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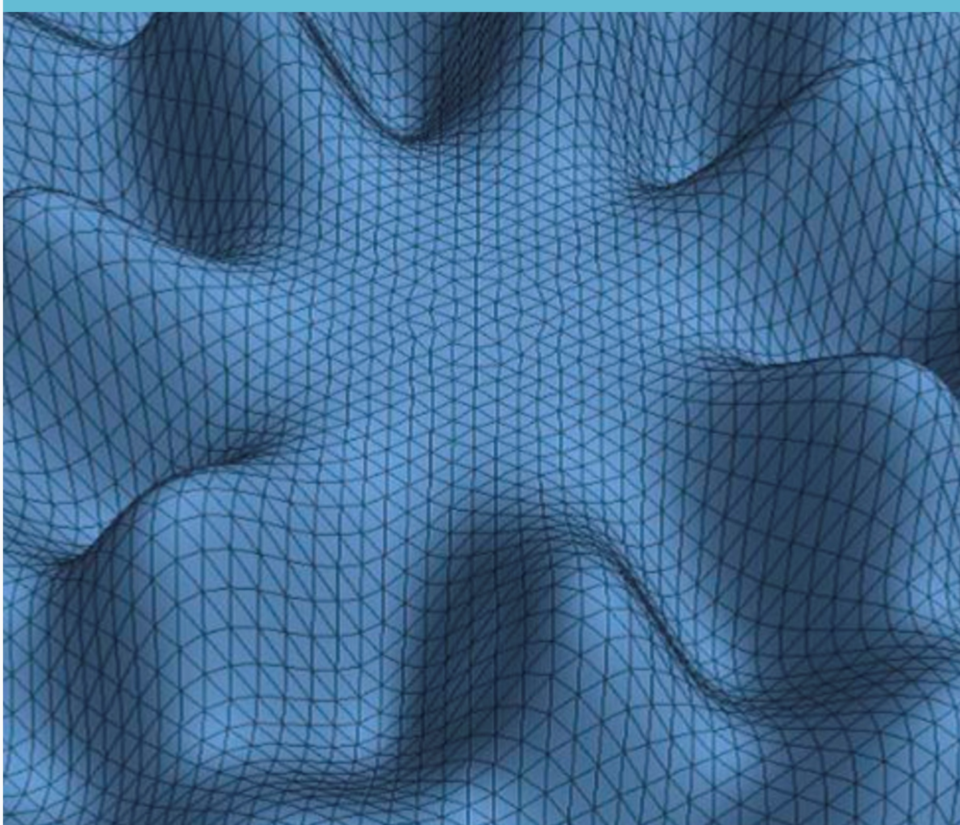

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Exact estimate of norm of integral operator with Oinarov condition

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Abstract. Criteria for the boundedness of integral operators satisfying the "Oinarov condition" in weighted Lebesgue spaces were obtained about thirty years ago. However, in these results, the norms of integral operators are estimated from below and from above by the same expressions, without exact calculation of the coefficients. For applications of these results to oscillatory and spectral problems of differential equations, the knowledge of these coefficients plays an important role. Therefore, this work is devoted to finding exact values of these coefficients.

Keywords. Integral operator, weight function, Hardy type inequality, kernel.

1 Introduction

Let $I = (0, \infty)$, $1 < p,\ q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let weight functions $v \geq 0$ and $\rho > 0$ satisfy the conditions $\rho,\ v \in L_1^{loc}(I)$ and $\rho^{1-p'} \in L_1^{loc}(I)$.

Consider the integral operator

$$Kf(x) = \int_0^x K(x,s) f(s) ds,\ x \in I,$$

where the kernel $K(\cdot,\cdot)$ is a continuous non-negative function increasing in the first argument, decreasing in the second argument and satisfying the condition: there exists a number $h \geq 1$ such that

$$K(x,s) \leq h(K(x,t) + K(t,s)) \tag{1}$$

for all $(x,t,s): \quad 0 < s \leq t \leq x < \infty$.

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Let $L_{p,\rho}(I)$, $1 < p < \infty$, be a space of all measurable and almost everywhere finite on I functions f with the norm

$$\|f\|_{p,\rho} = \left(\int_0^\infty \rho(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

We need the statement that follows from Theorem 5 given in [1, p. 48].

Theorem A. *Let $1 < p \leq q < \infty$ and μ be a Borel measure. Then the weighted Hardy inequality*

$$\left(\int_0^\infty v(x) \left| \int_0^x f(s)ds \right|^q d\mu(x) \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty \rho(t)|f(t)|^p dt \right)^{\frac{1}{p}}$$

holds for all functions $f \in L_{p,\rho}(I)$ if and only if

$$B = \sup_{x>0} [\mu([x,\infty)]^{\frac{1}{q}} \left(\int_0^x \rho^{1-p'}(t)dt \right)^{\frac{1}{p'}} < \infty.$$

Moreover, $B \leq C \leq p^{\frac{1}{q}} (p')^{\frac{1}{p'}} B$, where C is the best constant in the Hardy inequality.

Assume that

$$A_1 = \sup_{z>0} \left(\int_z^\infty v(x)dx \right)^{\frac{1}{q}} \left(\int_0^z K^{p'}(z,s)\rho^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$A_2 = \sup_{z>0} \left(\int_z^\infty v(x)K^q(x,z)dx \right)^{\frac{1}{q}} \left(\int_0^z \rho^{1-p'}(s)ds \right)^{\frac{1}{p'}}.$$

2 Main results

Theorem 1. *Let $1 < p \leq q < \infty$. The inequality*

$$\left(\int_0^\infty v(x) \left| \int_0^x K(x,s)f(s)ds \right|^q dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty \rho(t)|f(t)|^p dt \right)^{\frac{1}{p}} \tag{2}$$

holds for all functions $f \in L_{p,\rho}(I)$ if and only if $A = \max\{A_1, A_2\} < \infty$. Moreover, the estimate

$$A \leq C \leq (h+1)^3 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A \tag{3}$$

holds, where C is the best constant in (2).

Proof. *Necessity.* Let inequality (2) hold with the best constant $C > 0$.
Let $f_0(t) = \chi_{(\alpha,z)}(t)\rho^{1-p'}(t)$, where $\chi_{(\alpha,z)}(\cdot)$ is the characteristic function of the interval (α, z), $0 < \alpha < z < \infty$. Since

$$\int\limits_0^\infty \rho(t)|f_0(t)|^p dt = \int\limits_\alpha^z \rho(t)\rho^{p(1-p')}(t)dt = \int\limits_\alpha^z \rho^{1-p'}(t)dt < \infty,$$

then $f_0 \in L_{p,\rho}(I)$. Assuming $f \equiv f_0$ in (2), we have

$$C\left(\int\limits_0^\infty \rho(t)|f_0(t)|^p dt\right)^{\frac{1}{p}} = C\left(\int\limits_\alpha^z \rho^{1-p'}(t)dt\right)^{\frac{1}{p}}$$

$$\geq \left(\int\limits_0^\infty v(x)\left|\int\limits_0^x K(x,s)f_0(s)ds\right|^q dx\right)^{\frac{1}{q}} \geq \left(\int\limits_z^\infty v(x)\left|\int\limits_\alpha^z K(x,s)\rho^{1-p'}(s)ds\right|^q dx\right)^{\frac{1}{q}}$$

$$\geq \left(\int\limits_z^\infty v(x)K^q(x,z)dx\right)^{\frac{1}{q}} \int\limits_\alpha^z \rho^{1-p'}(s)ds.$$

This gives

$$\left(\int\limits_z^\infty v(x)K^q(x,z)dx\right)^{\frac{1}{q}} \left(\int\limits_\alpha^z \rho^{1-p'}(s)ds\right)^{\frac{1}{p'}} \leq C.$$

In the last inequality, the left-hand side does not depend on α and z, therefore, proceeding to limit when $\alpha \to 0$ and taking supremum with respect to z, we get

$$A_2 \leq C. \tag{4}$$

Now, we assume that $f_1(t) = \chi_{(\alpha,z)}(t)K^{p'-1}(z,t)\rho^{1-p'}(t)$. Then

$$\int\limits_0^\infty \rho(t)|f_1(t)|^p dt = \int\limits_\alpha^z \rho(t)K^{p'}(z,t)\rho^{p(1-p')}(t)dt$$

$$= \int\limits_\alpha^z \rho^{1-p'}(t)K^{p'}(z,t)dt \leq K^{p'}(z,\alpha)\int\limits_\alpha^z \rho^{1-p'}(t)dt < \infty.$$

Therefore, $f_1 \in L_{p,\rho}(I)$. Assuming $f \equiv f_1$ in (2), we have

$$\left(\int_z^\infty v(x)\left|\int_\alpha^z K(x,s)K^{p'-1}(z,s)\rho^{1-p'}(s)ds\right|^q dx\right)^{\frac{1}{q}} \le C\left(\int_\alpha^z K^{p'}(z,t)\rho^{1-p'}(t)dt\right)^{\frac{1}{p}}.$$

Using $K(x,s) \ge K(z,s)$, the latter yields

$$\left(\int_z^\infty v(x)dx\right)^{\frac{1}{q}}\left(\int_\alpha^z K^{p'}(z,s)\rho^{1-p'}(s)ds\right)^{\frac{1}{p'}} \le C$$

for all $0 < \alpha < z < \infty$. Hence, $A_1 \le C$, which, together with (4), gives

$$A \le C. \tag{5}$$

Sufficiency. Let $A < \infty$ and $f \in L_{p,\rho}(I)$, $f \ge 0$. Since the function $Kf(x)$, $x \in I$, is continuous and increasing, we find a sequence of points $\{x_k\}_{k>-\infty} \subset I$ such that $(h+1)^k = \int_0^{x_k} K(x_k,s)f(s)ds$. Then we have

$$\begin{aligned}
(h+1)^{k-1} &= (h+1)^k - h(h+1)^{k-1} \\
&= \int_0^{x_k} K(x_k,s)f(s)ds - h\int_0^{x_{k-1}} K(x_{k-1},s)f(s)ds \\
&= \int_{x_{k-1}}^{x_k} K(x_k,s)f(s)ds + \int_0^{x_{k-1}}\Big[K(x_k,s) - hK(x_{k-1},s)\Big]f(s)ds \\
&\le \int_{x_{k-1}}^{x_k} K(x_k,s)f(s)ds + hK(x_k,x_{k-1})\int_0^{x_{k-1}} f(s)ds.
\end{aligned} \tag{6}$$

Let us note that the last step in (6) is estimated by using (1). Now, using (6), we estimate the left-hand side of (2):

$$\left(\int_0^\infty v(x)\left|\int_0^x K(x,s)f(s)ds\right|^q dx\right)^{\frac{1}{q}} = \left(\sum_k \int_{x_k}^{x_{k+1}} v(x)\left|\int_0^x K(x,s)f(s)ds\right|^q dx\right)^{\frac{1}{q}}$$
$$\leq \left(\sum_k (h+1)^{q(k+1)} \int_{x_k}^{x_{k+1}} v(x)dx\right)^{\frac{1}{q}} = \left((h+1)^{2q}\sum_k (h+1)^{q(k-1)} \int_{x_k}^{x_{k+1}} v(x)dx\right)^{\frac{1}{q}}$$
$$= (h+1)^2\left(\sum_k \left(\int_{x_{k-1}}^{x_k} K(x_k,s)f(s)ds + hK(x_k,x_{k-1})\int_0^{x_{k-1}} f(s)ds\right)^q\right.$$
$$\left.\times \int_{x_k}^{x_{k+1}} v(x)dx\right)^{\frac{1}{q}} \leq (h+1)^2\left[\left(\sum_k \left(\int_{x_{k-1}}^{x_k} K(x_k,s)f(s)ds\right)^q \int_{x_k}^{x_{k+1}} v(x)dx\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\sum_k \int_{x_k}^{x_{k+1}} v(x)dx\, h^q K^q(x_k,x_{k-1})\left(\int_0^{x_{k-1}} f(s)ds\right)^q\right)^{\frac{1}{q}}\right]$$
$$= (h+1)^2\Big[J_1 + hJ_2\Big]. \tag{7}$$

Let us estimate J_1 and J_2 separately. To estimate J_1, we use Hölder's inequality and obtain

$$J_1 \leq \left(\sum_k \left(\int_{x_{k-1}}^{x_k} \rho(t)|f(t)|^p dt\right)^{\frac{q}{p}} \left(\int_{x_{k-1}}^{x_k} K^{p'}(x_k,s)\rho^{1-p'}(s)ds\right)^{\frac{q}{p'}} \int_{x_k}^{x_{k+1}} v(x)dx\right)^{\frac{1}{q}}$$
$$\leq \left(\sum_k \left(\int_{x_{k-1}}^{x_k} \rho(t)|f(t)|^p dt\right)^{\frac{q}{p}} \left(\int_0^{x_k} K^{p'}(x_k,s)\rho^{1-p'}(s)ds\right)^{\frac{q}{p'}} \int_{x_k}^{\infty} v(x)dx\right)^{\frac{1}{q}}$$
$$\leq A_1\left(\sum_k \left(\int_{x_{k-1}}^{x_k} \rho(t)|f(t)|^p dt\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \leq A_1\left(\sum_k \int_{x_{k-1}}^{x_k} \rho(t)|f(t)|^p dt\right)^{\frac{1}{p}} \tag{8}$$
$$\leq A_1\left(\int_0^\infty \rho(t)|f(t)|^p dt\right)^{\frac{1}{p}}.$$

We write the expression J_2 in the form

$$J_2 = \left(\sum_k \int\limits_{x_k}^{x_{k+1}} v(x) K^q(x_k, x_{k-1}) \left(\int\limits_0^{x_{k-1}} f(s)ds \right)^q dx \right)^{\frac{1}{q}} = \left(\int\limits_0^{\infty} \left(\int\limits_0^{x} f(s)ds \right)^q d\mu(x) \right)^{\frac{1}{q}}, \quad (9)$$

where $d\mu(x) = \sum\limits_k \int\limits_{x_k}^{x_{k+1}} v(x) K^q(x_k, x_{k-1}) \delta(x - x_{k-1}) dx$ and $\delta(\cdot)$ is the Dirac delta-function. Applying Theorem A to the right-hand side of (9), we have

$$\left(\int\limits_0^{\infty} \left(\int\limits_0^{x} f(s)ds \right)^q d\mu(x) \right)^{\frac{1}{q}}$$

$$\leq p^{\frac{1}{q}} (p')^{\frac{1}{p'}} \sup_{z>0} \left(\mu\big([z,\infty)\big)^{\frac{1}{q}} \left(\int\limits_0^{z} \rho^{1-p'}(s)ds \right)^{\frac{1}{p'}} \right) \left(\int\limits_0^{\infty} \rho(t)|f(t)|^p dt \right)^{\frac{1}{p}}. \quad (10)$$

Since

$$\mu\big([z,\infty)\big) = \int\limits_{[z,\infty)} d\mu(x) = \sum_{x_{k-1} \geq z} \int\limits_{x_k}^{x_{k+1}} v(x) K^q(x_k, x_{k-1}) dx$$

$$\leq \sum_{x_{k-1} \geq z} \int\limits_{x_k}^{x_{k+1}} v(x) K^q(x, z) dx \leq \int\limits_z^{\infty} K^q(x,z) v(x) dx,$$

from (9) and (10) we have

$$J_2 \leq p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A_2 \left(\int\limits_0^{\infty} \rho(t)|f(t)|^p dt \right)^{\frac{1}{p}}. \quad (11)$$

From (7), (8) and (11) we get

$$\left(\int\limits_0^{\infty} v(x) \left(\int\limits_0^{x} K(x,s) f(s)ds \right)^q dx \right)^{\frac{1}{q}} \leq (h+1)^3 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A \left(\int\limits_0^{\infty} \rho(t)|f(t)|^p dt \right)^{\frac{1}{p}}.$$

Therefore, inequality (2) holds with the estimate

$$C \leq (h+1)^3 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A \quad (12)$$

for the best constant C in (2). From (5) and (12) we get (3). The proof of Theorem 1 is complete.

Remark 1. *Theorem 1 was first announced in the paper [2]. Its complete proof was presented in the paper [3]. However, the value of the coefficient* $(h+1)^3 p^{\frac{1}{q}}(p')^{\frac{1}{p'}}$ *was not found in (3).*

Let us consider the operator

$$I_\alpha f(x) = \int_0^x (x-s)^\alpha f(s)ds, \ \alpha > 0.$$

The function $K(x,s) = (x-s)^\alpha \geq 0$ for $x \geq s$. Moreover, it increases with respect to x and decreases with respect to s for $0 < s \leq x$. In the case $0 < \alpha \leq 1$, we have

$$(x-s)^\alpha \leq (x-t)^\alpha + (t-s)^\alpha \ \text{ for } \ 0 < s \leq t \leq x \ \text{ and } \ h \equiv 1.$$

Hence, from Theorem 1 we have the following statement.

Corollary 1. *Let* $1 < p \leq q < \infty$ *and* $0 < \alpha \leq 1$. *The inequality*

$$\left(\int_0^\infty v(x) \left| \int_0^x (x-s)^\alpha f(s)ds \right|^q dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty \rho(t)|f(t)|^p dt \right)^{\frac{1}{p}}, \ \forall f \in L_{p,\rho} I, \tag{13}$$

holds if and only if $A_\alpha = \max\{A_{1,\alpha}, A_{2,\alpha}\} < \infty$. *Moreover, the estimate*

$$A_\alpha \leq C \leq 8 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A_\alpha \tag{14}$$

holds for the best constant C *in (13), where*

$$A_{1,\alpha} = \sup_{z>0} \left(\int_z^\infty v(x)dx \right)^{\frac{1}{q}} \left(\int_0^z (z-s)^{p'\alpha} \rho^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$A_{2,\alpha} = \sup_{z>0} \left(\int_z^\infty v(x)(x-z)^{q\alpha}dx \right)^{\frac{1}{q}} \left(\int_0^z \rho^{1-p'}(s)ds \right)^{\frac{1}{p'}}.$$

Let $\alpha > 1$. Then $(x-s)^\alpha \leq 2^{\alpha-1}\left[(x-t)^\alpha + (t-s)^\alpha\right]$ for $0 < s \leq t \leq x$ and $h = 2^{\alpha-1}$. In this case we get one more statement.

Corollary 2. *Let* $1 < p \leq q < \infty$ *and* $\alpha > 1$. *Inequality (13) holds if and only if* $A_\alpha = \max\{A_{1,\alpha}, A_{2,\alpha}\} < \infty$. *Moreover,*

$$A_\alpha \leq C \leq \left(2^{\alpha-1}+1\right)^3 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A_\alpha, \tag{15}$$

where C is the best constant in (13).

Remark 2. *The statements of Corollaries 1 and 2 were announced in [4] and proved in [5]. However, the numerical values of the coefficients were not found in (14) and (15).*

If $\alpha = n-1$, $n \geq 2$ and $p = q = 2$, then from Corollary 2 we have the following statement.

Corollary 3. *Let $n \geq 2$. The inequality*

$$\int_0^\infty v(x)\left|\int_0^x (x-s)^{n-1} f(s)ds\right|^2 dx \leq C\int_0^\infty \rho(t)|f(t)|^2 dt \tag{16}$$

holds if and only if $A_n = \max\{A_{1,n}, A_{2,n}\} < \infty$. *Moreover, the estimate*

$$A_n \leq C \leq 4\big(2^{n-2}+1\big)^6 A_n \tag{17}$$

holds for the best constant C in (16), where

$$A_{1,n} = \sup_{z>0}\int_z^\infty v(x)dx \int_0^z (z-s)^{2(n-1)}\rho^{-1}(s)ds,$$

$$A_{2,n} = \sup_{z>0}\int_z^\infty v(x)(x-z)^{2(n-1)}dx \int_0^z \rho^{-1}(s)ds.$$

For $n = 2$ inequality (16) with the estimate (17) has the form

$$\int_0^\infty v(x)\left|\int_0^x (x-s) f(s)ds\right|^2 dx \leq C\int_0^\infty \rho(t)|f(t)|^2 dt$$

with the estimate $A_2 \leq C \leq 256 A_2$.

Remark 3. *Let us note that in the mathematical literature condition (1) is often called "Oinarov condition".*

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Қалыбай А.А., Байарыстанов А.О. ОЙНАРОВ ШАРТЫ БАР ИНТЕГРАЛДЫ ОПЕРАТОРДЫҢ НОРМАСЫ ҮШІН НАҚТЫ БАҒА

Ядросы "Ойнаров шартын" қанағаттандыратын интегралдық оператордың салмақты Лебег кеңістерінде шенелімділігінің баламасы отыз жыл бұрын алынған. Бірақ та бұл нәтижеде интегралдық оператордың нормасы бір өрнекпен екі жақты бағаланып, бағалаудағы коэффициенттер шамасы есептелмеген болатын. Осы нәтижені дифференциалдық теңдеулердің тербелімдік, спектралдық есептеріне қолданғанда бұл коэффициенттердің мәндік шамасының орны өте зор. Сондықтан бұл мақала айтылған коэффициенттердің мәнін табуға арналған.

Кілттік сөздер. Интегралдық оператор, салмақты функция, Харди типті теңсіздік, өзек.

Калыбай А.А., Байарыстанов А.О. ТОЧНАЯ ОЦЕНКА НОРМЫ ИНТЕГРАЛЬНОГО ОПЕРАТОРА С УСЛОВИЕМ ОЙНАРОВА

Критерии ограниченности интегральных операторов в весовых пространствах Лебега, когда их ядра удовлетворяют "условию Ойнарова", были получены около тридцати лет назад. Однако в этих результатах нормы интегральных операторов оцениваются снизу и сверху одинаковыми выражениями, без точного подсчета коэффициентов. Для приложений этих результатов к осцилляционным и спектральным задачам дифференциальных уравнений знание этих коэффициентов играет важную роль. Поэтому данная работа посвящена нахождению точных значений этих коэффициентов.

Ключевые слова. Интегральный оператор, весовая функция, неравенство типа Харди, ядро.

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On a discrete Hilbert-Stieltjes inequality

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Abstract. In this paper we consider the problem of finding necessary and sufficient conditions for the fulfillment of a discrete inequality of the Hilbert-Stieltjes type. Moreover, an alternative proof of the discrete Hardy-type inequality with variable limits of summation is presented.

Keywords. Hardy-type inequality, boundedness, weighted Lebesgue spaces, Hilbert-Stieltjes type operator.

1 Introduction

Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $u = \{u_i\}_{i=1}^{\infty}$ be sequence of non-negative real numbers, $v = \{v_i\}_{i=1}^{\infty}$ be sequence of positive real numbers. Let l_{pv} be the space of sequences $f = \{f_i\}_{i=1}^{\infty}$ for which the following norm is finite

$$\|f\|_{p,v} = \|vf\|_p = \left(\sum_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty.$$

At the beginning of the 20th century, the famous Hilbert's double series inequality [1] of the following form was proved

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{f_k g_k}{k+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{n=1}^{\infty} f_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} g_n^{p'}\right)^{\frac{1}{p}}, \quad p > 1, \tag{1}$$

where $f_n, g_n \ge 0$, $\sum_{n=1}^{\infty} f_n^p < \infty$, $\sum_{n=1}^{\infty} g_n^{p'} < \infty$ and $\dfrac{\pi}{\sin(\pi/p)}$ is the best constant in (1) (see. [1]).

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The inequality (1) is equivalent to Hardy-Hilbert's inequality of the following form

$$\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{f_k}{n+k}\right)^p\right)^{\frac{1}{p}} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{n=1}^{\infty}f_n^p\right)^{\frac{1}{p}}, \quad f_n \geq 0. \tag{2}$$

The validity of the inequality (2) means the boundedness of the Hilbert operator:

$$(Hf)_n = \sum_{k=1}^{\infty}\frac{f_k}{k+n}$$

from l_p to l_p (see [2]). Note that a similar connection is kept between the integral analogues of the inequalities (1) and (2), with the best constant $\frac{\pi}{\sin(\pi/p)}$ (see [1], [2]).

The inequality (1) with its improvements has played a fundamental role in the development of many mathematical branches, and considerable attention has been paid to Hilbert's double series inequality with its integral version and inverse version, various improvements and extensions by many authors, see for instance [3–12]. In papers [13], [14] the boundedness of the Stieltjes integral operator of the following form

$$S_\gamma f(x) = \int_0^{\infty}\frac{f(t)}{(x+t)^\gamma}dt, \qquad x > 0, \qquad \gamma > 0,$$

in weighted Lebesgue spaces and the weighted estimates for its discrete analogue

$$(Sf)_n = \sum_{k=1}^{\infty}\frac{f_k}{(k+n)^\gamma}, \qquad \gamma > 0$$

are also established for cases $1 \leq p \leq q < \infty$ and $1 < q < p < \infty$, respectively. Moreover, in [15] the equivalence of four alternative conditions of the weighted integral inequality for Stieltjes operator is proved when $1 \leq p \leq q < \infty$. A similar result for the weighted integral Stieltjes inequality when $0 < q < p$, $1 < p < \infty$ was obtained in [16], where in particular, a new proof of a result of G. Sinnamon [14] is given, which also covers the case $0 < q < 1$.

In this paper we consider the generalized Hilbert-Stieltjes inequality of the following form

$$\left(\sum_{n=1}^{\infty}|u_n(Tf)_n|^q\right)^{\frac{1}{q}} \leq C\left(\sum_{n=1}^{\infty}|v_nf_n|^p\right)^{\frac{1}{p}}, \qquad \forall f \in l_{p,v}, \qquad 1 < p \leq q < \infty, \tag{3}$$

where

$$(Tf)_n = \sum_{k=1}^{\infty}\frac{f_k}{(b(k)+b(n))^\gamma} \tag{4}$$

is the Hilbert-Stieltjes type operator, $\gamma > 0$ and $b : N \to N$ is a non-decreasing mapping such that $b(1) = 1$, $\lim_{n\to\infty} b(n) = \infty$.

The aim of the work is to prove the weighted estimate (3) for the Hilbert-Stieltjes type operator (4).

Note that when $\gamma = 1$, $b(n) = n$, the inequality (3) coincides with the inequality (2). When $b(n) = n$, the operator T coincides with the discrete analogue of the Stieltjes operator.

The investigated operator (4) for $f = \{f\}_{i=1}^{\infty}$ non-negative sequences of real numbers can be represented as a sum of two discrete Hardy-type operators with upper and lower summation limits as follows

$$(Tf)_n = \sum_{k=1}^{\infty} \frac{f_k}{(b(k)+b(n))^{\gamma}} \approx \frac{1}{b^{\gamma}(n)} \sum_{k=1}^{b(n)} f_k + \sum_{k=b(n)}^{\infty} \frac{f_k}{b^{\gamma}(k)} = (T_1 f)_n + (T_2 f)_n, \tag{5}$$

then the inequality (3) is characterized by splitting it into two weighted Hardy-type inequalities for $f \geq 0$, and thus we obtain two different conditions.

In [17], [18] necessary and sufficient conditions of the validity of weighted inequalities (3) for discrete Hardy type operators of T_1, T_2 when $\gamma = 0$ are obtained. A similar problem for integral Hardy type operators was studied in a series papers [19–22].

From above-mentioned it follows that to obtain the main result of this paper (see Theorem 1), firstly we need to establish criteria for the fulfillment of discrete weighted Hardy type inequalities (see Theorems 2 and 3) for operators T_1, T_2 with variable summation limits of the following types

$$\left(\sum_{n=1}^{\infty} \left| u_n \sum_{k=1}^{b(n)} f_k \right|^q \right)^{\frac{1}{q}} \leq C \left(\sum_{k=1}^{\infty} |v_k f_k|^p\right)^{\frac{1}{p}}, \qquad \forall f \in l_{p,v}, \tag{6}$$

$$\left(\sum_{n=1}^{\infty} \left| u_n \sum_{k=b(n)}^{\infty} f_k \right|^q \right)^{\frac{1}{q}} \leq C \left(\sum_{k=1}^{\infty} |v_k f_k|^p\right)^{\frac{1}{p}}, \qquad \forall f \in l_{p,v}, \tag{7}$$

which have independent meanings. Note that in [17], [18] the condition on the sequence $\{b(n)\}$ differs from ours, where $\{b(n)\}$ is an increasing sequence of natural numbers and their method of the sufficiency proof is not applicable in our case. Here we use the method of localization, which previously was used in the paper [23].

Remark. In the sequel the symbol $M << K$ means that $M \leq cK$, where $c > 0$ is a constant depending only on unessential parameters. If $M << K << M$, then $M \approx K$. Also we will assume $\sum_{i=k}^{m} = 0$, if $m < k$.

2 Main results

Our result reads:

Theorem 1. *Let $1 < p \le q < \infty$. Then the inequality (4) holds if and only if $D = D_1 + D_2 < \infty$, where*

$$D_1 = \sup_{n\ge1} \left(\sum_{i=n}^{\infty} \left(\frac{u_i}{b^\gamma(i)} \right)^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{b(n)} v_k^{-p'} \right)^{\frac{1}{p'}},$$

$$D_2 = \sup_{n\ge1} \left(\sum_{i=1}^{n} u_i^q \right)^{\frac{1}{q}} \left(\sum_{k=b(n)}^{\infty} \left(\frac{v_k}{b^\gamma(k)} \right)^{-p'} \right)^{\frac{1}{p'}}.$$

Moreover $D \approx C$, where C is the best constant in (4).

Before to prove Theorem 1, we establish the criteria for the fulfillment of the inequalities (6) and (7).

Theorem 2. *Let $1 < p \le q < \infty$. Then the inequality (6) holds if and only if*

$$A = \sup_{n\ge1} \left(\sum_{i=n}^{\infty} u_i^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{b(n)} v_k^{-p'} \right)^{\frac{1}{p'}} < \infty. \tag{8}$$

Moreover $A \approx C$, where C is the best constant in (6).

Proof. *Necessity.* Let the inequality (6) hold for $\forall f \in l_{p,v}$ with a finite constant C, we show $A < \infty$. For $\forall m \in N$ take the test sequence $\bar{f}_k = \begin{cases} v_k^{-p'}, & 1 \le k \le b(m); \\ 0, & k > b(m). \end{cases}$

We substitute $\bar{f}$ in the inequality (6):

$$C\|\bar{f}\|_{p,v} = C \left(\sum_{k=1}^{b(n)} v_k^{-p'} \right)^{\frac{1}{p}} \ge \left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{b(n)} \bar{f}_k \right)^q \right)^{\frac{1}{q}}$$

$$\ge \left(\sum_{n=m}^{\infty} \left(u_n \sum_{k=1}^{b(m)} v_k^{-p'} \right)^q \right)^{\frac{1}{q}} = \left(\sum_{k=1}^{b(m)} v_k^{-p'} \right) \left(\sum_{n=m}^{\infty} u_n^q \right)^{\frac{1}{q}}. \tag{9}$$

From (9) it follows that

$$A = \sup_{m\ge1} \left(\sum_{k=1}^{b(m)} v_k^{-p'} \right)^{\frac{1}{p}} \left(\sum_{n=m}^{\infty} u_n^q \right)^{\frac{1}{q}} \le C < \infty. \tag{10}$$

Sufficiency. Let $A < \infty$ and $0 \le f \in l_{p,v}$.

For all $i \geq 1$ we define the following set: $T_i = \left\{k \in Z : 2^k \leq (Pf)_i := \sum_{j=1}^{b(i)} f_j\right\}$, $k_i = \max\ T_i$. Then

$$2^{k_i} \leq (Pf)_i < 2^{k_i+1}, \qquad i \geq 1. \tag{11}$$

Let $m_1 = 1$ and $M_1 = \{i \in N : k_i = k_1 = k_{m_1}\}$. Suppose that m_2 is such that $\sup M_1 + 1 = m_2$. Obviously $m_2 > m_1$ and if the set M_1 is upper bounded, then $m_2 < \infty$ and $m_2 - 1 = \max M_1 = \sup M_1$. Let us inductively define numbers $1 = m_1 < m_2 < ... < m_s < \infty$, $s \geq 1$. To define m_{s+1}, we assume that $m_{s+1} = \sup M_s + 1$, where $M_s = \{i \in N : k_i = k_{m_s}\}$.

Let $N_0 = \{s \in N : m_s < \infty\}$. Further, we assume $k_{m_s} = n_s$, $s \in N_0$. From the definition of m_s and from (11) it follows that, for $s \in N_0$

$$2^{n_s} \leq (Pf)_i < 2^{n_s+1}, m_s \leq i \leq m_{s+1} - 1 \tag{12}$$

and

$$N = \bigcup_{s \in N_0} [m_s, m_{s+1}), N = \bigcup_{s \in N_0} [b(m_s), b(m_{s+1})),$$

where $[m_s, m_{s+1}) \bigcap [m_l, m_{l+1}) = \emptyset, \quad [b(m_s), b(m_{s+1})) \bigcap [b(m_l), b(m_{l+1})) = \emptyset, \quad s \neq l.$

Hence

$$\|Pf\|_{q,u}^q = \sum_{s \in N_0} \sum_{j=m_s}^{m_{s+1}-1} (Pf)_j^q u_j^q = \sum_{j=m_1}^{m_2-1} (Pf)_j^q u_j^q + \sum_{j=m_2}^{m_3-1} (Pf)_j^q u_j^q + \sum_{s \geq 3} \sum_{j=m_s}^{m_{s+1}-1} (Pf)_j^q u_j^q. \tag{13}$$

We consider all three cases separately. Using (12) and Hölder's inequality, we obtain

$$\sum_{j=m_1}^{m_2-1} (Pf)_j^q u_j^q \leq \sum_{j=m_1}^{m_2-1} 2^{(n_1+1)q} u_j^q << 2^{qn_1} \sum_{j=m_1}^{m_2-1} u_j^q \leq (Pf)_{m_1}^q \sum_{j=m_1}^{m_2-1} u_j^q$$

$$\leq \left(\sum_{k=1}^{b(m_1)} f_k\right)^q \sum_{j=m_1}^{\infty} u_j^q \leq \left(\sum_{k=1}^{b(m_1)} v_k^{-p'}\right)^{\frac{q}{p'}} \sum_{j=m_1}^{\infty} u_j^q \left(\sum_{k=1}^{b(m_1)} (v_k f_k)^p\right)^{\frac{q}{p}} \leq A^q \|f\|_{p,v}^q. \tag{14}$$

$$\sum_{j=m_2}^{m_3-1} (Pf)_j^q u_j^q \leq 2^{(n_2+1)q} \sum_{j=m_2}^{m_3-1} u_j^q << 2^{qn_2} \sum_{j=m_2}^{m_3-1} u_j^q \leq (Pf)_{m_2}^q \sum_{j=m_2}^{m_3-1} u_j^q$$

$$\leq \left(\sum_{k=1}^{b(m_2)} f_k\right)^q \sum_{j=m_2}^{m_3-1} u_j^q \leq \left(\sum_{k=1}^{b(m_2)} v_k^{-p'}\right)^{\frac{q}{p'}} \sum_{j=m_2}^{\infty} u_j^q \left(\sum_{k=1}^{b(m_2)} (v_k f_k)^p\right)^{\frac{q}{p}} \leq A^q \|f\|_{p,v}^q. \tag{15}$$

To estimate the third term in (13) for $s \geq 3$, we first estimate 2^{n_s-1} using (12) and $n_{s-2} + 1 \leq n_s - 1$, which follows from $n_{s-2} < n_{s-1} < n_s$

$$2^{n_s-1} = 2^{n_s} - 2^{n_s-1} \leq 2^{n_s} - 2^{n_{s-2}+1} \leq (Pf)_{m_s} - (Pf)_{m_{s-1}-1} \leq \sum_{k=b(m_{s-1})}^{b(m_s)} f_k. \tag{16}$$

Using Hölder's and Jensen's inequalities, we get

$$\sum_{s\geq 3}\sum_{j=m_s}^{m_{s+1}-1}(Pf)_j^q u_j^q < \sum_{s\geq 3} 2^{(n_s+1)q}\sum_{j=m_s}^{m_{s+1}-1} u_j^q << \sum_{s\geq 3} 2^{(n_s-1)q}\sum_{j=m_s}^{m_{s+1}-1} u_j^q$$

$$\leq \sum_{s\geq 3}\left(\sum_{k=b(m_{s-1})}^{b(m_s)} f_k\right)^q \sum_{j=m_s}^{m_{s+1}-1} u_j^q \leq \sum_{s\geq 3}\left(\sum_{k=b(m_{s-1})}^{b(m_s)} |v_k f_k|^p\right)^{\frac{q}{p}}\left(\sum_{k=b(m_{s-1})}^{b(m_s)} v_k^{-p'}\right)^{\frac{q}{p'}}\sum_{j=m_s}^{m_{s+1}-1} u_j^q$$

$$\leq \left(\sum_{s\geq 3}\sum_{k=b(m_{s-1})}^{b(m_s)} |v_k f_k|^p\right)^{\frac{q}{p}}\left[\sup_{s\geq 1}\left(\sum_{k=1}^{b(m_s)} v_k^{-p'}\right)\left(\sum_{j=m_s}^{\infty} u_j^q\right)^{\frac{1}{q}}\right]^q << A^q\|vf\|_p^q. \tag{17}$$

Then from (13), (14), (15) and (17) it follows

$$\|Pf\|_{q,u} << A\|f\|_{p,v}, f\geq 0$$

and $C << A$, which together with (10) we get $C \approx A$.

Theorem 3. *Let $1 < p \leq q < \infty$. Then the inequality (7) holds if and only if*

$$B = \sup_{n\geq 1}\left(\sum_{i=1}^{n} u_i^q\right)^{\frac{1}{q}}\left(\sum_{k=b(n)}^{\infty} v_k^{-p'}\right)^{\frac{1}{p'}} < \infty. \tag{18}$$

Moreover $C \approx B$, where C is the best constant in (7).

Proof. *Necessity.*

Let us show that $B < \infty$, when inequality (7) holds $\forall f \in l_{p,v}$ $\forall m, M \in N : b(m) \leq M$, we assume that $\bar{f}_k = \begin{cases} v_k^{-p'}, & b(m)\leq k\leq M; \\ 0, & 1\leq k < b(m). \end{cases}$ Substituting $\bar{f}$ into the inequality (7), we have

$$C\|\bar{f}\|_{p,v} = C\left(\sum_{k=b(m)}^{M} v_k^{-p'}\right)^{\frac{1}{p}} \geq \left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=b(n)}^{\infty}\bar{f}_k\right)^q\right)^{\frac{1}{q}} \geq$$

$$\geq \left(\sum_{n=1}^{m}\left(u_n\sum_{k=b(m)}^{M} v_k^{-p'}\right)^q\right)^{\frac{1}{q}} = \left(\sum_{k=b(m)}^{M} v_k^{-p'}\right)\left(\sum_{n=1}^{m} u_n^q\right)^{\frac{1}{q}}. \tag{19}$$

It follows from (19) taking into account that $\forall m, M \in N$

$$B = \sup_{m\geq 1}\left(\sum_{i=1}^{m} u_i^q\right)^{\frac{1}{q}}\left(\sum_{k=b(m)}^{\infty} v_k^{-p'}\right)^{\frac{1}{p'}} \leq C < \infty. \tag{20}$$

Sufficiency. Let $f \geq 0$. For all $i \geq 1$ we define the following set

$$T_i = \{k \in Z : (Qf)_i := \sum_{j=b(i)}^{\infty} f_j \leq 2^{-k}\}$$

and we assume that $k_i = \max T_i$. Then

$$2^{-(k_i+1)} < (Qf)_i \leq 2^{-k_i}, \qquad i \geq 1. \tag{21}$$

Let $m_1 = 1$ and $M_1 = \{i \in N : k_i = k_1 = k_{m_1}\}$. Suppose m_2 such that $\sup M_1 + 1 = m_2$. Obviously, $m_2 > m_1$ and if the set M_1 is bounded from above, then $m_2 < \infty$ and $m_2 - 1 = \max M_1 = \sup M_1$. We define inductively the numbers $1 = m_1 < m_2 < ... < m_s < \infty$, $s \geq 1$. For the definition m_{s+1}, assume that $m_{s+1} = \sup M_s + 1$, where $M_s = \{i \in N : k_i = m_s\}$. Let $N_0 = \{s \in N : m_s < \infty\}$. Further, we put $k_{m_s} = n_s$, $s \in N_0$. It follows from the definition of m_s and (21) that for $s \in N_0$

$$2^{-(n_s+1)} < (Qf)_i \leq 2^{-n_s}, \quad m_s \leq i \leq m_{s+1} - 1 \tag{22}$$

and

$$N = \bigcup_{s \in N_0} [m_s, m_{s+1}), \quad N = \bigcup_{s \in N_0} [b(m_s), b(m_{s+1})),$$

where $[m_s, m_{s+1}) \bigcap [m_l, m_{l+1}) = \oslash$, $\quad [b(m_s), b(m_{s+1})) \bigcap [b(m_l), b(m_{l+1})) = \oslash$, $\quad s \neq l$.

Therefore

$$\|Qf\|_{q,u}^q = \sum_{s \in N_0} \sum_{j=m_s}^{m_{s+1}-1} (Qf)_j^q u_j^q \leq \sum_{s \in N_0} 2^{-n_s q} \sum_{j=m_s}^{m_{s+1}-1} u_j^q << \sum_{s \in N_0} 2^{-(n_s+2)q} \sum_{j=m_s}^{m_{s+1}-1} u_j^q. \tag{23}$$

Let us estimate the value 2^{n_s+2} using (22) and $n_s + 2 \leq n_{s+2}$, which follows from $n_s < n_{s+1} < n_{s+2}$

$$2^{-(n_s+2)} = 2^{-(n_s+1)} - 2^{-(n_s+2)} \leq 2^{-(n_s+1)} - 2^{-n_{s+2}} \leq$$

$$(Qf)_{m_{s+1}-1} - (Qf)_{m_{s+2}} \leq \sum_{j=b(m_{s+1}-1)}^{\infty} f_j - \sum_{j=b(m_{s+2})}^{\infty} f_j \leq \sum_{j=b(m_{s+1}-1)}^{b(m_{s+2})} f_j. \tag{24}$$

Applying (24) and Holder's inequality in (23), we obtain

$$\|Qf\|_{q,u}^q << \sum_{s \in N_0} 2^{-(n_s+2)q} \sum_{j=m_s}^{m_{s+1}-1} u_j^q \leq \sum_{s \in N_0} \left(\sum_{i=b(m_{s+1}-1)}^{b(m_{s+2})} f_i \right)^q \sum_{j=m_s}^{m_{s+1}-1} u_j^q$$

$$\leq \sum_{s \in N_0} \left(\sum_{i=b(m_{s+1}-1)}^{b(m_{s+2})} (v_i f_i)^p \right)^{\frac{q}{p}} \left(\sum_{b(m_{s+1}-1)}^{\infty} v_i^{-p'} \right)^{\frac{q}{p'}} \sum_{j=1}^{m_{s+1}-1} u_j^q << B^q \|vf\|_p^q.$$

Hence $C << B$ together with (20) we obtain $C \approx B$. Theorem 3 is proved.

Using Theorems 2 and 3, we present the proof of the main result:

Proof of Theorem 1. It follows from condition (5) that the fulfillment of inequality (3) is equivalent to the fulfillment of the weighted Hardy-type inequalities of the following form

$$\left(\sum_{n=1}^{\infty}\left|\frac{u_n}{b^{\gamma}(n)}\sum_{k=1}^{b(n)} f_k\right|^q\right)^{\frac{1}{q}} \leq C_1\left(\sum_{k=1}^{\infty}|v_k f_k|^p\right)^{\frac{1}{p}}, \qquad \forall f \in l_{p,v} \tag{25}$$

and

$$\left(\sum_{n=1}^{\infty}\left|u_n\sum_{k=b(n)}^{\infty} \frac{f_k}{b^{\gamma}(k)}\right|^q\right)^{\frac{1}{q}} \leq C_2\left(\sum_{k=1}^{\infty}|v_k f_k|^p\right)^{\frac{1}{p}}, \qquad \forall f \in l_{p,v} \tag{26}$$

and $C \approx C_1 + C_2$, where C_1, C_2 are the best constants of inequalities (25) and (26), respectively. Then, by Theorem 2 and Theorem 3, inequalities (25) and (26) hold, respectively, if and only if

$$C_1 \approx D_1 = \sup_{n\geq 1}\left(\sum_{i=n}^{\infty}\left(\frac{u_i}{b^{\gamma}(i)}\right)^q\right)^{\frac{1}{q}}\left(\sum_{k=1}^{b(n)} v_k^{-p'}\right)^{\frac{1}{p'}} < \infty,$$

$$C_2 \approx D_2 = \sup_{n\geq 1}\left(\sum_{i=1}^{n} u_i^q\right)^{\frac{1}{q}}\left(\sum_{k=b(n)}^{\infty}\left(\frac{v_k}{b^{\gamma}(k)}\right)^{-p'}\right)^{\frac{1}{p'}} < \infty.$$

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Темирханова А.М., Бесжанова А.Т. ДИСКРЕТТІ ГИЛЬБЕРТ-СТИЛЬТЬЕС ТЕҢСІЗДІГІ ТУРАЛЫ

Бұл жұмыста Гильберт-Стильтьес типті дискретті теңсіздіктің орындалуының қажетті және жеткілікті шарттары қарастырылады. Одан басқа, қосындылау шектері айнымалы болатын Харди типтес дискретті теңсіздіктің дәлелдеуінің балама әдісі келтірілген.

Кілттік сөздер. Харди типтес теңсіздік, оператордың шенелімдігі, Лебег салмақты кеңістіктері, Гильберт-Стильтьес типті оператор.

Темирханова А.М., Бесжанова А.Т. ОБ ОДНОМ ДИСКРЕТНОМ НЕРАВЕНСТВЕ ГИЛЬБЕРТА-СТИЛЬТЬЕСА

В работе рассматривается задача о нахождении необходимых и достаточных условий выполнения дискретного неравенства типа Гильберта-Стилтьеса. Кроме того, приводится альтернативный способ доказательства дискретного неравенства типа Харди с переменными пределами суммирования.

Ключевые слова. Неравенство типа Харди, ограниченность оператора, весовые пространства Лебега, оператор типа Гильберта-Стилтьеса.

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An impulsive system with unpredictable oscillations

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Abstract. A new type of oscillations of discontinuous unpredictable solutions for linear inhomogeneous systems with impulsive actions is considered. The moments of impulses of investigated systems constitute a newly determined unpredictable discrete set. The models under investigation admit unpredictable perturbations. Sufficient conditions for the existence and the uniqueness of asymptotically stable discontinuous unpredictable solutions are provided. For constructive definitions of unpredictable components in examples, randomly determined unpredictable sequences are utilized. The set of discontinuity moments is realized by the logistic map. Examples with simulations are given to illustrate the results.

Keywords. Discontinuous unpredictable function, Linear impulsive system, Bernoulli process, Asymptotic stability.

1 Introduction and preliminaries

Oscillations are functions, which are of indisputable importance for applications. This is why they are in the focus not only of specialists in the field of applied mathematics and differential equations, but also physicists and neural network experts. A new type of oscillation, called an unpredictable trajectory, was introduced in the paper [1]. An unpredictable trajectory is necessarily positively Poisson stable, and one of its distinctive features is the emergence of chaos in the corresponding quasi-minimal set. The type of chaos based on the presence of an unpredictable trajectory is called Poincaré chaos [1].

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Unpredictable functions continue the line of oscillations such as periodic, almost periodic, recurrent and Poisson stable motions, which are basic oscillations considered in theory of differential equations. Different types of differential equations with unpredictable solutions were studied in the papers [2–6] and in the book [7]. Recently it is proved in [8, 9] that unpredictable motions take place also in random processes.

There are papers [10–16], which investigate discontinuous periodic and almost periodic oscillations in various types of differential equations. In this paper, using the techniques presented in [2–4, 7] and results on the theory of impulsive differential equations [17, 18], the existence, uniqueness, and stability of discontinuous unpredictable solutions of linear impulsive systems are studied. To construct an unpredictable function, an unpredictable sequence resulting from a randomly defined discrete Bernoulli process [8] is utilized.

Throughout the work, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of natural numbers, integers and real numbers, respectively. Moreover, we make use of the usual Euclidean norm for vectors and the spectral norm for square matrices [19].

Denote by $\mathcal{R}$ the set of all functions defined on the real axis. They are continuous, except countable sets of points. At the points the functions admit one-sided limits. The sets of points do not necessarily coincide, if functions are different. The sets of points do not have finite accumulation points and are unbounded on both sides.

The functions $g(t)$ and $h(t)$ from $\mathcal{R}$, are said to be ϵ-equivalent on interval $S \subseteq \mathbb{R}$, if the points of discontinuity of the functions $g(t)$ and $h(t)$ in S can be respectively numerated θ_k^g and θ_k^h, $k = 1, 2, \ldots, p$, such that $|\theta_k^g - \theta_k^h| < \epsilon$ for each $k = 1, 2, \ldots, n$, and $||g(t) - h(t)|| < \epsilon$ for each $t \in S$, except those between θ_k^g and θ_k^h for each k. In the case that g and h are ϵ-equivalent on S, we also say that the functions are in ϵ-neighborhoods of each other. The topology defined with the aid of such neighborhoods is called the *B-topology* [17].

In what follows, we will denote by $\widehat{[d_1, d_2]}$, $d_1, d_2 \in \mathbb{R}$, the interval $[d_1, d_2]$, if $d_1 < d_2$ and the interval $[d_2, d_1]$, if $d_2 < d_1$.

Let θ_k, $k \in \mathbb{Z}$, be a sequence of real numbers such that $\underline{\theta} \leq \theta_{k+1} - \theta_k \leq \overline{\theta}$ for some positive numbers $\underline{\theta}$, $\overline{\theta}$, and $|\theta_k| \to \infty$ as $|k| \to \infty$.

Definition 1. *A piecewise continuous and bounded function $\varphi(t) : \mathbb{R} \to \mathbb{R}^p$ with the set of discontinuity points θ_k, $k \in \mathbb{Z}$, satisfying $\varphi(\theta_k-) = \varphi(\theta_k)$ for each $k \in \mathbb{Z}$ is called discontinuous unpredictable function (d.u.f.) if there exist positive numbers ϵ_0, σ, sequences t_n, s_n of real numbers and sequences l_n, m_n of integers all of which diverge to infinity such that*

- *(a) $|\theta_{k+l_n} - t_n - \theta_k| \to 0$ as $n \to \infty$ on each bounded interval of integers and $|\theta_{m_n+l_n} - t_n - \theta_{m_n}| \geq \epsilon_0$ for each natural number n;*
- *(b) for every positive number ϵ there exists a positive number δ such that $\|\varphi(t_1) - \varphi(t_2)\| < \epsilon$ whenever the points t_1 and t_2 belong to the same interval of continuity and $|t_1 - t_2| < \delta$;*
- *(c) $\varphi(t + t_n) \to \varphi(t)$ as $n \to \infty$ in B-topology on each bounded interval;*

(d) for each natural number n there exists an interval $[s_n-\sigma, s_n+\sigma] \subseteq [\theta_{m_n}, \widehat{(\theta_{m_n+l_n}} - t_n)]$ which does not contain any point of discontinuity of $\varphi(t)$ and $\varphi(t+t_n)$, and $\|\varphi(t+t_n)-\varphi(t)\| \geq \epsilon_0$ for each $t \in [s_n-\sigma, s_n+\sigma]$.

In Definition 1, we call the property (b) *conditional uniform continuity of* φ, the property (c) *Poisson stability of* φ, and the property (d) *unpredictability of* φ.

The sequence $\theta_k, k \in \mathbb{Z}$, is said to be *an unpredictable discrete set* if the condition (a) is satisfied.

Definition 2. *Suppose that $\psi(t) : \mathbb{R} \to \mathbb{R}^p$ is a piecewise continuous and bounded function with the set of discontinuity points θ_k, $k \in \mathbb{Z}$, satisfying $\psi(\theta_k-) = \psi(\theta_k)$ and Γ_k, $k \in \mathbb{Z}$, is a bounded sequence in $\mathbb{R}^p$. The couple $(\psi(t), \Gamma_k)$ is called unpredictable if there exist positive numbers ϵ_0, σ, sequences t_n, s_n of real numbers and sequences l_n, m_n of integers all of which diverge to infinity such that*

(a) $|\theta_{k+l_n} - t_n - \theta_k| \to 0$ as $n \to \infty$ on each bounded interval of integers and $|\theta_{m_n+l_n} - t_n - \theta_{m_n}| \geq \epsilon_0$ for each natural number n;

(b) for every positive number ϵ there exists a positive number δ such that $\|\psi(t_1) - \psi(t_2)\| < \epsilon$ whenever the points t_1 and t_2 belong to the same interval of continuity and $|t_1 - t_2| < \delta$;

(c) $\psi(t+t_n) \to \psi(t)$ as $n \to \infty$ in B-topology on each bounded interval;

(d) for each natural number n there exists an interval $[s_n-\sigma, s_n+\sigma] \subseteq [\theta_{m_n}, \widehat{(\theta_{m_n+l_n}} - t_n)]$ which does not contain any point of discontinuity of $\psi(t)$ and $\psi(t+t_n)$, and $\|\psi(t+t_n) - \psi(t)\| \geq \epsilon_0$ for each $t \in [s_n-\sigma, s_n+\sigma]$;

(e) $|\Gamma_{k+l_n} - \Gamma_k| \to 0$ as $n \to \infty$ for each k in bounded intervals of integers and $|\Gamma_{m_n+l_n} - \Gamma_{m_n}| \geq \epsilon_0$ for each natural number n.

If the couple $(\psi(t), \Gamma_k)$ is unpredictable in the sense of Definition 2, then $\psi(t)$ is a discontinuous unpredictable function in the sense of Definition 1.

Definition 1 does not follow from the Definition 2, since one cannot obtain the former just by diminishing the terms Γ_k. The sequence of zeros is not an unpredictable sequence. Consequently, both definitions are needed in the paper.

According to the purpose of the present study, we specify the discontinuity moments of the impulsive systems that will be investigated as

$$\theta_k = kT + \gamma_k, \ k \in \mathbb{Z}, \tag{1}$$

where γ_k, $k \in \mathbb{Z}$, is a sequence of real numbers which is unpredictable in the sense of Definition 3.1 [4], and $T \geq 4$ is a number such that $\sup\limits_{k\in\mathbb{Z}} |\gamma_k| < T/h$ for some number $h \geq 3$.

Since γ_k, $k \in \mathbb{Z}$, is an unpredictable sequence, there exist a positive number ϵ_0 and sequences ζ_n, η_n both of which diverge to infinity such that $|\gamma_{k+\zeta_n} - \gamma_k| \to 0$ as $n \to \infty$ for each k in bounded intervals of integers and $|\gamma_{\zeta_n+\eta_n} - \gamma_{\eta_n}| \geq \epsilon_0$ for each natural number n.

Let us show that the sequence θ_k, $k \in \mathbb{Z}$, is an unpredictable discrete set. More precisely, we will demonstrate that the property (a) mentioned in Definition 1 is valid for θ_k, $k \in \mathbb{Z}$, with $t_n = T\zeta_n$, $l_n = \zeta_n$, and $m_n = \eta_n$ for each natural number n. By these choices of the sequences t_n, l_n and m_n, we have that

$$|\theta_{k+l_n} - t_n - \theta_k| = |(k+\zeta_n)T + \gamma_{k+\zeta_n} - \zeta_n T - kT - \gamma_k| = |\gamma_{k+\zeta_n} - \gamma_k|.$$

Therefore, $|\theta_{k+l_n} - t_n - \theta_k| \to 0$ as $n \to \infty$ for each k in bounded intervals of integers. On the other hand,

$$\begin{aligned} |\theta_{m_n+l_n} - t_n - \theta_{m_n}| &= |(\eta_n + \zeta_n)T + \gamma_{\eta_n+\zeta_n} - \zeta_n T - \eta_n T - \gamma_{\eta_n}| \\ &= |\gamma_{\eta_n+\zeta_n} - \gamma_{\eta_n}| \geq \epsilon_0 \end{aligned}$$

for each natural number n.

Additionally, one can confirm that θ_k, $k \in \mathbb{Z}$, defined by (1) satisfies the inequality $\underline{\theta} \leq \theta_{k+1} - \theta_k \leq \overline{\theta}$ with $\underline{\theta} = T - \dfrac{2T}{h}$ and $\overline{\theta} = T + \dfrac{2T}{h}$.

2 Linear systems with non-unpredictable impulses

The main object of the present section is linear impulsive system,

$$\begin{aligned} &x'(t) = Ax(t) + f(t),\ t \neq \theta_k, \\ &\Delta x|_{t=\theta_k} = Bx(\theta_k), \end{aligned} \tag{2}$$

where $t \in \mathbb{R}$, the matrices $A \in \mathbb{R}^{p\times p}$ and $B \in \mathbb{R}^{p\times p}$ commute, the sequence θ_k, $k \in \mathbb{Z}$, of discontinuity moments is defined by equation (1), and $f(t) : \mathbb{R} \to \mathbb{R}^p$ is a d.u.f. in the sense of Definition 1. We suppose that $\det(I + B) \neq 0$, where I is the $p \times p$ identity matrix.

Let us denote by $X(t, u)$ the Cauchy matrix of the following linear impulsive system associated with (2),

$$\begin{aligned} &x'(t) = Ax(t),\ t \neq \theta_k, \\ &\Delta x|_{t=\theta_k} = Bx(\theta_k). \end{aligned} \tag{3}$$

Since the matrices A and B commute, we have for $t > u$ that

$$X(t, u) = e^{A(t-u)} (I + B)^{k([u,t))}, \tag{4}$$

where $k([u, t))$ denotes the number of the terms of the sequence θ_k, $k \in \mathbb{Z}$, which belong to the interval $[u, t)$, and $X(u, u) = I$ [18].

Let us denote by λ_j, $j = 1, 2, \dots, p$, the eigenvalues of the matrix $A + \frac{1}{T}\mathrm{Ln}(I + B)$. The following condition on the system (2) is required:

(C) $\max\limits_j \Re e\lambda_j = \lambda < 0$, where $\Re e\lambda_j$ is the real part of λ_j for each $j = 1, 2, \dots, p$.

In consequence of (4), under the condition (C), there exist numbers $K \geq 1$ and $0 < \alpha < -\lambda$ such that

$$\|X(t,u)\| \leq Ke^{-\alpha(t-u)} \tag{5}$$

for $t \geq u$ [17, 18].

Let us prove the following auxiliary assertion.

Lemma 1. *Assume that the condition (C) is fulfilled, then the following inequality*

$$\|X(t+t_n, u+t_n) - X(t,u)\| \leq K_0 e^{-\alpha(t-u)} \tag{6}$$

holds, where $K_0 = K \max(1, \|B\|)$.

Proof. By using (4) and (5), we can show that

$$\|X(t+t_n, u+t_n) - X(t,u)\| \leq \left\| e^{A(t-u)} (I+B)^{k([u+t_n, t+t_n))} - e^{A(t-u)} (I+B)^{k([u,t))} \right\|$$

$$\leq \left\| e^{A(t-u)} (I+B)^{k([u,t))} \right\| \left\| (I+B)^{|k([u+t_n, t+t_n)) - k([u,t))|} - I \right\|$$

$$\leq K \max(1, \|B\|)\, e^{-\alpha(t-u)}$$

for $t \geq u$. The lemma is proved. □

The following theorem is concerned with the discontinuous unpredictable solution of system (2).

Theorem 1. *Suppose that the condition (C) is valid. If $f(t)$ is a d.u.f. in the sense of Definition 1, then system (2) possesses a unique asymptotically stable discontinuous unpredictable solution.*

Proof. As it is known from the theory of impulsive differential equations [17, 18], according to the boundedness of the function $f(t)$, system (2) admits a unique solution $\varphi(t)$ which is bounded on the real axis and satisfies the equation

$$\varphi(t) = \int_{-\infty}^{t} X(t,u) f(u)\, du, \quad t \in \mathbb{R}. \tag{7}$$

One can verify for points of continuity that

$$\left\|\frac{d\varphi(t)}{dt}\right\| \leq \|A\| \int\limits_{-\infty}^{t} \|X(t,u)\| \, \|f(u)\| \, du + \|f(t)\| = \frac{\|A\| \, M_f K}{\alpha} + M_f, \tag{8}$$

where $M_f = \sup\limits_{t\in\mathbb{R}} \|f(t)\|$. Therefore, $\varphi(t)$ is a conditional uniform continuous function. The asymptotic stability of $\varphi(t)$ can be verified in a very similar way to the stability of a bounded solution mentioned in [17].

Since $f(t)$ is a d.u.f., there exist positive numbers ϵ_0, σ, sequences t_n, s_n of real numbers and sequences l_n, m_n of integers all of which diverge to infinity such that the properties (c) and (d) in Definition 1 hold for $f(t)$, i.e., when φ is replaced by f.

Let us check that the Poisson stability of $\varphi(t)$ is valid.

Fix an arbitrary positive number ϵ and an arbitrary compact interval $[a,b]$, where $b > a$. We will show for sufficiently large n that the inequality $\|\varphi(t+t_n) - \varphi(t)\| < \epsilon$ is satisfied for each t in $[a,b]$. Choose numbers $c < a$ and $\xi > 0$ such that

$$\frac{M_f(K_0 + 2K)}{\alpha} e^{-\alpha(a-c)} < \frac{\epsilon}{3}, \tag{9}$$

$$\frac{M_f(K_0 + 2K)\left(e^{\alpha\xi} - 1\right)}{\alpha\left(1 - e^{-\alpha\underline{\theta}}\right)} < \frac{\epsilon}{3}, \tag{10}$$

and

$$\frac{K\xi}{\alpha} < \frac{\epsilon}{3}. \tag{11}$$

Let n be a sufficiently large natural number such that $|\theta_{k+l_n} - t_n - \theta_k| < \xi$ for $\theta_k \in [c,b]$, $k \in \mathbb{Z}$, and $||f(t+t_n) - f(t)|| < \xi$ for $t \in [c,b]$. We assume without loss of generality that $\theta_k \leq \theta_{k+l_n}$. Additionally, suppose that

$$\theta_{m-1} \leq c \leq \theta_m < \cdots < \theta_q \leq t \leq \theta_{q+1}$$

for $m, q \in \mathbb{Z}$.

If $t \in [a,b]$, then we have

$$\|\varphi(t+t_n) - \varphi(t)\| \leq \int\limits_{-\infty}^{c} \|X(t+t_n, u+t_n) - X(t,u)\| \, \|f(u+t_n)\| \, du$$

$$+\int\limits_c^t \|X(t+t_n,u+t_n)-X(t,u)\|\,\|f(u+t_n)\|\,du$$

$$+\int\limits_{-\infty}^c \|X(t,u)\|\,\|f(u+t_n)-f(u)\|\,du+\int\limits_c^t \|X(t,u)\|\,\|f(u+t_n)-f(u)\|\,du.$$

Using (6), one can obtain

$$\int\limits_{-\infty}^c \|X(t+t_n,u+t_n)-X(t,u)\|\,\|f(u+t_n)\|\,du$$

$$\leq \int\limits_{-\infty}^c M_f K_0 e^{-\alpha(t-u)}du < \frac{M_f K_0}{\alpha} e^{-\alpha(a-c)}.$$

Moreover, we have

$$\int\limits_c^t \|X(t+t_n,u+t_n)-X(t,u)\|\,\|f(u+t_n)\|\,du$$

$$\leq \sum_{k=m}^{q} \int\limits_{\theta_k}^{\theta_{k+l_n}-t_n} M_f K_0 e^{-\alpha(t-u)}du \leq \sum_{k=m}^{q} \frac{M_f K_0}{\alpha} e^{-\alpha t}\left(e^{\alpha(\theta_{k+l_n}-t_n)}-e^{\alpha\theta_k}\right)$$

$$\leq \sum_{k=m}^{q} \frac{M_f K_0}{\alpha} e^{-\alpha(t-\theta_k)}\left(e^{\alpha\xi}-1\right)$$

$$< \frac{M_f K_0}{\alpha}\left(e^{\alpha\xi}-1\right)e^{-\alpha(t-\theta_q)}\sum_{k=m}^{q} e^{-\alpha(\theta_q-\theta_k)} < \frac{M_f K_0\left(e^{\alpha\xi}-1\right)}{\alpha\left(1-e^{-\alpha\underline{\theta}}\right)}.$$

Likewise, one can deduce that

$$\int\limits_{-\infty}^c \|X(t,u)\|\,\|f(u+t_n)-f(u)\|\,du+\int\limits_c^t \|X(t,u)\|\,\|f(u+t_n)-f(u)\|\,du$$

$$\leq \int_{-\infty}^{c} 2M_f K e^{-\alpha(t-u)} du + \int_{c}^{t} K\xi e^{-\alpha(t-u)} du + \sum_{k=m}^{q} \int_{\theta_k}^{\theta_{k+l_n}-t_n} 2M_f K e^{-\alpha(t-u)} du$$

$$< \frac{2M_f K}{\alpha} e^{-\alpha(a-c)} + \frac{K\xi}{\alpha} + \frac{2M_f K\left(e^{\alpha\xi}-1\right)}{\alpha\left(1-e^{-\alpha\underline{\theta}}\right)}.$$

Thus, $\|\varphi(t+t_n)-\varphi(t)\| < \epsilon$ for $t \in [a,b]$ in conformity with the inequalities (9)-(11) and therefore, $\varphi(t+t_n) \to \varphi(t)$ uniformly on each compact interval in B-topology.

The unpredictability property can be proved identically as in Theorem 2.2 [5].

3 Linear systems with unpredictable impulses

Consider the following linear impulsive system,

$$\begin{aligned} x'(t) &= Ax(t) + f(t),\ t \neq \theta_k, \\ \Delta x|_{t=\theta_k} &= Bx(\theta_k) + J_k, \end{aligned} \tag{12}$$

where $t \in \mathbb{R}$, the matrices $A \in \mathbb{R}^{p\times p}$ and $B \in \mathbb{R}^{p\times p}$ commute, the sequence θ_k, $k \in \mathbb{Z}$, of discontinuity moments is defined by equation (1), and $(f(t), J_k)$ is an unpredictable couple in the sense of Definition 2. Additionally, $\det(I+B) \neq 0$, where I is the $p \times p$ identity matrix.

It is worth noting that (12) is a linear impulsive system with unpredictable impulses, and it is not a particular case of system (2). Indeed, to introduce the perturbations J_k in the impulsive part, one must not only consider the sequence to be unpredictable but also assume that the sequences t_n and s_n proper for the unperturbed system have to be consistent with the new terms.

Theorem 2. *Suppose that the condition (C) is valid. If the couple $(f(t), J_k)$ is unpredictable in the sense of Definition 2, then system (12) possesses a unique asymptotically stable discontinuous unpredictable solution.*

The proof of the Theorem 2 is similar to that of Theorem 1.

4 Examples

Example 1. Consider the logistic map

$$\nu_{k+1} = \mu\nu_k(1-\nu_k),\ k \in \mathbb{Z} \tag{13}$$

with $\mu = 3.95$ in the interval $[0,1]$. Then there exists the unpredictable solution $\gamma_k, k \in \mathbb{Z}$, [2]. And there exist a positive number ϵ_0 and sequences ζ_n, η_n, both of which diverge to infinity such that $|\gamma_{k+\zeta_n} - \gamma_k| \to 0$ as $n \to \infty$, for each k in bounded intervals of integers and $|\gamma_{\eta_n+\zeta_n} - \gamma_{\eta_n}| \geq \epsilon_0$ for each $n \in \mathbb{N}$.

Let us consider the sequence $\theta_k, k \in \mathbb{Z}$ defined by

$$\theta_k = 4k + \gamma_k, \ k \in \mathbb{Z}. \tag{14}$$

Since the sequence (14) has the form (1) with $T = 4$, then there exist a positive number ϵ_0, a sequence $t_n = 4\zeta_n$ of real numbers and sequences $l_n = \zeta_n$, $m_n = \eta_n$ of integers all of which diverge to infinity such that $|\theta_{k+l_n} - t_n - \theta_k| \to 0$ as $n \to \infty$ for each k in bounded intervals of integers and $|\theta_{m_n+l_n} - t_n - \theta_{m_n}| \geq \epsilon_0$ for each natural number n. That is, the sequence is the unpredictable discrete set.

We utilize a realization of the Bernoulli process for construction of discontinuous unpredictable function by considering it as infinite sequences of two integers 3 and 5 with equal probability 1/2 such that according to Theorem 1, [8]. Then there exists an unpredictable sequence τ_k, $\tau_k = 3, 5,\ k \in \mathbb{Z}$, and there exist sequences ζ_n, η_n, $n \in \mathbb{N}$, of positive integers both of which diverge to infinity as $n \to \infty$ such that $\tau_{k+\zeta_n} = \tau_k$ for each k in bounded intervals of integers and $|\tau_{\zeta_n+\eta_n} - \tau_{\eta_n}| \geq \varepsilon_0 = |3-5| = 2$ for each natural number n.

Let $\chi(t) : \mathbb{R} \to \mathbb{R}$ be the function defined through the equation $\chi(t) = \tau_k$ for $t \in [\theta_k, \theta_{k+1})$, $k \in \mathbb{Z}$. We will show that $\chi(t)$ is a discontinuous unpredictable function in the sense of Definition 1.

One can show that if $t \in [\theta_k, \theta_{k+1})$, $k \in \mathbb{Z}$, then $t + t_n \in [\theta_{k+\zeta_n}, \theta_{k+1+\zeta_n})$, $k \in \mathbb{Z}$. For $t \in [\theta'_k, \theta'_{k+1})$, $k \in \mathbb{Z}$, it can be verified that $\theta'_k \leq t < \theta'_{k+1}$, and $\theta'_k + t_n \leq t + t_n < \theta'_{k+1} + t_n$, then $\theta'_{k+\zeta_n} \leq t + t_n < \theta'_{k+1+\zeta_n}$. That is, the discontinuity points of $\chi(t + t_n)$ are that ones for $\chi(t)$. Let us denote them $\theta'_k = \theta_{k+\zeta_n} - t_n$. Accordingly, for each $k \in \mathbb{Z}, n \in \mathbb{N}$ the value of function $\chi(t + t_n)$ is equal to $\tau_{k+\zeta_n}$. Hence, by using the unpredictability τ_k, we have that $\chi(t + t_n) \to \chi(t)$ as $n \to \infty$ in B-topology on each bounded interval. Moreover, the values of functions $\chi(t)$ and $\chi(t + t_n)$, on the corresponding intervals $[\theta_{\eta_n}, \theta_{\eta_n+1})$, and $[\theta_{\eta_n+\zeta_n}, \theta_{\eta_n+1+\zeta_n})$, for fixed n, are respectively equal to τ_{η_n} and $\tau_{\eta_n+\zeta_n}$. Consequently, we have that $|\chi(t + t_n) - \chi(t)| = |\tau_{\eta_n+\zeta_n} - \tau_{\eta_n}| \geq \epsilon_0 = 2$.

Thus, $\chi(t)$ is the discontinuous unpredictable function with positive numbers $\epsilon_0 = 2$, $\sigma = \frac{3}{2}$ and sequences $t_n = 4\zeta_n$, $s_n = \dfrac{\theta_{\eta_n} + \theta_{\eta_n+1}}{2}$.

Example 2. Let us consider the impulsive system,

$$\begin{aligned}
x'_1 &= -2x_2 + 4\chi^3(t),\ t \neq \theta_k, \\
x'_2 &= 2x_1 - 9\chi(t),\ t \neq \theta_k, \\
\Delta x_1|_{t=\theta_k} &= -\frac{80}{81}x_1 + 0.4\gamma_k, \\
\Delta x_2|_{t=\theta_k} &= -\frac{80}{81}x_2 - 0.6\gamma_k,
\end{aligned} \tag{15}$$

where $\chi(t)$ is the d.u.f. from Example 1, and

$$A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \ B = \begin{pmatrix} -\frac{80}{81} & 0 \\ 0 & -\frac{80}{81} \end{pmatrix}.$$

The couple $(f(t),\ J_k) = \left(\begin{pmatrix} 4\chi^3(t) \\ -9\chi(t) \end{pmatrix}, \begin{pmatrix} 0.4\gamma_k \\ -0.6\gamma_k \end{pmatrix} \right)$ is unpredictable in the sense of Definition 2 according to Lemmas 1.4 and 1.5 [5].

The matrices A and B commute, and the matrix

$$A + \frac{1}{T}\ \ln\,(I+B) = \begin{pmatrix} -\ln\ 3 & -2 \\ 2 & -\ln\ 3 \end{pmatrix},$$

has the eigenvalues $\lambda_{1,2} = -\ln\ 3 \pm 2i$. Inequality (5) is satisfied for system (15) with $\alpha = 1$ and $K = 2.5$. According to Theorem 2, there exists the unique asymptotically stable discontinuous unpredictable solution of system (15).

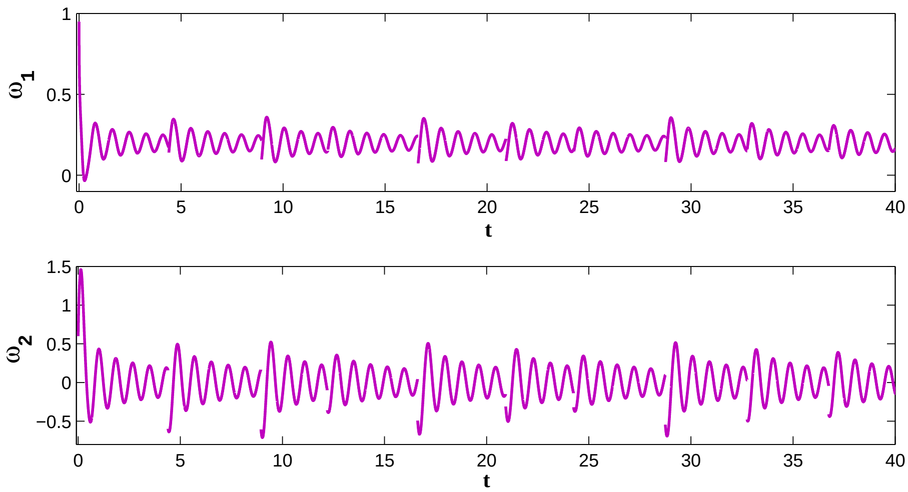

Figure 1 – The time series of the coordinates ω_1, ω_2 of the solution of system (15).

The simulation of the unpredictable solution $x(t)$ is not possible, because the initial value is not known precisely. For this reason, we will consider another solution $\omega(t) = (\omega_1(t), \omega_2(t))$, with initial value $\omega_1(0) = 0.95, \omega_2(0) = 0.6$. The graph of function $\omega(t)$ approaches to the discontinuous unpredictable solution $x(t)$ of the system (15), as t increases. Then, one can consider the graph of $\omega(t)$ instead of the curve of unpredictable solution $x(t)$. The coordinates of the solution $\omega(t)$ are depicted in Figure 1. Moreover, Figure 2 presents the trajectory of this solution.

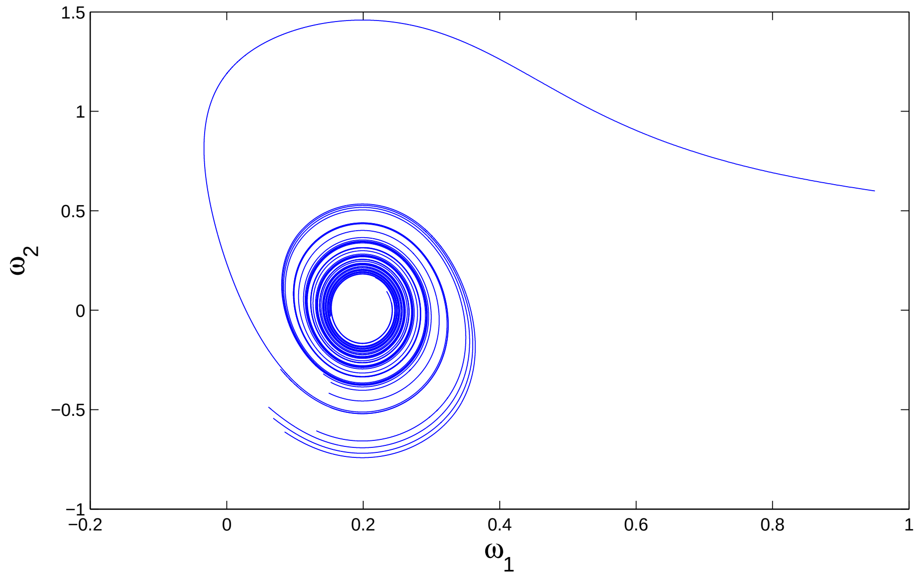

Figure 2 – The trajectory of the solution $\omega(t)$, which approximates the discontinuous unpredictable solution $x(t)$.

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Ахмет М., Тлеубергенова М., Нугаева З. БОЛЖАНБАЙТЫН ТЕРБЕЛІСТЕРІ БАР ИМПУЛЬСТІК ЖҮЙЕ

Импульстік әсері бар сызықтық біртекті емес жүйелер үшін үзілісті болжанбайтын шешімдердің тербелістерінің жаңа түрі қарастырылды. Бұл жүйелердің импульсті сәттері жаңадан анықталған болжанбайтын дискретті жиынтықты құрайды. Зерттелетін модельдер болжанбайтын қозуларға ие. Асимптотикалық орнықты үзілісті болжанбайтын шешімдердің бар болуы мен жалғыздығының жеткілікті шарттары келтірілді. Болжанбайтын компоненттерді жүйелі түрде анықтау үшін мысалдарда кездейсоқ анықталған болжанбайтын тізбектер қолданады. Үзіліс нүктелерінің жиыны логистикалық бейнелеуді қолдану арқылы жүзеге асырылады. Нәтижелердің орындалуын көрсету үшін мысалдар келтірілді.

Кілттік сөздер. Үзілісті болжанбайтын функция, сызықтық импульстік жүйе, Бернулли процесі, асимптотикалық орнықтылық.

Ахмет М., Тлеубергенова М., Нугаева З. ИМПУЛЬСНАЯ СИСТЕМА С НЕПРЕДСКАЗУЕМЫМИ КОЛЕБАНИЯМИ

Рассматривается новый тип колебаний разрывных непредсказуемых решений для линейных неоднородных систем с импульсными воздействиями. Моменты импульсов исследуемых систем являются нововведенным непредсказуемым дискретным множеством. Исследуемые модели допускают непредсказуемые возмущения. Приведены достаточные условия существования и единственности асимптотически устойчивых разрывных непредсказуемых решений. Для конструктивного определения непредсказуемых компонентов в примерах используются случайно определенные непредсказуемые последовательности. Множества моментов разрыва реализуется с помощью логистического отображения. Для иллюстрации результатов приведены примеры с моделированием.

Ключевые слова. Разрывная непредсказуемая функция, линейная импульсная система, процесс Бернулли, асимптотическая устойчивость.

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Transport problems of dynamics of multilayered shell in elastic half-space and their solutions

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Abstract. The problem of an action on a cavity supported by a multilayer circular cylindrical shell located in the elastic half-space of a stationary moving load of an arbitrary profile is under consideration. The motion of the shell layers and the elastic half-space is described by the dynamic equations of the theory of elasticity in a moving coordinate system. Analytical solution of the problem of determining components of stress-strain state of the array and the shell at subsonic speeds of load movement at different contact interaction of the shells between each other and the array is obtained. Results of numerical calculations of stress-strain state of steel shell and surface of elastic mass at transport loads are given.

Keywords. Elastic half-space, moving load, multilayer cylindrical shell, stress strain state.

1 Introduction

As the main model problems used to study the dynamics of transport underground structures under the influence of a transport load, the problems of acting on a circular cylindrical shell of a load uniformly moving along the inner surface of the shell along its generatrix located in an elastic space or half-space are usually considered. The first task simulates the dynamic behavior of a deep-laid structure, the second – a shallow one. The problems of the action of a movable axisymmetric normal load on a thin-walled and thick-walled circular cylindrical shell in elastic space are solved in articles [1,2], respectively. Similar problems under the action of various non-axisymmetric moving loads on the shell were considered in [3-5] and other works.

In contrast to these problems, similar problems for the elastic half-space are more complicated, since it becomes necessary to take into account the waves reflected by the boundary

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of the half-space. Therefore, the number of publications devoted to the study of this problem is small and covers mainly the recent years, in particular [6-14]. In these works, when constructing a mathematical model, the lining of the tunnel or pipeline was considered as a homogeneous elastic circular cylindrical shell.

In the present work, these constructions are presented in the form of an inhomogeneous, multilayered elastic shell, the layers of which are thick-walled circular cylindrical shells with different geometric parameters and physico-mechanical properties. This is typical for lining of modern underground structures, which are multilayer and have a thickness comparable to the diameter of the structures. A mathematical model of such structures is constructed on the basis of solving the problem by the contact grab based on the equations of motion of elastic multilayer elastic shells in the elastic half-space for different contact interaction of the layers. Transport solutions of this problem are obtained in the range of subcritical velocities at which there are no resonance phenomena.

In the particular case when the shell is a single-layer (homogeneous thick-walled shell), the results of a numerical experiment are presented and analyzed.

2 Statement of transport boundary value problems

Let consider an infinitely long circular cylindrical multilayer shell, consisting of N concentric layers with different physicomechanical and geometric characteristics, located in a linearly elastic, homogeneous and isotropic half-space (array). We enter fixed cylindrical and Cartesian coordinate systems whose z-axis coincides with the axis of the shell and parallel to the horizontal load-free horizontal boundary of the half-space, the x-axis is perpendicular to this boundary (Figure 1). The contact between the shell layers is assumed to be hard. The contact between the shell by an array will be assumed to be either rigid or sliding for two-way communication in the radial direction.

A load of intensity P moves along the inner surface of the shell in the direction of its z-axis at a constant speed c, the form of which does not change over time (*transport load*). The speed of the load is taken subsonic, i.e. lower propagation velocity of shear waves in the array and shell layers.

We number the shell layers sequentially, assigning the serial number 2 to the layer in contact with the array. The physical and mechanical properties of the material of the array and the shell layers are characterized by the following constants, respectively: ν_1, μ_1, ρ_1; ν_i, μ_i, ρ_i $(i = 2, 3, ..., N+1)$, where ν_k is Poisson's ratio, $\mu_k = E_k/2(1+\nu_k)$ is shear modulus, ρ_k is mass density, E_k is Young's modulus $(k = 1, 2, ..., N+1)$. Further, the index $k = 1$ refers to the array, and $k = 2, 3, ..., N+1$ to the layers of the shell.

Let us determine the reaction of the shell and its environment to a transport load, using the Lama's dynamic equations of elasticity theory in vector form to describe the motion of

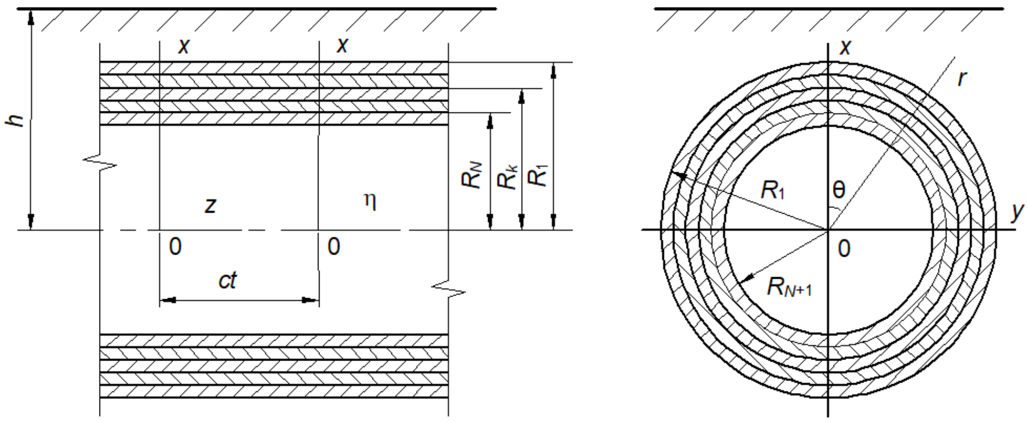


Figure 1: A multilayer shell in an elastic half-space

the array and shell layers:

$$(\lambda_k + \mu_k)\mathrm{graddiv}\mathbf{u}_k + \mu_k \Delta \mathbf{u}_k = \rho_k \frac{\partial^2 \mathbf{u}_k}{\partial t^2}, \; k = 1, 2, ..., N+1, \tag{1}$$

Here λ_k, μ_k are Lama's parameters, $\mathbf{u}_k$ are the displacement vectors of the points of the array and shell layers, Δ is Laplace operator.

Since a steady process is considered, the deformation pattern is stationary with respect to a moving load. Therefore, we can go to a load-related moving Cartesian $(x, y, \eta = z - ct)$ or cylindrical $(r, \theta, \eta = z - ct)$ coordinate systems. Then equations (1) take the next form:

$$(M_{pk}^{-2} - M_{sk}^{-2})\mathrm{graddiv}\mathbf{u}_k + M_{sk}^{-2} \Delta \mathbf{u}_k = \frac{\partial^2 \mathbf{u}_k}{\partial \eta^2}, \; k = 1, 2, ..., N+1, \tag{2}$$

where $M_{pk} = c/c_{pk}$, $M_{sk} = c/c_{sk}$ are Mach numbers; $c_{pk} = \sqrt{(\lambda_k + \mu_k)/\rho_k}$, $c_{sk} = \sqrt{\mu_k/\rho_k}$ are propagation velocity of dilatation and shear waves in the array and shell layers.

By use Lama's potentials [15]

$$\mathbf{u}_k = \mathrm{grad}\varphi_{1k} + \mathrm{rot}(\varphi_{2k}\mathbf{e}_\eta) + \mathrm{rotrot}(\varphi_{3k}\mathbf{e}_\eta), \; k = 1, 2, ..., N+1, \tag{3}$$

we transform equations (2) to the form

$$\Delta \varphi_{jk} = M_{jk}^2 \frac{\partial \varphi_{jk}}{\partial \eta^2}, \; j = 1, 2, 3, \; k = 1, 2, ..., N+1. \tag{4}$$

Here $\mathbf{e}_\eta$ is the unit vector of η -axis, $M_{1k} = M_{pk}$, $M_{2k} = M_{3k} = M_{sk}$.

Using (3) and Hooke's law, we obtain expressions for the components of the vectors and stress tensors in the array ($k=1$) and shell layers ($k=2,3,...,N+1$) in a moving cylindrical coordinate system:

$$
\begin{aligned}
u_{rk} &= \frac{\partial \varphi_{1k}}{\partial r} + \frac{1}{r}\frac{\partial \varphi_{2k}}{\partial \theta} + \frac{\partial^2 \varphi_{3k}}{\partial \eta \partial r},\\
u_{\theta k} &= \frac{1}{r}\frac{\partial \varphi_{1k}}{\partial \theta} - \frac{\partial \varphi_{2k}}{\partial r} + \frac{1}{r}\frac{\partial^2 \varphi_{2k}}{\partial \eta \partial \theta},\\
u_{\eta k} &= \frac{\partial \varphi_{1k}}{\partial \eta} + m_{sk}^2 \frac{\partial^2 \varphi_{3k}}{\partial \eta^2};
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
\sigma_{\eta\eta k} &= \left(2\mu_k + \lambda_k M_{pk}^2\right)\frac{\partial^2 \varphi_{1k}}{\partial \eta^2} + 2\mu_k m_{sk}^2 \frac{\partial^3 \varphi_{3k}}{\partial \eta^2},\\
\sigma_{\theta\theta k} &= \frac{2\mu_k}{r}\left(\frac{1}{r}\frac{\partial^2 \varphi_{1k}}{\partial \theta^2} + \frac{\partial \varphi_{1k}}{\partial r} + \frac{1}{r}\frac{\partial^2 \varphi_{2k}}{\partial \theta} - \frac{\partial^2 \varphi_{2k}}{\partial r \partial \theta} + \frac{1}{r}\frac{\partial^3 \varphi_{3k}}{\partial \theta^2 \partial \eta} + \frac{\partial^2 \varphi_{3k}}{\partial r \partial \eta}\right),\\
\sigma_{rrk} &= \lambda_k M_{pk}^2 \frac{\partial^2 \phi_{1k}}{\partial \eta^2} + 2\mu_k\left(\frac{\partial^2 \phi_{1k}}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \phi_{2k}}{\partial \theta \partial r} - \frac{1}{r^2}\frac{\partial \phi_{2k}}{\partial \theta} + \frac{\partial^3 \phi_{3k}}{\partial r^2 \partial \eta}\right),\\
\sigma_{r\eta k} &= \mu_k\left(2\frac{\partial^2 \varphi_{1k}}{\partial \eta \partial r} + \frac{1}{r}\frac{\partial^2 \varphi_{2k}}{\partial \theta \partial \eta} + \left(1+m_{sk}^2\right)\frac{\partial^3 \varphi_{3k}}{\partial \eta^2 \partial r}\right),\\
\sigma_{\eta\theta k} &= \mu_k\left(\frac{2}{r}\frac{\partial^2 \varphi_{1k}}{\partial \theta \partial \eta} - \frac{\partial^2 \varphi_{2k}}{\partial r \partial \eta} + \frac{\left(1+m_{sk}^2\right)}{r}\frac{\partial^3 \varphi_{3k}}{\partial \theta \partial \eta^2}\right),\\
\sigma_{r\theta k} &= 2\mu_k\left(\frac{1}{r}\frac{\partial^2 \varphi_{1k}}{\partial \theta \partial r} - \frac{1}{r^2}\frac{\partial \varphi_{1k}}{\partial \theta} - \frac{\partial^2 \varphi_{2k}}{\partial r^2} - \frac{m_{sk}^2}{r^2}\frac{\partial^2 \varphi_{2k}}{\partial \eta^2} + \frac{1}{r}\frac{\partial^3 \varphi_{3k}}{\partial r \partial \eta \partial \theta} - \frac{1}{r^2}\frac{\partial^2 \varphi_{3k}}{\partial \eta \partial \theta}\right),
\end{aligned}
\tag{6}
$$

where

$$m_{sk}^2 = 1 - M_{sk}^2 > 0.$$

In moving Cartesian coordinates, the expressions for the components of the stress-strain state (SSS) of the array have the form

$$
\begin{aligned}
u_{x1} &= \frac{\partial \varphi_{11}}{\partial x} + \frac{\partial \varphi_{21}}{\partial y} + \frac{\partial^2 \varphi_{31}}{\partial x \partial \eta},\\
u_{y1} &= \frac{\partial \varphi_{11}}{\partial y} - \frac{\partial \varphi_{21}}{\partial x} + \frac{\partial^2 \varphi_{31}}{\partial y \partial \eta},\\
u_{\eta 1} &= \frac{\partial \varphi_{11}}{\partial \eta} + m_{s1}^2 \frac{\partial^2 \varphi_{31}}{\partial \eta^2};
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\sigma_{\eta\eta 1} &= \left(2\mu_1 + \lambda_1 M_{p1}^2\right)\frac{\partial^2 \varphi_{11}}{\partial \eta^2} + 2\mu_1 m_{s1}^2 \frac{\partial^3 \varphi_{31}}{\partial \eta^3},\\
\sigma_{yy1} &= \lambda_1 M_{p1}^2 \frac{\partial^2 \varphi_{11}}{\partial \eta^2} + 2\mu_1\left(\frac{\partial^2 \varphi_{11}}{\partial y^2} + \frac{\partial^2 \varphi_{21}}{\partial x \partial y} + \frac{\partial^3 \varphi_{31}}{\partial y^2 \partial \eta}\right),\\
\sigma_{xx1} &= \lambda_1 M_{p1}^2 \frac{\partial^2 \varphi_{11}}{\partial \eta^2} + 2\mu_1\left(\frac{\partial^2 \varphi_{11}}{\partial x^2} + \frac{\partial \varphi_{21}}{\partial x \partial y} + \frac{\partial^3 \varphi_{31}}{\partial x^2 \partial \eta}\right),\\
\sigma_{x\eta 1} &= \mu_1\left(2\frac{\partial^2 \varphi_{11}}{\partial \eta \partial x} + \frac{\partial^2 \varphi_{21}}{\partial y \partial \eta} + \left(1+m_{s1}^2\right)\frac{\partial^3 \varphi_{31}}{\partial \eta^2 \partial x}\right),\\
\sigma_{\eta y1} &= \mu_1\left(2\frac{\partial^2 \varphi_{11}}{\partial y \partial \eta} - \frac{\partial^2 \varphi_{21}}{\partial x \partial \eta} + \left(1+m_{s1}^2\right)\frac{\partial^3 \varphi_{31}}{\partial y \partial \eta^2}\right),\\
\sigma_{xy1} &= 2\mu_1\left(\frac{\partial^2 \varphi_{11}}{\partial x \partial y} - \frac{\partial \varphi_{21}}{\partial x^2} - \frac{m_{s1}^2}{2}\frac{\partial^2 \varphi_{21}}{\partial \eta^2} + \frac{\partial^3 \varphi_{31}}{\partial x \partial \eta \partial y}\right).
\end{aligned}
\tag{8}
$$

Thus, to determine the components of array and shell layers SSS, it is necessary to solve equation (4) using the following boundary conditions:

– for a load-free half-space surface ($x = h$)

$$\sigma_{xx1} = \sigma_{xy1} = \sigma_{x\eta1} = 0; \tag{9}$$

– for sliding contact of the shell with the array

$$\begin{aligned} &by\ r = R_1\ u_{r1} = u_{r2},\ \sigma_{rr1} = \sigma_{rr2},\ \sigma_{r\eta1} = 0,\ \sigma_{r\theta1} = 0,\ \sigma_{r\eta1} = 0,\ \sigma_{r\theta2} = 0,\\ &by\ r = R_k\ u_{jk} = u_{jk+1},\ \sigma_{rjk} = \sigma_{rjk+1},\\ &by\ r = R_{N+1}\ \sigma_{rjN+1} = P_j(\theta,\eta),\ j = r,\theta,\eta,\ k = 2,3,...,N; \end{aligned} \tag{10}$$

– for hard contact of the shell with the array

$$\begin{aligned} &by\ r = R_k\ u_{jk} = u_{jk+1},\ \sigma_{rjk} = \sigma_{rjk+1},\\ &by\ r = R_{N+1}\ \sigma_{rjN+1} = P_j(\theta,\eta),\ j = r,\theta,\eta,\ k = 2,3,...,N. \end{aligned} \tag{11}$$

Here $P_j(\theta,\eta)$ are the intensity components of transport load $P(\theta,\eta)$.

3 The solution of BVP for the periodic loads

Let us consider the action on the shell of a sinusoidal over η moving load with an arbitrary dependence on the angular coordinate:

$$\begin{aligned} &P(\theta,\eta) = p(\theta,\xi)e^{i\xi\eta},\ p(\theta,\xi) = \sum_{n=-\infty}^{\infty} P_n(\theta,\xi)e^{in\theta},\\ &P_j(\theta,\eta) = p_j(\theta,\xi)e^{i\xi\eta},\ p_j(\theta,\xi) = \sum_{n=-\infty}^{\infty} P_n(\theta,\xi)e^{in\theta},\ j = r,\theta,\eta, \end{aligned} \tag{12}$$

where ξ defines the period $T = 2\pi/\xi$ of acting load.

In the steady state, the dependence of all quantities on η has the form (12), therefore

$$\varphi_{jk}(r,\theta,\eta) = \Phi_{jk}(r,\theta)e^{i\xi\eta}. \tag{13}$$

Substituting (13) to (4) we get

$$\Delta_2\Phi_{jk} - m_{jk}^2\xi^2\Phi_{jk} = 0,\ j = 1,2,3,\ k = 1,2,...,N+1, \tag{14}$$

where Δ_2 is two-dimensional Laplace operator, $m_{1k} \equiv m_{pk}$, $m_{2k} = m_{3k} \equiv m_{sk}$, $m_{jk}^2 = 1 - M_{jk}^2$.

At a subsonic speed of motion, solutions of equations (14) can be represented as

$$\Phi_{jk} = \Phi_{jk}^{(1)} + \Phi_{jk}^{(2)},\ j = 1, 2, 3,\ k = 1, 2, ..., N + 1, \tag{15}$$

where:

for an array $(k = 1)$

$$\begin{gathered} \Phi_{j1}^{(1)} = \sum_{n=-\infty}^{\infty} a_{nj} K_n(k_{j1} r) e^{in\theta}, \\ \Phi_{j1}^{(2)} = \int_{-\infty}^{\infty} g_j(\xi, \zeta) \exp\left(iy\zeta + (x - h)\sqrt{\zeta^2 + k_{j1}^2}\right) d\zeta; \end{gathered} \tag{16}$$

for shell layers $(k = 2, 3, ..., N + 1)$

$$\Phi_{jk}^{(1)} = \sum_{n=-\infty}^{\infty} a_{nj+3(2k-3)} K_n(k_{jk} r) e^{in\theta},\ \Phi_{jk}^{(2)} = \sum_{n=-\infty}^{\infty} a_{nj+6(k-1)} I_n(k_{jk} r) e^{in\theta}. \tag{17}$$

Here $I_n(kr)$, $K_n(kr)$ are modified Bessel functions and MacDonald functions, $k_{j1} = |m_{j1}\xi|$, $k_{jk} = |m_{jk}\xi|$. Unknown functions $g_j(\xi, \zeta)$ and coefficients $a_{n1}, ..., a_{n(6N+3)}$ must be determined.

As shown in [6,9], the representation of potentials for half-space in the form (15) leads to their following expressions in the Cartesian coordinate system:

$$\Phi_{j1} = \int_{-\infty}^{\infty} \left[\frac{e^{-xf_j}}{2f_j} \sum_{n=-\infty}^{\infty} a_{nj}\Phi_{nj} + g_j(\xi, \zeta) e^{(x-h)f_j}\right] e^{iy\zeta} d\zeta \tag{18}$$

where $f_j = \sqrt{\zeta^2 + k_{j1}^2}$, $\Phi_{nj} = \left[(\zeta + f_j)/k_{j1}\right]^n$, $j = 1, 2, 3, ...$

We use the boundary conditions (9), taking into account (8), (13), (18). By isolating the coefficients by $e^{iy\zeta}$ and equating, due to the arbitrariness of y, to zero, we obtain a system of three equations, from which we express functions $g_j(\xi, \zeta)$ in terms of unknown coefficients a_{n1}, a_{n2}, a_{n3}:

$$g_j(\xi, \zeta) = \frac{1}{\Delta_*} \sum_{l=1}^{3} \Delta_{jl}^* e^{-hf_l} \sum_{n=-\infty}^{\infty} a_{nl}\Phi_{nl}, \tag{19}$$

where $\Delta_* = (2\rho_*^2 - \beta^2)^2 - 4\rho_*^2\sqrt{\rho_*^2 - \alpha^2}\sqrt{\rho_*^2 - \beta^2}$,

$$\Delta_{11}^* = \frac{\Delta_*}{2\sqrt{\rho_*^2 - \alpha^2}} - \frac{(2\rho_*^2 - \beta^2)^2}{\sqrt{\rho_*^2 - \alpha^2}},\ \Delta_{12}^* = -2\zeta(2\rho_*^2 - \beta^2),$$

$$\Delta_{13}^* = 2\xi(2\rho_*^2 - \beta^2)\sqrt{\rho_*^2 - \beta^2},$$

$$\Delta^*_{21}=\frac{M^2_{s1}}{m^2_{s1}}\Delta^*_{12},\ \Delta^*_{22}=-\frac{\Delta_{**}}{2\sqrt{\rho^2_*-\beta^2}},\ \Delta^*_{23}=-4\xi\zeta\frac{M^2_{s1}}{m^2_{s1}}\sqrt{\rho^2_*-\alpha^2}\sqrt{\rho^2_*-\beta^2},$$

$$\Delta^*_{31}=\frac{\Delta^*_{13}}{m^2_{s1}\xi^2},\ \Delta^*_{32}=\frac{\Delta^*_{21}}{\beta^2},\ \Delta^*_{33}=-\frac{\Delta_{**}}{2\sqrt{\rho^2_*-\beta^2}}+\frac{(2\rho^2_*-\beta^2)^2}{\sqrt{\rho^2_*-\beta^2}},$$

$$\alpha=M_{p1}\xi,\ \beta=M_{s1}\xi,\ \rho^2_*=\xi^2+\zeta^2,\ \Delta_{**}=(2\rho^2_*-\beta^2)^2-4\rho^2_{**}\sqrt{\rho^2_*-\alpha^2}\sqrt{\rho^2_*-\beta^2},$$

$$\rho^2_{**}=\xi^2+(2/m^2_{s1}-1)\zeta^2.$$

Note, that $\Delta_*(\rho_*)$ is the Rayleigh determinant, which vanishes at $\rho^2_{*R}=\xi^2M^2_R$, or at two points $\pm\zeta_R=\pm|\xi|\sqrt{M^2_R-1}$, where $M_R=c/c_R$ is the Mach number, $M_R=c/c_R$, c_R is Rayleigh velocity, the velocity of the Rayleigh surface waves [15]. It follows from the latter that $\Delta_*(\rho_*)$ does not vanish on the real axis, if $M_R<1\,(c<c_R)$, that is at pre-Delay transport load speeds. In this case, potentials (18) can be represented as

$$\Phi_{j1}=\int\limits_{\infty}^{-\infty}\left[\frac{e^{-xf_j}}{2f_j}\sum_{n=-\infty}^{\infty}a_{nj}\Phi_{nj}+e^{(x-h)f_j}\sum_{l=1}^{3}\frac{\Delta^*_{jl}}{\Delta_*}e^{hf_i}\sum_{n=-\infty}^{\infty}a_{nl}\Phi_{nl}\right]e^{iy\zeta}d\zeta. \tag{20}$$

It should be noted that the Rayleigh velocity is large, but slightly lower than the shear wave velocity in a array.

Using the relation, which is known for $x<h$ [6],

$$exp\left(iy\zeta+(x-h)\sqrt{\zeta^2+k^2_j}\right)=$$

$$=\sum_{n=-\infty}^{\infty}I_n(k_jr)e^{in\theta}\left[(\zeta+\sqrt{\zeta^2+k^2_j})/k_j\right]^n e^{-h\sqrt{\zeta^2+k^2_j}},$$

we write Φ_{j1} (15) in a cylindrical coordinate system

$$\Phi_{j1}=\sum_{n=-\infty}^{\infty}\left(a_{nj}K_n(k_{j1}r)+I_n(k_{j1}r)\int\limits_{-\infty}^{\infty}g_j(\xi,\zeta)\Phi_{nj}e^{-hf_j}d\zeta\right)e^{in\theta}.$$

Substituting in the last expression $g_j(\xi,\zeta)$ from (19), we obtain

$$\Phi_{j1}=\sum_{n=-\infty}^{\infty}\left(a_{nj}K_n(k_{j1}r)+b_{nj}I_n(k_{j1}r)\right)e^{in\theta}, \tag{21}$$

where $b_{nj}=\sum\limits_{l=1}^{3}\sum\limits_{m=-\infty}^{\infty}a_{ml}A^{ml}_{nj},\ A^{ml}_{nj}=\int\limits_{-\infty}^{\infty}\frac{\Delta^*_{jl}}{\Delta_*}\Phi_{ml}\Phi_{nj}e^{-h(f_l+f_j)}d\zeta.$

Substituting (21) into (5), (6), taking into account (13), we obtain formulas for computing the components of the array SSS in cylindrical coordinates (by $c < c_R$)

$$u_{l1}^{*} = \sum_{n=-\infty}^{\infty} \sum_{j=1}^{3} \left[T_{lj1}^{(1)} \left(K_n(k_{j1}r) \right) a_{nj} + T_{lj1}^{(2)} \left(I_n(k_{j1}r) \right) b_{nj} \right] e^{i(\xi\eta + n\theta)},$$
$$\frac{\sigma_{lm1}^{*}}{\mu_1} = \sum_{n=-\infty}^{\infty} \sum_{j=1}^{3} \left[S_{lmj1}^{(1)} \left(K_n(k_{j1}r) \right) a_{nj} + S_{lmj1}^{(2)} \left(K_n(k_{j1}r) \right) b_{nj} \right] e^{i(\xi\eta + n\theta)}. \quad (22)$$

Here $l = r, \theta, \eta,\ m = r, \theta, \eta$;

$$T_{r11}^{(1)} = k_{11}K_n'(k_{11}r), \qquad T_{r21}^{(1)} = -\frac{n}{r}K_n(k_{21}r),\ T_{r31}^{(1)} = -\xi k_{31}K_n'(k_{31}r),$$

$$T_{\theta 11}^{(1)} = -\frac{n}{r}K_n(k_{11}r)i, \qquad T_{\theta 11}^{(1)} = -k_{21}K_n'(k_{21}r)i,\ T_{\theta 31}^{(1)} = -\frac{n}{r}\xi K_n(k_{31}r)i,$$

$$T_{\eta 31}^{(1)} = \xi K_n(k_{11}r)i,\ T_{\eta 21}^{(1)} = 0, \qquad T_{\eta 31}^{(1)} = -k_{31}^2 K_n(k_{31}r)i,$$

$$S_{rr11}^{(1)} = 2\left(k_{11}^2 + \frac{n^2}{r^2} - \frac{\lambda_1 M_{p1}^2 \xi^2}{2\mu_1}\right) K_n(k_{11}r) - \frac{2k_{11}K_n'(k_{11}r)}{r},$$

$$S_{rr21}^{(1)} = \frac{2n}{r^2}K_n(k_{11}r) - \frac{2k_{21}K_n'(k_{21}r)}{r},$$

$$S_{rr31}^{(1)} = -2\xi\left(k_{31}^2 + \frac{n^2}{r^2}\right) K_n(k_{31}r) + \frac{2\xi k_{31}K_n'(k_{31}r)}{r},$$

$$S_{\theta\theta 11}^{(1)} = -2\left(\frac{n^2}{r^2} + \frac{\lambda_1 M_{p1}^2 \xi^2}{2\mu_1}\right) K_n(k_{11}r) + \frac{2k_{11}K_n'(k_{11}r)}{r},$$

$$S_{\theta\theta 21}^{(1)} = -\frac{2nK_n(k_{21}r)}{r^2} + \frac{2nk_{21}K_n'(k_{21}r)}{r}, \qquad S_{\theta\theta 31}^{(1)} = \frac{2\xi n^2 K_n(k_{31}r)}{r^2} - \frac{2\xi k_{31}K_n'(k_{31}r)}{r},$$

$$S_{\eta\eta 11}^{(1)} = -2\xi^2\left(\frac{1 + \lambda_1 M_{p1}^2}{2\mu_1}\right) K_n(k_{11}r),\ S_{\eta\eta 21}^{(1)} = 0, \qquad S_{\eta\eta 31}^{(1)} = 2m_{31}^2 \xi^3 K_n(k_{31}r),$$

$$S_{r\theta 11}^{(1)} = \left(-\frac{2nK_n(k_{11}r)}{r^2} + \frac{2nk_{11}K_n'(k_{11}r)}{r}\right) i,$$

$$S_{r\theta 21}^{(1)} = \left(-\left(k_{21}^2 + \frac{2n^2}{r^2}\right) K_n(k_{21}r) + \frac{2k_{21}K_n'(k_{21}r)}{r}\right) i,$$

$$S_{r\theta 31}^{(1)} = \left(\frac{2n\xi K_n(k_{31}r)}{r^2} - \frac{2n\xi k_{31}K_n'(k_{31}r)}{r}\right) i,$$

$$S_{\theta\eta 21}^{(1)} = \xi k_{21}K_n'(k_{21}r), \qquad S_{\theta\eta 31}^{(1)} = \frac{2n\xi^2(1 + m_{31}^2)K_n(k_{31}r)}{r},$$

$$S^{(1)}_{r\eta 11} = 2\xi k_{11} K'_n(k_{11}r)i, \qquad S^{(1)}_{r\eta 21} = \frac{\xi n K_n(k_{21}r)i}{r},$$

$$S^{(1)}_{\theta\eta 11} = -\frac{2n\xi K_n(k_{11}r)}{r}, \qquad S^{(1)}_{r\eta 31} = -\xi^2 k_{31}(1+m_{31}^2)K'_n(k_{31}r)i;$$

$T^{(2)}_{lj1}$, $S^{(2)}_{lmj1}$ are obtained from $T^{(1)}_{lj1}$, $S^{(1)}_{lmj1}$ replacing K_n by I_n.

For $k = 2, 3, \ldots, N+1$, substituting (15) in (5), (6) taking into account (13), we obtain formulas for calculating the components of shell layers SSS at $c < c_R$:

$$\begin{aligned} &u^*_{lk} = \\ &= \sum_{n=-\infty}^{\infty} \sum_{j=1}^{3} [T^{(1)}_{ljk}(K_n(k_{jk}r))a_{nj+3(2k-3)} + T^{(2)}_{ljk}(K_n(k_{jk}r))a_{nj+6(k-1)}]e^{i(\xi\eta+n\theta)}, \\ &\frac{\sigma^*_{lmk}}{\mu_k} = \\ &= \sum_{n=-\infty}^{\infty} \sum_{j=1}^{3} [S^{(1)}_{lmjk}(K_n(k_{jk}r))a_{nj+3(2k-3)} + S^{(2)}_{lmjk}(I_n(k_{jk}r))a_{nj+6(k-1)}]e^{i(\xi\eta+n\theta)}. \end{aligned} \tag{23}$$

Here $l = r, \theta, \eta$, $m = rm\theta, \eta$, $k = 2, 3, ..., N+1$;

$$T^{(1)}_{r1k} = k_{1k}K'_n(k_{1k}r),\ T^{(1)}_{r2k} = -\frac{n}{r}K_n(k_{2k}r), \qquad T^{(1)}_{r3k} = -\xi k_{3k}K'_n(k_{3k}r),$$

$$T^{(1)}_{\theta 1k} = \frac{n}{r}K_n(k_{1k}r)i, \qquad T^{(1)}_{\theta 2k} = -k_{2k}K'_n(k_{2k}r)i,\ T^{(1)}_{\theta 3k} = -\frac{n}{r}\xi K_n(k_{3k}r)i,$$

$$T^{(1)}_{\eta 1k} = \xi K_n(k_{1k}r)i,\ T^{(1)}_{\eta 2k} = 0, \qquad T^{(1)}_{\eta 3k} = -k_{3k}^2 K_n(k_{3k}r)i,$$

$$S^{(1)}_{rr1k} = 2\left(k_{1k}^2 + \frac{n^2}{r^2} - \frac{\lambda_k M_{pk}^2\xi^2}{2\mu_k}\right)K_n(k_{1k}r) - \frac{2k_{1k}K'_n(k_{1k}r)}{r},$$

$$S^{(1)}_{rr2k} = \frac{2n}{r^2}K_n(k_{2k}r) - \frac{2k_{2k}K'_n(k_{2k}r)}{r},$$

$$S^{(1)}_{rr3k} = 2\xi\left(k_{3k}^2 + \frac{n^2}{r^2}\right)K_n(k_{3k}r) + \frac{2\xi k_{3k}K'_n(k_{3k}r)}{r},$$

$$S^{(1)}_{\theta\theta 1k} = -2\left(\frac{n^2}{r^2} + \frac{\lambda_k M_{pk}^2\xi^2}{2\mu_k}\right)K_n\left(k_{1k}r\right) + \frac{2k_{1k}K'_n\left(k_{1k}r\right)}{r},$$

$$S^{(1)}_{\theta\theta 2k} = -\frac{2nK_n\left(k_{2k}r\right)}{r^2} + \frac{2nk_{2k}K'_n\left(k_{2k}r\right)}{r},$$

$$S^{(1)}_{\theta\theta 3k} = \frac{2\xi n^2 K_n\left(k_{3k}r\right)}{r^2} - \frac{2\xi k_{3k}K'_n\left(k_{3k}r\right)}{r},$$

$$S^{(1)}_{\eta\eta 1k} = -2\xi^2\left(\frac{1+\lambda_k M_{pk}^2}{2\mu_k}\right)K_n\left(k_{1k}r\right),$$

$$S^{(1)}_{\eta\eta 2k} = 0,\ S^{(1)}_{\eta\eta 3k} = 2m^2_{3k}\xi^3 K_n\left(k_{3k}r\right),$$

$$S^{(1)}_{r\theta 1k} = \left(-\frac{2nK_n\left(k_{1k}r\right)}{r^2} + \frac{2nk_{1k}K'_n\left(k_{1k}r\right)}{r}\right)i,$$

$$S^{(1)}_{r\theta 2k} = \left(-\left(k^2_{2k} + \frac{2n^2}{r^2}\right)K_n\left(k_{2k}r\right) + \frac{2k_{2k}K'_n\left(k_{2k}r\right)}{r}\right)i,$$

$$S^{(1)}_{r\theta 3k} = \left(\frac{2n\xi K_n\left(k_{3k}r\right)}{r^2} - \frac{2n\xi k_{3k}K'_n\left(k_{3k}r\right)}{r}\right)i,$$

$$S^{(1)}_{\theta\eta 2k} = \xi k_{2k}K'_n\left(k_{2k}r\right), S^{(1)}_{\theta\eta 3k} = \frac{n\xi^2\left(1+m^2_{3k}\right)K_n\left(k_{3k}r\right)}{r},$$

$$S^{(1)}_{r\eta 1k} = 2\xi k_{1k}K'_n\left(k_{1k}r\right)i,\ S^{(1)}_{\theta\eta 1k} = -\frac{2n\xi K_n\left(k_{1k}r\right)}{r},$$

$$S^{(1)}_{r\eta 2k} = -\frac{\xi nK_n\left(k_{2k}r\right)i}{r},\ S^{(1)}_{r\eta 3k} = -i\xi^2 k_{3k}\left(1+m^2_{3k}\right)K'_n\left(k_{3k}r\right);$$

$T^{(2)}_{ljk}, S^{(2)}_{lmjk}$ are obtained from $T^{(1)}_{ljk}, S^{(1)}_{lmjk}$ replacing K_n in I_n.

To determine the coefficients $a_{n1}, ..., a_{n(6N+3)}$, we use the boundary conditions (10) or (11), depending on the condition of contact between the shell and the medium.

Substituting the corresponding expressions into the boundary conditions and equating the coefficients of the series with $e^{in\theta}$, we obtain an infinite system ($n = 0, \pm 1, \pm 2, \dots$) of linear algebraic equations, for the solution of which one can use the reduction method or the method of successive reflections. It is more convenient for solving such problem [6] because it allows for each successive reflection to solve a system of linear equations of block-diagonal form with determinants $\Delta_n(\xi, c)$ along the main diagonal.

After determining the coefficients, the components of the stress-strain state of the array and shell layers can be calculated by the formulas (22), (23).

If we have any periodic transport load, then it can be presented as Fourier series:

$$P_j(\theta,\eta) = \sum_k p_{kj}(\theta)e^{i\xi_k\eta}.$$

In this case we solve the problem for every member of this series by use this method. The sum of this solutions gives the solution of periodic transport problem.

4 Numerical experiment

As an example, we consider the dynamic behavior of an underground single-layer steel pipeline ($\nu_2 = 0,3$, $\mu_2 = 8,08\cdot 10^{10}\,Pa$, $\rho_2 = 7,8\cdot 10^3 kg/m^3$; $c_{s2} = 3218,54\,m/s$, $c_{p2} = 6021,33\,m/s$) under the action of a moving load in it.

The radius of the outer surfaces of the pipes is $R_1 = R = 1m$, the internal one is $R_2 = 0,95m$. The depth of the pipeline in the rock mass is $h = 2R_1$. The array has the following characteristics [16]: $\nu_1 = 0,25$, $\mu_1 = \mu = 4,0 \cdot 10^9 Pa, \rho_1 = 2,6 \cdot 10^3\, kg/m^3; c_{s1} = 1240,35\, m/s, c_{p1} = 2148,34\, m/s, c_R = 1140,42\, m/s.$

An axisymmetric cylindrical normal load of intensity q (Pa) is uniformly distributed in the interval $|\eta| \le l_0 = 0,2R$, moves in a pipeline with a subcritical and pre-Delay speed $c = 100\, m/s$. The load intensity is selected so that the total load along the entire length of the loading section $2l_0$ is equivalent to the concentrated normal ring load intensity $P^{\circ\circ}$ (N/m), that is $q = P^{\circ\circ}/2l_0$.

We use the designations: $u_r^\circ = u_r\mu/P^\circ$ (m), $\sigma_{\theta\theta}^\circ = \sigma_{\theta\theta}/P^\circ$, $\sigma_{\eta\eta}^\circ = \sigma_{\eta\eta}/P^\circ$, (m), $u_x^\circ = u_x\mu/P^\circ$ (m), $u_y^\circ = u_y\mu/P^\circ$, $\sigma_{yy}^\circ = \sigma_{yy}/P^\circ$ where $P^\circ = P^{\circ\circ}/m$ (Pa).

The calculation results in the cross section $\eta = 0$ of the pipeline (in the xy coordinate plane) are shown in Tables 1, 2 and in Figures 2, 3.

Table 1: Components of the array of strain–stress state at the contact points: $r = R_1$, $\eta = 0$

Components of strain-stress state	$\theta, hail$									
	0	20	40	60	80	100	120	140	160	180
Rigid contact of the pipeline with the array										
u_r°	0,30	0,29	0,28	0,26	0,25	0,25	0,24	0,25	0,25	0,25
$\sigma_{\theta\theta}^\circ$	0,18	0,17	0,17	0,16	0,17	0,16	0,15	0,14	0,14	0,14
$\sigma_{\eta\eta}^\circ$	-0,35	-0,36	-0,36	-0,36	-0,36	-0,36	-0,36	-0,36	-0,37	-0,37
Sliding pipe contact with the array										
u_r°	0,32	0,32	0,30	0,28	0,26	0,26	0,26	0,26	0,27	0,27
$\sigma_{\theta\theta}^\circ$	-0,08	-0,06	-0,07	0,01	0,0	-0,01	-0,03	-0,06	-0,07	-0,07
$\sigma_{\theta\theta}^\circ$	-1,36	-1,36	-1,34	-1,30	-1,29	1,29	-1,29	-1,29	-1,30	-1,31

Tables 1, 2 show the values of the components of the array SSS under various contact conditions of the array with the pipeline.

Figure 2, on the external ($r = R_1$) and internal ($r = R_2$) pipeline contours, shows the diagrams of radial displacements u_r° and normal stresses $\sigma_{\theta\theta}^\circ$, $\sigma_{\eta\eta}^\circ$. Curves 1 correspond to the hard contact of the pipeline with the array, curves 2 correspond to the sliding contact.

Calculations show that with a hard contact, the extreme radial displacements u_r and

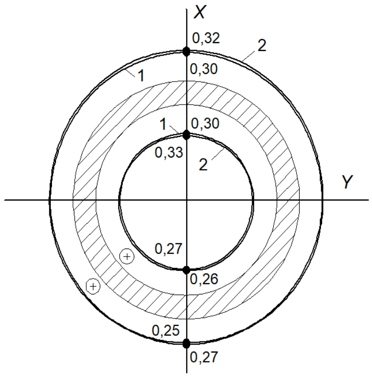
Table 2: Components of strain-stress state of earth surface ($x = h$, $\eta = 0$)

Components of strain-stress state	y/R$_1$								
	0,0	0,4	0,8	1,2	1,6	2,0	2,4	2,8	3,2
Rigid contact of the pipeline with the array									
u$_x^\circ$	0,11	0,10	0,08	0,06	0,04	0,03	0,02	0,01	0,01
u$_y^\circ$	0,0	0,02	0,03	0,04	0,04	0,03	0,03	0,02	0,02
s$_{yy}^\circ$	0,21	0,17	0,10	0,03	-0,01	-0,02	-0,03	-0,03	-0,02
s$_{\eta,\eta}^\circ$	0,25	0,23	0,17	0,12	0,08	0,05	0,03	0,02	0,02
Sliding pipe contact with the array									
u$_x^\circ$	0,13	0,12	0,09	0,07	0,05	0,03	0,02	0,01	0,01
u$_y^\circ$	0,0	0,03	0,04	0,04	0,04	0,03	0,02	0,02	0,01
s$_{yy}^\circ$	0,24	0,20	0,11	0,02	-0,02	-0,04	-0,03	-0,02	-0,02
s$_{\eta,\eta}^\circ$	0,29	0,26	0,20	0,13	0,08	0,05	0,04	0,02	0,02

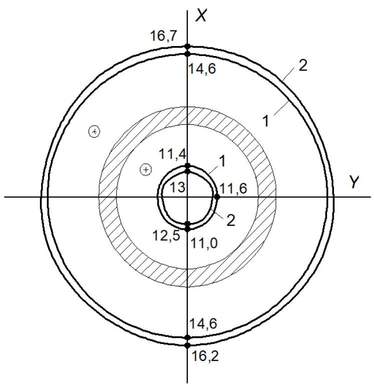
normal tangential stresses $\sigma_{\theta\theta}^\circ$ are positive and some less than by a sliding contact. Axial normal stresses $\sigma_{\eta\eta}^\circ$ are positive on the outer contour of the section and negative on the inner contour. Moreover, by a hard contact, the value $|\sigma_{\eta\eta}^\circ|$ on the external circuit is almost half as low as on the internal, and by a sliding contact they are almost the same. The highest normal stresses $\sigma_{\theta\theta}^\circ$ act on the external contour of the cross section and, under any contact conditions, are $2-3$ times higher than $\sigma_{\eta\eta}^\circ$.

Figure 3 shows the curves of changes in SSS of the earth's surface. The numbering of the curves has the same meaning as in Figure 1.

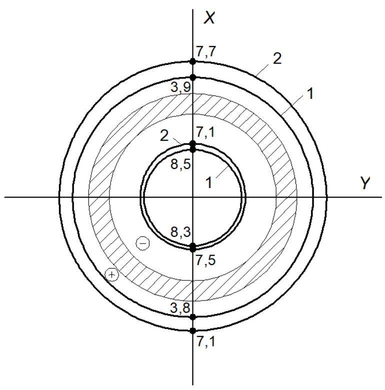
As follows from Figure 3 and Table 2, with an increase in $|y|$ the components of the earth's surface SSS rapidly attenuate, and for $|y| > 3R$, displacements and stresses become very small regardless of the interface between the pipeline and the array.



a



b



c

Figure 2: Plots u_r° (a), $\sigma_{\theta\theta}^\circ$ (b), $\sigma_{\eta\eta}^\circ$ (c) in the cross section of the pipeline

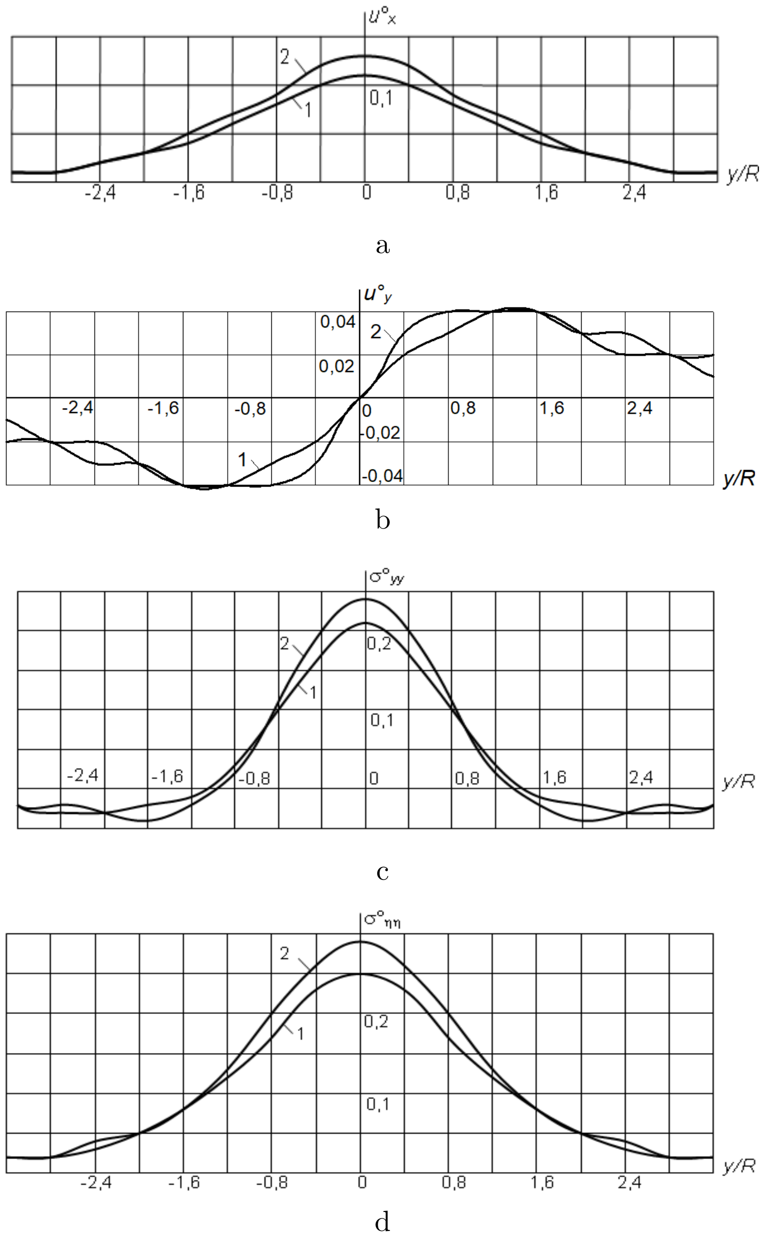


Figure 3: Changes of components of the earth's surface SSS in the xy coordinate plane ($x = h$, $\eta = 0$)

Conclusion

In a rigorous mathematical formulation, an analytical solution to the problem of the action of a moving load on a circular cylindrical multilayer shell in an elastic half-space with a free boundary is obtained. The solution was obtained for subcritical loading speeds.

When using the obtained solution, the dynamic behavior of an underground steel pipeline under the action of a load moving in it was investigated.

The developed calculation procedure is recommended to be used for the dynamic calculation of tunnels or layered and homogeneous shallow underground pipelines backed by layered (in particular, homogeneous) lining under the influence of transport loads.

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Алексеева Л.А., Украинец В.Н. ТРАНСПОРТНЫЕ ЗАДАЧИ ДИНАМИКИ МНОГОСЛОЙНЫХ ОБОЛОЧЕК В ЭЛАСТИЧНОМ ПОЛУПРОСТРАНСТВЕ И ИХ РЕШЕНИЯ

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On combinations of weakly o-minimal structures

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Abstract. In the present paper we study properties of combinations of countably categorical weakly o-minimal structures. The main result of the paper is a criterion for weak o-minimality of a linearly ordered disjoint P-combination of countably many countably categorical weakly o-minimal structures of finite convexity rank.

Keywords. Weak o-minimality, P-combination, countable categoricity, convexity rank.

1 Introduction

Let L be a countable first-order language. Throughout this paper we consider L-structures and suppose that L contains a binary relation symbol $<$ which is interpreted as a (strict) linear order in these structures. A subset A of a linearly ordered structure M is *convex* if for all $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$. This paper concerns the notion of *weak o-minimality* which was initially deeply studied by D. Macpherson, D. Marker and C. Steinhorn in [1]. A *weakly o-minimal structure* is a linearly ordered structure $M = \langle M, =, <, \ldots \rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in M. We recall that such a structure M is said to be *o-minimal* if any definable (with parameters) subset of M is a union of finitely many intervals and points in M. Thus, weak o-minimality generalizes the notion of o-minimality. Real closed fields with a proper convex valuation ring provide an important example of weakly o-minimal (not o-minimal) structures.

If $\langle M_1, <_1 \rangle$ and $\langle M_2, <_2 \rangle$ are linear orders then their *linearly ordered disjoint combination* (or *concatenation*), denoted by $M_1 + M_2$, is the linear order $\langle M_1 \cup M_2, < \rangle$, where

$$a < b \text{ iff } ([a, b \in M_1 \wedge a <_1 b] \text{ or } [a, b \in M_2 \wedge a <_2 b] \text{ or } [a \in M_1 \wedge b \in M_2]).$$

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Let M be an $\aleph_0$-categorical 1-indiscernible weakly o-minimal structure of convexity rank m for some $m < \omega$. If we consider a linearly ordered disjoint combination of n copies of M, each of which is distinguished by a unary predicate P_i (where $1 \leq i \leq n$), then we obtain an $\aleph_0$-categorical weakly o-minimal structure of the same convexity rank (we lose only 1-indiscernibility). If we consider a linearly ordered disjoint combination of ω copies of M, where each copy is distinguished by P_i, $i \in \omega$, then we can lose additionally both weak o-minimality and the $\aleph_0$-categoricity. We call such a combination as a *P-combination* (see [2]– [9]).

Let $M_i := \langle M_i; <_{M_i}, \Sigma_i\rangle$ be a linearly ordered structure of a relational language, $i \in \omega$. Further in this paragraph we denote by M' a linearly ordered disjoint P-combination of the structures M_i, $i \in \omega$, in the language $\{<, \Sigma, P_i^1\}_{i\in\omega}$, where $\Sigma = \cup_{i\in\omega}\Sigma_i$, and the universe of the combination is $\bigcup_{i\in\omega} M_i$; $P_i(M') = M_i$ for each $i \in \omega$; either $P_k(M') < P_m(M')$ or $P_m(M') < P_k(M')$ for any $k, m \in \omega$ with $k \neq m$. For definiteness we assume that each structure M_i together with its signature enters in a P-combination by the unique way, namely, every symbol S (but the order relation symbol) being in the signature Σ_i of the structure M_i receives the upper index i in the signature Σ of the P-combination, and the following holds:

(a) for every predicate n-ary symbol S of the signature Σ_i

$$M' \models \forall x_1 \dots \forall x_n [S^i(x_1, \dots, x_n) \to \wedge_{j=1}^n P_i(x_j)],$$

(b) for every functional m-ary symbol f of the signature Σ_i

$$M' \models \forall x_1 \dots \forall x_m [\exists x_{m+1} f^i(x_1, \dots, x_m) = x_{m+1} \to \wedge_{j=1}^{m+1} P_i(x_j)],$$

(c) for every constant symbol c of the signature Σ_i we have $M' \models P_i(c^i)$.

Thus, there are no coinciding relations (but the order relation) and functions acting in distinct P-predicates.

For any $i, j \in \omega$ the set $(P_i, P_j) := \{P_k \mid P_i(M') < P_k(M') < P_j(M')\}$ is said to be a *P-interval*. Similarly, we can define P-intervals $(P_i, P_j]$, $[P_i, P_j)$, $[P_i, P_j]$. If M' does not have the least P-predicate, then we can define a P-interval (∞, P_j), where

$$(\infty, P_j) := \{P_k \mid P_k(M') < P_j(M')\}.$$

We say that P_i is an *immediate P-predecessor* of P_j if

$$M' \models \forall x \forall y [(P_i(x) \wedge P_j(y) \to x < y) \wedge \forall z (x < z < y \to P_i(z) \vee P_j(z))].$$

In this case P_j is said to be an *immediate P-successor* of P_i.

Considering the predicates P_i instead of elements in M', we observe that cuts in M' are replaced by *P-cuts* (or *accumulation P-points*) consisting of partitions $(\mathcal{P}, \mathcal{P}')$ of the set of all predicates P_i with $P_j(M') < P_k(M')$ for $P_j \in \mathcal{P}$ and $P_k \in \mathcal{P}'$. We will also admit the possibility for $\mathcal{P} = \emptyset$ or $\mathcal{P}' = \emptyset$ replacing the intervals (P_j, P_k) by $(-\infty, P_k)$ or (P_i, ∞), respectively. We say that two P-cuts $\mathcal{C}_1$ and $\mathcal{C}_2$ are *orthogonal* if they are realized independently each other.

For a P-cut $\mathcal{C} = (\mathcal{P}, \mathcal{P}')$ the number of pairwise non-isomorphic countable models of $Th(M')$ in which $\mathcal{C}$ can be realized and all other P-cuts that are orthogonal to $\mathcal{C}$ are not realized is said to be the *$\mathcal{C}$-spectrum*.

We say that a P-cut $\mathcal{C}$ is *P-rational to right (left)* if there is a P-predicate P_i such that

$$\mathcal{C} = \{\neg P_j(x) \wedge \forall y[P_j(y) \to y < x] \mid P_j(M') < P_i(M')\} \cup \{\neg P_i(x) \wedge \forall y[P_i(y) \to x < y]\}$$

$$(\mathcal{C} = \{\neg P_i(x) \wedge \forall y[P_i(y) \to y < x]\} \cup \{\neg P_j(x) \wedge \forall y[P_j(y) \to x < y] \mid P_i(M') < P_j(M')\}).$$

A P-cut is said to be *P-rational* if it either P-rational to right or P-rational to left. A non-P-rational P-cut is said to be *P-irrational*.

The following theorem is a criterion for Ehrenfeuchtness of a P-combination of countably many structures in the ordered case.

Theorem 1 [10]. *Let M_i be a countably categorical linearly ordered structure for each $i \in \omega$, M' be a linearly ordered disjoint P-combination of these structures. Then $Th(M')$ is Ehrenfeucht iff there is no infinite partition of M' into infinite P-intervals, and for each P-cut $\mathcal{C}$ the $\mathcal{C}$-spectrum is finite.*

Let M be a linearly ordered structure. If $p, q \in S_1(\emptyset)$, we say that p is *weakly orthogonal* to q (denoting this by $p \perp^w q$) if $p(x) \cup q(y)$ has a unique extension to a complete 2-type over $\emptyset$. If $p_1, p_2, \ldots, p_s \in S_1(\emptyset)$, we say that a family of 1-types $\{p_1, \ldots, p_s\}$ is *weakly orthogonal over $\emptyset$* if every s–tuple $\langle a_1, \ldots, a_s\rangle \in p_1(M) \times \ldots \times p_s(M)$ satisfies the same type over $\emptyset$.

We extend the definition of the rank of convexity of a formula [11] on arbitrary (non-necessarily definable) sets:

Definition 1 [11]. *Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$. The rank of convexity of the set A ($RC(A)$) is defined as follows:*

1) $RC(A) = -1$ if $A = \emptyset$.

2) $RC(A) = 0$ if A is finite and non-empty.

3) $RC(A) \geq 1$ if A is infinite.

4) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:

- *For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;*
- *For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.*

5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha \leq \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A)) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$. The rank of convexity of an 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

The notion of (p, q)-splitting formula was introduced in [12] for non-algebraic isolated 1-types. Let $A \subseteq M$, $p, q \in S_1(A)$ be non-algebraic, $p \not\perp^w q$. Extending the definition of (p, q)-splitting formula to non-isolated case, we say that an A–definable formula $\phi(x, y)$ is a (p, q)–*splitting formula* if there is $a \in p(M)$ such that $\phi(a, M) \cap q(M) \neq \emptyset$, $\neg\phi(a, M) \cap q(M) \neq \emptyset$, $\phi(a, M) \cap q(M)$ is convex, and $[\phi(a, M) \cap q(M)]^- = [q(M)]^-$. If $\phi_1(x, y)$, $\phi_2(x, y)$ are (p, q)–splitting formulas then we say that $\phi_1(x, y)$ is not *greater than* $\phi_2(x, y)$ if there is $a \in p(M)$ such that $\phi_1(a, M) \cap q(M) \subseteq \phi_2(a, M) \cap q(M)$. We say that (p, q)–splitting formulas $\phi_1(x, y)$ and $\phi_2(x, y)$ are *equivalent* ($\phi_1(x, y) \sim \phi_2(x, y)$) if $\phi_1(a, M) \cap q(M) = \phi_2(a, M) \cap q(M)$ for some (any) $a \in p(M)$.

Obviously, if $p, q \in S_1(A)$ are non-algebraic and $p \not\perp^w q$, then there is at least one (p, q)–splitting formula, and the set of all (p, q)–splitting formulas is partitioned into a linearly ordered set of equivalence classes with respect to $\sim$. It is also obvious that for any (p, q)–splitting formula $\phi(x, y)$ the function $f(x) := \sup \phi(x, M)$ is not constant on $p(M)$.

Let $A, B, C \subseteq M$ and $f : B \to C$ be an A–definable function. The following notion was introduced in [1]. We say f is *locally increasing* (*locally decreasing, locally constant*) on B if for any $a \in B$ there is an infinite interval $J \subseteq B$ containing $\{a\}$ so that f is strictly increasing (strictly decreasing, constant) on J; we also say f is *locally monotonic* on B if it is locally increasing or locally decreasing on B.

In [13] countably categorical weakly o-minimal structures of finite convexity rank were completely described. Here we present a criterion for weak o-minimality of a linearly ordered disjoint P-combination of countably many countably categorical weakly o-minimal structures of finite convexity rank (Theorem 2).

2 Results

Let $\mathbf{k}$ denote a set consisting of k elements $a_1, a_2, \ldots, a_k$ such that $a_1 < a_2 < \ldots < a_k$, a_1 has no immediate predecessor and a_k has no immediate successor, and a_{i+1} is immediate successor of a_i for every $1 \leq i \leq k-1$. We also call $\mathbf{k}$ a finite ordering consisting of k elements. In particular, $\mathbf{2}$ denotes a duplet, where a *duplet* is a set consisting of two elements where one of them is immediate predecessor of the second one, the smaller element has no immediate predecessor and the second one has no immediate successor.

The following example shows that a linearly ordered disjoint P-combination of countably many countably categorical o-minimal structures is not weakly o-minimal in general.

Example 1. Let $M_i := \langle \mathbb{Q} + \mathbf{i} + \mathbb{Q}, < \rangle$ be a linearly ordered structure for every $i \in \omega$, where $\mathbb{Q}$ is the set of rational numbers. Obviously, M_i is a countably categorical o-minimal structure for every $i \in \omega$. Let M' be a linearly ordered disjoint P-combination of these structures ordered by ω. Consider the following formula:

$$\phi_1(x) := \exists y[x < y \wedge \forall z(x \leq z \leq y \to x = z \vee z = y)].$$

Obviously, $\phi_1(M')$ is a union of infinitely many $\neg\phi_1(M')$-separable convex sets, whence M' is not weakly o-minimal.

The following example shows that a linearly ordered disjoint P-combination of countably many copies of a countably categorical o-minimal structure is not weakly o-minimal in general.

Example 2. Let $M := \langle \mathbb{Q} + \mathbf{2} + \mathbb{Q}, < \rangle$ be a linearly ordered structure, where $\mathbb{Q}$ is the set of rational numbers. Obviously, M is a countably categorical o-minimal structure. Let M' be a linearly ordered disjoint P-combination of ω copies of M.

Similarly as in Example 1, M' also is not weakly o-minimal.

The following theorem is a criterion for weak o-minimality of a linearly ordered disjoint P-combination of countably many countably categorical weakly o-minimal structures of finite convexity rank.

Theorem 2. *Let M_i be a countably categorical weakly o-minimal structure of finite convexity rank for each $i \in \omega$, M' be a linearly ordered disjoint P-combination of these structures. Suppose that M_i has both left and right endpoints for almost all $i \in \omega$ (i.e. but finitely many structures M_i). Then $Th(M')$ is weakly o-minimal iff the following holds:*

(1) M_i is dense for almost all $i \in \omega$;

(2) there are only finitely many P-predicates having an immediate P-predecessor or an immediate P-successor.

Proof. ($\Rightarrow$) (1) follows by Proposition 1 [10]. For completeness of the proof we present it: if (1) does not hold, then there are infinitely many structures M_i that are not dense. Since every such M_i is not dense, then there are elements of the structure M_i having an immediate predecessor or an immediate successor. By the countable categoricity of M_i there is $n_i < \omega$ such that the length of any discretely ordered chain having at least two elements is less than n_i. By weak o-minimality of M_i there exists only finitely many such chains, whence there are only finitely many elements in M_i having an immediate predecessor or an immediate successor.

Consider the following formula:

$$\phi(x) := \exists y[x < y \wedge \forall z(x \leq z \leq y \to x = z \vee z = y)].$$

Obviously, by infinitely many non-dense structures M_i the set $\phi(M')$ is a union of infinitely many $\neg\phi(M')$-separable convex sets, whence M' is not weakly o-minimal.

Now we prove that (2) holds. Obviously, if P_i is immediate P-predecessor of P_j, then the maximal element in P_i has a successor which is the minimal element in P_j. If there are infinitely many P-predicates having immediate P-predecessor or immediate P-successor, then the formula saying "x has a successor" has infinitely many convex components that contradicts weak o-minimality of $Th(M')$.

($\Leftarrow$) By both (1) and (2) there are only finitely many elements in M' having immediate predecessor or immediate successor. Denote these elements by $d_1, \ldots, d_t$ for some $t < \omega$. Observe that $M' \upharpoonright_{P_i(M')} = M_i$, i.e. the restriction of the structure M' on an arbitrary P-predicate is a countably categorical weakly o-minimal structure of finite convexity rank. By Theorem 2.8 [13] $Th(M_i)$ admits quantifier elimination to the language

$$L := \{<\} \bigcup \{c_j^i \mid 1 \le j \le s_i\} \bigcup \{U_l^i(x) \mid U_l^i(M_i) = p_l^i(M_i), p_l^i \in S_1(\emptyset), 1 \le l \le r_i\}$$

$$\bigcup \{E_{l,j}^i(x,y) \mid E_{l,j}^i \text{ is an equivalence relation on } p_l^i(M_i), RC(p_l^i) = n_l^i, 1 \le j \le n_l^i, 1 \le l \le r_i\}$$

$$\bigcup \{f_{l,j}^i \mid f_{l,j}^i : p_l^i(M_i) \to p_j^i(M_i) \text{ is a locally monotonic bijection,}$$

$$\mathrm{dcl}(\{a\}) \cap p_j^i(M_i) \ne \emptyset \text{ for some } a \in p_l^i(M_i), RC(p_l^i) \ge RC(p_j^i)\}$$

$$\bigcup \{S_{l,j}^i(x,y) : p_l^i \not\perp^w p_j^i,\ \mathrm{dcl}(\{a\}) \cap p_j^i(M_i) = \emptyset \text{ for all } a \in p_l^i(M_i), RC(p_l^i) \ge RC(p_j^i),$$

$$S_{l,j}^i(x,y) \text{ is a basic } (p_l^i, p_j^i)\text{-splitting formula}\}.$$

Then $Th(M')$ admits quantifier elimination to the language

$$L \cup \{d_1 \ldots, d_t\} \cup \{P_i \mid i \in \omega\},$$

where $c_j^i \in P_i(M')$ and $U_l^i(M') \subseteq P_i(M')$ for all $1 \le j \le s_i, 1 \le l \le r$.

Recall that $\Sigma_i \cap \Sigma_j = \emptyset$ for any $i, j < \omega$ with $i \ne j$. Thus, any formula of the language $L \cup \{d_1 \ldots, d_t\} \cup \{P_i \mid i \in \omega\}$ is decomposed into a boolean combination of subformulas, each of which is a formula of the language Σ_i for some $i \in \omega$. By weak o-minimality of M_i for every $i \in \omega$ the set of realizations of any formula with one free variable of the signature Σ_i with constants from M_i is a union of finitely many convex sets in M_i. Since every predicate $P_i(x)$ is convex, we conclude that $Th(M')$ is weakly o-minimal.

Corollary 1. *Let M_i be a countably categorical weakly o-minimal structure of finite convexity rank for each $i \in \omega$, M' be a linearly ordered disjoint P-combination of these structures. Suppose that M_i has no endpoints or has only a left (right) endpoint for almost all $i \in \omega$. Then $Th(M')$ is weakly o-minimal iff M_i is dense for almost all $i \in \omega$.*

Proof. ($\Rightarrow$) It follows by Proposition 1 [10].

($\Leftarrow$) Since M_i is dense for almost all $i \in \omega$, there exist only finitely many structures M_i that are not dense. Consequently, by countable categoricity of M_i every non-dense M_i has only finitely many elements having immediate predecessor or immediate successor, whence M' also has only finitely many elements with this property. Denote these elements by $d_1, \dots, d_t$ for some $t < \omega$. Further by analogy with the proof of Theorem 2 we establish that $Th(M')$ is weakly o-minimal.

Recall that a theory T is *Ehrenfeucht* if T has finitely many countable models ($I(T, \omega) < \omega$) but is not countably categorical ($I(T, \omega) > 1$). A structure with an Ehrenfeucht theory is also *Ehrenfeucht*.

Proposition 1. *Let M be a countably categorical weakly o-minimal structure of finite convexity rank, M' be a linearly ordered disjoint P-combination of ω copies of M. Then $Th(M')$ has either 2^ω countable models or $Th(M')$ is Ehrenfeucht.*

Proof.

Case 1. M is not dense. Then consider the following formula:

$$\phi(x) := \exists y[x < y \wedge \forall z(x \le z \le y \to x = z \vee z = y)].$$

Obviously, $\phi(M) \neq \emptyset$. Since any countably categorical weakly o-minimal structure has only finitely many elements having an immediate predecessor or an immediate successor, $\phi(M)$ is finite. Consequently, $\neg\phi(M)$ is infinite.

Let us call the structures $M_r := \langle M_< + \mathbb{Q}, < \rangle$ and $M_l := \langle \mathbb{Q} + M_<, < \rangle$ M_r*-component* and M_l*-component*, respectively, where $M_<$ is the reduct of the structure M on $\{<\}$.

Obviously, for any ordering a P-combination of countably many copies of M there exists at least one P-cut $\mathcal{C}$. If $\mathcal{C}$ is rational to right (left), then it can be realized by any finite or infinite number of M_r-components (M_l-components), and between any discretely ordered chains of M_r-components (M_l-components) we can realize infinitely many densely ordered M_r-components (M_l-components). If $\mathcal{C}$ is P-irrational, then it can be realized by any finite or infinite number of either M_r-components or M_l-components and so on. Then the $\mathcal{C}$-spectrum is 2^ω, i.e. $Th(M')$ has 2^ω countable models.

Case 2. M is dense and it has both endpoints. If there are only finitely many P-predicates having immediate P-predecessor or immediate P-successor, then by Theorem 1 there are infinitely many pairwise orthogonal P-cuts, and consequently $Th(M')$ has 2^ω countable models.

Suppose now that M' has finitely many P-cuts and there are infinitely many P-predicates having immediate P-predecessor or immediate P-successor. Consider the following formula:

$$\theta(x) := \phi(x) \vee \exists y[y < x \wedge \forall z(y \le z \le x \to y = z \vee z = x)].$$

Then $\theta(M')$ is infinite, and $\theta(x)$ defines duplets in M'. Let us call the structures $M_r := \langle \mathbf{2} + \mathbb{Q}, < \rangle$ and $M_l := \langle \mathbb{Q} + \mathbf{2}, < \rangle$ M_r*-component* and M_l*-component*, respectively, where $\mathbf{2}$ denotes a duplet. By our supposition there is at least one P-cut $\mathcal{C}$. It can be showed similarly as in Case 1 that the $\mathcal{C}$-spectrum is 2^ω, i.e. $Th(M')$ has 2^ω countable models.

Case 3. M is dense and if M has a first element, then M has no a last element. In this case $\theta(M') = \emptyset$, and consequently the $\mathcal{C}$-spectrum of any P-cut $\mathcal{C}$ is finite. If there is no an infinite partition of M' into infinite P-intervals, then by Theorem 1 $Th(M')$ is Ehrenfeucht. If there is infinite partition of M' into infinite P-intervals, then by Theorem 1 there are infinitely many pairwise orthogonal P-cuts, and consequently $Th(M')$ has 2^ω countable models.

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Кулпешов Б.Ш., Торебекова Ф.А. ӘЛСІЗ О-МИНИМАЛДЫ ҚҰРЫЛЫМДАРДЫҢ КОМБИНАЦИЯЛАРЫ ТУРАЛЫ

Бұл мақалада біз саналымды категориялық әлсіз о-минималды құрылымдар комбинацияларының қасиеттерін зерттейміз. Мақаланың негізгі нәтижесі — дөңестік рангісі шектеулі саналымды категориялық әлсіз о-минималды құрылымдардың саналымды санды сызықты ретті қиылыспайтын P-комбинацияның әлсіз о-минималдық критерийі.

Түйінді сөздер. Әлсіз о-минималдық, P-комбинация, саналымды категориялық, дөңестік рангісі.

Кулпешов Б.Ш., Торебекова Ф.А. О КОМБИНАЦИЯХ СЛАБО О-МИНИМАЛЬНЫХ СТРУКТУР

В настоящей статье мы исследуем свойства комбинаций счетно категоричных слабо о-минимальных структур. Основной результат статьи — это критерий слабой о-минимальности линейно упорядоченной непересекающейся P-комбинации счетного числа счетно категоричных слабо о-минимальных структур конечного ранга выпуклости.

Ключевые слова. Слабая о-минимальность, P-комбинация, счетная категоричность, ранг выпуклости.

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Blow-up solutions to p-sub-Laplacian heat equations on the Heisenberg group

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Abstract. In this note, we prove the blow-up of solutions to the Dirichlet initial value problem for the p-sup-Laplacian heat equation on the Heisenberg group by using the concavity method.

Keywords. Blow-up, p-sub-Laplacian, Heisenberg group, Concavity method

1 Introduction

Let f be a locally Lipschitz continuous function on $\mathbb{R}$, $f(0) = 0$, and such that $f(u) > 0$ for $u > 0$. Furthermore, we suppose that u_0 is a non-negative and non-trivial function in $L^\infty(\Omega) \cap \mathring{S}^{1,p}$ and that $u_0(\xi) = 0$ on boundary $\partial\Omega$ of Ω, where $\mathring{S}^{1,p}$ is the Sobolev type space defined at the end of the Introduction.

We consider the following p-sub-Laplacian heat equation

$$\begin{cases} u_t(\xi,t) - \mathcal{L}_p u(\xi,t) = f(u(\xi,t)), & (\xi,t) \in \Omega \times (0,+\infty), \\ u(\xi,t) = 0, & (\xi,t) \in \partial\Omega \times [0,+\infty), \\ u(\xi,0) = u_0(\xi) \geq 0, & \xi \in \overline{\Omega}, \end{cases} \tag{1}$$

where $1 < p < \infty$ and Ω is a bounded domain in the Heisenberg group with smooth boundary $\partial\Omega$. Here

$$\mathcal{L}_p f := \sum_{j=1}^{n} \left(X_j(|\nabla_H f|^{p-2} X_j f) + Y_j(|\nabla_H f|^{p-2} Y_j f) \right), \ p > 1, \tag{2}$$

is the p-sub-Laplacian on the Heisenberg group, where $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are the left-invariant vector fields spanning the first stratum.

In the Euclidean setting, it is well-known that there often exists a solution of the p-Laplacian parabolic equation as the one in (1) for all times. There is a large literature on the

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sufficient conditions for the local existence of solutions to the p-Laplacian parabolic equation. For example, the sufficient conditions for the local existence of solutions to the p-Laplacian parabolic equations are derived by Ball [1] and Zhao [2] for $p = 2$ and $p > 2$, respectively. Then, blow-up solutions have been investigated by many authors such as Levine [3], Philippin and Proytcheva [4], Ding and Hu [5], Bandle and Brunner [6], with a more detailed review of their works presented in [7].

In this paper, we study the blow-up solutions of the p-sub-Laplacian heat equations on the Heisenberg group. Our proof is mainly based on the concavity method with a condition

$$c_1 \int_0^u f(s)ds \leq uf(u) + c_3u^p + c_1c_2, \text{ for } u > 0,$$

which is recently introduced by Chung and Choi [7].

Let us give a brief introduction to the Heisenberg group. Let $\mathbb{H}^n$ be the Heisenberg group, that is, the set $\mathbb{R}^{2n+1}$ equipped with the group law

$$\xi \circ \widetilde{\xi} := (x + \widetilde{x}, y + \widetilde{y}, s + \widetilde{s} + 2\sum_{i=1}^n (\widetilde{x}_i y_i - x_i \widetilde{y}_i)),$$

where $\xi := (x, y, s) \in \mathbb{H}^n$, $x := (x_1, \ldots, x_n)$, $y := (y_1, \ldots, y_n)$, and $\xi^{-1} = -\xi$ is the inverse element of ξ with respect to the group law (see, e.g. [8]). The dilation operation of the Heisenberg group with respect to the group law has the form

$$\delta_\lambda(\xi) := (\lambda x, \lambda y, \lambda^2 s) \text{ for } \lambda > 0.$$

The Lie algebra $\mathfrak{h}$ of the left-invariant vector fields on the Heisenberg group $\mathbb{H}^n$ is spanned by

$$X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial s} \text{ for } 1 \leq i \leq n,$$
$$Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial s} \text{ for } 1 \leq i \leq n,$$

and with their (non-zero) commutator

$$[X_i, Y_i] = -4\frac{\partial}{\partial s}.$$

The horizontal gradient of $\mathbb{H}^n$ is given by

$$\nabla_H := (X_1, \ldots, X_n, Y_1, \ldots, Y_n),$$

then we express

$$\mathcal{L} := \sum_{i=1}^n \left(X_i^2 + Y_i^2\right),$$

and for $p > 1$

$$\mathcal{L}_p f := \nabla_H \cdot (|\nabla_H f|^{p-2}\nabla_H f),$$

as the sub-Laplacian and p-sub-Laplacian on the Heisenberg group $\mathbb{H}^n$, respectively. We refer the recent open access book [9] for Heisenberg-type and more general Lie group dicussions. We also refer remarkable work by Kirane and his collaborators (see, e.g. [10]- [11]) in this direction.

Let $\Omega \subset \mathbb{H}^n$ be an open set, then we define the functional spaces

$$S^{1,p}(\Omega) = \{u : u, |\nabla_H u| \in L^p(\Omega)\}. \tag{3}$$

We consider the following functional

$$J_p(u) := \|\nabla_H u\|_{L^p(\Omega)}.$$

Thus, the functional class $\mathring{S}^{1,p}(\Omega)$ can be defined as the completion of $C_0^1(\Omega)$ in the norm generated by J_p, see e.g. [12].

2 Main results

2.1 Blow-up solutions to the sub-Laplacian heat equation. We consider the blow-up solutions to the sub-Laplacian heat equation on the Heisenberg group $\mathbb{H}^n$, that is,

$$\begin{cases} u_t(\xi,t) - \mathcal{L}u(\xi,t) = f(u(\xi,t)), & (\xi,t) \in \Omega \times (0,+\infty),\ \Omega \subset \mathbb{H}^n, \\ u(\xi,t) = 0, & (\xi,t) \in \partial\Omega \times [0,+\infty), \\ u(\xi,0) = u_0(\xi) \geq 0, & \xi \in \overline{\Omega}, \end{cases} \tag{4}$$

where f is locally Lipschitz continuous on $\mathbb{R}$, $f(0) = 0$, and such that $f(u) > 0$ for $u > 0$. Furthermore, we suppose that u_0 is a non-negative and non-trivial function in $C^1(\overline{\Omega})$ and that $u_0(\xi) = 0$ on the boundary $\partial\Omega$.

Lemma 1 [13]. *Let Ω be a bounded domain of the Heisenberg group $\mathbb{H}^n$. Then there exist $\lambda_1 > 0$ and $0 < v_1 \in \mathring{S}^{1,2}(\Omega)$ such that*

$$\begin{cases} -\mathcal{L}v_1(\xi) = \lambda_1 v_1(\xi), & \xi \in \Omega, \\ v_1(\xi) = 0, & \xi \in \partial\Omega,\ \Omega \subset \mathbb{H}^n, \end{cases} \tag{5}$$

where

$$\lambda_1 := \inf_{u \in \mathring{S}^{1,2}(\Omega)} \frac{\int_\Omega |\nabla_H u|^2 d\xi}{\int_\Omega |u|^2 d\xi}.$$

Recall that λ_1 is the principal frequency of $\mathcal{L}$ and v_1 is the associated eigenfunction.

Theorem 1. *Let Ω be a bounded domain of the Heisenberg group $\mathbb{H}^n$ with a smooth boundary $\partial\Omega$. Let a function f satisfy the condition that there exist constants $c_1 > 2$ and c_2 such that for all $u > 0$ we have*

$$c_1 \int_0^u f(s)ds \leq uf(u) + c_3 u^2 + c_1 c_2, \tag{6}$$

where $0 < c_3 \leq \frac{(c_1-2)\lambda_1}{2}$, where λ_1 is the principal frequency of the sub-Laplacian $\mathcal{L}$. If $u_0 \in C^1(\overline{\Omega})$ with $u_0 = 0$ on $\partial\Omega$ satisfies the inequality

$$-\frac{1}{2}\|\nabla_H u_0\|^2_{L^2(\Omega)} + \int_\Omega \left(\int_0^{u_0(\xi)} f(s)ds - c_2 \right) d\xi > 0, \tag{7}$$

then the nonnegative solution to the equation (4) blows up at a finite time T^* for

$$M := \frac{\left(1 + \sqrt{\frac{c_1}{2}}\right) \|u_0\|^4_{L^2(\Omega)}}{2(c_1 - 2)\left[-\frac{1}{2}\|\nabla_H u_0\|^2_{L^2(\Omega)} + \int_\Omega (\int_0^{u_0(\xi)} f(s)ds - c_2)d\xi \right]}, \tag{8}$$

such that

$$0 < T^* \leq \frac{M}{\left(\sqrt{c_1/2} - 1\right) \|u_0\|^2_{L^2(\Omega)}}, \tag{9}$$

that is,

$$\lim_{t \to T^*} \int_0^t \int_\Omega u^2(\xi, \tau) d\xi d\tau = +\infty. \tag{10}$$

Proof of Theorem 1. Following the standard procedure we define a new functional $\mathcal{F}$, that is,

$$\mathcal{F}(t) := -\frac{1}{2}\int_\Omega |\nabla_H u(\xi, t)|^2 d\xi + \int_\Omega (F(u(\xi, t)) - c_2) d\xi, \;\; t \in [0, +\infty), \tag{11}$$

where $F(u) := \int_0^u f(s)ds$. In the case $t = 0$ this functional has the form

$$\mathcal{F}(0) := -\frac{1}{2}\|\nabla_H u_0\|^2_{L^2(\Omega)} + \int_\Omega (F(u_0) - c_2)d\xi > 0, \tag{12}$$

in view of (7).

By (7) it is strictly positive. Now we have the following computations

$$\int_0^t \frac{d}{d\tau}\mathcal{F}(\tau)d\tau = -\int_0^t \int_\Omega \langle \nabla_H u(\xi,\tau), \nabla_H u_\tau(\xi,\tau)\rangle d\xi d\tau + \int_0^t \int_\Omega F_u(u(\xi,\tau))u_\tau(\xi,\tau)d\xi d\tau$$

$$= \int_0^t \int_\Omega \mathcal{L}u(\xi,\tau)u_\tau(\xi,\tau)d\xi d\tau + \int_0^t \int_\Omega f(u(\xi,\tau))u_\tau(\xi,\tau)d\xi d\tau = \int_0^t \int_\Omega u_\tau^2(\xi,\tau)d\xi d\tau,$$

and

$$\int_0^t \frac{d}{d\tau}\mathcal{F}(\tau)d\tau = -\frac{1}{2}\int_\Omega\int_0^t \frac{d}{d\tau}(|\nabla_H u(\xi,\tau)|^2)d\tau d\xi + \int_\Omega\int_0^t \frac{d}{d\tau}(F(u(\xi,\tau))-c_2)d\tau d\xi$$
$$= -\frac{1}{2}\int_\Omega(|\nabla_H u(\xi,t)|^2 - |\nabla_H u_0|^2)d\xi + \int_\Omega[F(u(\xi,t)) - F(u_0)]d\xi.$$

That allows one to write the functional $\mathcal{F}(t)$ in the following way

$$\mathcal{F}(t) = \mathcal{F}(0) + \int_0^t \frac{d}{d\tau}\mathcal{F}(\tau)d\tau = \mathcal{F}(0) + \int_0^t\int_\Omega u_\tau^2(\xi,\tau)d\xi d\tau. \tag{13}$$

We introduce a new function $\mathcal{I}$ as follows

$$\mathcal{I}(t) := \int_0^t\int_\Omega u^2(\xi,\tau)d\xi d\tau + M,\ t \geq 0, \tag{14}$$

where $M > 0$ is a constant to be determined later. By Leibniz's integral rule we get

$$\mathcal{I}'(t) = \frac{d}{dt}\mathcal{I}(t) = \frac{d}{dt}\left(\int_0^t\int_\Omega u^2(\xi,\tau)d\xi d\tau\right) = \int_\Omega u^2(\xi,t)d\xi,$$

and

$$\int_\Omega\int_0^t 2u(\xi,\tau)u_\tau(\xi,\tau)d\tau d\xi = \int_\Omega\int_0^t \frac{d}{d\tau}u^2(\xi,\tau)d\tau d\xi = \int_\Omega u^2(\xi,t)d\xi - \|u_0\|^2_{L^2(\Omega)}.$$

This gives the relation

$$\mathcal{I}'(t) = \int_\Omega u^2(\xi,t)d\xi = \int_\Omega\int_0^t 2u(\xi,\tau)u_\tau(\xi,\tau)d\tau d\xi + \|u_0\|^2_{L^2(\Omega)}. \tag{15}$$

Using the above computations, the condition (6) and Lemma 1, we compute the second derivative of $\mathcal{I}(t)$ with respect to time

$$\mathcal{I}''(t) = \frac{d}{dt}\mathcal{I}'(t) = \frac{d}{dt}\int_\Omega u^2(\xi,t)d\xi = 2\int_\Omega u(\xi,t)u_t(\xi,t)d\xi$$
$$= 2\int_\Omega u(\xi,t)\mathcal{L}u(\xi,t) + 2\int_\Omega u(\xi,t)f(u(\xi,t))d\xi$$
$$\geq -2\int_\Omega |\nabla_H u(\xi,t)|^2 d\xi + 2\int_\Omega \left[c_1F(u(\xi,t)) - c_3u^2(\xi,t) - c_1c_2\right]d\xi$$
$$= 2c_1\left[-\frac{1}{2}\int_\Omega |\nabla_H u(\xi,t)|^2 d\xi + \int_\Omega (F(u(\xi,t)) - c_2)d\xi\right]$$

$$+(c_1-2)\int_\Omega |\nabla_H u(\xi,t)|^2 d\xi - 2c_3\int_\Omega u^2(\xi,t)d\xi$$

$$\geq 2c_1\mathcal{F}(t) + ((c_1-2)\lambda_1 - 2c_3)\int_\Omega u^2(\xi,t)d\xi \geq 2c_1\mathcal{F}(t).$$

That can be rewritten as

$$\mathcal{I}''(t) \geq 2c_1\mathcal{F}(0) + 2c_1\int_0^t\int_\Omega u_\tau^2(\xi,\tau)d\xi d\tau. \tag{16}$$

Also, we compute by making use of Hölder and Schwartz's inequalities,

$$(\mathcal{I}'(t))^2 \leq 4(1+\sigma)\left(\int_\Omega\int_0^t u(\xi,\tau)u_\tau(\xi,\tau)d\tau d\xi\right)^2 + \left(1+\frac{1}{\sigma}\right)\|u_0\|^4_{L^2(\Omega)}$$

$$\leq 4(1+\sigma)\left(\int_\Omega\left(\int_0^t u^2(\xi,\tau)d\tau\right)^{\frac{1}{2}}\left(\int_0^t u_\tau^2(\xi,\tau)d\tau\right)^{\frac{1}{2}}d\xi\right)^2 + \left(1+\frac{1}{\sigma}\right)\|u_0\|^4_{L^2(\Omega)}$$

$$\leq 4(1+\sigma)\left(\int_\Omega\int_0^t u^2(\xi,\tau)d\tau d\xi\right)\left(\int_\Omega\int_0^t u_\tau^2(\xi,\tau)d\tau d\xi\right) + \left(1+\frac{1}{\sigma}\right)\|u_0\|^4_{L^2(\Omega)},$$

where $\sigma > 0$. Then by combining the above expressions and taking $\sigma = \sqrt{c_1/2} - 1 > 0$, we establish the estimate

$$\mathcal{I}''(t)\mathcal{I}(t) - (1+\sigma)(\mathcal{I}'(t))^2 \geq 2c_1\left(\mathcal{F}(0) + \int_0^t\int_\Omega u_\tau^2(\xi,\tau)d\xi d\tau\right)\left(\int_0^t\int_\Omega u^2(\xi,\tau)d\xi d\tau + M\right)$$

$$-4(1+\sigma)(1+\sigma)\left(\int_\Omega\int_0^t u^2(\xi,\tau)d\tau d\xi\right)\left(\int_\Omega\int_0^t u_\tau^2(\xi,\tau)d\tau d\xi\right)$$

$$-(1+\sigma)\left(1+\frac{1}{\sigma}\right)\|u_0\|^4_{L^2(\Omega)} \geq 2c_1 M\mathcal{F}(0) - (1+\sigma)\left(1+\frac{1}{\sigma}\right)\|u_0\|^4_{L^2(\Omega)}.$$

Since $\mathcal{F}(0) > 0$ and we choose $M > 0$ as large enough to satisfy

$$\mathcal{I}''(t)\mathcal{I}(t) - (1+\sigma)(\mathcal{I}'(t))^2 > 0. \tag{17}$$

We can see that the above expression for $t \geq 0$ implies

$$\frac{d}{dt}\left[\frac{\mathcal{I}'(t)}{\mathcal{I}^{\sigma+1}(t)}\right] > 0 \Rightarrow \begin{cases}\mathcal{I}' \geq \frac{\|u_0\|^2_{L^2(\Omega)}}{M^{\sigma+1}}\mathcal{I}^{1+\sigma}(t),\\ \mathcal{I}(0) = M.\end{cases}$$

Then we arrive at

$$\mathcal{I}(t) \geq \left(\frac{1}{M^\sigma} - \frac{\sigma\|u_0\|^2_{L^2(\Omega)}}{M^{\sigma+1}}t\right)^{-\frac{1}{\sigma}}.$$

From here we see that the solutions blow up in the finite time T^* which is

$$0 < T^* \leq \frac{M}{\sigma \|u_0\|^2_{L^2(\Omega)}},$$

where M can be estimated from (17), that is,

$$M := \frac{(1+\sigma)(1+\frac{1}{\sigma})\|u_0\|^4_{L^2(\Omega)}}{2c_1\mathcal{F}(0)} = \frac{\left(1+\sqrt{\frac{c_1}{2}}\right)\|u_0\|^4_{L^2(\Omega)}}{2(c_1-2)[-\frac{1}{2}\|\nabla_H u_0\|^2_{L^2(\Omega)} + \int_\Omega (F(u_0)-c_2)d\xi]}.$$

Therefore, it follows that $\mathcal{I}(t)$ cannot remain finite for all $t>0$. In other words, the solution u blows up in finite time T^*.

2.2 Blow-up solutions for p-sub-Laplacian heat equations. We consider now the blow-up solutions to the p-sub-Laplacian heat equation on the Heisenberg group $\mathbb{H}^n$, that is,

$$\begin{cases} u_t(\xi,t) - \mathcal{L}_p u(\xi,t) = f(u(\xi,t)), & (\xi,t) \in \Omega \times (0,+\infty),\ \Omega \subset \mathbb{H}^n, \\ u(\xi,t) = 0, & (\xi,t) \in \partial\Omega \times [0,+\infty), \\ u(\xi,0) = u_0(\xi) \geq 0, & \xi \in \overline{\Omega}, \end{cases} \tag{18}$$

where f is locally Lipschitz continuous on $\mathbb{R}$, $f(0)=0$, and such that $f(u)>0$ for $u>0$. Furthermore, we suppose that u_0 is a non-negative and non-trivial function in $L^\infty(\Omega)\cap \mathring{S}^{1,p}(\Omega)$ and that $u_0(\xi)=0$ on the boundary $\partial\Omega$.

Theorem 2. *Let Ω be a bounded domain of the Heisenberg group $\mathbb{H}^n$ with a smooth boundary $\partial\Omega$. Let a function f satisfy the condition that there exist constants $c_1>p$ and c_2 such that for all $u>0$ we have*

$$c_1 \int_0^u f(s)ds \leq uf(u) + c_3 u^p + c_1 c_2, \tag{19}$$

where $0 < c_3 \leq \frac{(c_1-p)\lambda_{1,p}}{p}$ and $\lambda_{1,p}$ is the principal eigenvalue of the p-sub-Laplacian $\mathcal{L}_p$. If $u_0 \in L^\infty(\Omega)\cap \mathring{S}^{1,p}(\Omega)$ satisfies for the inequality

$$-\frac{1}{p}\|u_0\|^p_{L^p(\Omega)} + \int_\Omega \left(\int_0^{u_0(\xi)} f(s)ds - c_2 \right) d\xi > 0, \tag{20}$$

then the nonnegative solution to the equation (18) blows up at a finite time T^* for

$$M := \frac{\left(1+\sqrt{\frac{c_1}{2}}\right)\|u_0\|^4_{L^2(\Omega)}}{2(c_1-2)[-\frac{1}{p}\|u_0\|^p_{L^p(\Omega)} + \int_\Omega(\int_0^{u_0(\xi)} f(s)ds - c_2)d\xi]}, \tag{21}$$

such that

$$0 < T^* \leq \frac{M}{\left(\sqrt{c_1/2} - 1\right) \|u_0\|^2_{L^2(\Omega)}}, \tag{22}$$

that is,

$$\lim_{t \to T^*} \int_0^t \int_\Omega u^2(\xi, \tau) d\xi d\tau = +\infty. \tag{23}$$

We introduce Lemma 2 and Lemma 3 that will be useful to proving Theorem 2.

Lemma 2 [13]. *For $1 < p < \infty$ there exist $\lambda_{1,p} > 0$ and $\phi_{1,p} \in S^{1,p}(\Omega)$ with $\phi_{1,p} > 0$ in Ω such that*

$$\begin{cases} -\mathcal{L}_p v_{1,p}(x) = \lambda_{1,p} |v_{1,p}(x)|^{p-2} v_{1,p}, & x \in \Omega, \ \Omega \subset \mathbb{H}^n, \\ v_{1,p}(x) = 0, & x \in \partial\Omega. \end{cases} \tag{24}$$

Moreover, $\lambda_{1,p}$ is given by

$$\lambda_{1,p} = \inf_{u \in S_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla_H u|^p d\xi}{\int_\Omega |u|^p d\xi} > 0.$$

Recall that $\lambda_{1,p}$ is the principal frequency and $v_{1,p}$ is the associated eigenfunction of the p-sub-Laplacian $\mathcal{L}_p$.

Remark. Note that the existence of solutions for the nonlinear eigenvalue problems with weights for the p-sub-Laplacian on the Heisenberg group was discussed in [13]. Lemma from [13] is useful in the proof of blow-up solutions for p-sub-Laplacian heat equations.

Lemma 3. *Let u be a weak solution to the equation* (18) *with $|\nabla_H u_0| \in L^p(\Omega)$. Then*

$$\frac{1}{2} \int_0^t \int_\Omega (u^2(\xi, \tau))_\tau d\xi d\tau = \frac{1}{2} \int_\Omega [u^2(\xi, t) - u_0^2(\xi)] d\xi \tag{25}$$

$$= \int_0^t \int_\Omega [-|\nabla_H u(\xi, t)|^p + u(\xi, t) f(u(\xi, t))] d\xi d\tau,$$

and

$$\int_0^t \int_\Omega u_\tau^2(\xi, \tau) d\xi d\tau = -\frac{1}{p} \int_\Omega [|\nabla_H u(\xi, t)|^p - |\nabla_H u_0(\xi)|^p] d\xi \tag{26}$$

$$+ \int_\Omega [F(u(\xi, t)) - F(u_0(\xi))] d\xi,$$

where $F(u) := \int_0^u f(s) ds$.

Proof of Lemma 2. We first prove the equality (25) by using the equation (18), that is,

$$\frac{1}{2}\int_0^t\int_\Omega (u^2(\xi,\tau))_\tau d\xi d\tau = \frac{1}{2}\int_\Omega u^2(\xi,\tau)|_0^t d\xi = \frac{1}{2}\int_\Omega [u^2(\xi,t)-u_0^2(\xi)]d\xi,$$

and

$$\frac{1}{2}\int_0^t\int_\Omega (u^2(\xi,\tau))_\tau d\xi d\tau = \int_\Omega\int_0^t u(\xi,\tau)u_\tau(\xi,\tau)d\xi d\tau$$
$$= \int_0^t\int_\Omega \mathcal{L}_p u(\xi,\tau)u(\xi,\tau)d\xi d\tau + \int_0^t\int_\Omega f(u(\xi,\tau))u(\xi,\tau)d\xi d\tau$$
$$= -\int_0^t\int_\Omega |\nabla_H u(\xi,\tau)|^p d\xi d\tau + \int_0^t\int_\Omega f(u(\xi,\tau))u(\xi,\tau)d\xi d\tau,$$

which proves the expression (25). Now we prove the inequality (26) by using the Leibniz integral rule, as follows

$$\int_0^t\int_\Omega u_\tau^2(\xi,\tau)d\xi d\tau = \int_0^t\int_\Omega \mathcal{L}_p u(\xi,\tau)u_\tau(\xi,\tau)d\xi d\tau + \int_0^t\int_\Omega f(u(\xi,\tau))u_\tau(\xi,\tau)d\xi d\tau$$
$$= -\int_0^t\int_\Omega \langle |\nabla_H u(\xi,\tau)|^{p-2}\nabla_H u(\xi,\tau), \nabla_H u_\tau(\xi,\tau)\rangle d\xi d\tau + \int_0^t\int_\Omega f(u(\xi,\tau))u_\tau(\xi,\tau)d\xi d\tau$$
$$= -\frac{1}{2}\int_\Omega\int_0^t \frac{d}{d\tau}\left(\int_0^{|\nabla_H u(\xi,\tau)|^2} s^{\frac{p-2}{2}}ds\right)d\tau d\xi + \int_\Omega\int_0^t \frac{d}{d\tau}\left(\int_0^{u(\xi,\tau)} f(s)ds\right)d\tau d\xi$$
$$= -\frac{1}{2}\int_\Omega\int_0^{|\nabla_H u(\xi,\tau)|^2} s^{\frac{p-2}{2}}ds d\xi|_0^t + \int_\Omega F(u(\xi,\tau))d\xi|_0^t$$
$$= -\frac{1}{2}\int_\Omega\left(\frac{2}{p}s^{\frac{p}{2}}|_0^{|\nabla_H u(\xi,\tau)|^2}\right)d\xi|_0^t + \int_\Omega F(u(\xi,\tau))d\xi|_0^t$$
$$= -\frac{1}{p}\int_\Omega [|\nabla_H u(\xi,t)|^p - |\nabla_H u_0(\xi)|^p]\,d\xi + \int_\Omega F(u(\xi,t)) - F(u_0(\xi))d\xi,$$

which proves the expression (26).

Proof of Theorem 2. We begin by defining the function $\mathcal{F}_p$ by

$$\mathcal{F}_p(t) := -\frac{1}{p}\int_\Omega |\nabla_H u(\xi,t)|^p d\xi + \int_\Omega [F(u(\xi,t)) - c_2]d\xi. \tag{27}$$

Then for $t=0$, by (20), we have

$$\mathcal{F}_p(0) = -\frac{1}{p}\|\nabla_H u_0\|^p_{L^2(\Omega)} + \int_\Omega [F(u_0(\xi)) - c_2]d\xi > 0.$$

We can rewrite $\mathcal{F}_p$ by using Lemma 3 as

$$\mathcal{F}_p(t) = \mathcal{F}_p(0) + \int_0^t \frac{d}{d\tau}\mathcal{F}_p(\tau)d\tau = \mathcal{F}_p(0) + \int_0^t \int_\Omega (u_\tau(\xi,\tau))^2 d\xi d\tau,$$

where

$$\int_0^t \frac{d}{d\tau}\mathcal{F}_p(\tau)d\tau = -\frac{1}{p}\int_\Omega \int_0^t \frac{d}{d\tau}(|\nabla_H u(\xi,\tau)|^p)d\tau d\xi + \int_0^t \int_\Omega \frac{d}{d\tau}F(u(\xi,\tau))d\tau d\xi$$

$$= -\int_\Omega \int_0^t \langle |\nabla_H u(\xi,\tau)|^{p-2}\nabla_H u(\xi,\tau), \nabla_H u_\tau(\xi,\tau)\rangle d\tau d\xi + \int_0^t \int_\Omega \frac{d}{d\tau}\left(\int_0^{u(\xi,\tau)} f(s)ds\right) d\tau d\xi$$

$$= \int_\Omega \int_0^t \mathcal{L}_p u(\xi,\tau) u_\tau(\xi,\tau) d\xi d\tau + \int_\Omega \int_0^t f(u(\xi,\tau)) u_\tau(\xi,\tau) d\tau d\xi = \int_0^t \int_\Omega (u_\tau(\xi,\tau))^2 d\xi d\tau.$$

Let us define the function $\mathcal{I}_p$ by

$$\mathcal{I}_p(t) := \int_0^t \int_\Omega u^2(\xi,\tau) d\xi d\tau + M, \ t \geq 0, \tag{28}$$

where M is a positive constant. Then we compute the derivative of $\mathcal{I}(t)$ with respect to time, which gives that

$$\mathcal{I}'(t) = \frac{d}{dt}\int_0^t \int_\Omega u^2(\xi,\tau) d\xi d\tau = \int_\Omega \frac{d}{dt}\left(\int_0^t u^2(\xi,\tau)d\tau\right) d\xi = \int_\Omega u^2(\xi,t)d\xi$$

$$= \int_0^t \int_\Omega 2u(\xi,\tau)u_\tau(\xi,\tau) d\tau d\xi + \|u_0\|^2_{L^2(\Omega)}.$$

The second derivative of $\mathcal{I}_p(t)$ with respect to time t can be calculated by Lemma 3, using the condition (19), and Lemma 2, so that we get the estimate for $\mathcal{I}''_p(t)$ as follows:

$$\mathcal{I}''_p(t) = \frac{d}{dt}\mathcal{I}'_p(t) = \frac{d}{dt}\int_\Omega u^2(\xi,t)d\xi = -2\int_\Omega |\nabla_H u(\xi,t)|^p d\xi + 2\int_\Omega u(\xi,t) f(u(\xi,t)) d\xi$$

$$\geq -2\int_\Omega |\nabla_H u(\xi,t)|^p d\xi + 2\int_\Omega \left(c_1 F(u(\xi,t)) - c_3 u^p(\xi,t) - c_1 c_2\right) d\xi$$

$$= 2c_1\left[-\frac{1}{p}\int_\Omega |\nabla_H u(\xi,t)|^p d\xi + \int_\Omega [F(u(\xi,t)) - c_2] d\xi\right]$$

$$+\frac{2(c_1 - p)}{p}\int_\Omega |\nabla_H u(\xi,t)|^p d\xi - 2c_3\int_\Omega u^p(\xi,t)d\xi$$

$$\geq 2c_1\mathcal{F}_p(t) + 2\left(\frac{(c_1-p)\lambda_{1,p}}{p} - c_3\right)\int_\Omega u^p(\xi,t)d\xi \geq 2c_1\mathcal{F}_p(t).$$

Then by using (28) the above estimate can be written in the following form

$$\mathcal{I}_p''(t) \geq 2c_1\mathcal{F}_p(0) + 2c_1\int_0^t\int_\Omega u_\tau^2(\xi,\tau)d\xi d\tau. \tag{29}$$

By making use of Schwartz's inequality and for arbitrary $\sigma > 0$ as in the case $p = 2$ we arrive at

$$\left(\mathcal{I}_p'(t)\right)^2 \leq 4(1+\sigma)\left(\int_\Omega\int_0^t u^2(\xi,\tau)d\tau d\xi\right)\left(\int_\Omega\int_0^t u_\tau^2(\xi,\tau)d\tau d\xi\right) + \frac{1+\sigma}{\sigma}\|u_0\|^4_{L^2(\Omega)}.$$

Now we use estimates of $\mathcal{I}_p'(t)$ and $\mathcal{I}_t''(t)$ with $\sigma = \left(\sqrt{c_1/2}-1\right)$ to obtain

$$\begin{aligned}&\mathcal{I}_p''(t)\mathcal{I}_p(t) - (1+\sigma)(\mathcal{I}_p'(t))^2\\
&\geq 2c_1\left[\mathcal{F}_p(0) + \int_0^t\int_\Omega u_\tau^2(\xi,\tau)d\xi d\tau\right]\left[\int_0^t\int_\Omega u^2(\xi,\tau)d\xi d\tau + M\right]\\
&-4(1+\sigma)(1+\sigma)\left(\int_\Omega\int_0^t u^2(\xi,\tau)d\tau d\xi\right)\left(\int_\Omega\int_0^t u_\tau^2(\xi,\tau)d\tau d\xi\right)\\
&-(1+\sigma)\left(\frac{1+\sigma}{\sigma}\right)\|u_0\|^4_{L^2(\Omega)} > 2c_1M\mathcal{F}_p(0) - (1+\sigma)\left(\frac{1+\sigma}{\sigma}\right)\|u_0\|^4_{L^2(\Omega)}.\end{aligned} \tag{30}$$

Noting that $\mathcal{F}_p(0) > 0$ and taking $M > 0$ as large as necessary, we obtain the following estimate

$$\mathcal{I}_p''(t)\mathcal{I}_p(t) - (1+\sigma)(\mathcal{I}_p'(t))^2 > 0. \tag{31}$$

For $t \geq 0$ the above expression can be written as

$$\frac{d}{dt}\left(\frac{\mathcal{I}_p'(t)}{\mathcal{I}_p^{1+\sigma}(t)}\right) > 0,$$

which implies

$$\begin{cases}\mathcal{I}_p'(t) \geq \frac{\|u_0\|^2_{L^2(\Omega)}}{M^{\sigma+1}}\mathcal{I}_p^{1+\sigma}(t),\ t > 0,\\ \mathcal{I}_p(0) = M.\end{cases} \tag{32}$$

Then we arrive at

$$\mathcal{I}_p(t) \geq \left(\frac{1}{M^\sigma} - \frac{\sigma\|u_0\|^2_{L^2(\Omega)}}{M^{\sigma+1}}t\right)^{-\frac{1}{\sigma}},$$

From here we see that the solutions blow up in the finite time T^* which is

$$0 < T^* \leq \frac{M}{\sigma \|u_0\|^2_{L^2(\Omega)}},$$

where M can be estimated from (30) as follows

$$M = \frac{(1+\sigma)\left(\frac{1+\sigma}{\sigma}\right)\|u_0\|^4_{L^2(\Omega)}}{2c_1 \mathcal{F}_p(0)}. \tag{33}$$

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Әбілқасым А., Сәбитбек Б. ГЕЙЗЕНБЕРГ ТОБЫНДАҒЫ p-СУБ-ЛАПЛАСИАН ҮШІН ЖЫЛУ ӨТКІЗГІШТІК ТЕҢДЕУІНІҢ ШЕШІМДЕРІНІҢ ҚИРАУЫ

Бұл мақалада біз Гейзенберг тобындағы p-суб-Лапласиан үшін жылу өткізгіштік теңдеуіне арналған Дирихле бастапқы есебінің шешімдерінің қирауын ойысу әдісі арқылы дәлелдедік.

Кілттік сөздер. Қирау, p-суб-Лапласиан, Гейзенберг тобы, ойысу әдісі.

Абилкасым А., Сабитбек Б. РАЗРУШЕНИЕ РЕШЕНИЙ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ ДЛЯ p-СУБ-ЛАПЛАСИАНА НА ГРУППЕ ГЕЙЗЕНБЕРГА

В этой статье мы доказали разрушение решений начальной задачи Дирихле уравнение теплопроводности для p-суб-Лапласиана на группе Гейзенберга с использованием метода вогнутости.

Ключевые слова. Разрушение, p-суб-Лапласиан, группа Гейзенберга, метод вогнутости.

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On a numerical method for solving a nonlinear boundary value problem with parameter

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Abstract. Differential equations encountered in applications, as a rule, contain numerical parameters that characterize certain properties of the described processes. Finding their values requires additional information on the solution and quite often leads to boundary value problems with a parameter, such as the equation of electron motion around a nucleus, the problem of a harmonic oscillator, and the electron motion in Hal's magnetron. To date, mainly, the regular linear boundary value problems for linear ordinary differential equations containing a parameter have been studied. In this paper, we study a boundary value problem for a linear differential equation with a parameter under nonlinear two-point boundary conditions. The problem is investigated by the parametrization method of D.S. Dzhumabaev with a modified algorithm, which was originally proposed to establish the unique solvability criteria for a linear two-point boundary value problem for a linear system of ordinary differential equations without a parameter. The present work proposes a numerical method for solving the boundary value problem under investigation, based on solving the Cauchy problems for functions of a special type and solving a system of nonlinear algebraic equations with respect to the introduced parameters, which arises when the parametrization method is applied. Also, to demonstrate the effectiveness of the proposed numerical method, a test example for finding a numerical solution to a nonlinear two-point boundary value problem for a system of linear differential equations with a parameter is given.

Keywords. Nonlinear boundary value problem, equation with parameter, numerical solution

1 Introduction

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Differential equations encountered in applications, as a rule, contain numerical parameters that characterize certain properties of the described processes. Finding their values requires additional information on the solution and often leads to boundary value problems with a parameter.

The theory of boundary value problems with a parameter goes back to the works of Hikosaka-Nobory [1], S. Takahaschi [2], K. Zawischa [3], G. Zwirner [4].

A significant contribution to the theory of boundary value problems with parameters was made by Kazakh mathematicians. For infinite systems of differential equations containing an infinite number of parameters, the two-point boundary value problems in spaces $l_{2,m}$ were considered in the works of O.A. Zhautykov, M.E. Esmukhanov [5], M.E. Esmukhanov [6, 7]. Necessary and sufficient conditions for the solvability of regular nonlinear boundary value problems with a parameter for differential equations in a Banach space were established by D.S. Dzhumabaev [8], [9]. Works of B.B. Minglibayeva [10, 11] were devoted to the establishment of coefficient criteria for the unique solvability and well-posedness of linear two-point boundary value problems with parameter.

This paper investigates a boundary value problem for a linear differential equation with a parameter under nonlinear two-point boundary conditions. The problem is investigated by using the Dzhumabaev parametrization method [12] with a modified algorithm. A numerical method for solving the boundary value problem under investigation is proposed. We also provide a test example that demonstrates the effectiveness of the proposed numerical method for finding a solution.

2 Modification of the parametrization method algorithms

We consider a boundary value problem for a linear differential equation with a parameter obeying the nonlinear two-point boundary conditions with a parameter

$$\frac{dx}{dt} = A(t)x + B(t)\lambda_0 + f(t), \quad t \in (0,T), \quad x \in R^n, \quad \lambda_0 \in R^m, \tag{1}$$

$$g(\lambda_0, x(0), x(T)) = 0, \tag{2}$$

where the $(n \times n)$-matrix $A(t)$, $(n \times m)$-matrix $B(t)$, and n-vector-function $f(t)$ are continuous on $[0,T]$, $g : R^m \times R^n \times R^n \to R^{m+n}$ is a continuous function, $\|x\| = \max\limits_{i=1:n} |x_i|$, $\|A(t)\| = \max\limits_{i=1:n} \sum\limits_{j=1}^{n} |a_{ij}(t)| \leqslant \alpha$, $\|B(t)\| = \max\limits_{i=1:n} \sum\limits_{j=1}^{m} |b_{ij}(t)| \leqslant \beta$, α, β are constants.

We need to determine a pair $(\lambda_0^*, x^*(t))$ with function $x^*(t)$ satisfying at $\lambda_0 = \lambda_0^*$ the differential equation (1) and boundary conditions (2). Note that the unknown parameter λ_0 is contained both in the differential equation and in the boundary condition.

Let us introduce the notation:

Δ_N is a partition of interval $[0,T) = \bigcup_{r=1}^{N}[t_{r-1},t_r)$ by the points $t_s = sh$, $s = 0 : N$, $h = T/N$ $(N = 1, 2, \ldots)$;

$C([0,T], R^n)$ is the space of continuous on $[0,T]$ functions $x : [0,T] \to R^n$ with the norm $\|x\|_1 = \max_{t\in[0,T]} \|x(t)\|$;

$C([0,T], \Delta_N, R^{nN})$ is the space of function systems $x[t] = (x_1(t), x_2(t), \ldots, x_N(t))$ with function $x_r(t) \in C[t_{r-1}, t_r)$ which has a finite limit $\lim_{t\to t_r-0} x_r(t)$ $(r = 1 : N)$ with the norm $\|x[\cdot]\|_2 = \max_{r=1:N} \sup_{t\in[t_{r-1},t_r)} \|x_r(t)\|$.

Denote the restriction of function $x(t)$ to $[t_{r-1}, t_r)$ by $x_r(t)$, $r = 1 : N$, and reduce the problem (1), (2) to the equivalent multipoint boundary value problem

$$\frac{dx_r(t)}{dt} = A(t)x_r(t) + B(t)\lambda_0 + f(t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : N, \tag{3}$$

$$g\left(\lambda_0, x_1(0), \lim_{t\to t_N-0} x_N(t)\right) = 0, \tag{4}$$

$$\lim_{t\to t_r-0} x_r(t) = x_{r+1}(t_s), \quad s = 1 : (N-1), \tag{5}$$

where (5) are the conditions for matching the solution at the interior points of partition of the interval $[0,T]$.

The solution to the problem (3)-(5) is the system of functions

$$x^*[t] = (x_1^*(t), x_2^*(t), \ldots, x_N^*(t)) \in C([0,T], h, R^{nN}),$$

with functions $x_r^*(t)$, $r = 1 : N$, continuously differentiable on $[t_{r-1}, t_r)$, satisfying the system of differential equations with a parameter (3) and conditions (4), (5) at $\lambda_0 = \lambda_0^*$.

The boundary value problem (3)-(5) is equivalent to a multipoint boundary value problem with parameters

$$\frac{du_r}{dt} = A(t)(\lambda_r + u_r) + B(t)\lambda_0 + f(t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : N, \tag{6}$$

$$u_r(t_{r-1}) = 0, \quad r = 1 : N, \tag{7}$$

$$g(\lambda_0, \lambda_1, \lambda_{N+1}) = 0, \tag{8}$$

$$\lambda_r + \lim_{t\to t_r-0} u_r(t) = \lambda_{r+1}, \quad r = 1 : N, \tag{9}$$

where $\lambda_r = x_r(t_{r-1})$, $r = 1 : N$, $\lambda_{N+1} = \lim_{t\to t_N-0} x_N(t)$, $u_r(t) = x_r(t) - \lambda_r$ at $t \in [t_{r-1}, t_r)$, $r = 1 : N$.

Let us introduce the linear operator [13, p. 145]

$$X_r(t) = I + \int_{t_{r-1}}^{t} A(\tau_1)d\tau_1 + \sum_{j=2}^{\infty} \int_{t_{r-1}}^{t} A(\tau_1) \int_{t_{r-1}}^{\tau_1} A(\tau_2) \dots \int_{t_{r-1}}^{\tau_{j-1}} A(\tau_j)d\tau_j \dots d\tau_2 d\tau_1,$$
$$t \in [t_{r-1}, t_r), \quad r = 1 : N,$$

where I is the identity matrix of dimension $(n \times n)$. The operator $X_r(t)$ satisfies the problem

$$\frac{dX_r}{dt} = A(t)X_r, \qquad X_r(t_{r-1}) = I, \qquad t \in [t_{r-1}, t_r), \quad r = 1 : N. \tag{10}$$

For fixed values of the parameters $\lambda_r \in R^n$ $(r = 1 : N)$ and $\lambda_0 \in R^m$, using the notation

$$a_r(P, t) = X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\xi)P(\xi)d\xi, \quad t \in [t_{r-1}, t_r], \quad r = 1 : N, \tag{11}$$

we write down the unique solution of the Cauchy problem (6), (7)

$$u_r(t) = a_r(A, t)\lambda_r + a_r(B, t)\lambda_0 + a_r(f, t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : N, \tag{12}$$

and compose a system of functions $u[t] = (u_1(t), u_2(t), \dots, u_N(t))$.

Note that the function $a_r(P, t)$ satisfies the Cauchy problem

$$\frac{d}{dt}a_r(P, t) = A(t) \cdot a_r(P, t) + P(t), \quad a_r(P, t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \quad r = 1 : N. \tag{13}$$

Determine $\lim\limits_{t \to t_r - 0} u_r(t)$, $r = 1 : N$, from (12), substitute them in (8), (9), then multiplying (8) by $h > 0$, we write down the system of nonlinear equations with respect to unknown parameters

$$Q_{*,\Delta_N}(\lambda) = 0, \quad \lambda = (\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_N, \lambda_{N+1}) \in R^{m+n(N+1)}, \tag{14}$$

where the operator $Q_{*,\Delta_N}(\lambda)$ has the form

$$Q_{*,\Delta_N}(\lambda) = \begin{pmatrix} h \cdot g\,(\lambda_0, \lambda_1, \lambda_{N+1}) \\ a_1(B, t_1)\lambda_0 + (I + a_1(A, t_1))\lambda_1 - \lambda_2 + a_1(f, t_1) \\ a_2(B, t_2)\lambda_0 + (I + a_2(A, t_2))\lambda_2 - \lambda_3 + a_2(f, t_2) \\ \dots \\ a_N(B, t_N)\lambda_0 + (I + a_N(A, t_N))\lambda_N - \lambda_{N+1} + a_N(f, t_N) \end{pmatrix}.$$

Let us choose the vector $\lambda^0 = (\lambda_0^0, \lambda_1^0, \lambda_2^0, \dots, \lambda_N^0, \lambda_{N+1}^0) \in R^{m+n(N+1)}$ and numbers $\rho_\lambda > 0$, $\rho_x > 0$ and define the sets:

$$S(\lambda^0, \rho_\lambda) = \Big\{ \lambda = (\lambda_0, \lambda_1, \dots, \lambda_{N+1}) \in R^{m+n(N+1)} :$$

$$\|\lambda - \lambda^0\| = \max\left\{\|\lambda_0 - \lambda_0^0\|, \max_{r=1:(N+1)} \|\lambda_r - \lambda_r^{(0)}\|\right\} < \rho_\lambda\right\},$$

$$G_0(\rho_\lambda, \rho_x) = \left\{(w_0, w_1, w_2) \in R^{m+2n} : \|w_0 - \lambda_0^0\| < \rho_\lambda, \|w_1 - \lambda_1^0\| < \rho_x, \|w_2 - \lambda_{N+1}^0\| < \rho_x\right\}.$$

Condition B**.** The function $g(v, w)$ is continuous in $G_0(\rho_\lambda, \rho_x)$ and has uniformly continuous partial derivatives $g'_{w_0}(w_0, w_1, w_2)$, $g'_{w_1}(w_0, w_1, w_2)$, and $g'_{w_2}(w_0, w_1, w_2)$.

When solving the equation (14) with respect to $\lambda \in R^{m+n(N+1)}$, we use iterative processes with damping factors. It is known that to expand the range of initial approximations at which the iterative process converges, damping factors are used. The following statement, formulated on the basis of Theorem 1 from [14], establishes the conditions for the convergence of iterative processes with different damping factors to the same solution of the equation (14) with the same initial approximations, as well as an estimate of the difference between the solution and the initial approximation.

Theorem 1. *Let the following conditions be satisfied:*
1) the Jacobi matrix $\dfrac{\partial Q_{*,\Delta_N}(\lambda)}{\partial\lambda} : R^{m+n(N+1)} \to R^{m+n(N+1)}$ *is uniformly continuous in* $S(\lambda^0, \rho)$,
2) $\dfrac{\partial Q_{*,\Delta_N}(\lambda)}{\partial\lambda} : R^{m+n(N+1)} \to R^{m+n(N+1)}$ *is boundedly invertible for all* $\lambda \in S(\lambda^0, \rho)$ *and*

$$\left\|\left(\frac{\partial Q_{*,\Delta_N}(\lambda)}{\partial\lambda}\right)^{-1}\right\| \le \gamma,$$

γ *is a constant,*
3) $\gamma \cdot \|Q_{*,\Delta_N}(\lambda^0)\| < \rho$.

Then there exists a number $\alpha_0 \ge 1$ *such that for any* $\alpha_1 \ge \alpha_0$*, the sequence* $\{\lambda^{(p+1)}\}$, $p = 0, 1, 2, \ldots,$ *determined by the iterative process:*

$$\begin{aligned} \lambda^{(0)} &= \lambda^0, \\ \lambda^{(p+1)} &= \lambda^{(p)} - \frac{1}{\alpha_1}\left(\frac{\partial Q_{*,\Delta_N}(\lambda^{(p)})}{\partial\lambda}\right)^{-1} \cdot Q_{*,\Delta_N}(\lambda^{(p)}), \qquad p = 0, 1, 2, \ldots, \end{aligned} \tag{15}$$

is contained in $S(\lambda^0, \rho)$*, converges to* λ^**, is the solution of the equation* (14) *in* $S(\lambda^0, \rho)$*, and the following estimate holds:*

$$\|\lambda^* - \lambda^0\| \le \gamma \cdot \|Q_{*,\Delta_N}(\lambda^0)\|. \tag{16}$$

Moreover, any solution to the equation (14) *in* $S(\lambda^0, \rho)$ *is isolated.*

3 Numerical method for solving a nonlinear two-point boundary value problem with parameter (1), (2)

In this section, we propose the following numerical method for solving a nonlinear two-point boundary value problem with a parameter (1), (2).

1) Divide the interval $[t_{r-1}, t_r]$ into M equal parts ($M = 1, 2, \ldots$) and determine the numerical solution to the Cauchy problem (13) at points $t_{r,k} = t_{r-1} + k \cdot \dfrac{t_r - t_{r-1}}{M}$ ($r = 1 : N$, $k = 0 : M$).

2) Compile a system of nonlinear algebraic equations (14) with respect to the parameter $\lambda \in R^{m+n(N+1)}$.

3) Choose the vector $\lambda^0 \in R^{m+n(N+1)}$ such way that $Q_{*,\Delta_N}(\lambda^0) \neq 0$ and using the iterative process (15), find a solution λ^* to the equation (14).

4) Define $\lambda^{(p)}$ such way that $Q_{*,\Delta_N}(\lambda^{(p)}) = 0$, $p = 1, 2, \ldots$.

5) Use the values of the numerical solution to the Cauchy problems (13) and according to the equality (12), find $u_r(t_{r,k})$ ($r = 1 : N$, $k = 0 : M$).

The numerical solution to the problem (1), (2) is a pair $(\lambda_0^{(p)}, \tilde{x}(\hat{t}))$, where

$$\tilde{x}(\hat{t}) = \begin{cases} \lambda_1^{(p)} + u_1(t_{1,k}), & \text{if} \quad \hat{t} = k\dfrac{t_1 - t_0}{M}, & k = 0 : (M-1), \\ \lambda_2^{(p)} + u_2(t_{2,k}), & \text{if} \quad \hat{t} = t_1 + k\dfrac{t_2 - t_1}{M}, & k = 0 : (M-1), \\ \ldots & & \\ \lambda_N^{(p)} + u_N(t_{N,k}), & \text{if} \quad \hat{t} = t_{N-1} + k\dfrac{t_N - t_{N-1}}{M}, & k = 0 : (M-1), \\ \lambda_{N+1}^{(p)}, & \text{if} \quad \hat{t} = t_N. & \end{cases}$$

4 Example

It is required to find the numerical solution to the nonlinear boundary value problem for a differential equation with parameter:

$$\frac{dx}{dt} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix} x + \frac{1}{8} \cdot \begin{pmatrix} 0 & \frac{1}{2}t & -\frac{1}{4}t^2 \\ -\frac{1}{5}t^2 & \frac{1}{4}t^3 & \frac{1}{2} \end{pmatrix} \cdot \lambda_0$$

$$+ \begin{pmatrix} \frac{39}{80}t - \frac{9}{64}t^2 - \frac{1}{2} \\ \frac{9}{80}t^2 - \frac{1}{160}t^3 - \frac{1}{4}t - \frac{17}{32} \end{pmatrix}, \quad x \in R^2, \quad \lambda_0 \in R^3, \tag{17}$$

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{3} & 1 \\ -\frac{1}{2} & -\frac{1}{3} & \frac{1}{2} \\ 0 & -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \cdot \lambda_0 + \begin{pmatrix} \frac{1}{5} & \frac{1}{2} \\ -\frac{1}{8} & 1 \\ -\frac{1}{5} & \frac{1}{8} \\ \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{3} & -\frac{1}{9} \end{pmatrix} \cdot x(0) + \begin{pmatrix} -\frac{1}{2} & \frac{1}{8} \\ -\frac{2}{5} & \frac{1}{16} \\ -\frac{1}{5} & \frac{3}{4} \\ \frac{7}{8} & -\frac{1}{8} \\ \frac{2}{5} & \frac{1}{4} \end{pmatrix} \cdot x(1)$$

$$+ \begin{pmatrix} \lambda_{0,1} \cdot x_1(0) \\ \lambda_{0,2} \cdot x_1(1) \\ \lambda_{0,3} \cdot x_2(0) \\ x_1(0) \cdot x_2(1) \\ x_2(0) \cdot x_1(1) \end{pmatrix} - \begin{pmatrix} -\frac{139}{192} \\ \frac{2497}{1920} \\ \frac{1259}{480} \\ \frac{6161}{960} \\ \frac{9379}{1440} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{18}$$

Let us make a partition of the interval $[0,1)$: $\Delta_{10}: \quad [0,\,1) = \bigcup\limits_{r=1}^{10} [0.1(r-1),\ 0.1r)$. Construct a system of nonlinear equations with respect to parameters of the form (14):

$$Q_{*,\Delta_{10}}(\lambda) = 0, \quad \lambda = (\lambda_0, \lambda_1, \lambda_2, ..., \lambda_{10}, \lambda_{11}) \in R^{25}, \quad \lambda_0 \in R^3, \ \lambda_r \in R^2, \ r = 1:11, \tag{19}$$

where $Q_{*,\Delta_{10}}(\lambda) = \begin{pmatrix} Q_{*,\Delta_{10}}(\lambda)_0 & Q_{*,\Delta_{10}}(\lambda)_1 & Q_{*,\Delta_{10}}(\lambda)_2 & \ldots & Q_{*,\Delta_{10}}(\lambda)_{10} \end{pmatrix}^T$,

$$Q_{*,\Delta_{10}}(\lambda)_0 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{3} & 1 \\ -\frac{1}{2} & -\frac{1}{3} & \frac{1}{2} \\ 0 & -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \cdot \lambda_0 + \begin{pmatrix} \frac{1}{5} & \frac{1}{2} \\ -\frac{1}{8} & 1 \\ -\frac{1}{5} & \frac{1}{8} \\ \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{3} & -\frac{1}{9} \end{pmatrix} \cdot \lambda_1 + \begin{pmatrix} -\frac{1}{2} & \frac{1}{8} \\ -\frac{2}{5} & \frac{1}{16} \\ -\frac{1}{5} & \frac{3}{4} \\ \frac{7}{8} & -\frac{1}{8} \\ \frac{2}{5} & \frac{1}{4} \end{pmatrix} \cdot \lambda_{11}$$

$$+\begin{pmatrix} \lambda_{0,1}\cdot\lambda_{1,1} \\ \lambda_{0,2}\cdot\lambda_{11,1} \\ \lambda_{0,3}\cdot\lambda_{1,2} \\ \lambda_{1,1}\cdot\lambda_{11,2} \\ \lambda_{1,2}\cdot\lambda_{11,1} \end{pmatrix} - \begin{pmatrix} -\frac{139}{192} \\ \frac{2497}{1920} \\ \frac{1259}{480} \\ \frac{6161}{960} \\ \frac{9379}{1440} \end{pmatrix},$$

$$Q_{*,\Delta_{10}}(\lambda)_1 = \begin{pmatrix} 5.313926e-8 & 0.000318 & -0.000091 \\ -8.438817e-6 & -1.881198e-6 & 0.006410 \end{pmatrix}\cdot\lambda_0$$

$$+\begin{pmatrix} 1.051600 & -0.026285 \\ -0.026285 & 1.051600 \end{pmatrix}\cdot\lambda_1 - \lambda_2 + \begin{pmatrix} -0.048148 \\ -0.055089 \end{pmatrix},$$

$$Q_{*,\Delta_{10}}(\lambda)_2 = \begin{pmatrix} 5.898982e-7 & 0.000959 & -0.000155 \\ -0.000060 & 1.176714e-6 & 0.006410 \end{pmatrix}\cdot\lambda_0$$

$$+\begin{pmatrix} 1.051600 & -0.026285 \\ -0.026285 & 1.051600 \end{pmatrix}\cdot\lambda_2 - \lambda_3 + \begin{pmatrix} -0.043406 \\ -0.057485 \end{pmatrix},$$

$$Q_{*,\Delta_{10}}(\lambda)_3 = \begin{pmatrix} 1.772920e-6 & 0.001599 & -0.000283 \\ -0.000162 & 0.000033 & 0.006412 \end{pmatrix}\cdot\lambda_0$$

$$+\begin{pmatrix} 1.051600 & -0.026285 \\ -0.026285 & 1.051600 \end{pmatrix}\cdot\lambda_3 - \lambda_4 + \begin{pmatrix} -0.038955 \\ -0.059653 \end{pmatrix},$$

$$Q_{*,\Delta_{10}}(\lambda)_4 = \begin{pmatrix} 3.602205e-6 & 0.002239 & -0.000475 \\ -0.000315 & 0.000113 & 0.006414 \end{pmatrix}\cdot\lambda_0$$

$$+\begin{pmatrix} 1.051600 & -0.026285 \\ -0.026285 & 1.051600 \end{pmatrix}\cdot\lambda_4 - \lambda_5 + \begin{pmatrix} -0.034795 \\ -0.061596 \end{pmatrix},$$

$$Q_{*,\Delta_{10}}(\lambda)_5 = \begin{pmatrix} 6.077754\mathrm{e}-6 & 0.002878 & -0.000731 \\ -0.000520 & 0.000260 & 0.006417 \end{pmatrix}\cdot\lambda_0$$

$$+\begin{pmatrix} 1.051600 & -0.026285 \\ -0.026285 & 1.051600 \end{pmatrix}\cdot\lambda_5 - \lambda_6 + \begin{pmatrix} -0.030927 \\ -0.063318 \end{pmatrix},$$

$$Q_{*,\Delta_{10}}(\lambda)_6 = \begin{pmatrix} 9.199565\mathrm{e}-6 & 0.003516 & -0.001051 \\ -0.000777 & 0.000493 & 0.006421 \end{pmatrix}\cdot\lambda_0$$

$$+\begin{pmatrix} 1.051600 & -0.026285 \\ -0.026285 & 1.051600 \end{pmatrix}\cdot\lambda_6 - \lambda_7 + \begin{pmatrix} -0.027350 \\ -0.064822 \end{pmatrix},$$

$$Q_{*,\Delta_{10}}(\lambda)_7 = \begin{pmatrix} 0.000013 & 0.004153 & -0.001436 \\ -0.001084 & 0.000832 & 0.006426 \end{pmatrix} \cdot \lambda_0$$

$$+ \begin{pmatrix} 1.051600 & -0.026285 \\ -0.026285 & 1.051600 \end{pmatrix} \cdot \lambda_7 - \lambda_8 + \begin{pmatrix} -0.024063 \\ -0.066114 \end{pmatrix},$$

$$Q_{*,\Delta_{10}}(\lambda)_8 = \begin{pmatrix} 0.000017 & 0.004789 & -0.001884 \\ -0.001443 & 0.001297 & 0.006431 \end{pmatrix} \cdot \lambda_0$$

$$+ \begin{pmatrix} 1.051600 & -0.026285 \\ -0.026285 & 1.051600 \end{pmatrix} \cdot \lambda_8 - \lambda_9 + \begin{pmatrix} -0.021068 \\ -0.067196 \end{pmatrix},$$

$$Q_{*,\Delta_{10}}(\lambda)_9 = \begin{pmatrix} 0.000022 & 0.005422 & -0.002397 \\ -0.001853 & 0.001905 & 0.006438 \end{pmatrix} \cdot \lambda_0$$

$$+ \begin{pmatrix} 1.051600 & -0.026285 \\ -0.026285 & 1.051600 \end{pmatrix} \cdot \lambda_9 - \lambda_{10} + \begin{pmatrix} -0.018364 \\ -0.068073 \end{pmatrix},$$

$$Q_{*,\Delta_{10}}(\lambda)_{10} = \begin{pmatrix} 0.000028 & 0.006054 & -0.002973 \\ -0.002314 & 0.002676 & 0.006445 \end{pmatrix} \cdot \lambda_0$$

$$+ \begin{pmatrix} 1.051600 & -0.026285 \\ -0.026285 & 1.051600 \end{pmatrix} \cdot \lambda_{10} - \lambda_{11} + \begin{pmatrix} -0.015950 \\ -0.068748 \end{pmatrix}.$$

When solving the system of nonlinear algebraic equations (19), we take the vector

$$\lambda^0 = (\lambda_0^0, \lambda_1^0, \lambda_2^0, ..., \lambda_{10}^0, \lambda_{11}^0) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \in R^{25}$$

as an initial approximation.

The operator $Q_{*,\Delta_{10}}(\lambda)$ in the sphere $S(\lambda^0, 3)$ satisfies all conditions of the Theorem 1. Using the iterative process (15) at the 100th iteration, we find the parameter $\lambda^* = (\lambda_0^*, \lambda_1^*, \lambda_2^*, ..., \lambda_{10}^*, \lambda_{11}^*) \in R^{25}$, where

$$\lambda_0^* = \begin{pmatrix} -0.50000 \\ 0.20000 \\ 2.00000 \end{pmatrix}, \ \lambda_1^* = \begin{pmatrix} 2.00000 \\ 2.00000 \end{pmatrix}, \lambda_2^* = \begin{pmatrix} 2.00250 \\ 1.99875 \end{pmatrix}, \lambda_3^* = \begin{pmatrix} 2.01000 \\ 1.99500 \end{pmatrix},$$

$$\lambda_4^* = \begin{pmatrix} 2.02250 \\ 1.98875 \end{pmatrix}, \lambda_5^* = \begin{pmatrix} 2.04000 \\ 1.98000 \end{pmatrix}, \lambda_6^* = \begin{pmatrix} 2.06250 \\ 1.96875 \end{pmatrix}, \lambda_7^* = \begin{pmatrix} 2.09000 \\ 1.95500 \end{pmatrix},$$

$$\lambda_8^* = \begin{pmatrix} 2.12250 \\ 1.99500 \end{pmatrix}, \lambda_9^* = \begin{pmatrix} 2.16000 \\ 1.92000 \end{pmatrix}, \lambda_{10}^* = \begin{pmatrix} 2.20250 \\ 1.89875 \end{pmatrix}, \lambda_{11}^* = \begin{pmatrix} 2.25000 \\ 1.87500 \end{pmatrix}.$$

The exact solution to the problem (17), (18) is a pair $(\lambda_0^*, x^*(t))$, where $x^*(t) = \begin{pmatrix} \frac{1}{4}t^2 + 2 \\ \\ 2 - \frac{1}{8}t^2 \end{pmatrix}$ and $\lambda^* = (-1/2, 1/5, 2)$.

From the estimation

$$\max_{r=1:11} \|\lambda_r^* - x^*(0.1 \cdot (r-1))\| \leq 3.307576 \cdot 10^{-12} < 10^{-11}$$

it can be seen that the values of the required vector function $\tilde{x}(t) = \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix}$ at the points of the partition Δ_{10}, found by the proposed method, differ from the values of the exact solution $x^*(t)$ at the same points by no more than 10^{-11}. Table 1 shows the values of the found numerical solution $\tilde{x}(t)$ at points $\hat{t} = 0.05k$ $k = 0:20$, and the difference between the values of the obtained solution and the exact solution at these points.

Table 1. Comparison of the values of the numerical solution and the exact solution at the points of the interval $[0,1]$

k	$\hat{t}$	$\tilde{x}_1(\hat{t})$	$\tilde{x}_1(\hat{t}) - x_1^*(\hat{t})$	$\tilde{x}_2(\hat{t})$	$\tilde{x}_2(\hat{t}) - x_2^*(\hat{t})$
0	0.00	2.000000	-1.599165e-12	2.000000	2.520428e-12
1	0.05	2.000625	5.196696e-10	1.999688	-5.060261e-10
2	0.10	2.002500	-1.639577e-12	1.998750	2.577272e-12
3	0.15	2.005625	5.196275e-10	1.997188	-5.059675e-10
4	0.20	2.010000	-1.685763e-12	1.995000	2.638778e-12
5	0.25	2.015625	5.195786e-10	1.992188	-5.059038e-10
6	0.30	2.022500	-1.735945e-12	1.988750	2.704503e-12
7	0.35	2.030625	5.195258e-10	1.984688	-5.058356e-10
8	0.40	2.040000	-1.791900e-12	1.980000	2.774669e-12
9	0.45	2.050625	5.194671e-10	1.974688	-5.057630e-10
10	0.50	2.062500	-1.853628e-12	1.968750	2.849720e-12
11	0.55	2.075625	5.194023e-10	1.962188	-5.056855e-10
12	0.60	2.090000	-1.922462e-12	1.955000	2.929879e-12
13	0.65	2.105625	5.193304e-10	1.947188	-5.056024e-10
14	0.70	2.122500	-1.996181e-12	1.938750	3.015366e-12
15	0.75	2.140625	5.192531e-10	1.929688	-5.055143e-10
16	0.80	2.160000	-2.076561e-12	1.920000	3.106848e-12
17	0.85	2.180625	5.191692e-10	1.909688	-5.054197e-10
18	0.90	2.202500	-2.164491e-12	1.898750	3.204104e-12
19	0.95	2.225625	5.190777e-10	1.887188	-5.053196e-10
20	1.00	2.250000	1.060075e-9	1.875000	-1.033438e-9

The Table 1 shows that $\left\|x^*(\hat{t}) - \tilde{x}(\hat{t})\right\| < 10^{-8}$. The result obtained clearly shows the rather high efficiency of the proposed method for finding the numerical solution to the nonlinear two-point boundary value problem for the system of ordinary differential equations with parameter. Calculations have been performed in the mathematical package MathCAD 15.

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Темешева С.М., Дүйсен А.Қ., Алиханова Г.С. СЫЗЫҚТЫҚ ЕМЕС ПАРАМЕТРІ БАР ШЕТТІК ЕСЕПТІ ШЕШУДІҢ БІР САНДЫҚ ӘДІСІ ТУРАЛЫ

Қосымшаларда кездесетін дифференциалдық теңдеулерде, әдетте, сипатталған процестердің белгілі бір қасиеттерін сипаттайтын сандық параметрлер болады. Олардың мәндерін табу шешім туралы қосымша ақпаратты қажет етеді және көбінесе параметрі бар шеттік есептерге әкеледі. Мысалы, ядроның айналасындағы электронның қозғалыс теңдеуі, гармоникалық осциллятор туралы есеп, Хэл магнетронындағы электронның қозғалысы. Негізінен, осы уақытқа дейін, параметрі бар сызықтық жәй дифференциалдық теңдеулер үшін регулярлық сызықтық шеттік есептер зерттелген. Бұл жұмыста параметрі бар сызықтық дифференциалдық теңдеу үшін сызықтық емес екі нүктелі шеттік шарттарды қанағаттандыратын шеттік есеп зерттеледі. Есеп Д.С. Джумабаевтың параметрлеу әдісімен зерттеледі. Бұл әдіс параметрі жоқ жәй сызықтық дифференциалдық теңдеулер жүйесі үшін сызықтық екі нүктелі шеттік есептің бірмәнді шешілуінің белгілерін анықтау үшін ұсынылған. Зерттелетін шеттік есепті шешудің сандық әдісі ұсынылады, бұл әдіс арнайы типтегі функциялар үшін Коши есептерін шешуге және параметрлеу әдісін қолдану кезінде пайда болатын енгізілген параметрлерге қатысты сызықтық емес алгебралық теңдеулер жүйесін шешуге негізделген. Шешімді табудың ұсынылған сандық әдісінің тиімділігін көрсететін параметрі бар сызықтық дифференциалдық теңдеулер жүйесі үшін сызықты емес екі нүктелі шеттік есептің сандық шешімін табудың сынақ мысалы келтірілген.

Кілттік сөздер. Сызықтық емес шеттік есеп, параметрі бар теңдеу, сандық шешім.

Темешева С.М., Дүйсен А.Қ., Алиханова Г.С. ОБ ОДНОМ ЧИСЛЕННОМ МЕТОДЕ РЕШЕНИЯ НЕЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ С ПАРАМЕТРОМ

Дифференциальные уравнения, встречающиеся в приложениях, как правило, содержат числовые параметры, характеризующие те или иные свойства описываемых процессов. Нахождение их значений требует дополнительной информации о решении и часто приводит к краевым задачам с параметром. Например, уравнение движения электрона вокруг ядра, задача о гармоническом осцилляторе, движения электрона в магнетроне Хэла. В основном на сегодняшний день изучены регулярные линейные краевые задачи для линейных обыкновенных дифференциальных уравнений, содержащих параметр. В данной работе исследуется краевая задача для линейного дифференциального уравнения с параметром с нелинейными двухточечными краевыми условиями. Задача исследуется методом параметризации Д.С. Джумабаева с модифицированным алгоритмом, который изначально был предложен для установления признаков однозначной разрешимости линейной двухточечной краевой задачи для линейной системы обыкновенных дифференциальных уравнений без параметра. Предлагается численный метод решения исследуемой краевой задачи, который основан на решении задач Коши для функций специального вида и решении системы нелинейных алгебраических уравнений, возникающей при применении метода параметризации, относительно вводимых параметров. Приведен тестовый пример нахождения численного решения нелинейной двухточечной краевой задачи для системы линейных дифференциальных уравнений, содержащих параметр, демонстрирующий эффективность предложенного численного метода нахождения решения.

Ключевые слова. Нелинейная краевая задача, уравнение с параметром, численное решение.

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On the Green function of the Cauchy-Neumann problem for the hyperbolic equation in the quarter plane

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Abstract. The definition of Green's function of the Cauchy-Neumann problem for the hyperbolic equation in a quarter plane is given. Its existence and uniqueness have been proven. Representation of the Green's function is given. It is shown that Green's function can be represented by Riemann-Green function.

Keywords. hyperbolic equation, second initial-boundary value problem, boundary condition, Green function.

1 Introduction

An explicit form of the Green's function in the sector for biharmonic and triharmonic equations is given in [1], [2]. The Green's function of the Neumann problem for the Poisson equation in the half-space R_n^+ is explicitly constructed in [3], and the Green's function for the Robin problem in the circle in [4], [5], [6]. We also note the articles [7], [8], which are devoted to the construction of the Green's function for the Dirichlet problem for the polyharmonic equation in the unit ball. In [9], [10] a representation of the Green's function for the classical external and internal Neumann problems for the Poisson equation in the unit ball is given.

2 Formulation of the problem

Let $Q = \{(x,t): \, x>0, \, t>0\}$. The following hyperbolic equation is considered in Q:

$$Lu \equiv \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + a_1(x,t)\cdot\frac{\partial u(x,t)}{\partial x}$$

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$$+b_1(x,t)\cdot\frac{\partial u(x,t)}{\partial x}+c_1(x,t)\cdot u(x,t)=F(x,t),\ (x,t)\in Q, \tag{1}$$

with the initial conditions

$$u(x,0)=T(x),\ \frac{\partial u}{\partial t}(x,0)=N(x),\ x>0, \tag{2}$$

and the boundary condition

$$\frac{\partial u}{\partial x}(t,0)=\Phi(t),\ t>0. \tag{3}$$

It is well known that this problem is correct, both in the sense of classical and generalized solutions. We are interested in the question of the integral form of the solution of this problem. We show that the solution of the problem can be written in terms of the Green function, the definition of which we introduce.

In the characteristic coordinates $\xi=x+t,\ \eta=x-t$ equation (1) has the form

$$\frac{\partial^2 u}{\partial\xi\partial\eta}+a(\xi,\eta)\frac{\partial u}{\partial\xi}+b(\xi,\eta)\frac{\partial u}{\partial\eta}+c(\xi,\eta)u=f(\xi,\eta),\quad (\xi,\eta)\in\Omega, \tag{4}$$

and the initial conditions (2) have the form

$$u(\xi,\xi)=\tau(\xi),\ \left(\frac{\partial u}{\partial\xi}-\frac{\partial u}{\partial\eta}\right)(\xi,\xi)=\nu(\xi),\ \xi>0, \tag{5}$$

and the boundary condition (3) will change to

$$\left(\frac{\partial u}{\partial\xi}+\frac{\partial u}{\partial\eta}\right)(-\eta,\eta)=\varphi(\eta),\ \eta\le 0. \tag{6}$$

We will assume that $a,b\in C^1\left(\overline{\Omega}\right);\ \ c,f\in C\left(\overline{\Omega}\right);\ \varphi\in C^1\left((-\infty,0]\right);\ \tau\in C^1\left([0,+\infty)\right);\ \nu\in C^1\left([0,+\infty)\right);\ \varphi'(0)=-\nu'(0),\ \varphi(0)=\tau'(0)$.

The task is to build a Green's function and a solution of the problem (4)-(6).

3 On the Riemann function of the equation (4).

It is well known that the Riemann-Green function $R(\xi,\eta;\xi_1,\eta_1)$ is not defined in the entire domain $\Omega\times\Omega$, but only for those points $(\xi_1,\eta_1)\in\Omega$, which $|\eta|<\xi_1, -\xi<\eta_1<\xi$. And for the remaining points of the domain $\Omega\times\Omega$, the Riemann-Green function is not uniquely determined. For our further constructions, it is important for us to use the Riemann-Green function defined at all points of the domain $\Omega\times\Omega$, for which $\eta_1<-\xi$.

For further reasoning, we need to fulfill some relations between the coefficients $a(\xi,\eta)$ and $b(\xi,\eta)$ on the border $\xi=-\eta$. For this purpose, in equation (4) let us replace the function

$$u(\xi,\eta)=U(\xi,\eta)\cdot\gamma(\eta)\cdot\mu(\xi). \tag{7}$$

Then with respect to the new unknown function $U(\xi,\eta)$ we get the equation

$$\frac{\partial^2 U}{\partial\xi\partial\eta}+\widehat{a}(\xi,\eta)\frac{\partial U}{\partial\xi}+\widehat{b}(\xi,\eta)\frac{\partial U}{\partial\eta}+\widehat{c}(\xi,\eta)U=\widehat{f},\ (\xi,\eta)\in\Omega, \tag{8}$$

where

$$\widehat{a}=\frac{1}{\gamma(\eta)}\cdot\left(\gamma'(\eta)+a(\xi,\eta)\gamma(\eta)\right),\ \widehat{b}=\frac{1}{\mu(\xi)}\cdot\left(\mu'(\xi)+b(\xi,\eta)\mu(\xi)\right),$$

$$\widehat{c}=\frac{\gamma'(\eta)\mu'(\xi)}{\gamma(\eta)\mu(\xi)}+a(\xi,\eta)\frac{\mu'(\xi)}{\mu(\xi)}+b(\xi,\eta)\frac{\gamma'(\eta)}{\gamma(\eta)}+c(\xi,\eta),\ \widehat{f}=\frac{f}{\gamma(\eta)\mu(\xi)}. \tag{9}$$

Let us take functions $\gamma(\eta),\mu(\xi)$ so that equalities

$$\widehat{a}(-\eta,\eta)=-\widehat{b}(-\eta,\eta),\ \widehat{a_\xi}(-\eta,\eta)=\widehat{b_\eta}(-\eta,\eta),\ \eta\le 0. \tag{10}$$

is performed. Then from (10) we have the next system of equations

$$\begin{cases}\dfrac{\gamma'(\eta)}{\gamma(\eta)}=-\dfrac{\mu'(-\eta)}{\mu(-\eta)}-a(-\eta,\eta)-b(-\eta,\eta), & \eta\le 0,\\ \dfrac{\gamma'(\eta)}{\gamma(\eta)}=\dfrac{\mu'(-\eta)}{\mu(-\eta)}-a_\xi(-\eta,\eta)+b_\eta(-\eta,\eta), & \eta\le 0.\end{cases} \tag{11}$$

This system (11) has a solution that can be written as

$$\gamma(\eta)=\exp\left[\frac{1}{2}\int_0^\eta\left(b_\eta(-t,t)-a_\xi(-t,t)-a(-t,t)-b(-t,t)\right)dt\right],$$

$$\mu(\xi)=\exp\left[\frac{1}{2}\int_0^\xi\left(-b_\eta(t,-t)+a_\xi(t,-t)-a(t,-t)-b(t,-t)\right)dt\right].$$

Thus, if $\gamma(\eta),\mu(\xi)$ are selected in this way, condition (10) is met at $\eta\le 0$. For values $\eta>0$, we continue the function $\gamma(\eta)$ in such a way that it is continuously differentiable and the condition $\gamma(\eta)>0$ is met .

To introduce the Riemann-Green function at all points of the domain $\Omega\times\Omega$ we continue the coefficients of equation (8) in the domain $\Omega^-=\{(\xi,\eta)\in\mathbb{R}^2:\eta<-|\xi|\}$ as follows

$$A(\xi,\eta)=\begin{cases}\widehat{a}(\xi,\eta), & (\xi,\eta)\in\Omega,\\ -\widehat{b}(-\eta,-\xi), & (\xi,\eta)\in\Omega^-,\end{cases} \tag{12}$$

$$B(\xi,\eta)=\begin{cases}\widehat{b}(\xi,\eta), & (\xi,\eta)\in\Omega,\\ -\widehat{a}(-\eta,-\xi), & (\xi,\eta)\in\Omega^-,\end{cases} \tag{13}$$

$$C(\xi,\eta)=\begin{cases}\widehat{c}(\xi,\eta), & (\xi,\eta)\in\Omega,\\ \widehat{c}(-\eta,-\xi). & (\xi,\eta)\in\Omega^-.\end{cases} \tag{14}$$

If the coefficients $a(\xi,\eta), b(\xi,\eta) \in C^1\left(\overline{\Omega}\right);\ c(\xi,\eta) \in C\left(\overline{\Omega}\right)$, then in virtue of (9), (10) coefficients $A(\xi,\eta),\ B(\xi,\eta),\ C(\xi,\eta)$ in the domain $\widetilde{\Omega} = \Omega \cup \Omega^- = \{(\xi,\eta) \in \mathbb{R}^2 : \xi > \eta\}$ have the following smoothness

$$A(\xi,\eta), B(\xi,\eta) \in C^1\left(\overline{\Omega}\right); C(\xi,\eta) \in C\left(\overline{\Omega}\right), \tag{15}$$

and satisfies the following symmetry conditions:

$$A(\xi,\eta) = -B(-\eta,-\xi),\ A_\xi(\xi,\eta) = B_\eta(-\eta,-\xi),$$

$$C(\xi,\eta) = C(-\eta,-\xi),\quad (\xi,\eta) \in \widetilde{\Omega}. \tag{16}$$

Actually, show that (16) is true. From (12) we have that

$$A(-\eta,-\xi) = \begin{cases} \widehat{a}(-\eta,-\xi), & (-\eta,-\xi) \in \Omega, \\ -\widehat{b}(\xi,\eta), & (-\eta,-\xi) \in \Omega^-, \end{cases}$$

$$= -\begin{cases} \widehat{b}(\xi,\eta), & (\xi,\eta) \in \Omega, \\ -\widehat{a}(-\eta,-\xi), & (\xi,\eta) \in \Omega^-, \end{cases} = -B(\xi,\eta).$$

Also in the same way, from (13), (14) we get

$$A_\xi(\xi,\eta) = \begin{cases} \widehat{a_\xi}(\xi,\eta), & (\xi,\eta) \in \Omega, \\ \widehat{b_\eta}(-\eta,-\xi), & (\xi,\eta) \in \Omega^-, \end{cases} = B_\eta(-\eta,-\xi),$$

$$C(\xi,\eta) = \begin{cases} \widehat{c}(\xi,\eta), & (\xi,\eta) \in \Omega, \\ \widehat{c}(-\eta,-\xi), & (\xi,\eta) \in \Omega^-, \end{cases} = C(-\eta,-\xi).$$

If we have chosen (ξ,η) from Ω then $(-\eta,-\xi)$ will be from Ω^-.

In $\widetilde{\Omega}$ we consider the equation

$$\frac{\partial^2 U}{\partial\xi\partial\eta} + A(\xi,\eta)\cdot\frac{\partial U}{\partial\xi} + B(\xi,\eta)\cdot\frac{\partial U}{\partial\eta} + C(\xi,\eta)\cdot U = F,\ (\xi,\eta) \in \widetilde{\Omega}, \tag{17}$$

Due to smoothness (14), it is well known that for the equation (17) a Riemann-Green function [11] exists in $\widetilde{\Omega}$, that for any $(\xi,\eta) \in \widetilde{\Omega}$ satisfies equation

$$\frac{\partial^2}{\partial\xi_1\partial\eta_1}R(\xi,\eta;\xi_1,\eta_1) - \frac{\partial}{\partial\xi_1}\big(A(\xi_1,\eta_1)R(\xi,\eta;\xi_1,\eta_1)\big)$$

$$-\frac{\partial}{\partial\eta_1}\big(B(\xi_1,\eta_1)R(\xi,\eta;\xi_1,\eta_1)\big) + C(\xi_1,\eta_1)R(\xi,\eta;\xi_1,\eta_1) = 0,\ (\xi_1,\eta_1) \in \widetilde{\Omega}; \tag{18}$$

and the conditions on the characteristics

$$\frac{\partial R(\xi,\eta;\xi_1,\eta_1)}{\partial \eta_1} - A(\xi_1,\eta_1)\cdot R(\xi,\eta;\xi_1,\eta_1) = 0, \text{ when } \ \xi_1 = \xi; \tag{19}$$

$$\frac{\partial R(\xi,\eta;\xi_1,\eta_1)}{\partial \xi_1} - B(\xi_1,\eta_1)\cdot R(\xi,\eta;\xi_1,\eta_1) = 0, \text{ when } \ \eta_1 = \eta; \tag{20}$$

$$R(\xi,\eta;\xi,\eta) = 1. \tag{21}$$

Thus, with this choice of the method of continuation of the coefficients of the equation (16), we determined the values of the Riemann-Green function for all points of the domain $\Omega \times \Omega$.

Lemma 1. *If conditions* (16) *are met, then the Riemann-Green function has symmetry such that*

$$R(\xi,\eta;\xi_1,\eta_1) = R(-\eta,-\xi;-\eta_1,-\xi_1), \ (\xi,\eta)\in\widetilde{\Omega}, \ (\xi_1,\eta_1)\in\widetilde{\Omega}. \tag{22}$$

Proof. Denote

$$R_1(\xi,\eta;\xi_1,\eta_1) = R(-\eta,-\xi;-\eta_1,-\xi_1), \ (\xi,\eta)\in\widetilde{\Omega}, \ (\xi_1,\eta_1)\in\widetilde{\Omega}.$$

Show that $R_1(\xi,\eta;\xi_1,\eta_1)$ satisfies equation (18) and conditions (19)-(21). Indeed, substituting the representation of $R_1(\xi,\eta;\xi_1,\eta_1)$ in equation (18), at first entering a new designation $-\xi_1 = \eta_2, -\eta_1 = \xi_2$, and then also entering the new symbols

$$-\eta = \widetilde{\xi}, \quad -\xi = \widetilde{\eta}, \quad \xi_2 = \widetilde{\xi_1}, \quad \eta_2 = \widetilde{\eta_1}$$

again and using conditions (16) we get

$$\frac{\partial^2}{\partial\xi_1\partial\eta_1}R(-\eta,-\xi;-\eta_1,-\xi_1) - \frac{\partial}{\partial\xi_1}\big(A(\xi_1,\eta_1)R(-\eta,-\xi;-\eta_1,-\xi_1)\big)$$
$$-\frac{\partial}{\partial\eta_1}\big(B(\xi_1,\eta_1)R(-\eta,-\xi;-\eta_1,-\xi_1)\big) + C(\xi_1,\eta_1)R(-\eta,-\xi;-\eta_1,-\xi_1) =$$
$$\frac{\partial^2}{\partial\xi_2\partial\eta_2}R(-\eta,-\xi;\xi_2,\eta_2) + \frac{\partial}{\partial\eta_2}\big(A(-\eta_2,-\xi_2)R(-\eta,-\xi;\xi_2,\eta_2)\big)$$
$$+\frac{\partial}{\partial\xi_2}\big(B(-\eta_2,-\xi_2)R(-\eta,-\xi;\xi_2,\eta_2)\big) + C(-\eta_2,-\xi_2)\cdot R(-\eta,-\xi;\xi_2,\eta_2) =$$
$$\frac{\partial^2}{\partial\widetilde{\xi_1}\partial\widetilde{\eta_1}}R(\widetilde{\xi},\widetilde{\eta};\widetilde{\xi_1},\widetilde{\eta_1}) - \frac{\partial}{\partial\widetilde{\eta_1}}\big(B(\widetilde{\xi_1},\widetilde{\eta_1})R(\widetilde{\xi},\widetilde{\eta};\widetilde{\xi_1},\widetilde{\eta_1})\big)$$
$$-\frac{\partial}{\partial\widetilde{\xi_1}}\big(A(\widetilde{\xi_1},\widetilde{\eta_1})R(\widetilde{\xi},\widetilde{\eta};\widetilde{\xi_1},\widetilde{\eta_1})\big) + C(\widetilde{\xi_1},\widetilde{\eta_1})\cdot R(\widetilde{\xi},\widetilde{\eta};\widetilde{\xi_1},\widetilde{\eta_1}) = 0. \tag{23}$$

Thus $R_1(\xi,\eta;\xi_1,\eta_1)$ satisfies equation (18). Also substituting the representation of $R_1(\xi,\eta;\xi_1,\eta_1)$ into conditions (19)-(21) and using all the notation at the top we have

$$-\frac{\partial R(-\eta,-\xi;-\eta_1,-\xi_1)}{\partial \xi_1} - A(\xi_1,\eta_1)\cdot R(-\eta,-\xi;-\eta_1,-\xi_1)$$

$$= \frac{\partial R(\widetilde{\xi},\widetilde{\eta};\widetilde{\xi}_1,\widetilde{\eta}_1)}{\partial \widetilde{\xi}_1} - B(\widetilde{\xi}_1,\widetilde{\eta}_1)\cdot R(\widetilde{\xi},\widetilde{\eta};\widetilde{\xi}_1,\widetilde{\eta}_1) = 0, \text{ when } \widetilde{\eta}_1 = \widetilde{\eta}; \tag{24}$$

$$-\frac{\partial R(-\eta,-\xi;-\eta_1,-\xi_1)}{\partial \eta_1} - B(\xi_1,\eta_1)\cdot R(-\eta,-\xi;-\eta_1,-\xi_1)$$

$$= \frac{\partial R(\widetilde{\xi},\widetilde{\eta};\widetilde{\xi}_1,\widetilde{\eta}_1)}{\partial \widetilde{\eta}_1} - A(\widetilde{\xi}_1,\widetilde{\eta}_1)\cdot R(\widetilde{\xi},\widetilde{\eta};\widetilde{\xi}_1,\widetilde{\eta}_1) = 0, \text{ when } \widetilde{\xi}_1 = \widetilde{\xi}; \tag{25}$$

$$R(-\eta,-\xi;-\eta,-\xi) = R(\widetilde{\xi},\widetilde{\eta};\widetilde{\xi},\widetilde{\eta}) = 1. \tag{26}$$

Due to (23)-(26) easy to see that the function $R(-\eta,-\xi;-\eta_1,-\xi_1)$ is also Riemann-Green function of the same equation (17). But, it is well-known that Riemann-Green function is unique. It follows that equality (22) is true. □

Corollary 1. *On the line* $\xi = -\eta,\ \eta \le 0$, *the next equality holds*

$$R(-\eta,\eta;\xi_1,\eta_1) = R(-\eta,\eta;-\eta_1,-\xi_1). \tag{27}$$

4 Green's function of the problem (4)-(6).

Let us build a Green's function to the first initial-boundary value problem in the quarter plane

$$\frac{\partial^2 U}{\partial\xi\partial\eta} + A(\xi,\eta)\cdot\frac{\partial U}{\partial\xi} + B(\xi,\eta)\cdot\frac{\partial U}{\partial\eta} + C(\xi,\eta)\cdot U = F,\ (\xi,\eta)\in\Omega, \tag{28}$$

$$U(\xi,\xi) = T_1(\xi),\ \left(\frac{\partial U}{\partial\xi} - \frac{\partial U}{\partial\eta}\right)(\xi,\xi) = M_1(\xi),\ \xi > 0, \tag{29}$$

$$\left(\frac{\partial U}{\partial\xi} - \frac{\partial U}{\partial\eta}\right)(-\eta,\eta) = P(\eta),\ \eta \le 0. \tag{30}$$

Definition 1. *Green's function of the problem* (28)-(30) let us call the function $G(\xi,\eta;\xi_1,\eta_1)$, which for every fixed $(\xi_1,\eta_1)\in\Omega$, satisfies the homogeneous equation

$$L_{(\xi,\eta)}G(\xi,\eta;\xi_1,\eta_1) = 0,\ (\xi,\eta)\in\Omega, \text{ at } \xi \ne \xi_1,\ \eta \ne \eta_1,\ \eta \ne -\xi_1; \tag{31}$$

and the next boundary conditions

$$G(\xi,\xi;\xi_1,\eta_1) = 0,\ \xi \ge 0,\ (\xi_1,\eta_1)\in\Omega; \tag{32}$$

$$\left(\frac{\partial G}{\partial \xi}-\frac{\partial G}{\partial \eta}\right)(\xi,\xi;\xi_1,\eta_1)=0,\ \xi\geq 0,\ (\xi_1,\eta_1)\in\Omega,\ \text{at}\ \xi\neq\xi_1,\ \ \eta\neq\eta_1; \tag{33}$$

$$\left(\frac{\partial G}{\partial \xi}+\frac{\partial G}{\partial \eta}\right)(-\eta,\eta;\xi_1,\eta_1)=0,\ \eta\leq 0,\ (\xi_1,\eta_1)\in\Omega, \tag{34}$$

and on the above characteristic lines, the following conditions must be met: the values of the derivatives of the Green function in directions parallel to these characteristics must coincide in adjacent regions; i.e.,

$$\frac{\partial G(\xi_1+0,\eta;\xi_1,\eta_1)}{\partial \eta}+A(\xi_1,\eta)G(\xi_1+0,\eta;\xi_1,\eta_1)$$

$$=\frac{\partial G(\xi_1-0,\eta;\xi_1,\eta_1)}{\partial \eta}+A(\xi_1,\eta)G(\xi_1-0,\eta;\xi_1,\eta_1),\ \text{at}\ \eta\neq\eta_1; \tag{35}$$

$$\frac{\partial G(\xi,\eta_1+0;\xi_1,\eta_1)}{\partial \xi}+B(\xi,\eta_1)G(\xi,\eta_1+0;\xi_1,\eta_1)$$

$$=\frac{\partial G(\xi,\eta_1-0;\xi_1,\eta_1)}{\partial \xi}+B(\xi,\eta_1)G(\xi,\eta_1-0;\xi_1,\eta_1),\ \text{at}\ \xi\neq\xi_1; \tag{36}$$

$$\frac{\partial G(\xi,-\xi_1+0;\xi_1,\eta_1)}{\partial \xi}+B(\xi,-\xi_1)G(\xi,-\xi_1+0;\xi_1,\eta_1)$$

$$=\frac{\partial G(\xi,-\xi_1-0;\xi_1,\eta_1)}{\partial \xi}+B(\xi,-\xi_1)G(\xi,-\xi_1-0;\xi_1,\eta_1); \tag{37}$$

$$G(\xi_1,-\xi_1-0;\xi_1,\eta_1)=2G(\xi_1,-\xi_1+0;\xi_1,\eta_1); \tag{38}$$

and the "corner condition"

$$G(\xi_1-0,\eta_1-0;\xi_1,\eta_1)-G(\xi_1+0,\eta_1-0;\xi_1,\eta_1)$$

$$+G(\xi_1+0,\eta_1+0;\xi_1,\eta_1)-G(\xi_1-0,\eta_1+0;\xi_1,\eta_1)=1. \tag{39}$$

must be satisfied as the regions meet at $(\xi,\eta)=(\xi_1,\eta_1)$.

5 Existence and uniqueness of the Green's function of the problem (4)-(6).

Theorem 1. *The function $G(\xi,\eta;\xi_1,\eta_1)$ that satisfies the conditions* (31)-(19) *exists and is unique.*

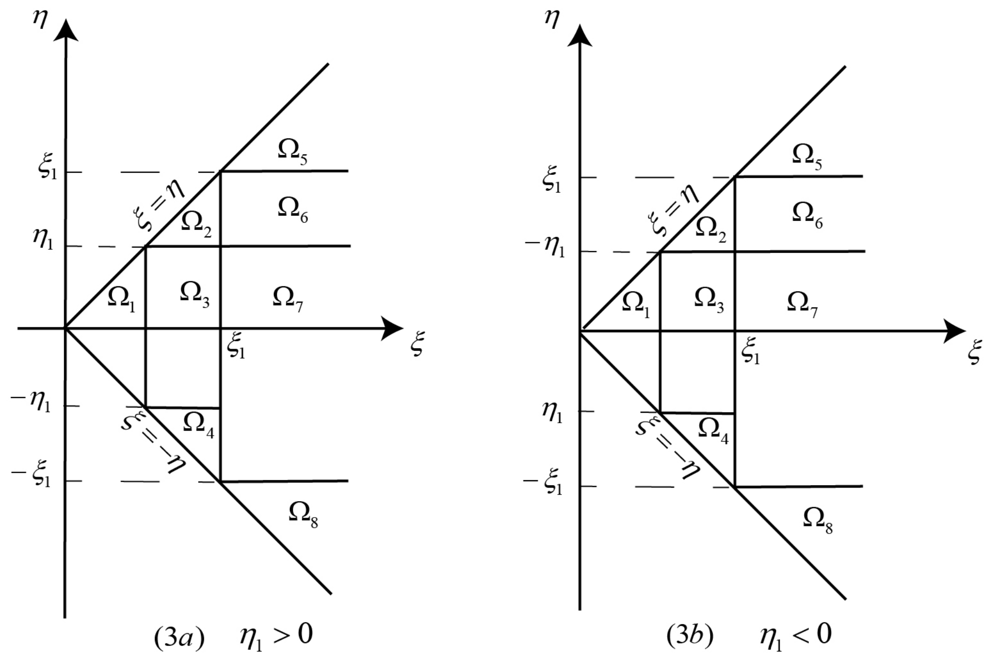

Figure 1: (3a) - splitting the domain Ω, when $\eta_1 > 0$; (3b) - splitting the domain Ω, when $\eta_1 < 0$.

Proof. To show that a function $G(\xi,\eta;\xi_1,\eta_1)$ which satisfies the conditions (31)-(39) exists and unique, we divide the domain Ω into several subdomains (see Figure (1)) and consider the following problems sequentially. Let (ξ_1,η_1) be an arbitrary point of the domain Ω. Consider the case of $\eta_1>0$, the case of $\eta_1<0$ is considered similarly.

In the domain $\Omega_1=\{(\xi,\eta):0<\xi<\eta_1,-\xi<\eta<\xi\}$ we consider the problem

$$L_{(\xi,\eta)}G=0,\ (\xi,\eta)\in\Omega_1; \tag{40}$$

$$G(\xi,\xi;\xi_1,\eta_1)=0,\ \xi\geq 0; \tag{41}$$

$$\left(\frac{\partial G}{\partial\xi}-\frac{\partial G}{\partial\eta}\right)(\xi,\xi;\xi_1,\eta_1)=0,\ \xi\geq 0; \tag{42}$$

$$\left(\frac{\partial G}{\partial\xi}+\frac{\partial G}{\partial\eta}\right)(-\eta,\eta;\xi_1,\eta_1)=0,\ \eta\leq 0. \tag{43}$$

The problem (40)-(43) is a Cauchy-Neumann problem and has a unique solution

$$G(\xi,\eta;\xi_1,\eta_1)\equiv 0,\ (\xi,\eta)\in\Omega_1. \tag{44}$$

In the domain $\Omega_2=\{(\xi,\eta):\eta_1<\xi<\xi_1,\eta_1<\eta<\xi\}$ let us consider the problem

$$L_{(\xi,\eta)}G=0,\ (\xi,\eta)\in\Omega_2; \tag{45}$$

$$G(\xi,\xi;\xi_1,\eta_1)=0,\ \xi\geq 0; \tag{46}$$

$$\left(\frac{\partial G}{\partial\xi}-\frac{\partial G}{\partial\eta}\right)(\xi,\xi;\xi_1,\eta_1)=0,\ \xi\geq 0. \tag{47}$$

The problem (45)-(47) is a Cauchy problem and has a unique solution

$$G(\xi,\eta;\xi_1,\eta_1)\equiv 0,\ (\xi,\eta)\in\Omega_2. \tag{48}$$

Therefore from (36), (44), (48) in the domain $\Omega_3=\{(\xi,\eta):\eta_1<\xi<\xi_1,-\eta_1<\eta<\eta_1\}$, we get the problem

$$L_{(\xi,\eta)}G=0,\ (\xi,\eta)\in\Omega_3; \tag{49}$$

$$G(\eta_1,\eta;\xi_1,\eta_1)=0,-\eta_1\leq\eta\leq\eta_1; \tag{50}$$

$$\frac{\partial G(\xi,\eta_1-0;\xi_1,\eta_1)}{\partial\xi}+B(\xi,\eta_1)\cdot G(\xi,\eta_1-0;\xi_1,\eta_1)=0,\ \eta_1<\xi<\xi_1. \tag{51}$$

Integrating (51) by ξ we have

$$G(\xi,\eta_1-0;\xi_1,\eta_1)=\exp\left(-\int_{\eta_1}^{\xi}B(t,\eta_1)dt\right)C_1(\xi_1,\eta_1),\ \eta_1<\xi<\xi_1. \tag{52}$$

Substituting $\xi = \eta_1 - 0$ in (52), using condition (32) we have that $C_1(\xi_1, \eta_1) \equiv 0$ and

$$G(\xi, \eta_1 - 0; \xi_1, \eta_1) = 0,\ \eta_1 \leq \xi \leq \xi_1. \tag{53}$$

Therefore, the problem (49)-(51) is equivalent to the problem (49), (50), (53), which is a Goursat problem and has a unique solution

$$G(\xi, \eta; \xi_1, \eta_1) \equiv 0,\ (\xi, \eta) \in \Omega_3. \tag{54}$$

Since Green's function is continuous at $\eta = -\eta_1$, then from (54) in the domain $\Omega_4 = \{(\xi, \eta) : \eta_1 < \xi < \xi_1, -\xi < \eta < -\eta_1\}$ we get the problem

$$L_{(\xi,\eta)}G = 0,\ (\xi, \eta) \in \Omega_4; \tag{55}$$

$$\left(\frac{\partial G}{\partial \xi} + \frac{\partial G}{\partial \eta}\right)(-\eta, \eta; \xi_1, \eta_1) = 0,\ \eta \leq 0; \tag{56}$$

$$G(\xi, -\eta_1; \xi_1, \eta_1) = 0,\ \eta_1 \leq \xi \leq \xi_1. \tag{57}$$

This problem (55)-(57) is a Darboux problem and has a unique solution

$$G(\xi, \eta; \xi_1, \eta_1) \equiv 0,\ (\xi, \eta) \in \Omega_4. \tag{58}$$

In the domain $\Omega_5 = \{(\xi, \eta) : \xi_1 < \xi, \eta > \xi_1\}$ our problem is the Cauchy problem

$$L_{(\xi,\eta)}G = 0,\ (\xi, \eta) \in \Omega_5; \tag{59}$$

$$G(\xi, \xi; \xi_1, \eta_1) = 0,\ \xi \geq 0; \tag{60}$$

$$\left(\frac{\partial G}{\partial \xi} - \frac{\partial G}{\partial \eta}\right)(\xi, \xi; \xi_1, \eta_1) = 0,\ \xi \geq 0. \tag{61}$$

which has a unique solution

$$G(\xi, \eta; \xi_1, \eta_1) \equiv 0,\ (\xi, \eta) \in \Omega_5. \tag{62}$$

Therefore from (35), (48), (62) in the domain $\Omega_6 = \{(\xi, \eta) : \xi_1 < \xi, \eta_1 < \eta < \xi_1\}$ we have the next problem

$$L_{(\xi,\eta)}G = 0,\ (\xi, \eta) \in \Omega_6; \tag{63}$$

$$G(\xi, \xi_1; \xi_1, \eta_1) = 0,\ \xi \geq \xi_1; \tag{64}$$

$$\frac{\partial G(\xi_1 + 0, \eta; \xi_1, \eta_1)}{\partial \eta} + A(\xi_1, \eta)G(\xi_1 + 0, \eta; \xi_1, \eta_1) = 0,\ \eta_1 < \eta < \xi_1. \tag{65}$$

Integrating (65) by η we get

$$G(\xi_1+0,\eta;\xi_1,\eta_1)=\exp\left(-\int_{\eta_1}^{\eta}A(\xi_1,t)dt\right)C_2(\xi_1,\eta_1),\ \eta_1<\eta<\xi_1; \tag{66}$$

Substituting $\eta=\xi_1+0$ in (66), using condition (32) we have that $C_2(\xi_1,\eta_1)\equiv 0$ and

$$G(\xi_1+0,\eta;\xi_1,\eta_1)=0,\ \eta_1\le\eta\le\xi_1. \tag{67}$$

Therefore, the problem (63)-(65) is equivalent to the problem (63), (64), (67), which is a Goursat problem and has a unique solution

$$G(\xi,\eta;\xi_1,\eta_1)\equiv 0,\ (\xi,\eta)\in\Omega_6. \tag{68}$$

From (35), (36), (38), (54), (58), (68) in the domain $\Omega_7=\{(\xi,\eta):\xi_1<\xi,-\xi_1<\eta<\eta_1\}$ we have the problem

$$L_{(\xi,\eta)}G=0,\ (\xi,\eta)\in\Omega_7; \tag{69}$$

$$\frac{\partial G(\xi_1+0,\eta;\xi_1,\eta_1)}{\partial\eta}+A(\xi_1,\eta)G(\xi_1+0,\eta;\xi_1,\eta_1)=0,\ -\xi_1<\eta<\eta_1. \tag{70}$$

$$\frac{\partial G(\xi,\eta_1-0;\xi_1,\eta_1)}{\partial\xi}+B(\xi,\eta_1)G(\xi,\eta_1-0;\xi_1,\eta_1)=0,\ \xi_1<\xi. \tag{71}$$

$$G(\xi_1+0,\eta_1-0;\xi_1,\eta_1)=-1. \tag{72}$$

The problem (69)-(72) is a Goursat problem and it has a unique solution, and it is easy to see that its solution coincides with the Riemann-Green function, that is,

$$G(\xi,\eta;\xi_1,\eta_1)=-R(\xi,\eta;\xi_1,\eta_1),\ (\xi,\eta)\in\Omega_7. \tag{73}$$

Therefore from (73) in the domain $\Omega_8=\{(\xi,\eta):\xi_1<\xi,-\xi<\eta<-\xi_1\}$ we get the problem

$$L_{(\xi,\eta)}G=0,\ (\xi,\eta)\in\Omega_8; \tag{74}$$

$$\left(\frac{\partial G}{\partial\xi}+\frac{\partial G}{\partial\eta}\right)(-\eta,\eta;\xi_1,\eta_1)=0,\ \eta\le 0; \tag{75}$$

$$\frac{\partial G(\xi,-\xi_1-0;\xi_1,\eta_1)}{\partial\xi}+B(\xi,-\xi_1)G(\xi,-\xi_1-0;\xi_1,\eta_1)$$
$$=-\frac{\partial R(\xi,-\xi_1;\xi_1,\eta_1)}{\partial\xi}-B(\xi,-\xi_1)R(\xi,-\xi_1-0;\xi_1,\eta_1),\ \xi_1<\xi. \tag{76}$$

Let us rewrite condition (76) in the following form

$$\left[\frac{\partial}{\partial\xi}\left(G(\xi,-\xi_1-0;\xi_1,\eta_1)\exp\left(\int_{\xi_1}^{\xi}B(t,-\xi_1)dt\right)\right)\right]\exp\left(\int_{\xi}^{\xi_1}B(t,-\xi_1)dt\right)$$

$$= \left[\frac{\partial}{\partial\xi}\left(-R(\xi,-\xi_1;\xi_1,\eta_1)\exp\left(\int_{\xi_1}^{\xi}B(t,-\xi_1)dt\right)\right)\right]\exp\left(\int_{\xi}^{\xi_1}B(t,-\xi_1)dt\right). \quad (77)$$

Integrating (77) by ξ we get

$$G(\xi,-\xi_1-0;\xi_1,\eta_1) = -R(\xi,-\xi_1;\xi_1,\eta_1) + C(\xi_1,\eta_1)\exp\left(\int_{\xi}^{\xi_1}B(t,-\xi_1)dt\right). \quad (78)$$

Using condition (38) we have that

$$C(\xi_1,\eta_1) = -R(\xi_1,-\xi_1;\xi_1,\eta_1) = \exp\left(\int_{-\xi_1}^{\eta_1}A(\xi_1,t)dt\right). \quad (79)$$

Substituting (79) in (78) and using condition (16) we have

$$G(\xi,-\xi_1-0;\xi_1,\eta_1) = -R(\xi,-\xi_1;\xi_1,\eta_1) - \exp\left(\int_{\xi}^{-\eta_1}B(t,-\xi_1)dt\right),\ \xi > \xi_1. \quad (80)$$

The problem (74), (75), (80) is a Darboux problem and has a unique solution.

Thus, we have shown that for any $(\xi_1,\eta_1) \in \Omega$ and $(\xi,\eta) \in \Omega$ the Green's function that satisfies the conditions (31)-(39) exists and unique. The theorem is proved. □

Corollary 2. *In the course of proving the existence of the Green's function, we obtained that* $G(\xi,\eta;\xi_1,\eta_1) \equiv 0$ *in the domains* Ω_1, Ω_2, Ω_3, Ω_4, Ω_5, Ω_6. *That is,* $G(\xi,\eta;\xi_1,\eta_1) \equiv 0$ *for* $\xi_1 > \xi$.

6 Construction of the Green's function of the problem (4)-(6).

As can be seen from the proof of Theorem 6.1, the Green's function $G(\xi,\eta;\xi_1,\eta_1) = 0$ in the domains Ω_1, Ω_2, Ω_3, Ω_4, Ω_5, Ω_6. And in the domain Ω_7 it coincides with the Riemann function (73).

Let us find a representation of the Green's function in the domain Ω_8. To construct the Green's functions, we assume that the coefficients of equation (80) satisfy the symmetry conditions of (16).

Let (ξ_1,η_1) be an arbitrary point of the domain Ω. In order to construct the Green function in the domain Ω_8, consider the problem:

$$\frac{\partial^2 G_1}{\partial\xi\partial\eta} + A(\xi,\eta)\frac{\partial G_1}{\partial\xi} + B(\xi,\eta)\frac{\partial G_1}{\partial\eta} + C(\xi,\eta)G_1 = 0,\ (\xi,\eta) \in \widetilde{\Omega}_8, \quad (81)$$

where $\widetilde{\Omega}_8 = \Omega_8 \cup \Omega_8^-$, $\Omega_8^- = \{(\xi,\eta) : \xi_1 < \xi, \eta < -\xi\}$.

$$G(\xi,-\xi_1-0;\xi_1,\eta_1) = -R(\xi,-\xi_1;\xi_1,\eta_1) - \exp\left(\int_{\xi}^{-\eta_1}B(t,-\xi_1)dt\right),\ \xi_1 < \xi; \quad (82)$$

$$G(\xi_1+0,\eta;\xi_1,\eta_1) = -R(-\eta,-\xi_1;\xi_1,\eta_1) - \exp\left(\int_{\eta}^{\eta_1} B(t,-\xi_1)dt\right),\ \eta < -\xi_1; \tag{83}$$

$$G_1(\xi_1,-\xi_1-0;\xi_1,\eta_1) = -2R(\xi_1,-\xi_1;\xi_1,\eta_1). \tag{84}$$

The problem (81)-(84) is a Goursat problem. Its solution exists and unique. We are interested in the representation of the function $G_1(\xi,\eta;\xi_1,\eta_1)$.

Lemma 2. *If the function $G_1(\xi,\eta;\xi_1,\eta_1)$ is the solution to the problem* (81)-(84)*, then for any $(\xi,\eta)\in\widetilde{\Omega}_8$ we have $G_1(\xi,\eta;\xi_1,\eta_1) = G_1(-\eta,-\xi;\xi_1,\eta_1)$.*

Proof. To show that the function $G_1(-\eta,-\xi;\xi_1,\eta_1)$ satisfies the equation (81), in (82) replace $\xi=-\eta_2$, $\eta=-\xi_2$, $(-\eta_2,-\xi_2)\in\Omega_8^-$ and after using the conditions (12)-(14), we get that $G_1(-\eta,-\xi;\xi_1,\eta_1)$ satisfies the equation (81).

Also doing the substitution of $\xi=-\eta_2$, $\eta_2<-\xi_1$ in (81) and using the conditions (12), (13) we get the condition (83). Similarly, by replacing $-\eta=\xi_2$, $\eta<-\xi_1$ in (83) and using the conditions (12), (13) we get the condition (82).

Thus, we have shown that the function $-G_1(-\eta,-\xi;\xi_1,\eta_1)$ is also a solution to the problem (81)-(84). Since the solution to problem (81)-(84) is unique, then

$$G_1(\xi,\eta;\xi_1,\eta_1) = G_1(-\eta,-\xi;\xi_1,\eta_1),\ (\xi,\eta)\in\widetilde{\Omega}_8.$$

□

Solution of the problem (81)-(84) we search in the following form

$$G_1(\xi,\eta;\xi_1,\eta_1) = g(\xi,\eta;\xi_1,\eta_1) - R(\xi,\eta;\xi_1,\eta_1), (\xi,\eta)\in\widetilde{\Omega}_8.$$

Then we get the following problem

$$\frac{\partial^2 g}{\partial\xi\partial\eta} + A(\xi,\eta)\frac{\partial g}{\partial\xi} + B(\xi,\eta)\frac{\partial g}{\partial\eta} + C(\xi,\eta)g = 0,\ (\xi,\eta)\in\widetilde{\Omega}_8; \tag{85}$$

$$g(\xi,-\xi_1;\xi_1,\eta_1) + R(\xi,-\xi_1;\xi_1,\eta_1) = 0,\ \xi_1<\xi; \tag{86}$$

$$g(\xi_1,\eta;\xi_1,\eta_1) + R(-\eta,-\xi_1;\xi_1,\eta_1) = 0,\ \eta<-\xi_1; \tag{87}$$

It is easy to see that the solution to the problem (85)-(87) has the form

$$g(\xi,\eta;\xi_1,\eta_1) = -R(-\eta,-\xi;\xi_1,\eta_1),\ (\xi,\eta)\in\widetilde{\Omega}_8. \tag{88}$$

Then from (88) we get

$$G_1(\xi,\eta;\xi_1,\eta_1) = -R(-\eta,-\xi;\xi_1,\eta_1) - R(\xi,\eta;\xi_1,\eta_1),\ (\xi,\eta)\in\widetilde{\Omega}_8. \tag{89}$$

Thus the next theorem is proved.

Theorem 2. *The Green's function of the equation* (28)-(30) *exists and unique. Representation of the solution of the first initial boundary value problem* (28)-(30) *in integral form with Green's function*

$$G(\xi,\eta;\xi_1,\eta_1) \equiv 0,\ \textit{if } (\xi,\eta) \in \Omega_1,\Omega_2,\Omega_3,\Omega_4,\Omega_5,\Omega_6;$$

$$G(\xi,\eta;\xi_1,\eta_1) = -R(\xi,\eta;\xi_1,\eta_1),\quad \textit{if } (\xi,\eta) \in \Omega_7;$$

$$G(\xi,\eta;\xi_1,\eta_1) = -R(-\eta,-\xi;\xi_1,\eta_1) - R(\xi,\eta;\xi_1,\eta_1),\ \textit{if } (\xi,\eta) \in \Omega_8.$$

It is well known that for self-adjoint problems (for example, for elliptic equation), the Green function is symmetric with respect to external and internal variables. In our case, for the Green's function of the hyperbolic first initial-boundary value problem, this is not the case.

Lemma 3. *Let* (ξ,η) *be an arbitrary point of the domain* Ω. *By internal variables* (ξ_1,η_1) *the Green's function of the problem* (28)-(30) *has the following properties:*

$$L^*_{(\xi_1,\eta_1)} G(\xi,\eta;\xi_1,\eta_1) = 0,\ (\xi_1,\eta_1) \in \Omega,\ \textit{at } \xi_1 \neq \xi,\ \eta_1 \neq \eta,\ \xi_1 \neq -\eta; \tag{90}$$

$$\begin{gathered}\left(\frac{\partial G}{\partial \xi_1} + \frac{\partial G}{\partial \eta_1}\right)(\xi,\eta;\xi_1,-\xi_1) \\ -\big(A(\xi_1,-\xi_1) + B(\xi_1,-\xi_1)\big)G(\xi,\eta;\xi_1,-\xi_1) = 0,\ \xi_1 < -\eta;\end{gathered} \tag{91}$$

$$\frac{\partial G(\xi,\eta;\xi-0,\eta_1)}{\partial \eta_1} - A(\xi,\eta_1)G(\xi,\eta;\xi-0,\eta_1) = 0,\ \textit{at } \eta_1 \neq \eta; \tag{92}$$

$$\frac{\partial G(\xi,\eta;\xi_1,\eta+0)}{\partial \xi_1} - B(\xi_1,\eta)G(\xi,\eta;\xi_1,\eta+0) = 0,\ \textit{at } \xi_1 \neq \xi; \tag{93}$$

$$\begin{gathered}\frac{\partial G(\xi,\eta;-\eta-0,\eta_1)}{\partial \xi_1} - B(-\eta,\eta_1)\ G(\xi,\eta;-\eta-0,\eta_1) \\ = \frac{\partial G(\xi,\eta;-\eta+0,\eta_1)}{\partial \xi_1} - B(-\eta,\eta_1)G(\xi,\eta;-\eta+0,\eta_1);\end{gathered} \tag{94}$$

$$\begin{gathered}G(\xi,\eta;\xi-0,\eta-0) - G(\xi,\eta;\xi+0,\eta-0) \\ +G(\xi,\eta;\xi+0,\eta+0) - G(\xi,\eta;\xi-0,\eta+0) = 1.\end{gathered} \tag{95}$$

Proof. Properties (90)-(95) are easy to get out of the construction of the Green's function of problem (28)-(30). Under these conditions (90)-(95) it is possible to uniquely restore the Green's function of problem (28)-(30). □

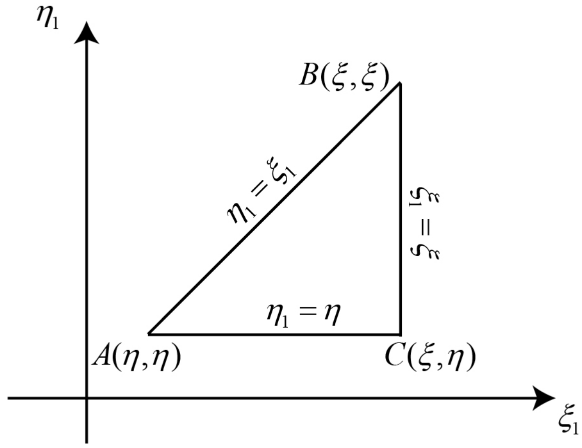

Figure 2: The domain $\Omega_{(\xi\eta)}$, when $\eta < 0$.

Using properties (90)-(95) we can use it to write the integral representation of the solution to problem (28)-(30). To do this, we consider the following integral

$$\iint_{\Omega_{(\xi\eta)}} G(\xi,\eta;\xi_1,\eta_1)F(\xi_1,\eta_1)d\xi_1 d\eta_1$$

$$= \iint_{\Omega_{(\xi\eta)}} G(\xi,\eta;\xi_1,\eta_1)\left(\frac{\partial^2 U}{\partial\xi_1\partial\eta_1} + A\frac{\partial U}{\partial\xi_1} + B\frac{\partial U}{\partial\eta_1} + CU\right)d\xi_1 d\eta_1. \tag{96}$$

Applying Green's theorem in a plane [12] and using the initial conditions (29), properties of Green's function (90)-(95), from (96) we get the following representation of the solution to problem (28)-(30) in the domain $\Omega_{(\xi\eta)}$, at $\eta > 0$ (see Figure (2))

$$U(\xi,\eta) = -\frac{1}{2}G(\xi,\eta;\eta,\eta)T_1(\eta) - \frac{1}{2}G(\xi,\eta;\xi,\xi)T_1(\xi)$$

$$-\frac{1}{2}\int_\xi^\eta\left(\frac{\partial G}{\partial N_1}(\xi,\eta;\xi_1,\xi_1) + 2(A-B)(\xi_1,\xi_1)G(\xi,\eta;\xi_1,\xi_1)\right)T_1(\xi_1)d\xi_1$$

$$+\frac{1}{2}\int_\xi^\eta G(\xi,\eta;\xi_1,\xi_1)M_1(\xi_1)d\xi_1 + \iint_{\Omega_{(\xi\eta)}} G(\xi,\eta;\xi_1,\eta_1)F(\xi_1,\eta_1)d\xi_1 d\eta_1. \tag{97}$$

Also, at $\eta < 0$ applying Green's theorem in a plane [12] and using the initial conditions (29), boundary condition (30), properties of Green's function (89)-(94), from (95) we get the following representation of the solution to problem (28)-(30) in the domain $\Omega_{(\xi\eta)}$ (see Figure (3))

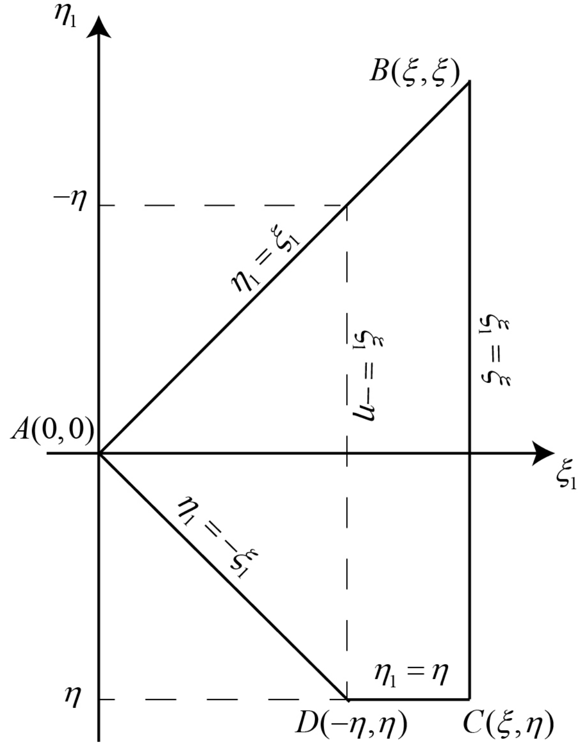

Figure 3: The domain $\Omega_{(\xi\eta)}$, when $\eta < 0$.

$$U(\xi,\eta)=-\frac{1}{2}\big(G(\xi,\eta;-\eta-0,-\eta-0)-G(\xi,\eta;-\eta+0,-\eta+0)\big)T_1(-\eta)$$
$$-\frac{1}{2}G(\xi,\eta;\xi,\xi)T_1(\xi)+\frac{1}{2}\int_0^{-\eta}G(\xi,\eta;\xi_1,-\xi_1)P(-\xi_1)d\xi_1$$
$$+\frac{1}{2}\int_0^{\xi}\left(\frac{\partial G}{\partial N_1}(\xi,\eta;\xi_1,\xi_1)+2(A-B)(\xi_1,\xi_1)G(\xi,\eta;\xi_1,\xi_1)\right)T_1(\xi_1)d\xi_1$$
$$-\frac{1}{2}\int_0^{\xi}G(\xi,\eta;\xi_1,\xi_1)M_1(\xi_1)d\xi_1+\iint_{\Omega_{(\xi\eta)}}G(\xi,\eta;\xi_1,\eta_1)F(\xi_1,\eta_1)d\xi_1. \qquad (98)$$

It is easy to see that (97), (98) are solutions to problem (28)-(30). Substituting $U(\xi,\eta),\gamma(\eta),\mu(\xi)$ for (28) we get a solution to problem (4)-(6).

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Дербісалы Б.О., Садыбеков М.А. ШИРЕК ЖАЗЫҚТЫҚТАҒЫ ГИПЕРБОЛАЛЫҚ ТЕҢДЕУ ҮШІН КОШИ-НЕЙМАН ЕСЕБІНІҢ ГРИН ФУНКЦИЯСЫ

Ширек жазықтықтағы гиперболалық теңдеу үшін Коши-Нейман есебінің Грин функциясының анықтамасы берілді. Оның бар екендігі және жалғыздығы дәлелденді. Грин функцияның анықтамасы берілді. Грин функциясы Риман-Грин функциясы арқылы берілетіні көрсетілді.

Кілттік сөздер. Гиперболалық теңдеу, екінші бастапқы шекаралық есеп, шекаралық шарт, Грин функциясы.

Дербисалы Б.О., Садыбеков М.А. О ФУНКЦИИ ГРИНА ЗАДАЧИ КОШИ-НЕЙМАНА ДЛЯ ГИПЕРБОЛИЧЕСКОГО УРАВНЕНИЯ В ЧЕТВЕРТИ ПЛОСКОСТИ

Дано определение функции Грина задачи Коши-Неймана для гиперболического уравнения в четверти плоскости. Доказаны ее существование и единственность. Дано представление функции Грина. Показано, что функция Грина может быть представлена через функцию Римана-Грина.

Ключевые слова. Гиперболическое уравнение, вторая начально-краевая задача, граничное условие, функция Грина.

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