ramazanov M.I. *On the solvability of nonhomogeneous boundary value problem for the Burgers equation in the angular domain and related integral equations*, Springer Proceedings in Mathematics and Statistics, 216 (2017), 123-141.

[23] Lions J.-L., Magenes E. *Problemes aux limites non homogenes et applications,* Vol. 1, Dunod, Paris, 1968.

[24] Riesz F., Sz.-Nagy Bela. *Lecons d'Analyse Fonctionelle*, Akademiai Kiado, Budapest, 1972.

[25] Adams R.A., Fournier J.J.F. *Sobolev spaces*, 2nd ed., Pure and Applied Mathematics (Amsterdam), V. 140, Elsevier/Academic Press, Amsterdam, 2003.

Жиеналиев М. Т., Ерғалиев М.Г. БҰРЫШТЫ ОБЛЫСТАҒЫ БЮРГЕРС ТЕҢДЕУІНЕ ҚОЙЫЛҒАН ДИНАМИКАЛЫҚ ШАРТТАРЫ БАР ШЕКАРАЛЫҚ ЕСЕП ТУРАЛЫ

Бұған дейін [1] және [2] жұмыстарында Бюргерс теңдеуі үшін Дирихле шекаралық есебінің қисынды шешімділігі орнатылған. Ол жұмыстардан осы жұмыстың ерекшелігі біз Соболев кеңістігінде және бұрышты облыста Бюргерс теңдеуі үшін динамикалық шекаралық шарттары бар шекаралық есептің қисынды шешімділігін көрсетеміз. Функционалдық талдау, априорлы бағалаулар және Фаедо-Галеркин әдісі қолданылады.

Кілттік сөздер. Бюргерс теңдеуі, Соболев кеңістігі, азғындалатын облыс, динамикалық шекаралық шарттар, априорлы бағалаулар.

Дженалиев М.Т., Ергалиев М.Г. О ГРАНИЧНОЙ ЗАДАЧЕ С ДИНАМИЧЕСКИМИ УСЛОВИЯМИ ДЛЯ УРАВНЕНИЯ БЮРГЕРСА В УГЛОВОЙ ОБЛАСТИ

Ранее в работах [1] и [2] была установлена корректность граничной задачи Дирихле для уравнения Бюргерса. В отличие от этих работ, в пространствах Соболева и в угловой области мы показываем корректность граничной задачи для уравнения Бюргерса с динамическими граничными условиями. Используются методы функционального анализа, априорных оценок и Фаэдо-Галеркина.

Ключевые слова. Уравнение Бюргерса, пространство Соболева, вырождающаяся область, динамические граничные условия, априорные оценки.

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Some inequalities for best trigonometric approximation and Fourier coefficients in weighted Orlicz spaces

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Abstract. In this work the best approximation of functions in weighted Orlicz spaces have been investigated. Also, we study the inverse problem of approximation theory in weighted Orlicz spaces.

Keywords. Orlicz spaces, weighted Orlicz spaces, Boyd indices, modulus of smoothness, best approximation, inverse theorem.

1. Introduction

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$, $\mathbb{C}$ the complex plane. We denote by $L^p(\mathbb{T})$, $1 \le p < \infty$, the Lebesgue space of all measurable 2π-periodic functions, for which the norm

$$\|f\|_p = \left(\int_{\mathbb{T}} |f(x)|^p \, dx \right)^{1/p} < \infty.$$

A convex and continuous function $M : [0,\infty) \to. [0,\infty)$ which satisfies the four conditions $M(0) = 0$, $M(u) > 0$ for $u > 0$, $M(u)/u \to 0$ if $u \to 0$, and $M(u)/u \to \infty$ if $u \to \infty$ is called an *N-function.* The *complementary* *N-function* to M is defined by $N(v) = \max\{uv - M(u) : u \ge 0\}$ if $v \ge 0$. We will say that M satisfies the Δ_2-condition if $M(2u) \le cM(u)$ for any $u \ge u_0 \ge 0$ with some constant c, independent of u.

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \le p \le \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$.

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For a given Young function M, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f:\mathbb{T}\to\mathbb{C}$ for which

$$\int\limits_{\mathbb{T}} M\left(|f(x)|\right)dx<\infty.$$

Let N be the complementary Young function of M. It is well-known [1, p. 69], [2, pp. 52-68] that the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(\mathbb{T})}:=\sup\left\{\int\limits_{\mathbb{T}}|f(x)g(x)|\,dx:g\in\widetilde{L}_N(\mathbb{T}),\ \int\limits_{\mathbb{T}}N\left(|g(x)|\right)dx\le 1\right\},$$

becomes a Banach space. This space is denoted by $L_M(\mathbb{T})$ and is called an *Orlicz space* [1, p. 26]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces $L_p(\mathbb{T}),\ 1<p<\infty$.

If we choose $M(u)=u^p/p$ $(1<p<\infty)$, then the complementary function is $N(u)=u^q/q$ with $1/p+1/q=1$ and we have the relation

$$p^{-1/p}\|u\|_{L_p(\mathbb{T})}\le\|u\|_{L_M(\mathbb{T})}\le q^{1/q}\|u\|_{L_p(\mathbb{T})},$$

where $\|u\|_{L_p(\mathbb{T})}=\left(\int\limits_{\mathbb{T}}|u(x)|^p\,dx\right)^{1/p}$ denotes the usual norm of the space $L_p(\mathbb{T})$.

The Orlicz space $L_M(\mathbb{T})$ is *reflexive* if and only if the N-function M and its complementary function N both satisfy the Δ_2-condition [2, p. 113].

Let $M^{-1}:[0,\infty)\to[0,\infty)$ be the inverse function of the N-function M. The *lower* and *upper indices*

$$\alpha_M:=\lim_{t\to+\infty}-\frac{\log h(t)}{\log t},\ \beta_M:=\lim_{t\to o^+}-\frac{\log h(t)}{\log t}$$

of the function

$$h:(0,\infty)\to(0,\infty],\quad h(t):=\lim_{y\to\infty}\sup\frac{M^{-1}(y)}{M^{-1}(ty)},\ \ t>0,$$

first considered by Matuszewska and Orlicz [3], are called the *Boyd indices* of the Orlicz space $L_M(\mathbb{T})$.

It is known that the indices α_M and β_M satisfy $0\le\alpha_M\le\beta_M\le 1,\alpha_N+\beta_M=1$, $\alpha_M+\beta_N=1$ and the space $L_M(\mathbb{T})$ is reflexive if and only if $0<\alpha_M\le\beta_M<1$. The detailed information about the Boyd indices can be found in [4] - [7].

A measurable function $\omega:\mathbb{T}\to[0,\infty]$ is called a *weight function* if the set $\omega^{-1}\left(\{0,\infty\}\right)$ has Lebesgue measure zero. With any given weight ω we associate the ω-*weighted Orlicz space* $L_M(\mathbb{T},\omega)$ consisting of all measurable functions f on $\mathbb{T}$ such that

$$\|f\|_{L_M(\mathbb{T},\omega)}:=\|f\omega\|_{L_M(\mathbb{T})}<\infty.$$

Let $1<p<\infty,\ 1/p+1/p'=1$ and let ω be a weight function on $\mathbb{T}$. ω is said to satisfy *Muckenhoupt's A_p-condition* on $\mathbb{T}$ [8] - [10] if

$$\sup_J \left(\frac{1}{|J|}\int_J \omega^p(t)\,dt\right)^{1/p}\left(\frac{1}{|J|}\int_J \omega^{-p'}(t)\,dt\right)^{1/p'}<\infty,$$

where J is any subinterval of $\mathbb{T}$ and $|J|$ denotes its length.

Let us denote by $A_p(\mathbb{T})$ the set of all weight functions satisfying Muckenhoupt's A_p-condition on $\mathbb{T}$.

Note that by [11, Lemma 3.3], and [12, Section 2.3] if $L_M(\mathbb{T})$ is reflexive and ω weight function satisfies the condition $\omega\in A_{1/\alpha_M}(\mathbb{T})\cap A_{1/\beta_M}(\mathbb{T})$, then the space $L_M(\mathbb{T},\omega)$ is also reflexive.

Let $L_M(\mathbb{T},\omega)$ be a weighted Orlicz space, let $0<\alpha_M\le\beta_M<1$ and let $\omega\in A_{\frac{1}{\alpha_M}}(\mathbb{T})\cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. For $f\in L_M(\mathbb{T},\omega)$ we set

$$(\sigma_h f)(x):=\frac{1}{2h}\int_{-h}^{h} f(x+t)\,dt,\ 0<h<\pi,\ x\in\mathbb{T}.$$

By [10, Lemma 1] the shift operator σ_h is a bounded linear operator on $L_M(\mathbb{T},\omega)$:

$$\|\sigma_h(f)\|_{L_M(\mathbb{T},\omega)}\le C\|f\|_{L_M(\mathbb{T},\omega)}.$$

The function

$$\Omega^k_{M,\omega}(\delta,f):=\sup_{\substack{0<h_i\le\delta\\1\le i\le k}}\left\|\prod_{i=1}^{k}(I-\sigma_{h_i})f\right\|_{L_M(\mathbb{T},\omega)},\ \delta>0,\ k=1,2,...,$$

is called *k-th modulus of smoothness* of $f\in L_M(\mathbb{T},\omega)$, where I is the identity operator.

It can easily be shown that $\Omega^k_{M,\omega}(\cdot,f)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to0}\Omega^k_{M,\omega}(\delta,f)=0,\ \Omega^k_{M,\omega}(\delta,f+g)\le\Omega^k_{M,\omega}(\delta,f)+\Omega^k_{M,\omega}(\delta,g),\ \delta>0,$$

for $f,g\in L_M(\mathbb{T},\omega)$.

Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}A_k(x,f) \tag{1}$$

be the Fourier series of the function $f\in L_1(\mathbb{T})$, where

$A_k(x,f) := (a_k(f)\cos kx + b_k(f)\sin kx)$,
$a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f \in L_1(\mathbb{T})$.

We denote by $\prod_n$the class of trigonometric polynomials of degree at most n. The best approximation of $f \in L_M(\mathbb{T},\omega)$ by trigonometric polynomials is defined as

$$E_n(f)_{M,\omega} := \inf\left\{\|f - T_n\|_{L_M(\mathbb{T},\omega)} : T_n \in \prod\nolimits_n\right\}.$$

We use the constants $c, c_1, c_2, \ldots$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest.

To prove the main results we need the following theorem [10].

Theorem 1.1. *Let $L_M(\mathbb{T},\omega)$ be a weighted, Orlicz space with Boyd indices $0 < \alpha_M \le \beta_M < 1$, and let $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$.*

If $f \in L_M(\mathbb{T},\omega)$, then the inequality

$$\Omega^k_{M,\omega}\left(\frac{1}{n}, f\right) \le \frac{c_1}{n^{2k}}\left\{E_0(f)_{M,\omega} + \sum_{\nu=1}^{n} \nu^{2k-1} E_\nu(f)_{M,\omega}\right\} \tag{2}$$

holds with a constant $c_1 > 0$, independent of n.

2. Main Results

The problems of approximation theory in weighted and non-weighted Orlicz spaces have been investigated by several authors (see, for example, [10], [13] - [26]).

In this work we investigate the problem of the best approximation in the weighted Orlicz spaces. Also, we prove the inverse theorem of approximation theory in weighted Orlicz spaces. Similar approximation problems in the space of continuous functions have been investigated in [27], [28], [30] and [32]. Also, similar results in weighted generalized grand Lebesgue spaces and weighted Smirnov classes have been obtained in [29] and in [31], respectively.

Our main results are the following.

Theorem 2.1. *Let $L_M(\mathbb{T},\omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \le \beta_M < 1$, and let $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$, let*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{3}$$

be its Fourier series and let

$$\sum_{n=1}^{\infty} E_n(f)_{M,\omega} n^{\alpha-1} < \infty, \tag{4}$$

where $\alpha \in \mathbb{R}$.

Then the series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} n^{\alpha}\left(a_n \cos nx+b_n \sin nx\right) \tag{5}$$

is the Fourier series of the some function $\widetilde{f} \in L_M\ (\mathbb{T},\,\omega)$ *and for this* $\widetilde{f} \in L_M\,(\mathbb{T},\,\omega)$ *the estimates*

$$E_n(\widetilde{f})_{M,\omega} \;\leq\; c_2\left[E_n(f)_{M,\omega} n^{\alpha}+\sum_{k=n+1}^{\infty} E_k(f)_{M,\omega} k^{\alpha-1}\right] \qquad (n=1,2,\ldots), \tag{6}$$

$$E_0(\widetilde{f})_{M,\omega} \;\leq\; c_3\left[E_0(f)_{M,\omega}+\sum_{k=1}^{\infty} E_k(f)_{M,\omega} k^{\alpha-1}\right], \tag{7}$$

hold with a constant $c_2>0$, *which does not depend on* f *and* n.

Note that in [14, Theorem 5] we can find the proof of the inequality (6) for $\alpha=2r,\ r\in\mathbb{N}$, with another restrictions on function M and weight ω.

Corollary 2.2. *Under the conditions of Theorem 1.1 the estimate*

$$\Omega_{M,\omega}^{k}\left(\frac{1}{n},\widetilde{f}\right)\leq c_4\left\{\frac{1}{n^{2k}}\sum_{\nu=1}^{n}\nu^{2k+\alpha-1}E_{\nu-1}(f)_{M,\omega}+\sum_{s=n+1}^{\infty}s^{\alpha-1}E_s(f)_{M,\omega}\right\} \tag{8}$$

holds with a constant $c_4>0$, *which depends on* α *and* k.

Note that a similar estimate in the Lebesgue spaces for modulus of continuity was proved in [30]. Also, in [18, Theorem 1] was proved inequality (8) for $\alpha=r\in\mathbb{N}$, with the same restrictions on functions M and ω.

3. Proofs of the theorems

Proof of Theorem 2.1. Let s_n and $\widetilde{s}_n$ be the n-th partial sums of (3) and (5), respectively, and let $\mu_n=n^{\alpha}$ $(n=1,2,\ldots)$. Using Abel transformation, we find that

$$\widetilde{s}_m-f=\sum_{i=1}^{m-1}(s_i-f)\Delta\mu_i+(s_m-f)\ \mu_m \qquad (m=1,2,\ldots),$$

where $\Delta\mu_i=\mu_i-\mu_{i+1}$.
It is clear that $|\Delta\mu_i|\leq ci^{\alpha-1}$. Then for a fixed $n=1,2\ldots$ and for every $k=0,1\ldots$ we have

$$\widetilde{s}_{2^{k+1}n}-\widetilde{s}_{2^{k}n}=\sum_{i=2^{k}n}^{2^{k+1}n-1}(s_i-f)\Delta\mu_i+(s_{2^{k+1}n}-f)\mu_{2^{k+1}n}-(s_{2^{k}n}-f)\mu_{2^{k}n}. \tag{9}$$

Considering [14], the inequality

$$\|f - s_n\|_{L_M(\mathbb{T},\,\omega)} \leq c_5 E_n(f)_{M,\omega} \tag{10}$$

holds. Then from (9) and (10) we obtain

$$\left\|\widetilde{s}_{2^{k+1}n} - \widetilde{s}_{2^k n}\right\|_{L_M(\mathbb{T},\omega)}$$

$$\leq c_6 \sum_{l=2^k n}^{2^{k+1}n-1} E_l(f)_{M,\omega} l^{\alpha-1} + c_7 E_{2^k n}(f)_{M,\omega}(2^k n)^\alpha$$

$$\leq c_8 2^k n E_{2^k n}(f)_{M,\omega}(2^k n)^{\alpha-1} + c_9 (2^k n)^\alpha E_{2^k n}(f)_{M,\omega}$$

$$= c_{10}(2^k n)^\alpha E_{2^k n}(f)_{M,\omega}.$$

The last inequality yields

$$\sum_{k=0}^{\infty} \left\|\widetilde{s}_{2^{k+1}n} - \widetilde{s}_{2^k n}\right\|_{L_M(\mathbb{T},\omega)} \leq c_{11} \sum_{k=0}^{\infty} (2^k n)^\alpha E_{2^k n}(f)_{M,\omega}. \tag{11}$$

On the other hand the following inequality holds:

$$\sum_{k=1}^{\infty} (2^k n)^\alpha E_{2^k n}(f)_{M,\omega} \leq c_{12} \sum_{k=n+1}^{\infty} k^{\alpha-1} E_k(f)_{M,\omega}. \tag{12}$$

Consideration of (11) and (12) gives us

$$\sum_{k=0}^{\infty} \left\|\widetilde{s}_{2^{k+1}n} - \widetilde{s}_{2^k n}\right\|_{L_M(\mathbb{T},\omega)} \leq c_{13} \left[E_n(f)_{M,\omega} n^\alpha + \sum_{k=n+1}^{\infty} k^{\alpha-1} E_k(f)_{M,\omega} \right]. \tag{13}$$

By (4), it follows that the series

$$\widetilde{s}_n + \sum_{k=o}^{\infty} (\widetilde{s}_{2^{k+1}n} - \widetilde{s}_{2^k n})$$

converges in the sense of the metric $L_M(\mathbb{T},\omega)$ to some function $\widetilde{f} \in L_M(\mathbb{T},\omega)$. It is clear that the series (5) is the Fourier series of the function $\widetilde{f}$. We can write the following inequality

$$E_n(\widetilde{f})_{M,\omega} \leq \left\|\widetilde{f} - \widetilde{s}_n\right\|_{L_M(\mathbb{T},\omega)} \leq \sum_{k=0}^{\infty} \left\|\widetilde{s}_{2^{k+1}n} - \widetilde{s}_{2^k n}\right\|_{L_M(\mathbb{T},\,\omega)}.$$

Now combining (13) and last relation, we obtain the inequality (6) of Theorem 2.1.

Now, we estimate $E_0(\widetilde{f})_{M,\omega}$. The inequality

$$E_0(\widetilde{f})_{M,\omega} \leq \left\| \widetilde{f} - \frac{a_0}{2} \right\|_{L_M(\mathbb{T},\omega)} \leq \left\| \widetilde{f} - \widetilde{s}_1 \right\|_{L_M(\mathbb{T},\omega)} + \left\| \widetilde{s}_1 - \frac{a_0}{2} \right\|_{L_M(\mathbb{T},\omega)} \tag{14}$$

holds. From (10) and (6) we have

$$\left\| \widetilde{f} - \widetilde{s}_1 \right\|_{L_M(\mathbb{T},\omega)} \leq c_{14} E_1(\widetilde{f})_{M,\omega} \leq c_{15} \left[E_1(f)_{M,\omega} + \sum_{k=2}^{\infty} E_k(f)_{M,\omega} k^{\alpha-1} \right]. \tag{15}$$

It is known that

$$\left\| \widetilde{s}_1 - \frac{a_0}{2} \right\|_{L_M(\mathbb{T},\omega)} = \|a_1 \cos x + b_1 \sin x\|_{L_M(\mathbb{T},\omega)} \leq 2\pi \left(|a_1| + |b_1| \right). \tag{16}$$

We choose a number t_0, such that $\|f - t_0\|_{L_M(\mathbb{T},\omega)} = E_0(f)_{M,\omega}$. Then we obtain

$$\begin{aligned} \pi |a_1| &= \left| \int_0^{2\pi} f(x) \cos x dx \right| = \left| \int_0^{2\pi} [f(x) - t_0] \cos x dx \right| \\ &\leq c_{16} \|f - t_0\|_{L_M(\mathbb{T},\omega)} = c_{16} E_0(f)_{M,\omega}. \end{aligned}$$

The last inequality yields

$$|a_1| \leq \frac{c_{16}}{\pi} E_0(f)_{M,\omega}. \tag{17}$$

Similar to the above, we obtain

$$|b_1| \leq \frac{c_{17}}{\pi} E_0(f)_{M,\omega}. \tag{18}$$

Using (14),(15)-(18), we obtain the inequality (7) of Theorem 2.2.

Proof of Corollary 2.2. Taking into account the relations (2), (6) and (7), we get

$$\begin{aligned} \Omega^k_{M,\omega}\left(\frac{1}{n}, \widetilde{f}\right) &\leq \frac{c_{19}}{n^{2k}} \left\{ E_0(\widetilde{f})_{M,\omega} + \sum_{\nu=1}^{n} \nu^{2k-1} E_\nu(\widetilde{f})_{M,\omega} \right\} \\ &\leq \frac{c_{20}}{n^{2k}} \left[E_0(f)_{M,\omega} + \sum_{\nu=1}^{\infty} E_\nu(f)_{M,\omega} \nu^{\alpha-1} \right] \\ &+ \frac{c_{21}}{n^{2k}} \sum_{\nu=1}^{n} \nu^{2k-1} \left[E_\nu(f)_{M,\omega} \nu^{\alpha} + \sum_{s=\nu+1}^{\infty} s^{\alpha-1} E_s(f)_{M,\omega} \right] \end{aligned}$$

$$\leq \frac{c_{22}}{n^{2k}} \left[\sum_{\nu=1}^{n} \nu^{2k+\alpha-1} E_{\nu-1}(f)_{M,\omega} + \sum_{\nu=1}^{n} \nu^{2k-1} \sum_{s=\nu}^{\infty} s^{\alpha-1} E_s(f)_{M,\omega} \right]$$

$$\leq \frac{c_{23}}{n^{2k}} \left[\sum_{\nu=1}^{n} \nu^{2k+\alpha-1} E_{\nu-1}(f)_{M,\omega} \right]$$

$$+ \frac{c_{24}}{n^{2k}} \sum_{\nu=1}^{n} \nu^{2k-1} \left[\sum_{s=\nu}^{n} s^{\alpha-1} E_s(f)_{M,\omega} + \sum_{s=n+1}^{\infty} s^{\alpha-1} E_s(f)_{M,\omega} \right]$$

$$\leq c_{25} \left\{ \frac{1}{n^{2k}} \sum_{\nu=1}^{n} \nu^{2k+\alpha-1} E_{\nu-1}(f)_{M,\omega} + \frac{1}{n^{2k}} \sum_{s=1}^{n} s^{\alpha-1} E_s(f)_{M,\omega} \sum_{\nu=1}^{s} \nu^{2\alpha-1} \right\}$$

$$+ \sum_{s=n+1}^{\infty} s^{\alpha-1} E_s(f)_{M,\omega}$$

$$\leq c_{26} \left\{ \frac{1}{n^{2k}} \sum_{\nu=0}^{n} (\nu+1)^{2k+\alpha-1} E_{\nu}(f)_{M,\omega} + \frac{1}{n^{2k}} \sum_{s=1}^{n} s^{2k+\alpha-1} E_{\nu}(f)_{M,\omega} \right\}$$

$$+ \sum_{s=n+1}^{\infty} s^{\alpha-1} E_s(f)_{M,\omega}$$

$$\leq c_{27} \left\{ \frac{1}{n^{2k}} \sum_{\nu=1}^{n} \nu^{2k+\alpha-1} E_{\nu-1}(f)_{M,\omega} + \sum_{s=n+1}^{\infty} s^{\alpha-1} E_s(f)_{M,\omega} \right\},$$

which finishes the proof.

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References

[1] Krasnoselskii M.A. and Rutickii Ya.B. *Convex Functions and Orlicz Spaces*, P. Norrdhoff Ltd., Groningen, 1961.

[2] Rao M.M. and Ren Z.D. *Theory of Orlicz spaces,* Marcel Dekker, New York, 1991.

[3] Matuszewska W. and Orlicz W. *On certain properties of ϕ -functions*, Bull. Acad. Polon. Sci., Ser. Math. Aster. et Phys. 8:7 (1960), 439-443.

[4] Bottcher A. and Karlovich Yu.I. *Carleson curves, Muckenhoupt weights and Teoplitz operators*, Birkhauser - Verlag, 1997.

[5] Boyds D.W. *İndices for the Orlicz spaces*, Pacific J. Math., 38 (1971).

[6] Karlovich A.Yu. *Algebras of Singular integral operators with piecewise continuous coefficients on reflexive Orlicz spaces,* Math. Nachr., 179 (1996), 187-222.

[7] Maligranda L. *Indices and interpolation,* Dissertation Math., 234 (1985).

[8] Garnett J. B. *Bounded analytic functions*, Academic Press, New York, 1981.

[9] Gadjieva E.A. *Investigation of the properties of functions with quasimonotone Fourier coefficients in generalized Nikolskii-Besov spaces*, Author's summary of dissertation, Tbilisi, 1986 (in Russian).

[10] Israfilov D. M., Guven A. *Approximation by trigonometric polynomials in weighted Orlicz spaces*, Studia Mathematica, 174:2 (2006), 147-168.

[11] Karlovich A.Yu. *Singular Integral Operators with PC Coefficients in Reflexive Rearrangement Invariant Spaces,* Integ. Eq. and Oper. Th., 32 (1998), 436-481.

[12] Karlovich A.Yu. *Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces*, J. Integ. Eq. Appl., 15 (2003), 263-320.

[13] Akgün R. *Improved converse theorems and fractional moduli of smoothness in Orlicz spaces,* Bull. Malays. Math. Sci. Soc. (2), 36:1, (2013), 49-62.

[14] Akgün R., Israfilov D.M. *Approximation in weighted Orlicz spaces,* Math. Slovaca, 61:4 (2011), 601-618.

[15] Akgün R., Israfilov D.M. *Approximation and moduli of fractional orders in Smirnov-Orlicz classes*, Glas. Mat., Ser. III, 43(63):1 (2008), 121-136.

[16] Akgün R., Koç H. *Simultaneous approximation of functions in Orlicz spaces*, Complex Var. Elliptic Equ., 61:8 (2016), 1107-1115.

[17] Guven A., Israfilov D.M. *Polynomial approximation in Smirnov Orlicz classes*, Comput. Methods Funct. Theory, 2:2 (2002), 509-517.

[18] Guven A., Israfilov D.M. *On approximation in weighted Orlicz spaces*, Math. Slovaca, 62:1 (2012), 77-86.

[19] Jafarov S.Z. *Approximation by rational functions in Smirnov-Orlicz classes,* J. Math. Anal. Appl., 379 (2011), 870-877.

[20] Jafarov S.Z. *Approximation by polynomials and rational functions in Orlicz spaces,* J. Comp. Anal. Appl. (JoCAAA), 13-5 (2011), 953-962.

[21] Jafarov S.Z. *On approximation in weighted Smirnov- Orlicz classes*, Complex Var. Elliptic Equa., 57:5 (2012), 567-577.

[22] Jafarov S.Z. *The inverse theorem of approximation of the function in Smirnov-Orlicz classes*, Math. Inequal. Appl., 12:4 (2012), 835-844.

[23] Jafarov S.Z. *Approximation by Fejér sums of Fourier trigonometric series in weighted Orlicz spaces*, Hacet. J. Math. Stat., 42:3 (2013), 259-268.

[24] Kokilashvili V.M. *On analytic functions of Smirnov-Orlicz classes,* Studia Mathematica, 31 (1968), 43-59.

[25] Ponomarenko V.G. *Approximation of periodic functions in a Orlicz space*, Sibirsk. Math. J., 7 (1966), 1337-1346 (in Russian).

[26] Ramazanov A.-R.K. *On approximation by polynomials and rational functions in Orlicz spaces,* Analysis Mathematica, 10 (1984), 117-132.

[27] Ditzian Z. and Totik V. Moduli of smoothness, Springer Ser. Comput. Math., 9, Springer, New York, 1987.

[28] Devore R.A. and Lorentz G.G. *Constructive approximation*, Springer-Verlag, 1993.

[29] Jafarov S.Z. *Approximation in weighted generalized grand Lebesgue spaces*, Applied Mathematics E-Notes, 18 (2018), 140-147.

[30] Konyushkov A.A. *Best approximation by trigonometric polynomials and Fourier coefficients,* Mat. Sb. (N.S.), 44(86):1 (1958), 53-84 (in Russian).

[31] Israfilov D.M. *Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G,\omega)$ and the Bieberbach polynomials,* Constructive Approximation, 17:3 (2001), 335-351.

[32] Timan A.F. *Theory of approximation of functions of a real variable*, Pergamon Press and MacMillan, 1963 (Russian orginal published by Fizmatgiz, Moscow, 1960).

Жафаров С.З. САЛМАҚТЫ ОРЛИЧ КЕҢІСТІКТЕРІНДЕГІ ЕҢ ЖАҚСЫ ТРИГОНОМЕТРИЯЛЫҚ ЖУЫҚТАУ ҮШІН КЕЙБІР ТЕҢСІЗДІКТЕР ЖӘНЕ ФУРЬЕ КОЭФФИЦИЕНТТЕРІ

Бұл жұмыста салмақты Орлич кеңістіктеріндегі функцияның ең жақсы жуыаулары зерттеледі. Сонымен бірге салмақты Орлич кеңістіктеріндегі жуықтаулар теориясының кері есебі зерделенеді.

Кілттік сөздер. Орлич кеңістіктері, салмақты Орлич кеңістіктері, Бойд индексі, тегістік модулі, ең жақсы жуықтау, кері теорема.

Джафаров С.З. НЕКОТОРЫЕ НЕРАВЕНСТВА ДЛЯ НАИЛУЧШЕГО ТРИГОНОМЕТРИЧЕСКОГО ПРИБЛИЖЕНИЯ И КОЭФФИЦИЕНТЫ ФУРЬЕ В ВЕСОВЫХ ПРОСТРАНСТВАХ ОРЛИЧА

В этой работе исследуются наилучшие приближения функции в весовых пространствах Орлича. Также изучается обратная задача теории приближений в весовых пространствах Орлича.

Ключевые слова. Пространства Орлича, весовые пространства Орлича, индексы Бойда, модуль гладкости, наилучшее приближение, обратная теорема.

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