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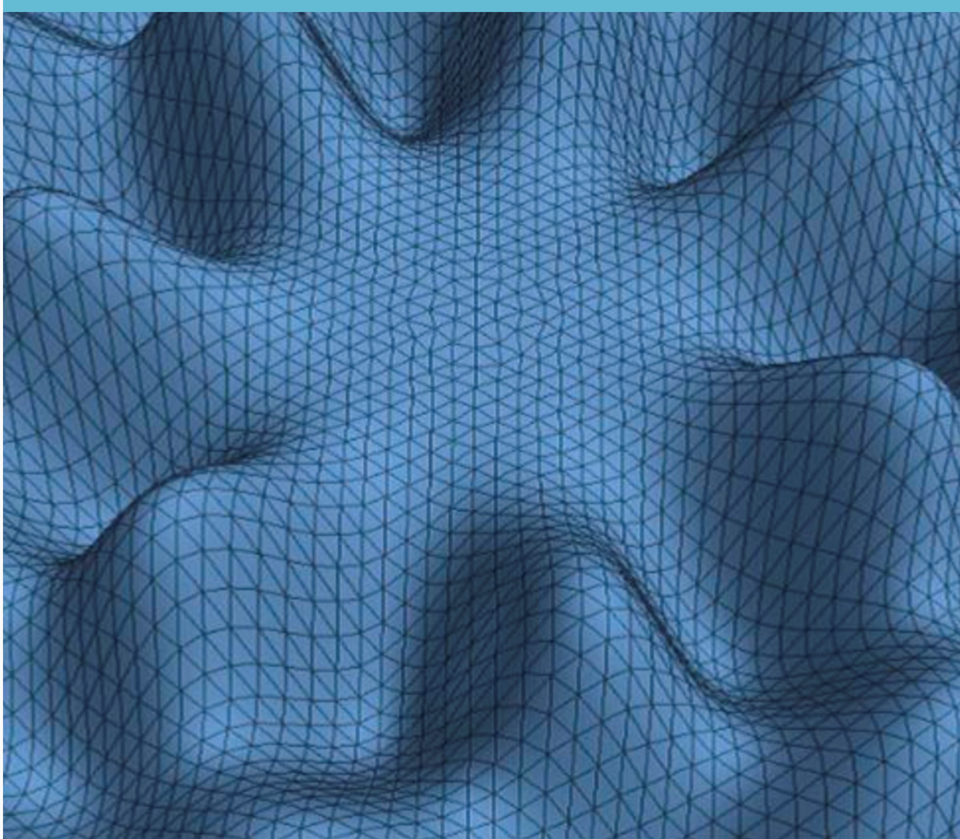

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Dulat Syzdykbekovich Dzhumabaev. Life and scientific activity (devoted to his memory)

Professor Dulat Syzdykbekovich Dzhumabaev, Doctor of Physical and Mathematical Sciences, was a prominent scientist, a well-known specialist in the field of the qualitative theory of differential and integro-differential equations, the theory of nonlinear operator equations, numerical and approximate methods for solving boundary value problems.

Dzhumabaev D.S. was born in Kantagi, Turkistan district, South Kazakhstan region, on April 11, 1954. From 1961 to 1971, he attended secondary school in Turkistan. In 1971, he entered Faculty of Mechanics and Mathematics of Kazakh State University named after S.M. Kirov (now Al-Farabi Kazakh National University). After graduating with honors from the Department of Mathematics in 1976, he continued to pursue postgraduate studies at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR. His scientific activity began under the guidance of Academician Orymbek Akhmetbekovich Zhautykov, an outstanding scientist and mathematician, who made a huge contribution to the creation and development of the mathematical science in Kazakhstan. After successful completion of postgraduate studies in 1979, Dzhumabaev D.S. joined the Laboratory of Ordinary Differential Equations headed by Academician Zhautykov O.A. He went from being a junior researcher to becoming the head of the Laboratory of Differential Equations, one of the leading divisions of the Institute of Mathematics. He chaired the laboratory from 1996 to 2012.

Dzhumabaev D.S. was a successful scientist and versatile specialist in the field of mathematics and its applications. He devoted his talent and hard work to the study of nonlinear operator equations, to the creation and development of qualitative methods in the theory of boundary value problems for differential equations.

The main research areas and the results obtained by Professor Dzhumabaev can be divided into several groups. The most significant and important scientific results are presented below in chronological order.

1. Boundary value problems for ordinary differential equations with a parameter in a Banach space

During postgraduate studies, his research was focused on nonlinear boundary value problems with parameter for ordinary differential equations of the following form:

$$\frac{dx}{dt} = f(t, x, \lambda), \qquad x(0) = x^0, \tag{1}$$

$$x(T) = x^1, \tag{2}$$

where $f : [0, T] \times B \times B \to B$ is a continuous function satisfying the existence conditions for the Cauchy problem (1) on $[0, T]$ for all values of a parameter λ from some set $G \subset B$; here B is a Banach space.

The problem is to find a pair $(\lambda^*, x^*(t))$, where $\lambda^* \in G$ and $x^*(t)$ is a solution to Cauchy problem (1) with $\lambda = \lambda^*$, satisfying the boundary condition (2).

Let the right-hand part of the differential equation be defined on the set

$$D^0 = \{(t, x, \lambda) : 0 \le t \le T, ||x - x^{(0)}(t)|| \le R(t)\rho, ||\lambda - \lambda^0|| \le \rho\}.$$

Here $\lambda^0 \in G$, $x^{(0)}(t)$ is a solution to Cauchy problem (1) with $\lambda = \lambda^0$, $R(t)$ is a positive function continuously differentiable on $[0, T]$, and ρ is a nonnegative number. Let $M(f)$ denote a set of triples $(\lambda^0 \in G, R(t) > 0, \rho \ge 0)$ for which the Lipschitz condition $||f(t, x, \lambda) - f(t, \tilde{x}, \tilde{\lambda})|| \le \alpha(t) \cdot (||x - \tilde{x}|| + ||\lambda - \tilde{\lambda}||)$ is satisfied on the set D^0, and the inequality

$$(a_1) \quad \exp\Big\{\int_0^t \alpha(\tau)d\tau\Big\} - 1 \le R(t)$$

holds $(\alpha(t) \in C([0, T]))$.

The set $M(f)$ is non-empty if so is the set G.

For a triple $(\lambda^0, R(t), \rho)$, a solution of problem (1), (2) is sought in the set $\alpha^0 = \alpha^0_\lambda \times \alpha^0_{x(t)}$, where $\alpha^0_\lambda = \{\lambda : ||\lambda - \lambda^0|| \le \rho\}$ and $\alpha^0_{x(t)} = \{x(t) : ||x(t) - x^{(0)}(t)|| \le R(t)\rho\}$.

Theorem 1. *Problem (1), (2) is solvable if and only if, given some* $(\lambda^0, R(t), \rho) \in M(f)$, *for any two pairs* $(\lambda, x(t))$ *and* $(\tilde{\lambda}, x(t))$ *from the set* α^0, *there exist an invertible operator* $A \in L(B,B)$ *and a number* $\theta > 0$ *satisfying the inequality*

$$(a_2) \quad \Big|\Big|\lambda - \tilde{\lambda} - A\Big[\int_0^T \{f(t,x(t),\lambda) - f(t,x(t),\tilde{\lambda})\}dt\Big]\Big|\Big| \le (1-\theta)||\lambda - \tilde{\lambda}||,$$

and the following inequality is true

$$(a_3) \quad \frac{1}{\theta}\Big|\Big|A\Big[\int_0^T f(t, x^{(0)}(t), \lambda^0)dt - (x^1 - x^0)\Big]\Big|\Big| \le \rho(1-q),$$

where $q = \frac{||A||}{\theta} \cdot \Big[\exp\Big\{\int\limits_0^T \alpha(t)dt\Big\} - 1 - \int\limits_0^T \alpha(t)dt\Big] < 1$. *Here* $L(B,B)$ *is a space of linear bounded operators mapping* B *into* B.

Under the conditions of Theorem 1, problem (1), (2) is uniquely solvable on the domain α^0.

For the linear boundary value problem

$$\frac{dx}{dt} = Q_1(t)x + Q_2(t)\lambda + f(t), \qquad x(0) = x^0, \qquad x(T) = x^1,$$

the conditions of Theorem 1 are reduced to the bounded invertibility of the operator $\bar{Q} = \int\limits_0^T Q_2(t)dt$.

The inequality (a_3) guarantees the existence and uniqueness of a solution to problem (1), (2) on the domain α^0.

The proposed approach was applied to semi-explicit differential equations with nonlinear boundary conditions:

$$\frac{dx}{dt} = f\Big(t, x, \frac{dx}{dt}, \lambda\Big), \qquad x(0) = x^0, \tag{3}$$

$$\Phi[x(T), \dot{x}(T), \lambda] = 0. \tag{4}$$

Here $f : [0,T] \times B \times B \times B \to B$ is a continuous function satisfying the conditions for the existence of a solution to the Cauchy problem (3) on $[0,T]$ for all $\lambda \in G$; $G \subset B$, $\Phi : B \times B \times B \to B$.

Analogously, the right-hand side of the differential equation is considered on the set $\tilde{D}^0 = \{(t,x,y,\lambda) : 0 \le t \le T, ||x - x^{(0)}(t)|| \le R(t)\rho, ||y - \dot{x}^{(0)}(t)|| \le \dot{R}(t)\rho, ||\lambda - \lambda^0|| \le \rho\}$, where $\lambda^0 \in G$, $x^{(0)}(t)$ is a solution to the Cauchy problem (3) with $\lambda = \lambda^0$, $R(t)$ is a positive function continuously differentiable on $[0,T]$, and ρ is a nonnegative number. Let $\tilde{M}(f)$ denote the set of triples $(\lambda^0 \in G, R(t) > 0, \rho \ge 0)$ for which the following inequalities are satisfied:

$$||f(t,x,y,\lambda) - f(t,\tilde{x},\tilde{y},\tilde{\lambda})|| \le \alpha_1(t) \cdot (||x - \tilde{x}|| + ||\lambda - \tilde{\lambda}||) + \alpha_2(t) \cdot ||y - \tilde{y}||,$$

$$\alpha_2(t) < 1 \quad \big(\alpha_i(t) \in C([0,T]), i = 1,2\big); \quad c(t)\exp\Big\{\int\limits_0^t c(\tau)d\tau\Big\} \le \dot{R}(t) \quad \big(c(t) = \frac{\alpha_1(t)}{1-\alpha_2(t)}\big).$$

For a triple $(\lambda^0, R(t), \rho)$, the following sets are introduced:

$$\tilde{\alpha}^0_{x(t)} = \{x(t) : ||x(t) - x^{(0)}(t)|| \le R(t)\rho, ||\dot{x}(t) - \dot{x}^{(0)}(t)|| \le \dot{R}(t)\rho\},$$

$$\tilde{D}^0(T) = \{(u, v, \lambda) : ||u - x^{(0)}(T)|| \le R(T)\rho, ||v - \dot{x}^{(0)}(T)|| \le \dot{R}(T)\rho, ||\lambda - \lambda^0|| \le \rho\}.$$

Let the boundary function in (4) satisfy the Lipschitz condition $||\Phi(u,v,\lambda) - \Phi(\tilde{u},\tilde{v},\tilde{\lambda})|| \le \Phi_u||u-\tilde{u}|| + \Phi_v||v-\tilde{v}|| + \Phi_\lambda||\lambda-\tilde{\lambda}||$ on the set $\tilde{D}^0(T)$.

Theorem 2. *Problem (3), (4) is solvable if and only if, given some $(\lambda^0, R(t), \rho) \in \tilde{M}(f)$, for any two pairs $(\lambda, x(t))$ and $(\tilde{\lambda}, x(t))$ from the set $\tilde{\alpha}^0 = \alpha^0_\lambda \times \tilde{\alpha}^0_{x(t)}$, there exist an invertible operator $A \in L(B,B)$ and a number $\theta > 0$ satisfying the inequality $||\lambda - \tilde{\lambda} - A\{\tilde{K}_1[\lambda, x(t)] - \tilde{K}_1[\tilde{\lambda}, x(t)]\}|| \le (1-\theta)||\lambda - \tilde{\lambda}||$, and the following inequality is true:*

$$\frac{1}{\theta}||A\tilde{K}_1[\lambda^0, x^{(0)}(t)]|| \le \rho(1-q),$$

where $q = \frac{||A||}{\theta} \cdot \Big[\Phi_u \cdot \Big\{\exp\Big\{\int\limits_0^T c(t)dt - 1 - \int\limits_0^T \alpha_1(t)dt\Big\} + \Phi_v \cdot \Big\{c(T)\exp\Big\{\int\limits_0^T c(t)dt - \alpha_1(T)\Big\}\Big] < 1,$
$\tilde{K}_1[\lambda, x(t)] = \Phi\Big[x^0 + \int\limits_0^T f(t, x(t), \dot{x}(t), \lambda), f(T, x(T), \dot{x}(T), \lambda), \lambda\Big].$

Conditions for the continuous dependence of a solution on the initial data and a criterion for the existence of an isolated solution to problem (3), (4) were established.

Dzhumabaev D.S. justified a new version of the shooting method for nonlinear two-point boundary value problems of the following form

$$\frac{dz}{dt} = f(t, z), \tag{5}$$

$$g[z(0), z(T)] = 0, \tag{6}$$

where $f : [0,T] \times B \to B$ is continuous in t and z, $g : B \times B \to B$.

Let λ denote the value of $z(t)$ at the point $t = 0$. By the substitution $x(t) = z(t) - \lambda$, problem (5), (6) is reduced to the following boundary value problem with parameter

$$\frac{dx}{dt} = f(t, x + \lambda), \qquad x(0) = 0, \tag{7}$$

$$g[\lambda, \lambda + x(T)] = 0. \tag{8}$$

Assume that in the closed regions $D^0 = \{(t, x, \lambda) : 0 \le t \le T, ||x - x^{(0)}(t)|| \le R(t)\rho, ||\lambda - \lambda^0|| \le \rho\}$ and $D^0_1 = \{(\lambda, u) : ||\lambda - \lambda^0|| \le \rho, ||u - \lambda^0 - x^{(0)}(T)|| \le [1 + R(T)]\rho\}$ (here $x^{(0)}(t)$ is a solution to Cauchy problem (7) for $\lambda = \lambda^0$, $R(t) > 0$ for $t \in [0, T]$, and $\rho > 0$), the following inequalities hold:

$$||f(t, x + \lambda) - f(t, \tilde{x} + \tilde{\lambda})|| \leq \alpha(t)(||x - \tilde{x}|| + ||\lambda - \tilde{\lambda}||),$$

$$||g(\lambda, u) - g(\tilde{\lambda}, \tilde{u})|| \leq g_\lambda ||\lambda - \tilde{\lambda}|| + g_u ||u - \tilde{u}||,$$

and $\exp\Big\{\int\limits_0^t \alpha(\tau) d\tau\Big\} - 1 \leq R(t)$.

Theorem 3. *If for any two pairs $(\lambda, x(t))$ and $(\tilde{\lambda}, x(t))$ from the domain $\alpha^0 = \alpha^0_\lambda \times \alpha^0_{x(t)}$ and for some $N \geq 0$, there exist an invertible operator $A \in L(B, B)$ and a number $\theta > 0$ satisfying the inequality $||\lambda - \tilde{\lambda} - A\{K^{(1)}_N[\lambda, x(t)] - K^{(1)}_N[\tilde{\lambda}, x(t)]\}|| \leq (1 - \theta)||\lambda - \tilde{\lambda}||$, and the following inequality holds*

$$\frac{1}{\theta}||A\{K^{(1)}_N[\lambda^0, x^{(0)}(t)]\}|| \leq \rho(1 - q^{(1)}_N),$$

where $q^{(1)}_N = g_u \cdot \frac{||A||}{\theta} \cdot \Big[\exp\Big\{\int\limits_0^T \alpha(t) dt\Big\} - 1 - \int\limits_0^T \alpha(t) dt - ... - \frac{1}{N!}\Big(\int\limits_0^T \alpha(t) dt\Big)^N\Big] < 1$, then the boundary value problem (7), (8) has a unique solution in α^0.

Here $K^{(1)}_N[\lambda, x(t)] = g\Big[\lambda, \lambda + \int\limits_0^T f(t, \lambda + ... + \int\limits_0^{\tau_{N-3}} f(\tau_{N-2}, \lambda + x(\tau_{N-2})) d\tau_{N-2}) ...) dt\Big]$, $N = 0, 1, 2, ...$

For different values of N, various sufficient conditions for the unique solvability to problem (7), (8) can be derived from Theorem 3. The problem of choosing an initial approximation and other replacement versions in problems with parameter were also considered.

Dzhumabaev D.S. also studied nonlinear infinite systems of equations

$$Q_j(\lambda_1, \lambda_2, ..., \lambda_i, ...) = b_j, \qquad j = 1, 2, ..., \tag{9}$$

where $\lambda = (\lambda_1, \lambda_2, ...)$ and $b = (b_1, b_2, ...)$ are elements of l_p $(1 \leq p \leq \infty)$. It is supposed that in the domain $D' = \{\lambda : ||\lambda - \lambda^0|| < \rho\} \subset l_p$, for all $i (i = 1, 2, ...)$, functions $Q_i(\lambda_1, \lambda_2, ...)$ have continuous partial derivatives with respect to all arguments and 1) $\sum\limits_{j=1}^{\infty} \sup\limits_{\lambda \in D'} \Big|\frac{\partial Q_i(\lambda)}{\partial \lambda_j}\Big| \leq k_1 < \infty$; 2) $\sum\limits_{k=1}^{\infty} \sup\limits_{\lambda \in D'} \Big|\frac{\partial Q_k(\lambda)}{\partial \lambda_i}\Big| \leq k_2 < \infty$. Then there exist numbers θ_1 and θ_2 satisfying the inequalities 3) $\Big|\frac{\partial Q_i(\lambda)}{\partial \lambda_i}\Big| \geq \sum\limits_{j \neq i} \Big|\frac{\partial Q_i(\lambda)}{\partial \lambda_j}\Big| + \theta_1$; 4) $\Big|\frac{\partial Q_i(\lambda)}{\partial \lambda_i}\Big| \geq \sum\limits_{k \neq i} \sup\limits_{\lambda \in D'} \Big|\frac{\partial Q_k(\lambda)}{\partial \lambda_i}\Big| + \theta_2$, for all $\lambda \in D'$ and $i = 1, 2, ...$

The following definition extends the concept of complete regularity to the case of nonlinear infinite systems in l_p.

Definition 1. *An operator $Q = (Q_1, Q_2, ...)$ is called completely regular in the domain D', if it satisfies conditions 1)-4) wherein the numbers θ_1 and θ_2 are such that 5) $\frac{p-1}{p}\theta_1 + \frac{1}{p}\theta_2 = \theta > 0$.*

Lemma 1. *If Q is a completely regular operator in the domain D' and $\frac{1}{\theta}||Q(\lambda^0) - b|| < \rho$, then the infinite system of nonlinear equations (9) has a unique solution in D'.*

Using Lemma 1, the results obtained for problems (1)-(2), (3)-(4), and (5)-(6) were concretized for infinite systems of differential equations. Effective conditions were established for the unique solvability of nonlinear boundary value problems for infinite systems of differential equations in the space l_p.

The findings described in this Section were published in [1-5] and formed the basis of his candidate thesis. In 1980, Dzhumabaev D.S. defended his dissertation "Boundary value problems with a parameter for ordinary differential equations in a Banach space" and earned a degree of Candidate of Physical and Mathematical Sciences in the specialty 01.01.02 - Differential Equations.

The methods and results of [1-5] were applied to nonlinear differential equations of various classes [6-11, 14]. Dzhumabaev's research was then focused on various problems for nonlinear operator equations [12-13, 15-17].

2. A linearizer and iterative processes for unbounded non-smooth operators.

Consider the nonlinear operator equation

$$A(x) = 0, \tag{10}$$

where $x \in B_1$, $A(x) \in B_2$, and each B_i is a Banach space with norm $||\cdot||_i$, $i = 1, 2$. Let $D(A)$ and $R(A)$ denote the domain and range of A, respectively.

For a point $x^0 \in D(A)$, the following sets are constructed: $S(x^0, r) = \{x \in B_1 : ||x - x^0||_1 \leq r\}$, $U^0 = \{x \in D(A) : ||A(x)||_2 \leq ||A(x^0)||_2 = u^0\}$, and $\Omega = S(x^0, r) \cap U^0$. Assume that the operator A is closed on Ω. As is known, iterative methods, that allow one to find a solution under some sufficient conditions for its existence, rely on certain linearizations of the nonlinear operator. Linearization of an unbounded operator naturally leads to unbounded linear operators.This motivated Dzhumabaev D.S. to introduce the concept of a linearizer of an operator A at a point $\hat{x} \in D(A)$ that generalizes the Frechet derivative for unbounded non-smooth operators.

Definition 2. *A linear operator $C : B_1 \to B_2$ is called a linearizer of an operator A at a point $\hat{x} \in D(A)$, if $D(A) \subseteq D(C)$ and there exist numbers $\epsilon \geq 0$ and $\delta > 0$ such that*

$$||A(x) - A(\hat{x}) - C(x - \hat{x})||_2 \leq \epsilon ||x - \hat{x}||_1$$

for all $x \in D(A)$ satisfying $||x - \hat{x}||_1 < \delta$.

If $C \in L(B_1, B_2)$ is the Frechet derivative of A at a point $\hat{x} \in D(A)$, then it is also a linearizer. However, the definition of a linearizer, unlike that of the Frechet derivative, does not require: a) the boundedness of the operator C and 2) the dependence of ϵ on δ ($\epsilon(\delta) \to 0$ as $\delta \to 0$ for the Frechet derivative).

While the Frechet derivative of an operator A is uniquely determined, there can be infinitely many linearizers of this operator.

Distinctive advantages of linearizers make it possible to extend the domain of application of iterative methods to solving nonlinear operator equations. Dzhumabaev D.S. proposed a method for proving the convergence of iterative processes that takes into account the specificities of unbounded operator equations.

Theorem 4. *Suppose that at each point $x \in \Omega$ the operator A has a linearizer C_x with constants ϵ_x and δ_x such that: 1) C_x is a one-to-one mapping of $D(C)$ onto $R(C)$, and $||C_x^{-1}|| \le \gamma_x \le \bar{\gamma}$; 2) $\epsilon_x \cdot \delta_x \le \Theta < 1$; and 3) $\frac{\gamma_x}{\delta_x} \cdot ||A(x)||_2 \le K$. If $\frac{\bar{\gamma}}{1-\Theta} \cdot ||A(x)||_2 < r$, then (10) has a solution $x^* \in \Omega$, to which the iteration process*

$$x^{(n+1)} = x^{(n)} - \frac{1}{\alpha} C^{-1}_{x^{(n)}} \{A(x^{(n)})\} \tag{11}$$

converges, here $\alpha = \max\{1, K\}$, $n = 0, 1, 2, ...$

In the case when for a given $\delta > 0$ there exists $\epsilon(\delta)$ independent of x, the following assertion is true.

Theorem 5. *Suppose that at each point $x \in \Omega$ and for each $\delta \in (0, h)$ the operator A has a linearizer C_x with constants δ and $\epsilon(\delta) \ge 0$ satisfying the following conditions: 1) C_x^{-1} exists on $R(C)$, and $||C_x^{-1}|| \le \gamma$, 2) $\lim\limits_{\delta \to 0} \epsilon(\delta) = 0$.*

Then (10) has a solution $x^* \in \Omega$, if the following inequality holds: 3) $\gamma \cdot ||A(x)||_2 < r$.

Theorem 5 generalizes the local theorem of Hadamard to unbounded operator equations. This made it possible to extend the well-known Newton-Kantorovich method to unbounded nonsmooth operator equations and apply it to nonlinear boundary value problems for differential equations.

Consider the closed operator equation

$$A(x) \equiv Cx + F(x) = 0, \tag{12}$$

where $C : X \to Y$ is a closed linear operator, $F : X \to Y$ is a continuous operator, and X and Y are Banach spaces with respective norms $|| \cdot ||_1$ and $|| \cdot ||_2$.

Assume that F has a Frechet derivative in some domain containing the ball $\bar{S}(x^0, r) = \{x \in X : ||x - x^0||_1 \le r\}$, $x^0 \in D(C)$, and $R(C + F'(x)) = Y$ for $x \in S(x^0, r)$. Then in $D(A) = D(C) \cap \bar{S}(x^0, r)$ the operator A has the linearizer $C_1(x) = C + F'(x)$, and $D(C_1) = D(C) \cap X = D(C)$.

Theorem 6. *Assume that the following conditions hold:*

(1) For all $x \in D(A)$, the linearizer $C_1(x)$ has a bounded inverse, and $||C_1^{-1}(x)||_{L(Y,X)} \le \gamma$;

(2) $||F'(x) - F'(y)||_{L(X,Y)} \le L \cdot ||x - y||_1$, $x, y \in \bar{S}(x^0, r)$;

(3) $\frac{m}{L\gamma} + \gamma \frac{b_m}{b_0} ||A(x^0)||_2 \sum\limits_{s=0}^{\infty} (b_m)^{2^s - 1} < r$, where $b_0 = \frac{L}{2}\gamma^2 u_0$, $u_0 = ||A(x^0)||_2$, $\beta_k = 1 - \frac{1}{4b_{k-1}}$, $b_k = \beta_k \cdot b_{k-1}$, $k = 1, 2, ..., m$, where m is a nonnegative number such that $b_m < 1$ and $b_{m-1} \ge 1$.

Then the damped Newton-Kantorovich method

$$x^{(k+1)} = x^{(k)} - \frac{1}{\alpha_k}[C + F'(x^k)]^{-1}[Cx^k + F(x^k)], \quad k = 0, 1, 2, ..., \tag{13}$$

where $\alpha_k = 2b_k$ *for* $k = 0, ..., m-1$ *and* $\alpha_k = 1$ *for* $k = m, m+1, ...,$ *converges to a solution of (12).*

Theorem 7. *Assume that the following conditions hold:*

(1) For all $x \in D(A)$*, the linearizer* $C_1(x)$ *has a bounded inverse, and* $||C_1^{-1}(x)||_{L(Y,X)} \le \gamma$*;*

(2) The Frechet derivative $F'(x)$ *is uniformly continuous in* $\bar{S}(x^0, r)$*;*

(3) $\gamma \cdot ||A(x^0)||_2 < r$.

Then there exist numbers $\alpha_n \ge 1$*,* $n = 0, 1, ...,$ *such that the iteration process*

$$x^{(m+s+1)} = x^{(m+s)} - [C + F'(x^{m+s})]^{-1}[Cx^{m+s} + F(x^{m+s})], \quad s = 0, 1, 2, ...,$$

converges to an isolated solution $x^* \in D(A)$ *of (12). Furthermore, starting with some* k^0*, we can take* α_n $(n \ge k^0)$ *equal to 1, and the convergence rate becomes superlinear.*

These results were published in "News of the Academy of Sciences of Kazakh SSR. Series Physical and Mathematical", 1984 [12,13], and, at the request of the American Mathematical Society, were translated and published in "American Mathematical Society Translations", 1989 [16-17], as well as in "Mathematical Notes" [15]. Various aspects of applications of these results were considered in [18, 20, 22].

3. The parameterization method for solving boundary value problems

Dzhumabaev D.S. developed the parametrization method for investigation and solving boundary value problems. The method was originally offered in [21, 23] for solving two-point boundary value problems for a linear differential equation of the following form

$$\frac{dx}{dt} = A(t)x + f(t), \qquad x \in \mathbb{R}^n, \tag{14}$$

$$Bx(0) + Cx(T) = d, \tag{15}$$

where $A(t)$ and $f(t)$ are continuous in $(0, T]$, B and C are $n \times n$ matrices, $d \in \mathbb{R}^n$.

Consider a partition dividing the interval $[0, T)$ into N equal parts with step size $h > 0$: $[0, T) = \bigcup_{r=1}^{N} [(r-1)h, rh)$. Let $x_r(t)$ denote the restriction of the function $x(t)$ to the r-th subinterval, i.e. $x_r(t)$, $r = \overline{1, N}$, is a vector function of dimension n defined on $[(r-1)h, rh)$ and coinciding there with $x(t)$. Problem (14), (15) is thus transformed into an equivalent multipoint boundary-value problem

$$\frac{dx_r}{dt} = A(t)x_r + f(t), \qquad t \in [(r-1)h, rh), \quad r = 1, 2, ..., N, \tag{16}$$

$$Bx_1(0) + C \lim_{t\to T-0} x_N(t) = d, \tag{17}$$

$$\lim_{t\to sh-0} x_s(t) = x_{s+1}(sh), \qquad s = 1,2,...,N-1. \tag{18}$$

Here (18) are the matching conditions for the solution at the interior points of the partition.

Obviously, if $x(t)$ is a solution of problem (14), (15), then the set of restrictions $(x_r(t))$, $r = 1,2,...,N$, is a solution of the multipoint problem (16)-(18). Conversely, if a set of vector functions $(x_r(t))$, $r = 1,2,....N$, is a solution of problem (16)-(18), then the function $x(t)$ obtained by piecing together $x_r(t)$ is a solution of the original boundary value problem.

On each subinterval $[(r-1)h, rh)$, the substitution $u_r(t) = x_r(t) - \lambda_r$ is made, where λ_r denotes the value of $x_r(t)$ at the point $t = (r-1)h$. Problem (16)-(18) is then reduced to the boundary value problem with parameter

$$\frac{du_r}{dt} = A(t)u_r + A(t)\lambda_r + f(t), \quad t\in[(r-1)h, rh), \quad u_r[(r-1)h] = 0, \quad r = 1,2,...,N, \tag{19}$$

$$B\lambda_1 + C\lambda_N + C \lim_{t\to T-0} u_N(t) = d, \tag{20}$$

$$\lambda_s + \lim_{t\to sh-0} u_s(t) = \lambda_{s+1}, \qquad s = 1,2,...,N-1. \tag{21}$$

An advantage of problem (19)-(21) is that it involves the initial conditions $u_r[(r-l)h] = 0$, so that one can determine $u_r(t)$ from the integral equations

$$u_r(t) = \int\limits_{(r-1)h}^{t} [A(\tau)u_r + A(\tau)\lambda_r]d\tau + \int\limits_{(r-1)h}^{t} f(\tau)d\tau. \tag{22}$$

In (22), replacing $u_r(\tau)$ by the right-hand side of (22) and repeating the process ν ($\nu = 1,2,...$) times, one obtains a representation of $u_r(t)$ by a sum of iterated integrals. Letting $t \to rh-0$ and substituting $\lim\limits_{t\to rh-0} u_r(t)$, $r = 1,2,\ldots,N$, into (20) and (21) results in a system of nN algebraic equations in the parameters λ_{ri}, $r = 1,2,...,N$, $i = 1,2,...,n$:

$$Q_\nu(h)\lambda = -F_\nu(h) - G_\nu(u,h), \qquad \lambda\in\mathbb{R}^{Nn}. \tag{23}$$

The basic idea behind the method is to reduce the problem in question to an equivalent problem with a parameter (19)-(21) whose solution is determined as the limit of a sequence of systems of pairs consisting of the parameter λ and the function u. The parameter is found from the system of linear equations (23) determined by the matrices of the differential equation (14) and boundary conditions (15). The functions u_r are solutions of Cauchy problems (19) on the partition subintervals $[(r-1)h, rh)$, $r = 1,2,...,N$, for the found values of the parameter. The introduction of parameters made it possible to obtain conditions for the convergence of proposed algorithms and, at the same time, for the existence of a solution, in terms of the input data. This makes the parameterization method different from the shooting method and its modifications, where shooting parameters are found from some equations constructed via general solutions of differential equations, and convergence conditions are also given in terms of general solutions.

Theorem 8. *Suppose that for some* $h > 0 \quad (Nh = T)$ *and* $\nu \quad (\nu = 1, 2, \ldots)$ *the matrix* $Q_\nu(h) : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ *is invertible and the following inequalities hold:*

(a) $||[Q_\nu(h)]^{-1}|| \le \gamma_\nu(h)$;

(b) $q_\nu(h) = \gamma_\nu(h)\max(1, h||C||)[e^{\alpha h} - 1 - \alpha h - ... - \frac{(\alpha h)^\nu}{\nu!}] < 1$, *where* $\alpha = \max\limits_{t\in[0,T]} ||A(t)||$.

Then the boundary-value problem (14), (15) has a unique solution $x^*(t)$, *and the estimate*

$$||x^*(t) - x^{(k)}(t)|| \le \gamma_\nu(h)\max(1, h||C||)\frac{(\alpha h)^\nu}{\nu!}e^{\alpha h}\frac{[q_\nu(h)]^\nu}{1 - q_\nu(h)}M(h), \qquad t \in [0, T], \tag{24}$$

holds true, where

$$M(h) = \gamma_\nu(h)[e^{\alpha h} - 1]\max\Big\{1 + h||C||\sum_{j=0}^{\nu-1}\frac{(\alpha h)^j}{j!}, \sum_{j=0}^{\nu-1}\frac{(\alpha h)^j}{j!}\Big\}\max(||d||, \max_{t\in[0,T]}||f(t)||)h$$

$$+e^{\alpha h}\max_{t\in[0,T]}||f(t)||h,$$

and $x^{(k)}(t)$ *is a piecewise-continuously differentiable function on* $[0, T]$, *for which* $\lambda_r^{(k)} + u_r^{(k)}(t)$ *is the restriction to* $[(r-l)h, rh)$, $r = 1, 2, ..., N$.

It was shown that the conditions of Theorem 8 are also necessary and sufficient for the unique solvability of problem (14),(15).

The parametrization method was then applied to the study of singular problems for which the problem of approximation by regular two-point boundary value problems was solved [19, 24-27]. Necessary and sufficient conditions were obtained for the well-posed solvability of the problem of finding a solution to the system of differential equations (14), that is bounded on the whole axis $\mathbb{R}$. For systems whose matrices and right-hand sides are constant in the limit, approximating regular two-point boundary value problems were constructed. The connection between the well-posed solvability of the original singular problem and that of an approximating problem was established. In the general case, Lyapunov transformations possessing certain properties were used to construct regular two-point boundary value problems as approximations to the problem of determining a solution bounded on the entire real line. The concept of a solution "in the limit as $t \to \infty$" was introduced and the behaviour of solutions of linear ordinary differential equations as $t \to \infty$ was investigated. Necessary and sufficient conditions were derived under which a singular boundary value problem with conditions assigned at infinity is uniquely solvable, and an appropriate approximating problem was constructed. These results were developed to the system of differential equations on the real axis:

$$\frac{dx}{dt} = f(t,x), \qquad x \in \mathbb{R}^n. \tag{24}$$

In [29, 34] the results of Section 2 were also extended to system (24) with the nonlinear boundary condition

$$g[x(0), x(T)] = 0. \tag{25}$$

Results of Sections 2 and 3 were included in the doctoral dissertation.

The doctoral dissertation by Dzhumabaev D.S. titled "Singular boundary value problems for ordinary differential equations and their approximation" is a fundamental scientific work that underwent comprehensive approbation in leading scientific centers, such as the Computing Center of the Russian Academy of Sciences (A.A. Abramov, N.B. Konyukhova), the Institute of Applied Mathematics of the Russian Academy of Sciences (K.I. Babenko), Lomonosov Moscow State University (V.M. Millionshchikov, V.A. Kondratiev, N.Kh. Rozov), Institute of Mathematics NAS of Ukraine (Y.A. Mitropol'skii, A.M. Samoilenko, V.L. Makarov, V.L. Kulik), Voronezh State University (V.I. Perov), I. Vekua Institute of Applied Mathematics of Tbilisi State University (I.T. Kiguradze), Kiev State University named after T. Shevchenko (N.I. Perestyuk). Doctoral dissertation was defended at the Specialized Council of the Institute of Mathematics of the NAS of Ukraine in 1994.

The parameterization method was extended to various linear and nonlinear boundary value problems for ordinary differential equations on a finite interval and on the whole real line; necessary and sufficient solvability conditions for those problems were obtained in [28-32, 34, 47, 50-51, 53, 55-56, 62, 65, 70, 74, 77, 83, 85, 92, 96].

4. Nonlocal problems for systems of second-order hyperbolic equations

The results obtained in Sections 2 and 3 provided a basis for solving nonlocal boundary value problems for systems of second-order hyperbolic equations [33, 36-46, 48, 49, 52, 58, 63, 67, 71, 73, 86].

In the domain $\Omega = [0,T] \times [0,\omega]$, consider the following nonlocal boundary value problem for the system of hyperbolic equations with two independent variables:

$$\frac{\partial^2 u}{\partial t \partial x} = A(t,x)\frac{\partial u}{\partial x} + B(t,x)\frac{\partial u}{\partial t} + C(t,x)u + f(t,x), \tag{26}$$

$$P_2(x)\frac{\partial u(t,x)}{\partial x}\Big|_{t=0} + P_1(x)\frac{\partial u(t,x)}{\partial t}\Big|_{t=0} + P_0(x)u(t,x)|_{t=0}$$

$$+ S_2(x)\frac{\partial u(t,x)}{\partial x}\Big|_{t=T} + S_1(x)\frac{\partial u(t,x)}{\partial t}\Big|_{t=T} + S_0(x)u(t,x)|_{t=T} = \varphi(x), \quad x \in [0,\omega], \tag{27}$$

$$u(t,0) = \psi(t), \qquad t \in [0,T], \tag{28}$$

where $u(t,x) = col(u_1(t,x),...,u_n(t,x))$ is an unknown function, the $n\times n$ matrices $A(t,x)$, $B(t,x)$, $C(t,x)$, $P_i(x)$, $S_i(x)$, $i=\overline{0,2}$, and the n-vector functions $f(t,x)$, $\varphi(x)$ are continuous on Ω and $[0,\omega]$, respectively; the n-vector function $\psi(t)$ is continuously differentiable on $[0,T]$.

Sufficient coefficient conditions for the existence and uniqueness of a classical solution of problem (26)–(28) were established by a modification of the parametrization method [33, 38, 40, 45, 46]. A relationship with the following family of boundary value problems for ordinary differential equations was established:

$$\frac{\partial v}{\partial t} = A(t,x)v + F(t,x), \qquad x\in[0,\omega], \tag{29}$$

$$P_2(x)v(0,x) + S_2(x)v(T,x) = \Phi(x), \tag{30}$$

here n-vector functions $F(t,x)$ and $\Phi(x)$ are continuous on Ω and $[0,\omega]$, respectively.

For fixed $x\in[0,\omega]$ problem (29), (30) is a linear boundary value problem for the system of ordinary differential equations. Suppose the variable x is changed on $[0,\omega]$; then we obtain a family of boundary value problems for ordinary differential equations.

Sufficient and necessary conditions for the well-posedness of nonlocal boundary value problem for the system of hyperbolic equations (28)-(30) were obtained in [44, 52, 63, 67].

Let $C([0,\omega],R^n)$ be a space of continuous on $[0,\omega]$ vector functions $\varphi(x)$ with the norm $||\varphi||_{0,1} = \max\limits_{x\in[0,\omega]} ||\varphi(x)||$;

$C^1([0,T],R^n)$ be a space of continuously differentiable on $[0,T]$ vector functions $\psi(t)$ with the norm $||\psi||_{1,0} = \max\Big(\max\limits_{t\in[0,T]}||\psi(t)||, \max\limits_{t\in[0,T]}||\dot{\psi}(t)||\Big)$;

$C^{1,1}(\Omega,R^n)$ be a space of functions $u(t,x)\in C(\Omega,R^n)$ with continuous on Ω partial derivatives $\frac{\partial u(t,x)}{\partial x}$, $\frac{\partial u(t,x)}{\partial t}$, $\frac{\partial^2 u(t,x)}{\partial t\partial x}$ with the norm $||u||_{1,1} = \max\Big(||u||_0, \Big|\Big|\frac{\partial u}{\partial x}\Big|\Big|_0, \Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_0, \Big|\Big|\frac{\partial^2 u}{\partial t\partial x}\Big|\Big|_0\Big)$.

Lemma 2. *If problem (29), (30) has a solution for arbitrary $F(t,x)\in C(\Omega,R^n)$ and $\Phi(x)\in C([0,\omega],R^n)$, then this solution is unique.*

Definition 3. *Problem (29), (30) is called well-posed if for arbitrary $F(t,x)\in C(\Omega,R^n)$ and $\Phi(x)\in C([0,\omega],R^n)$ it has a unique solution $v(t,x)\in C(\Omega,R^n)$ and for it the estimate holds*

$$\max_{t\in[0,T]}||v(t,x)|| \le K\max\Big(\max_{t\in[0,T]}||F(t,x)||, ||\Phi(x)||\Big), \tag{31}$$

where the constant K is independent of $F(t,x)$ and $\Phi(x)$, and $x\in[0,\omega]$.

Lemma 3. *If $v(t,x)$ is a solution to problem (29), (30), and the estimate holds*

$$||v||_0 \le K\max\Big(||F||_0, ||\Phi||_{0,1}\Big), \tag{32}$$

where K is a constant independent of the functions $F(t,x)$ and $\Phi(x)$, then for every $x\in[0,\omega]$ the inequality (31) is valid.

Denote by $\Omega_\eta = [0,T] \times [0,\eta]$ and $||u||_{0,\eta} = \max\limits_{(t,x)\in\Omega_\eta} ||u(t,x)||$.

Definition 4. *Boundary value problem (26)-(28) is called well-posed if for arbitrary $f(t,x) \in C(\Omega, R^n)$ and $\psi(t) \in C^1([0,T], R^n)$ and $\varphi(x) \in C([0,\omega], R^n)$ it has a unique classical solution $u(t,x)$ and this solution satisfies the following estimate*

$$\max\Big(||u||_{0,\eta}, \Big|\Big|\frac{\partial u}{\partial x}\Big|\Big|_{0,\eta}, \Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{0,\eta}\Big) \le \tilde{K} \max\Big(||f||_{0,\eta}, ||\psi||_{1,0}, \max_{x\in[0,\eta]} ||\varphi(x)||\Big),$$

where constant $\tilde{K}$ is independent of $f(t,x)$ and $\psi(t)$ and $\varphi(x)$ and $\eta \in [0,\omega]$.

Theorem 9. *The boundary value problem (26)-(28) is well-posed if and only if so is problem (29), (30).*

From Theorem 9 it follows that the well-posedness of problem (26)-(28) are equivalent to the well-posedness of problem (29), (30).

These results were extended to a nonlocal problem with an integral condition for system (28) (see [76]).

The problem of finding bounded solutions of system (26) and the families of systems (29) was solved in [35, 39, 41-43, 46, 48, 49, 58].

The parametrization method was further developed to nonlinear nonlocal problems for a system of hyperbolic equations [71, 73, 86, 88].

5. Boundary value problems for loaded and integro-differential equations

On the basis of the parametrization method, constructive algorithms were developed for finding solutions to various boundary value problems for integro-differential and loaded equations [35, 54, 57, 59-61, 68, 69, 91, 94].

In the interval $[0,T]$, consider the following linear two-point boundary value problem for an integro-differential equation:

$$\frac{dx}{dt} = A(t)x + \int_0^T K(t,s)x(s)ds + f(t), \qquad x \in \mathbb{R}^n, \tag{33}$$

$$Bx(0) + Cx(T) = d, \qquad d \in \mathbb{R}^n, \tag{34}$$

where $A(t)$ and $K(t,s)$ are continuous matrices on $[0,T]$ and $[0,T]\times[0,T]$, respectively; $f(t)$ is continuous on $[0,T]$.

It is well known that the basic techniques for analysis and solving boundary value problems for integro-differential equations are the Nekrasov method and the Green's function method. Nekrasov's method applies to problem (33), (34), if we assume the unique solvability of the second-kind Fredholm integral equation

$$x(t) = \int_0^T M(t,s)x(s)ds + F(t), \qquad t \in [0,T], \tag{35}$$

with the kernel $M(t,s) = \int\limits_0^t X(t)X^{-1}(\tau)K(\tau,s)d\tau$, where $X(t)$ is the fundamental matrix of the differential part of equation (33) and $F(t) \in C([0,T],\mathbb{R}^n)$. The Green's function method applies to problem (33), (34) under assumption that the boundary value problem for the differential part of (33) is uniquely solvable; i.e., this method assumes the unique solvability of problem (33), (34) with $K(t,s) = 0$.

However, the assumptions of neither Nekrasov's method nor Green's function method are necessary conditions for the solvability of problem (33), (34).

In [66], a coefficient criterion for the well-posedness of problem (33), (34) was established in terms of approximating boundary value problems for the loaded differential equation

$$\frac{dx}{dt} = A(t)x + \sum_{i=1}^{m} K_i(t)x(\theta_i) + f(t), \qquad x \in \mathbb{R}^n, \tag{35}$$

subject to condition (34), by the parameterization method.

In [72], Dzhumabaev proposed a method for solving the problem (33), (34) that is based on the parameterization method and properties of a fundamental matrix of the differential part of (33). The interval $[0,T]$ is divided into N equal parts with step size $h > 0$: $[0,T) = \bigcup\limits_{r=1}^{N} [(r-1)h, rh)$. Let $x_r(t)$ be the restriction of $x(t)$ to the rth subinterval $[(r-1)h, rh)$. The values of the solution at the left-endpoints of the subintervals are assumed as additional parameters $\lambda_r = x_r[(r-1)h]$. By the substitution $u_r(t) = x_r(t) - \lambda_r$ at every rth subinterval, the problem (33), (34) is reduced to the multi-point boundary value problem for a system of integro-differential equations with parameters

$$\frac{du_r}{dt} = A(t)u_r + A(t)\lambda_r + \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K(t,s)[u_j(s) + \lambda_j]ds + f(t), \qquad t \in [(r-1)h, rh), \tag{37}$$

$$u_r[(r-1)h] = 0, \qquad r = 1,2,...,N, \tag{38}$$

$$B\lambda_1 + C\lambda_N + C \lim_{t \to T-0} u_N(t) = d, \tag{39}$$

$$\lambda_p + \lim_{t \to ph-0} u_p(t) - \lambda_{p+1} = 0, \qquad p = 1,2,...,N-1. \tag{40}$$

The introduction of additional parameters resulted in the emergence of the initial data (38) for the unknown functions $u_r(t)$, $r = 1,2,...,N$. For fixed parameter values $\lambda \in \mathbb{R}^{nN}$, the system of functions $u[t] = (u_1(t), u_2(t), ..., u_N(t))$ is determined from problem (37), (38), which is a special Cauchy problem for the system of integro-differential equations. Problem (37), (38) is equivalent to the system of integral equations

$$u_r(t) = X(t) \int\limits_{(r-1)h}^{t} X^{-1}(\tau)A(\tau)d\tau\lambda_r + X(t) \int\limits_{(r-1)h}^{t} X^{-1}(\tau) \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K(\tau,s)[u_j(s) + \lambda_j]dsd\tau$$

$$+X(t) \int\limits_{(r-1)h}^{t} X^{-1}(\tau)f(\tau)d\tau, \qquad t \in [(r-1)h, rh), \qquad r = 1, 2, ..., N. \tag{41}$$

By solving (41), one can find the representations of $u_r(t)$ in terms of $\lambda \in \mathbb{R}^{nN}$ and $f(t)$. Substituting them into (39) and (40) yields a system of equations for finding the unknown parameters. Thus, when applying the parameterization method to problem (33), (34), one has to solve an auxiliary problem, namely, the special Cauchy problem (37), (38), or the equivalent system of integral equations (41). However, unlike the auxiliary problem of Nekrasov's method, the special Cauchy problem is uniquely solvable for any sufficiently small partition step size $h > 0$. Let a number $h_0 > 0$ satisfy the inequality

$$\sigma(h_0) = \beta T h_0 e^{\alpha h_0} < 1, \tag{42}$$

where $\beta = \max\limits_{(t,s)\in[0,T]\times[0,T]} ||K(t,s)||$ and $\alpha = \max\limits_{t\in[0,T]} ||A(t)||$. It was shown that, for any $h \in (0, h_0] : Nh = T$, system (41) is uniquely solvable. This property of the auxiliary problem of the parameterization method made it possible to establish solvability criteria for the boundary value problem considered.

Necessary and sufficient conditions for the solvability, including the unique solvability, of problem (33), (34) were obtained in terms of a matrix $Q_{*,*}(h)$ constructed via the fundamental matrix of the differential part of system (33), the matrices of boundary conditions (34), and the resolvent of an auxiliary Fredholm integral equation of the second kind.

In [78], a family of algorithms was proposed for solving problem (33), (34). The numerical parameters of the family are the partition step $h > 0 : Nh = T$, the number $\nu \in \mathbb{R}^n$ of iterated integrals used in the algorithm, and a nonnegative integer m specifying how many terms of the resolvent of the corresponding Fredholm integral equation of the second kind are used in the algorithm. The basic condition for the feasibility and convergence of the algorithm is that the matrix $Q_\nu^m(h)$ is invertible for chosen numerical parameters. The unknown parameters are found at the first stage of each step in the algorithm by using the invertibility of this matrix. The special Cauchy problem (37), (38) with the found parameter values is solved at the second stage of the algorithm. Necessary and sufficient conditions for the well-posedness of problem (33), (34) were established in terms of the input data without using the fundamental matrix or the resolvent.

In [82], the method and results of [72] were generalized to the case of an arbitrary partition. Let Δ_N denote a partition of $[0, T]$ into N parts: $t_0 = 0 < t_1 < \ldots < t_N = T$; the case of no partitioning is denoted by Δ_1. Each partition Δ_N is associated with a homogeneous Fredholm integral equation of the second kind. The partition Δ_N is called regular if the corresponding equation has only the trivial solution. The regularity of Δ_N leads to a unique solvability of the special Cauchy problem mentioned above. The solvability criteria for linear two-point boundary value problem for Eq. (33) obtained in [82] are applicable for arbitrary regular partition Δ_N. The algorithms of the parameterization method for solving linear boundary value problems for Fredholm integro-differential equations were offered in [86].

These results were extended to boundary value problems for impulsive integro-differential equations in [84].

6. New general solutions to linear Fredholm integro-differential equations and their applications in solving boundary value problems

It is known that Volterra integro-differential equations are solvable for any right-hand side and have classical general solutions. However, there exist linear loaded differential equations and Fredholm integro-differential equations that do not admit classical general solutions. The question arises as to whether it is possible to construct such general solutions that exist for all differential and integro-differential equations and would allow solving boundary value problems for these equations.

Dzhumabaev D.S. proposed a novel approach to the concept of the general solution for a linear ordinary Fredholm integro-differential equation based on the parametrization method in [97]. The domain interval is partitioned and the values of the solution at the left endpoints of the subintervals are considered as additional parameters. By introducing new unknown functions on the partition subintervals, a special Cauchy problem for a system of integro-differential equations with parameters is obtained. Using the solution of this problem, a new general solution of the linear Fredholm integro-differential equation was constructed.

Suppose Δ_N is a partition $t_0 = 0 < t_1 < \ldots < t_N = T$. Let $x(t)$ be a function, piecewise continuous on $[0, T]$ with the possible points of discontinuity: $t = t_p$, $p = 1, 2, ..., N-1$. Let $x_r(t)$ be the restriction of $x(t)$ to the rth subinterval $[t_{r-1}, t_r)$, i.e. $x_r(t) = x(t)$, $t \in [t_{r-1}, t_r)$, $r = 1, 2, \ldots, N$. For definiteness, assume that $x_r(t_{r-1}) = \lim\limits_{t \to t_{r-1}+0} x_r(t)$, $r = 1, 2, ..., N$. If $x(t)$ is piecewise continuously differentiable on $(0, T)$ and satisfies the Fredholm integro-differential equation (33) for each $t \in (0, T) \backslash \{t_p, p = 1, 2, ..., N-1\}$, then the system of its restrictions $x[t] = (x_1(t), ..., x_N(t))$ satisfies the following system of integro-differential equations:

$$\frac{dx_r}{dt} = A(t)x_r + \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} K(t,\tau)x_j(\tau)d\tau + f(t), \qquad t \in [t_{r-1}, t_r), \quad r = 1, 2, ..., N. \tag{43}$$

Let $C([0, T], \Delta_N, \mathbb{R}^{nN})$ denote the space of function systems $x[t] = (x_1(t), x_2(t), ..., x_N(t))$, where $x_r : [t_{r-1}, t_r) \to \mathbb{R}^n$ is continuous and has the finite left-sided limit $\lim\limits_{t \to t_r - 0} x_r(t)$ for any $r = 1, 2, ..., N$, with the norm $x[\Delta]_2 = \max\limits_{r=1,2,...,N} \sup\limits_{t \in [t_{r-1}, t_r)} ||x_r(t)||$.

A function system $x[t] = (x_1(t), x_2(t), ..., x_N(t)) \in C([0, T], \Delta_N, \mathbb{R}^{nN})$ is called a solution to the system of integro-differential equations (41) if the functions $x_r(t)$, $r = 1, 2, ..., N$, are continuously differentiable on (t_{r-1}, t_r) and satisfy equations (43).

Suppose that the function system $x^*[t] = (x_1^*(t), x_2^*(t), ..., x_N^*(t))$ is a solution to (43). Then the function $x^*(t)$, defined as $x^*(t) = x_r^*(t)$ for $t \in [t_{r-1}, t_r)$, $r = 1, 2, ..., N$, and $x^*(T) = \lim\limits_{t \to T-0} x_N^*(t)$, is piecewise continuously differentiable and consistent with Eq. (33) for $t \in (0,T)\backslash\{t_p, p = 1, 2, ..., N-1\}$. The introduction of the parameters $\lambda_r = x_r(t_{r-1})$, $r = 1, 2, ..., N$, and substituting new unknown functions $u_r(t) = x_r(t) - \lambda_r$ on each subinterval $[t_{r-1}, t_r)$, yields the system of integro-differential equations with parameters

$$\frac{du_r}{dt} = A(t)u_r + A(t)\lambda_r + \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} K(t,\tau)[u_j(\tau) + \lambda_j]d\tau + f(t), \quad t \in [t_{r-1}, t_r),\ r = 1, ..., N, \quad (44)$$

subject to the initial conditions

$$u_r(t_{r-1}) = 0, \qquad r = 1, 2, ..., N. \quad (45)$$

Problem (44), (45) is called a special Cauchy problem for the system of integro-differential equations with parameters. Without the interval's partition, problem (44), (45) is the Cauchy problem with the initial condition at $t = 0$ for the Fredholm integro-differential equation with parameter.

A solution to the special Cauchy problem (44), (45) with fixed values of parameters $\lambda_r^* \in \mathbb{R}^n$, $r = 1, ..., N$, is a function system $u[t, \lambda^*] = (u_1(t, \lambda^*), u_2(t, \lambda^*), ..., u_N(t, \lambda^*)) \in C([0,T], \Delta_N, \mathbb{R}^{nN})$, which satisfies the system of integro-differential equations (44) with $\lambda = \lambda^*$ and initial conditions (45).

Let $X_r(t)$ be a fundamental matrix of the differential equation $\frac{dx}{dt} = A(t)x$ on the interval $[t_{r-1}, t_r]$. Then problem (44),(45) is equivalent to the system of integral equations

$$u_r(t) = X_r(t) \int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau_1)A(\tau_1)d\tau_1\lambda_r + X_r(t) \int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau_1) \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} K(\tau_1,\tau)[u_j(\tau) + \lambda_j]d\tau d\tau_1$$

$$+ X_r(t) \int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau_1)f(\tau_1)d\tau_1, \qquad t \in [t_{r-1}, t_r), \quad r = 1, 2, ..., N. \quad (46)$$

Take an arbitrary partition Δ_N and consider the corresponding homogeneous Fredholm integral equation of the second kind

$$y(t) = \int\limits_0^T M(\Delta_N, t, \tau)y(\tau)d\tau, \qquad t \in [0,T], \quad (47)$$

where $M(\Delta_N,t,\tau)=\int\limits_{\tau}^{t_1} K(t,\tau_1)X_1(\tau_1)d\tau_1 X_1^{-1}(\tau)$, $t\in[0,T]$, $\tau\in[0,t_1]$,

$M(\Delta_N,t,\tau)=\int\limits_{\tau}^{t_j} K(t,\tau_1)X_j(\tau_1)d\tau_1 X_j^{-1}(\tau)$, $t\in[0,T]$, $\tau\in(t_{j-1},t_j]$, $j=2,...,N$.

Definition 5. *A partition Δ_N is called regular for Eq. (33) if the integral equation (47) has only the trivial solution.*

Let $\sigma([0,T])$ denote the set of regular partitions of the interval $[0,T]$. The set $\sigma([0,T])$ is not empty.

Definition 6. *The special Cauchy problem (44), (45) is called uniquely solvable if it has a unique solution for any pair $(f(t),\lambda)$ with $f(t)\in C([0,T],\mathbb{R}^n)$ and $\lambda\in\mathbb{R}^{nN}$.*

Definition 7. *Suppose that $\Delta_N\in\sigma([0,T])$, $\lambda=(\lambda_1,\lambda_2,...,\lambda_N)\in\mathbb{R}^{nN}$, and the function system $u[t,\lambda]=(u_1(t,\lambda),u_2(t,\lambda),...,u_N(t,\lambda))$ is a solution to the special Cauchy problem for the system of integro-differential equations with parameters (44), (45). Then the function $x(\Delta_N,t,\lambda)$ defined by the equalities $x(\Delta_N,t,\lambda)=\lambda_r+u_r(t,\lambda)$, $t\in[t_{r-1},t_r)$, $r=1,2,...,N$, and $x(\Delta_N,T,\lambda)=\lambda_N+\lim\limits_{t\to T-0}u_N(t,\lambda)$ is called the Δ_N general solution to the integro-differential equation (33).*

Theorem 10. *For any $\Delta_N\in\sigma([0,T])$, there exists a unique Δ_N general solution to the linear Fredholm integro-differential equation (33).*

In contrast to the classical general solution, the Δ_N general solution exists for all linear nonhomogeneous Fredholm integro-differential equations and contains N arbitrary parameters $\lambda_r\in\mathbb{R}^n$.

The concept of new general solution, introduced by Dzhumabaev, made it possible to derive the solvability criteria for the linear Fredholm integro-differential equations and boundary value problems for this equation. The method proposed consists the construction of Δ_N general solutions and solving linear algebraic equations with respect to parameters of those solutions. The Cauchy problems for ordinary differential equations and problems of evaluation of the definite integrals on the subintervals are used as auxiliary problems. Depending on the choice of methods for solving auxiliary problems, either numerical or approximate methods were obtained in order to solve the linear boundary value problems for Fredholm integro-differential equations [81, 93, 99].

The new general solution made it possible to propose new numerical and approximate methods for solving boundary value problems with and without parameter for nonlinear ordinary differential equations [101, 102, 104, 106, 108, 111]. These methods are based on the construction and solving a system of algebraic equations for arbitrary vectors of the new general solution. The coefficients and the right-hand sides of this system are determined using solutions of the Cauchy problems for ordinary differential equations on the subintervals. Using the new general solution, solvability criteria were established for boundary value problems for nonlinear ordinary differential equations.

The results and methods were extended to linear nonlocal boundary value problems for systems of loaded hyperbolic equations and Fredholm hyperbolic integro-differential equations [100].

The new approach to the general solution became the basis of methods for research and solving nonlinear boundary value problems for loaded differential and integro-differential equations [103, 105, 107, 109, 110, 112]. The methods are based on the construction and solving systems of nonlinear algebraic equations for arbitrary vectors of new general solutions. To solve nonlocal boundary value problems for nonlinear partial differential and integro-differential equations, a modification of Euler's broken lines method was developed.

These results were further extended to multipoint problems, periodic problems with impulse effects, and control problems for various classes of differential, loaded differential, integro-differential, and partial differential equations [75, 80, 87].

Conclusion

Dzhumabaev D.S. was a highly qualified expert in the theory of differential, integral and nonlinear operator equations, computer and mathematical modeling of applied problems. He has published over 300 papers in scientific journals, including authoritative periodicals like *Journal of Mathematical Analysis and Applications, Journal of Computational and Applied Mathematics, Mathematical Methods in Applied Sciences, Mathematical Notes, Computational Mathematics and Mathematical Physics, Differential Equations, Ukrainian Mathematical Journal, Journal of Integral Equations and Applications, Journal of Mathematical Sciences, Eurasian Mathematical Journal*, etc. The list of his major publications is given below.

The research findings were presented and discussed at many international symposia and conferences. His scientific results were widely recognized in Kazakhstan and at the international level by experts in the field of differential equations and computational mathematics. The scientific direction formed by Dzhumabaev D.S. has been further developed by his students, who successfully work at the Institute of Mathematics and Mathematical Modeling and leading universities in Kazakhstan.

In 1998, Dzhumabaev D.S. was awarded the title of professor (specialty 01.01.00 - Mathematics). Under his supervision, two doctoral, twenty candidate dissertations, and one PhD thesis were defended. He supervised five PhD students. In 2004-2005, Dzhumabaev D.S. was the chair of the Expert Commission on Mathematics and Computer Science of the Committee on Supervision and Certification in Education and Science of the Ministry Education and science of the Republic of Kazakhstan.

Professor Dzhumabaev made a great contribution to academic community. He led a scientific seminar on the qualitative theory of differential equations at the Institute of Mathematics and Mathematical Modeling. He was a scientific expert of the State Expertise of the Ministry of Education and Science of the Republic of Kazakhstan. For many years, Dzhumabaev D.S. was a member of Dissertation Councils at the Institute of Mathematics, Al-Farabi Kazakh National University, Abai Kazakh National Pedagogical

University, K.Zhubanov Aktobe Regional State University.

In 2014, at the invitation of the university authorities, Professor Dzhumabaev began to deliver lectures at the International University of Information Technology. He taught such courses as "Mathematical Analysis" , "Methods of solving linear and nonlinear boundary value problems for ordinary differential equations" , "Problems for integro-differential equations of processes with consequences" , "Boundary value problems, their applications and methods for solving" . It should be noted that his scientific results of recent years were obtained under the influence of teaching at the International University of Information Technology. While giving lectures and conducting practical classes, he realized with great clarity the importance of developing numerical methods for solving applied problems. Having set himself the goal of bringing to the final numerical implementation the theoretical results and algorithms of the parameterization method, he made a breakthrough in the field of mathematical and computer modeling. Under scientific supervision of Professor Dzhumabaev, master students and undergraduates of the International University of Information Technology carried out research in the area of numerical methods for solving boundary value problems for differential and integro-differential equations.

Professor Dzhumabaev chaired the Mathematics Section of Academic Council of the Institute of Mathematics and Mathematical Modeling. He was a member of the editorial board of the scientific journals *News of NAS RK. Series: Physics and Mathematics, Kazakh Mathematical Journal, Bulletin of Karaganda State University. Series: Mathematics.*

Dzhumabaev D.S. was awarded the lapel badge "For Contribution to the Development of Science and Technology" and the Certificate of Merit of the Ministry of Education and Science of the Republic of Kazakhstan.

Since 2018, Dzhumabaev D.S. headed the Department of Mathematical Physics and Mathematical Modeling at the Institute of Mathematics and Mathematical Modeling. In 2019, his research team, together with mathematicians from Ukraine, Belarus, Uzbekistan, Azerbaijan, Germany, and the Czech Republic, received funding from the European Union's Horizon 2020 research and innovation programme under EC grant agreement 873071-H2020-MSCA-PISE-2019 (Marie Sklodowska-Curie Research and Innovation Staff Exchange), project titled "Spectral Optimization: From Mathematics to Physics and Advanced Technology"(SOMPATY).

The first publication in the framework of this project is devoted to the application of the parameterization method to multipoint problems for Fredholm integro-differential equations and was published in *Kazakh Mathematical Journal* (2020, Vol. 20, No. 1).

At the end of 2019, having applied for the competition from the International University of Information Technology, Professor Dzhumabaev became the owner of the grant "The Best University Teacher 2019" of the Ministry of Education and Science of the Republic of Kazakhstan.

A prominent scientist, an outstanding teacher, and a talented organizer, Dulat

Syzdykbekovich Dzhumabaev passed away on February 20, 2020. He will be lovingly remembered by his wife Klara Kabdygalymovna, daughters Dana and Damira, son Anuar, and three grandchildren. His memory will live in the hearts of his friends, colleagues, as well as generations of grateful and adoring students. His research, scientific ideas and plans will be continued and implemented by his students.

The major publications by Dzhumabaev D.S.

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On the absolute stability of a program manifold of non-autonomous control systems with non-stationary nonlinearities

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Abstract. The absolute stability of a program manifold of non-autonomous control systems with nonstationary nonlinearities is considered. Conditions of the stability of the control systems with variable coefficients are investigated in the neighborhood of a given program manifold. Nonlinearities satisfy to the conditions of local quadratic connections and they are differentiable in all variables. Sufficient conditions of the absolute stability of the program manifold with respect to the given vector functions are obtained by constructing the Lyapunov function, in the type of "quadratic form plus an integral from nonlinearity". Estimates for the integral part of the Lyapunov function are obtained by representing nonlinearities in a linear form.

Keywords. Program manifold, absolute stability, non-stationary nonlinearity, non-autonomous control systems, Lyapunov functions, local quadratic connection.

Dedicated to the memory of
Professor Dulat Dzhumabaev

1 Introduction

The construction problem was formulated and solved for systems of ordinary differential equations on a given integral curve by Yerugin in [1]. Later, this problem was developed by Galiullin, Mukhametzyanov, Mukharlyamov and others (see [2-24]) to constructing of systems of differential equations by a given integral manifold, to solving various inverse problems of the dynamics, and to constructing systems of the program motion. The integral manifold is

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defined as an intersection of hypersurfaces. It should be noted that, the construction of stable systems developed into an independent theory. The problem of the construction of automatic control systems for a vector nonlinear function with locally quadratic relations was solved in [6, 7]. Analysis of works in this direction shows that an essential part of the literature is devoted to the study of the program manifold of control systems with constant coefficients (see [3-20]).

At the same time, in mathematical modelling of various physical, chemical, biological and environmental and other phenomena in most cases it leads to the need of the research of control systems with variable coefficients. This is the movement of a point of variable mass, moving objects in which there is a change in mass and momentum over time, in particular, jet thrust aircraft with variable mass (see [21 - 27]). Linearization of nonlinear equations of motion of automatic control systems with respect to non-stationary behaviour leads to the need to study linear systems with variable coefficients. Depending on the functional features of the control system and the technical means that implement the selected control scheme, the linearized equations have different specific features. In modern theory, when constructing stable dynamical systems with feedback for a given program manifold, it is extremely important to be able to determine the stability of the manifold itself with respect to some function.

We introduce into consideration the class of continuously-differentiable on time t and bounded in the norm matrices Ξ and define a program manifold as follows $\Omega\left(t\right)\equiv\omega(t,x)=0$, which is integral for the system

$$\dot{x}=f\left(t,x\right)-B(t)\xi,\quad \xi=\varphi\left(t,\sigma\right),\quad \sigma=P^{T}(t)\omega,\quad t\in I=[0,\ \infty)\,, \tag{1}$$

where $x\in R^{n}$ is a state vector of the object, $f\in R^{n}$ is a vector-function satisfying conditions of the existence of the solution $x(t)=0$; $B(t)\in\Xi^{n\times r},\ P(t)\in\Xi^{s\times r}$ are continuous matrices, $\omega\in R^{s}(s\leq n)$ is a vector, $\varphi(t,\sigma)\in R^{r}$ is a vector-function of control on deviation from given program manifold satisfying conditions of local quadratic connection

$$\varphi(t,0)=0\ \wedge\ \varphi^{T}(t,\sigma)\theta(t)(\sigma-K^{-1}(t)\varphi(t,\sigma))>0, \tag{2}$$

$$K_1(t)\leq\frac{d\varphi(t,\sigma)}{dt}\leq K_2(t), \tag{3}$$

$$\left[\theta(t)=diag\|\theta_1,\ldots,\theta_r\|\right]\in\Xi^{r\times r},\left[K(t)=K^{T}(t)\right]\in\Xi^{r\times r},$$

$$\left[K_i(t)=K_i^{T}(t)>0,\ i=1,2\right]\in\Xi^{r\times r}.$$

Note, that the following estimate

$$\|\omega\|^2\frac{\beta_1}{\nu_2}\leq\|\varphi\|^2\frac{\beta_1}{\nu_2}\leq\|\omega\|^2, \tag{4}$$

can be obtained from the condition (2), where $\beta_1, \nu_1; \beta_2, \nu_2$ are the smallest and the largest eigenvalues of matrices $P\theta P^T, \theta K^{-1}$.

In the space R^n we select the domain $G\left(R\right)$:

$$G\left(R\right) = \left\{(t,x) : t \in I \wedge \|\omega\left(t,x\right)\| \leq \rho < \infty\right\}. \tag{5}$$

Due to the fact that $\Omega(t)$ is the integral manifold for the system (1)-(2), we have

$$\dot{\omega} = \frac{\partial \omega}{\partial t} + Hf\left(t,x\right) = F(t,x,\omega),$$

where $H = \dfrac{\partial \omega}{\partial x}$ is the Jacobi matrix and $F(t,x,\omega)$ is a certain s-dimensional Erugin vector function, satisfying conditions $F(t,x,0) \equiv 0$ [1].

Taking into account that $\Omega(t)$ is the integral manifold for the system (1), and by choosing the Erugin function as the following

$$F(t,x,\omega) = -A(t)\omega, \tag{6}$$

where $-A(t) \in \Xi^{s\times s}$ is Hurwitz matrix and differentiating the manifold $\Omega(t)$ with respect to time t along the solutions of system (1), we get [2]:

$$\dot{\omega} = -A(t)\omega - H(t)B(t)\xi, \quad \xi = \varphi\left(t,\sigma\right), \quad \sigma = P^T(t)\omega, \tag{7}$$

$$\varphi(t,0) = 0 \ \wedge \ \varphi^T(t,\sigma)\theta(t)(\sigma - K^{-1}(t)\varphi(t,\sigma)) > 0, \tag{8}$$

$$K_1(t) \leq \frac{d\varphi(t,\sigma)}{dt} \leq K_2(t). \tag{9}$$

Definition 1. *A program manifold $\Omega(t)$ of the non-autonomous basic control system is called absolutely stable with respect to a vector-function ω, if it is asymptotically stable on the whole at all functions $\omega(t_0,x_0)$ and $\varphi(t,\sigma)$ satisfying the conditions (8), (9).*

Statement of the problem. *To get a condition of absolute stability of a program manifold $\Omega(t)$ of the non-autonomous basic control systems with non-stationary nonlinearity with respect to the given vector-function ω.*

First, we consider the following system with variable coefficients as a linear approximation of the system (7)-(8) with respect to the vector function ω :

$$\dot{\omega} = -A(t)\omega, \quad t \in I = [0,\infty). \tag{10}$$

The following theorem is valid [21]:

Theorem 1. Let the Erugin function $F(t,x,\omega)$ have the form (6). Then, for asymptotic stability in the whole of the program manifold $\Omega(t)$ of the linear system (10) with variable coefficients with respect to the vector function ω it is sufficient fulfillment of relations

$$L(t) = M(t)A^{-1}(t) >> 0 \wedge G(t) >> 0 \quad t \in I = [0,\infty),$$

where $G(t)$ is determined by the formula

$$G(t) = M(t) + M^T(t) - \frac{dL(t)}{dt}A^{-1}(t) - M(t)\frac{dA^{-1}(t))}{dt}.$$

Based on the generalized theorem of A.M. Lyapunov (see [6, p.226]), the following theorem is true:

The Basic Theorem 1. Let the Erugin function $F(t, x, \omega)$ have the form (6) and there exist a real, continuously differentiable function $V(t, \omega)$ in the domain (5) and positive-definite and allowing the highest limit in whole such that its derivative

$$-\frac{dV}{dt}\bigg|_{(7)} = W(t, \omega)$$

would be definitely positive for any function $\varphi(t, \sigma)$ satisfying conditions (8), (9), then the program manifold $\Omega(t)$ of basic control system is absolutely stable with respect to vector functions $\omega(t, x)$.

2 Absolute stability of program manifold of the basic control system

Theorem 2. *Let the Erugin function $F(t, x, \omega)$ have the form (6) and suppose that there exist matrices*

$$L(t) = L^T(t) > 0,\ \beta(t) = diag\,(\beta_1(t), \ldots, \beta_r(t)) > 0$$

and non-linear function $\varphi(t, \sigma)$ satisfies the conditions (8), (9). Then, for the absolute stability of the program manifold $\Omega\,(t)$ of the non-autonomous basic control system with non-stationary nonlinearity with respect to the vector function ω it is sufficient performing of the following conditions

$$l_1\|\omega\|^2 \leq V \leq l_2\|\omega\|^2, \tag{11}$$

$$\eta_1\|\omega\|^2 \leq W \leq \eta_2(\|\omega\|^2, \tag{12}$$

where l_1, l_2, η_1, η_2 are positive constants.

Proof. Let there exist matrices

$$L(t) = L^T(t) >> 0,\ \beta(t) = diag\,(\beta_1(t), \ldots, \beta_r(t)) >> 0,$$

then for the system (7) we can construct Lyapunov function of the form

$$V(\omega,\ \xi) = \omega^T L(t)\omega + \int\limits_0^\sigma \varphi^T(t, \sigma)\beta(t)d\sigma >> 0. \tag{13}$$

Taking into account the properties (4),(8), (9) and making the substitution

$$\varphi(t, \sigma) = h(t)\sigma \quad (0 \leq h(t) \leq K(t)),$$

we obtain the estimate

$$l_1(t)\|\omega\|^2 \leq V(\omega,\sigma) \leq l_2(t)\|\omega\|^2, \tag{14}$$

where

$$l_1(t) = l^{(1)}(t) + \lambda_1(t), \quad l_2(t) = l^{(2)}(t) + \lambda_2(t);$$

$$\lambda_1(t)\|\omega\|^2 \leq \int\limits_0^{\sigma} \varphi^T(t,\sigma)\beta(t)d\sigma \leq \lambda_2(t)\|\omega\|^2.$$

Here $l^{(1)}(t), \lambda_1(t); l^{(2)}(t), \lambda_2(t)$ are the smallest and the largest eigenvalues of the matrices $L, \Lambda : \Lambda = \Lambda P(t)\beta(t)P^T(t)$. The diagonal elements of the matrix Λ are divided by 2. On the basis of properties (4),(8), (9) the derivative of the function (13) takes the form

$$-\dot{V} = \omega^T G(t)\omega + 2\omega^T G_1(t)\xi + \xi^T G_2(t)\xi >> 0, \tag{15}$$

where

$$G(t) = -L(t) + A^T(t)L(t) + L(t)A(t) + P(t)\big[N_1(t) + N_2(t) - N_3(t)\big]P^T(t);$$

$$G_1(t) = \frac{1}{2}A^T(t)P(t)\beta(t) + L(t)H(t)B(t) + \frac{1}{2}\beta(t)P^T(t);$$

$$G_2(t) = \beta(t)P^T(t)H(t)\beta(t);$$

$$\int\limits_0^{\sigma} \frac{\partial\varphi^T(t,\sigma)}{\partial t}\beta(t)d\sigma \leq \int\limits_0^{\sigma} \sigma^T K_2(t)\beta(t)d\sigma = \int\limits_0^{\sigma} \sigma^T M_1(t)d\sigma$$

$$+ \int\limits_0^{\sigma} \sigma^T M_2(t)d\sigma = \sigma^T\big[N_1(t) + N_2(t)\big]\sigma;$$

$$\int\limits_0^{\sigma} \sigma^T M_3(t)d\sigma = \sigma^T N_3(t)\sigma; \quad M_1(t) = P^T(t)\big(P(t)P^T(t)\big)^{-1}\big(A^T(t)P^T(t) - \beta(t)K_2(t)\big)\beta;$$

$$M_2(t) = h(t)\beta(t)LP^T K_2(t)\beta(t); \quad M_3(t) = h(t)\frac{\partial\beta(t)}{\partial t}.$$

Due to the fact that $-\dot{V} >> 0$ the following estimates hold

$$q_1(t)\big(\|\omega\|^2 + \|\xi\|^2\big) \leq z^T Q(t)z \leq q_2(t)\big(\|\omega\|^2 + \|\xi\|^2\big), \tag{16}$$

where

$$Q(t) = \left\| \begin{matrix} G(t) & G_1(t) \\ G_1^T(t) & G_2(t) \end{matrix} \right\|, \quad z = \left\| \begin{matrix} \omega \\ \xi \end{matrix} \right\|,$$

$q_1(t), q_2(t)$ are the smallest and the largest eigenvalues of matrix $Q(t)$. Taking into account the estimates (4) from (16), we get

$$\eta_1(t)\|\omega\|^2 \leq -\dot{V}(\omega,\sigma) \leq \eta_2(t)\|\omega\|^2, \tag{17}$$

where

$$\eta_1(t) = q_1(t)\Big(1+\frac{\beta_1(t)}{\nu_2(t)}\Big); \quad \eta_2(t) = q_2(t)\Big(1+\frac{\beta_2(t)}{\nu_1(t)}\Big).$$

If we assume that

$$\eta_1 = \inf_t \eta_1(t) \wedge \eta_2 = \sup_t \eta_2(t), \tag{18}$$

$$l_1 = \inf_t l_1(t) \wedge l_2 = \sup_t l_2(t), \tag{19}$$

we receive estimates (11), (12).

Based on Theorem 1 and the Basic Theorem 1, we conclude: when the nonlinearity $\varphi(t,\sigma)$ satisfies the conditions (4), (8), (9), from estimates (14) and (17) it follows that conditions of Theorem 2 hold, in case (18) and (19), then the program manifold $\Omega(t)$ of the basic control system with non-stationary nonlinearities is absolutely stable with respect to the vector function ω. Therefore, the proof is complete.

3 Absolute stability of program manifold of the indirect control system

In the class of continuously-differentiable on time t and bounded in the norm matrices Ξ we consider the program manifold $\Omega\,(t) \equiv \omega(t,x) = 0$, which is integral for the system

$$\dot{x} = f\,(t,x) - B_1(t)\xi, \quad \dot{\xi} = \varphi\,(t,\sigma)\,, \quad \sigma = P^T(t)\omega - Q(t)\xi, \quad t \in I = [0,\ \infty)\,, \tag{20}$$

provided $Q(t) >> 0$, where $x \in R^n$ is a state vector of the object, $f \in R^n$ is a vector-function satisfying conditions of the existence of the solution $x(t) = 0$, $B_1(t) \in \Xi^{n\times r}$, $P(t) \in \Xi^{s\times r}$ are continuous matrices, $\omega \in R^s (s \leq n)$ is a vector, $\varphi(t,\sigma) \in R^r$ is a vector-function of control on deviation from given program manifold satisfying conditions of local quadratic connection

$$\varphi(t,0) = 0 \ \wedge \ 0 < \sigma^T\varphi(t,\sigma) \leq \sigma^T K(t)\sigma, \tag{21}$$

$$K_1(t) \leq \frac{d\varphi(t,\sigma)}{dt} \leq K_2(t), \tag{22}$$

$$\big[K_i(t) = K_i^T(t) > 0,\ i = 1,2\big] \in \Xi^{r\times r},$$
$$K(t) = diag\,\|\,k_1(t),\ldots,k_r(t)\|\ , \quad K(t) >> 0.$$

Taking into account that $\Omega(t)$ is the integral manifold for the system (20), and by choosing the Erugin function in the form (6) and differentiating the manifold $\Omega(t)$ with respect to time t along the solutions of system (20), we get [2]:

$$\begin{cases} \dot{\omega} = -A(t)\omega - B(t)\xi, \quad B((t) = H(t)B_1(t), \\ \dot{\xi} = \varphi\left(t, \sigma\right), \quad \sigma = P^T(t)\omega - Q(t)\xi \end{cases}, \tag{23}$$

$$\varphi(t, 0) = 0 \ \wedge \ 0 < \sigma^T \varphi(t, \sigma) \leq \sigma^T K(t)\sigma \tag{24}$$

$$K_1(t) \leq \frac{d\varphi(t, \sigma)}{dt} \leq K_2(t). \tag{25}$$

Definition 2. *A program manifold $\Omega(t)$ of the indirect control system is called absolutely stable with respect to a vector-function ω, if it is asymptotically stable on the whole at all functions $\omega(t_0, x_0)$ and $\varphi(t, \sigma)$ satisfying the conditions (24), (25).*

Statement of the problem. *To get the condition of absolute stability of a program manifold $\Omega(t)$ of the non-autonomous indirect control systems with non-stationary nonlinearity (24), (25) with respect to the given vector-function ω.*

Based on the generalized theorem of A.M. Lyapunov (see [6, p.226]), the following theorem is true:

The Basic Theorem 2. *Let the Erugin function $F(t, x, \omega)$ have the form (6) and there be a real, continuous differentiable function of $V(t, \omega)$ in the domain (5) and positive-definite and allowing the highest limit in whole such that its derivative*

$$-\frac{dV}{dt}\bigg|_{(23)} = W(t, \omega)$$

would be definitely positive for any function $\varphi(t, \sigma)$ satisfying conditions (24), (25), then the program manifold $\Omega(t)$ of non-autonomous indirect control system is absolutely stable with respect to vector functions $\omega(t, x)$.

Theorem 3. *Let the Erugin function $F(t, x, \omega)$ have the form (6), $Q(t) >> 0$ and suppose that there exist matrices*

$$L(t) = L^T(t) > 0, \ \beta(t) = diag\left(\beta_1(t), \ldots, \beta_r(t)\right) > 0$$

and non-linear function $\varphi(t, \sigma)$ satisfies the conditions (24), (25). Then, for the absolute stability of the program manifold $\Omega\left(t\right)$ with respect to the vector function ω it is sufficient fulfillment of the following conditions

$$l_1(\|\omega\|^2 + \|\xi\|^2) \leq V \leq l_2(\|\omega\|^2 + \|\xi\|^2), \tag{26}$$

$$g_1(\|\omega\|^2 + \|\xi\|^2) \leq W \leq g_2(\|\omega\|^2 + \|\xi\|^2), \tag{27}$$

where l_1, l_2, g_1, g_2 are positive constants.

Proof. Let there exist matrices

$$L(t) = L^T(t) > 0, \ \beta(t) = diag\left(\beta_1(t), \ldots, \beta_r(t)\right) > 0,$$

then for the system (23) we can construct a Lyapunov function of the form

$$V(\omega,\ \xi) = \omega^T L(t)\omega + \int\limits_0^\sigma \varphi^T(t,\sigma)\beta(t)d\sigma > 0. \tag{28}$$

The second term in (28) is equal to $J = \dfrac{\sigma^T h\beta(t)\sigma}{2}$ in the case $\varphi(t,\sigma) = h(t)\sigma, \quad h(t) \leq K(t)$.

For this case we have estimates:

$$l_1(t)\|z\|^2 \leq V(\omega,\xi) \leq l_2(t)\|z\|^2, \tag{29}$$

here $l_1(t), l_2(t)$ are real, positive, continuous, smallest and largest roots of the characteristic equation $\det \|\Lambda(t) - l(t)E\| = 0$:

$$\Lambda(t) = \left\| \begin{array}{cc} L_1(t) & L_2(t) \\ L_2^T(t) & L_3(t) \end{array} \right\|, \quad z = \left\| \begin{array}{c} \omega \\ \xi \end{array} \right\|;$$

$$L_1(t) = L(t) + P(t)K_0(t)P^T(t); \quad L_2(t) = P(t)K_0(t)Q(t);$$

$$L_3(t) = Q^T(t)K_0(t)Q(t);$$

$$\sigma^T K_0(t)\sigma = \int\limits_0^\sigma \sigma^T h(t)\beta(t)d\sigma; \quad h(t) \leq K(t);$$

$$K_0(t) = \left\| \begin{array}{ccc} \big(K_{11}^{(0)}\big)/2 & \dots & K_{1r}^{(0)} \\ . & . & . \\ K_{r1}^{(0)} & \dots & \big(K_{rr}^{(0)}\big)/2 \end{array} \right\|.$$

The derivative on time t of this function in view of the system (23) will take the following form

$$-\dot{V}(\omega,\xi) = \omega^T G(t)\omega + 2\omega^T G_1(t)\xi + \xi^T G_2\xi >> 0, \tag{30}$$

where

$$G(t) = A^T(t)L(t) + L(t)A(t) + F(t); \quad G_1(t) = L(t)B(t) + 1/2F_1(t); \quad G_2(t) = F_2(t);$$

$$F(t) = -D(t) - H_0(t) - P(t)N_1(t)P^T(t) - \dot{L}; \quad F_1(t) = -D_1(t) - H_1(t) + P(t)\big(N_1(t) + N_1^T\big)Q(t);$$

$$F_2(t) = -D_2(t) - H_2(t) - Q^T(t)N_1(t)Q(t);$$

$$D(t) = \dot{P}(t)K_2(t)\beta(t)P^T(t) - A^T(t)P(t)K_2(t)P^T(t);$$

$$D_1(t) = -\dot{P}(t)K_2(t)\beta(t)Q(t) - P(t)\beta(t)K_2(t)\dot{Q}(t) - P(t)\beta(t)K_2(t)P^T(t)B(t)$$

$$+A^T(t)P(t)K_2(t)\beta(t)Q(t) - P(t)\big(N_2(t) + N_2^T\big)Q(t);$$

$$D_2(t) = \dot{Q}(t)K_2(t)\beta(t)Q(t) + B^T(t)P(t)K_2(t)\beta(t)Q(t) - Q^T(t)N_2(t)Q(t);$$

$$\int\limits_0^\sigma \frac{\partial\varphi^T(t,\sigma)}{\partial t}\beta(t)d\sigma \leq \int\limits_0^\sigma \sigma^T K_2(t)\beta(t)d\sigma = \omega D(t)\omega^T + \omega^T D_1(t)\xi + \xi^T D_2(t)\xi;$$

$$\int\limits_0^\sigma \sigma^T M^{(2)}(t)d\sigma = \sigma^T N^2(t)\sigma; \quad \int\limits_0^\sigma \sigma^T M^{(1)}(t)d\sigma = \sigma^T N_1(t)\sigma;$$

$$M^{(2)}(t) = h(t)Q^T(t)K_2(t)\beta(t); \quad M^{(1)}(t) = h(t)\frac{\partial\beta(t)}{\partial t};$$

$$N_i(t) = \left\| \begin{array}{ccc} (m_{11}^{(i)})/2 & \dots & m_{1r}^{(i)} \\ . & . & . \\ m_{r1}^{(i)} & \dots & (m_{rr}^{(i)})/2 \end{array} \right\|, i = 1, 2;$$

$$\varphi^T(t,\sigma)\beta(t)\sigma = \omega^T H_0(t)\omega + \omega^T H_1(t)\xi + \xi^T H_2\xi;$$

$$H_0(t) = P(t)h(t)\beta(t) - (P^T(t) - P^T(t)A(t)) - P(t)h(t)\beta(t)Q(t)h(t)P^T(t);$$

$$H_1(t) = P(t)h(t)\beta(t)P^T(t)B(t) - P(t)h(t)\beta(t)Q(t) + P(t)h(t)\beta(t)Q(t)h(t)Q(t)$$
$$-(P^T(t) - P^T(t)A(t))\beta(t)h(t)Q(t) + P(t)h(t)Q^T(t)\beta(t)h(t)Q(t);$$

$$H_2(t) = Q(t)h(t)\beta(t)P^T(t)B(t) + Q^T(t)h(t)\beta(t)Q(t) - Q^T(t)h(t)\beta(t)Q(t)h(t)Q(t).$$

Based on inequality (30), the following estimates are valid

$$g_1(t)\|z\|^2 \leq -\dot{V} \leq g_2(t)\|z\|^2, \tag{31}$$

here $g_1(t), g_2(t)$ are real, positive, continuous, smallest and largest roots of the characteristic equation

$$\det\|\widetilde{G}(t) - g(t)E\| = 0, \quad \widetilde{G}(t) = \left\| \begin{array}{cc} G(t) & G_1(t) \\ G_1^T(t) & G_2(t) \end{array} \right\|.$$

If we assume that

$$g_1 = \inf_t g_1(t) \wedge g_2 = \sup_t g_2(t), \tag{32}$$

$$l_1 = \inf_t l_1(t) \wedge l_2 = \sup_t l_2(t), \tag{33}$$

we receive estimates (26), (27).

Based on Theorem 1 and the Basic Theorem 2, we conclude: when the nonlinearity $\varphi(\sigma)$ satisfies the conditions (24), (25), from estimates (29) and (31) it follows that conditions of

Theorem 2 hold, in case (32) and (33), then the program manifold $\Omega(t)$ of the non-autonomous indirect control system with non-stationary nonlinearities is absolutely stable with respect to the vector function ω. Therefore, the proof is complete.

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Жұматов С.С. СТАЦИОНАР ЕМЕС СЫЗЫҚСЫЗДЫҚТАРЫ БАР АВТОНОМДЫ ЕМЕС БАСҚАРУ ЖҮЙЕЛЕРІНІҢ БАҒДАРЛАМАЛЫҚ КӨПБЕЙНЕСІНІҢ ОРНЫҚТЫЛЫҒЫ

Стационар емес сызықсыздықтары бар автономды емес басқару жүйелерінің бағдарламалық көпбейнесінің абсолют орнықтылығы қарастырылады. Айнымалы коэффициентті басқару жүйелерінің орнықтылық шарттары берілген бағдарламалық көпбейне маңайында зерттелді. Сызықсыздықтар локалды квадраттық байланыс шарттарын қанағаттандырады және олар барлық айнымалылары бойынша дифференциалданады. Берілген вектор-функцияға қатысты бағдарламалық көпбейненің абсолют орнықтылығының жеткілікті шарттары "квадраттық форма қосу сызықсыздықтан алынған интеграл" түріндегі Ляпунов функциясын тұрғызу арқылы алынды. Ляпунов функциясының интегралдық бөлігінің бағалаулары сызықсыздықты сызықты форма түрінде кейіптеу арқылы алынды.

Кілттік сөздер. Бағдарламалық көпбейне, абсолют орнықтылық, стационар емес сызықсыздық, автономды емес басқару жүйелері, Ляпунов функциялары, локалды квадраттық байланыс.

Жуматов С.С. УСТОЙЧИВОСТЬ ПРОГРАММНОГО МНОГООБРАЗИЯ НЕАВТОНОМНЫХ СИСТЕМ УПРАВЛЕНИЙ С НЕСТАЦИОНАРНЫМИ НЕЛИНЕЙНОСТЯМИ

Рассматривается абсолютная устойчивость программного многообразия неавтономных систем управлений с нестационарными нелинейностями. Условия устойчивости систем управлений с переменными коэффициентами исследованы в окрестности заданного программного многообразия. Нелинейности удовлетворяют условиям локальной квадратичной связи и они дифференцируемы по всем переменным. Достаточные условия абсолютной устойчивости программного многообразия, относительно заданной вектор-функции получены с помощью построения функции Ляпунова типа "квадратичная форма плюс интеграл от нелинейности". Оценки интегральной части функции Ляпунова получены с помощью представления нелинейности в линейной форме.

Ключевые слова. Программное многообразие, абсолютная устойчивость, нестационарная нелинейность, неавтономные системы управлений, функции Ляпунова, локальная квадратичная связь.

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Numerical solution of multi-point boundary value problems for essentially loaded ordinary differential equations

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Abstract. A linear multi-point boundary value problem for essentially loaded ordinary differential equations is investigated. Using the properties of essentially loaded ordinary differential equation and assuming the invertibility of the matrix compiled through the coefficients at the values of the derivative of the desired function at load points, we reduce the considering problem to a multi-point boundary value problem for ordinary loaded differential equations. The method of parameterization is used for solving this problem. Numerical method for finding solution of the considering problem is suggested.

Keywords. Essentially loaded differential equation, multi-point condition, parameterization method, algorithm.

Dedicated to the bright memory of an outstanding scientist,
Doctor of Physical and Mathematical Sciences, Professor,
our scientific supervisor Dzhumabaev Dulat Syzdykbekovich

1 Introduction

Loaded differential equations are used in solving the problems of long-term forecasting and regulation of the level of groundwater and soil moisture [1]-[3]. Many phenomena in complex evolutionary systems with memory substantially depend on the background of this system. These phenomena are usually described by loaded differential equations. Note that loaded

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differential equations in the literature are also called boundary differential equations [4]. Also, a loaded differential equation was called a differential equation that includes the values of the desired function and its derivatives at fixed points in the domain [5]. In [6], [7], loaded differential equations are interpreted as perturbations of differential equations. It is worth paying attention to the works [8]-[11], where a numerical method for solving the problem of optimal control of a system of linear, phase-variable, loaded ordinary differential equations of the first order with the Cauchy condition and with undivided multipoint conditions is proposed. In [12], [13], a class of loaded ordinary differential equations with non-local integral boundary conditions is studied in terms of an abstract operator equation. In [14] numerical solution of systems of loaded ordinary differential equations with multipoint conditions is investigated.

When studying a moving observation point in feedback devices, essentially loaded differential equations often appear, where the order of the derivative in the loaded term is equal to or higher than the order of the differential part of the equation. In contrast to the previously studied loaded differential equations, the loaded term in the equation will not be a certain perturbation of its differential part [4]. Various types of problems for essentially loaded parabolic equations and loaded equations of hyperbolic type of the first order were investigated in [4], [15]-[17]. In these works, new properties are obtained for loaded differential equations, containing as loaded terms the values of derivatives, the order of which is equal to the order of the differential part.

In the present paper, a linear multi-point boundary value problem for essentially loaded ordinary differential equations is investigated. The significance is that the loading members of the equation appear in the form of derivatives of solutions at loaded points of the interval i.e., the order of the loaded term is equal to the order of the differential part of the equation. The presence of derivatives of solutions in loaded points has a strong influence on the properties of equations.

We consider a linear multi-point boundary value problem for essentially loaded ordinary differential equations

$$\frac{dx}{dt} = A_0(t)x + \sum_{j=1}^{N} A_j(t)\dot{x}(\theta_j) + f(t), \quad t \in (0,T), \tag{1}$$

$$\sum_{i=0}^{N+1} C_i x(\theta_i) = d, \quad d \in R^n, \quad x \in R^n, \tag{2}$$

where $(n \times n)$-matries $A_i(t)$, $(i = \overline{0,N})$, and n-vector-function $f(t)$ are continuous on $[0,T]$, C_i $(i = \overline{0,N+1})$ are constant $(n \times n)$-matrices, and $0 = \theta_0 < \theta_1 < \theta_2 < \ldots < \theta_{N-1} < \theta_N < \theta_{N+1} = T$; $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

Let $C([0,T],R^n)$ denote the space of continuous on $[0,T]$ functions $x(t)$ with norm $||x||_1 = \max\limits_{t\in[0,T]} ||x(t)||$.

A solution to problem (1), (2) is a continuously differentiable on $(0,T)$ function $x(t) \in C([0,T],R^n)$ satisfying the essentially loaded differential equations (1) and multi-point condition (2).

We can find the values of derivatives at the loading points $t=\theta_j$, $j=\overline{1,N}$, from the system of differential equations (1). Using the system (1), we consequentially define $\dot{x}(\theta_j)$, $j=\overline{1,N}$:

$$\dot{x}(\theta_k) - \sum_{j=1}^{N} A_j(\theta_k)\dot{x}(\theta_j) = A_0(\theta_k)x(\theta_k) + f(\theta_k), \quad k=\overline{1,N}. \tag{3}$$

We can rewrite (3) in the following form

$$G(\theta)\mu = R(\theta). \tag{4}$$

Here $G(\theta) = \Big(G_{p,k}(\theta)\Big)$, $p,k=\overline{1,N}$, i.e.

$$G_{p,k}(\theta) = -A_k(\theta_p), \quad p\neq k, \quad p,k=\overline{1,N}, \quad G_{p,p}(\theta) = I - A_p(\theta_p),$$

where I is the identity matrix of dimension n,

$$\mu = \Big(\dot{x}(\theta_1), \dot{x}(\theta_2), \ldots, \dot{x}(\theta_N)\Big)', \quad R(\theta) = \Big(R_1(\theta), R_2(\theta), \ldots, R_N(\theta)\Big)',$$

$$R_k(\theta) = A_0(\theta_k)x(\theta_k) + f(\theta_k), \quad k=\overline{1,N}.$$

We assume that the matrix $G(\theta)$ is invertible. Denote by $S(\theta)$ the inverse matrix $G(\theta)$, i.e. $S(\theta) = [G(\theta)]^{-1}$, where $S(\theta) = s_{p,k}(\theta)$, $p,k=\overline{1,N}$. Then from (4) we can uniquely determine μ: $\mu = [G(\theta)]^{-1}R(\theta) = S(\theta)R(\theta)$.

Thus, the components of the vector μ allow us to find the values of the derivative $x(t)$ at the points $t=\theta_j$, $j=\overline{1,N}$.

We consider a linear multi-point boundary value problem for loaded differential equations

$$\frac{dx}{dt} = A_0(t)x + \sum_{j=1}^{N} B_j(t)x(\theta_j) + F(t), \quad t\in(0,T), \tag{5}$$

$$\sum_{i=0}^{N+1} C_i x(\theta_i) = d, \quad d\in R^n, \quad x\in R^n, \tag{6}$$

where $B_j(t) = \sum\limits_{p=1}^{N} A_p(t)s_{p,j}(\theta)A_0(\theta_j), \quad j=\overline{1,N},$

$F(t) = \sum\limits_{p=1}^{N} \sum\limits_{k=1}^{N} A_p(t) s_{p,k}(\theta) f(\theta_k) + f(t).$

We use the approach offered in [18]-[24] to solve the boundary value problem (5), (6). This approach is based on the algorithms of the parameterization method [25] and numerical methods for solving Cauchy problems.

2 Scheme of parametrization method

The interval $[0, T]$ is divided into subintervals load points:
$[0, T) = \bigcup\limits_{r=1}^{N+1} [\theta_{r-1}, \theta_r).$

Define the space $C([0, T], \theta_N, R^{n(N+1)})$ of system functions $x[t] = (x_1(t), x_2(t), \ldots, x_{N+1}(t))$, where x_r: $[\theta_{r-1}, \theta_r) \to R^n$ are continuous on $[\theta_{r-1}, \theta_r)$ and have finite left-sided limits $\lim\limits_{t \to \theta_r - 0} x_r(t)$ for all $r = \overline{1, N+1}$, with the norm $||x[\cdot]||_2 = \max\limits_{r=\overline{1,N+1}} \sup\limits_{t \in [\theta_{r-1}, \theta_r)} ||x_r(t)||.$

The restriction of the function $x(t)$ to the r-th interval $[\theta_{r-1}, \theta_r)$ is denoted by $x_r(t)$, i.e. $x_r(t) = x(t)$ for $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N+1}$. Then we reduce problem (5), (6) to the equivalent multi-point boundary value problem

$$\frac{dx_r}{dt} = A_0(t) x_r + \sum_{j=1}^{N} B_j(t) x_{j+1}(\theta_j) + F(t), \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, N+1}, \tag{7}$$

$$\sum_{i=0}^{N} C_i x_{i+1}(\theta_i) + C_{N+1} \lim_{t \to T-0} x_{N+1}(t) = d, \tag{8}$$

$$\lim_{t \to \theta_p - 0} x_p(t) = x_{p+1}(\theta_p), \qquad p = \overline{1, N}, \tag{9}$$

where (9) are conditions for matching the solution at the interior points of partition.

The solution of problem (7)–(9) is a system of functions $x^*[t] = (x_1^*(t), x_2^*(t), \ldots, x_{N+1}^*(t)) \in C([0, T], \theta_N, R^{n(N+1)})$, where functions $x_r^*(t)$, $r = \overline{1, N+1}$, are continuously differentiable on $[\theta_{r-1}, \theta_r)$, satisfy system (7) and conditions (8), (9).

Problems (5), (6) and (7)–(9) are equivalent. If a system of functions $\widetilde{x}[t] = (\widetilde{x}_1(t), \widetilde{x}_2(t), \ldots, \widetilde{x}_{N+1}(t)) \in C([0, T], \theta_N, R^{n(N+1)})$, is a solution of problem (7)–(9), then the function $\widetilde{x}(t)$ defined by the equalities $\widetilde{x}(t) = \widetilde{x}_r(t)$, $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N+1}$, $\widetilde{x}(T) = \lim\limits_{t \to T-0} \widetilde{x}_{N+1}(t)$, is a solution of the original problem (5), (6). Conversely, if $x(t)$ is a solution of problem (5), (6), then the system of functions $x[t] = (x_1(t), x_2(t), \ldots, x_{N+1}(t))$, where $x_r(t) = x(t)$, $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N+1}$, and $\lim\limits_{t \to T-0} x_{N+1}(t) = x(T)$, is a solution of problem (7)–(9).

We introduce additional parameters $\lambda_r = x_r(\theta_{r-1})$, $r=\overline{1,N+1}$. Making the substitution $x_r(t) = u_r(t)+\lambda_r$ on every r-th interval $[\theta_{r-1},\theta_r)$, $r=\overline{1,N+1}$, we obtain multi-point boundary value problem with parameters

$$\frac{du_r}{dt} = A_0(t)(u_r+\lambda_r) + \sum_{j=1}^{N} B_j(t)\lambda_{j+1} + F(t), \quad t\in[\theta_{r-1},\theta_r), \tag{10}$$

$$u_r(\theta_{r-1}) = 0, \qquad r=\overline{1,N+1}, \tag{11}$$

$$\sum_{i=0}^{N} C_i\lambda_{i+1} + C_{N+1}\lambda_{N+1} + C_{N+1}\lim_{t\to T-0} u_{N+1}(t) = d, \tag{12}$$

$$\lambda_p + \lim_{t\to\theta_p-0} u_p(t) = \lambda_{p+1}, \qquad p=\overline{1,N}. \tag{13}$$

A pair $(u^*[t],\lambda^*)$, with elements $u^*[t] = \big(u_1^*(t),u_2^*(t),\ldots,u_{N+1}^*(t)\big) \in C([0,T],\theta_N,R^{n(N+1)})$, $\lambda^* = (\lambda_1^*,\lambda_2^*,\ldots,\lambda_{N+1}^*) \in R^{n(N+1)}$, is said to be a solution to problem (10)–(13) if the functions $u_r^*(t)$, $r=\overline{1,N+1}$, are continuously differentiable on $[\theta_{r-1},\theta_r)$ and satisfy (10) and additional conditions (12), (13) with $\lambda_j=\lambda_j^*$, $j=\overline{1,N+1}$, and initial conditions (11).

Problems (5), (6) and (10)–(13) are equivalent. If the $x^*(t)$ is a solution of problem (5), (6), then the pair $(u^*[t],\lambda^*)$, where $u^*[t] = \Big(x^*(t)-x^*(\theta_0), x^*(t)-x^*(\theta_1),\ldots,x^*(t)-x^*(\theta_N)\Big)$, and $\lambda^* = \Big(x^*(\theta_0),x^*(\theta_1),\ldots,x^*(\theta_N)\Big)$, is a solution to problem (10)–(13). Conversely, if the pair $(\widetilde{u}[t],\widetilde{\lambda})$ with elements $\widetilde{u}[t] = \big(\widetilde{u}_1(t),\widetilde{u}_2(t),\ldots,\widetilde{u}_{N+1}(t)\big) \in C([0,T],\theta_N,R^{n(N+1)})$, $\widetilde{\lambda} = (\widetilde{\lambda}_1,\widetilde{\lambda}_2,\ldots,\widetilde{\lambda}_{N+1}) \in R^{n(N+1)}$, is a solution to problem (10)–(13), then the function $\widetilde{x}(t)$ defined by the equalities $\widetilde{x}(t) = \widetilde{u}_r(t)+\widetilde{\lambda}_r$, $t\in[\theta_{r-1},\theta_r)$, $r=\overline{1,N+1}$, will be the solution of the original problem (5), (6).

Let $X_r(t)$ be a fundamental matrix to the differential equation $\dfrac{dx}{dt} = A(t)x$ on $[\theta_{r-1},\theta_r]$, $r=\overline{1,N+1}$.

Then the unique solution to the Cauchy problem for the system of ordinary differential equations (10), (11) at the fixed values $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{N+1})$, has the following form

$$u_r(t) = X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)A_0(\tau)d\tau\lambda_r + X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\sum_{j=1}^{N} B_j(\tau)d\tau\lambda_{j+1}$$

$$+X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)F(\tau)d\tau, \quad t\in[\theta_{r-1},\theta_r), \quad r=\overline{1,N+1}. \tag{14}$$

Substituting the corresponding right-hand sides of (14) into the conditions (12), (13), we obtain a system of linear algebraic equations with respect to the parameters λ_r, $\quad r=\overline{1,N+1}$:

$$\sum_{i=0}^{N} C_i\lambda_{i+1}+C_{N+1}\lambda_{N+1}+C_{N+1}X_{N+1}(T)\int\limits_{\theta_N}^{T}X_{N+1}^{-1}(\tau)A_0(\tau)d\tau\lambda_{N+1}$$

$$+C_{N+1}X_{N+1}(T)\int\limits_{\theta_N}^{T}X_{N+1}^{-1}(\tau)\sum_{j=1}^{N}B_j(\tau)d\tau\lambda_{j+1}$$

$$=d-C_{N+1}X_{N+1}(T)\int\limits_{\theta_N}^{T}X_{N+1}^{-1}(\tau)F(\tau)d\tau, \tag{15}$$

$$\lambda_p+X_p(\theta_p)\int\limits_{\theta_{p-1}}^{\theta_p}X_p^{-1}(\tau)A_0(\tau)d\tau\lambda_p+X_p(\theta_p)\int\limits_{\theta_{p-1}}^{\theta_p}X_p^{-1}(\tau)\sum_{j=1}^{N}B_j(\tau)d\tau\lambda_{j+1}$$

$$-\lambda_{p+1}=-X_p(\theta_p)\int\limits_{\theta_{p-1}}^{\theta_p}X_p^{-1}(\tau)F(\tau)d\tau, \qquad p=\overline{1,N}. \tag{16}$$

We denote the matrix corresponding to the left-hand side of the system of equations (15), (16) by $Q_*(\theta)$ and write the system in the form

$$Q_*(\theta)\lambda=F_*(\theta), \quad \lambda\in R^{n(N+1)}, \tag{17}$$

where $\quad F_*(\theta)=\Big(d-X_{N+1}(T)\int\limits_{\theta_N}^{T}X_{N+1}^{-1}(\tau)F(\tau)d\tau, -X_1(\theta_1)\int\limits_{0}^{\theta_1}X_1^{-1}(\tau)F(\tau)d\tau,$

$$\ldots,-X_N(\theta_N)\int\limits_{\theta_{N-1}}^{\theta_N}X_N^{-1}(\tau)F(\tau)d\tau\Big)'.$$

The solution of the system (17) is a vector $\lambda^*=(\lambda_1^*,\lambda_2^*,...,\lambda_{N+1}^*)\in R^{n(N+1)}$, consists of the values of the solutions of the original problem (5), (6) in the initial points of subintervals, i.e. $\lambda_r^*=x^*(\theta_{r-1})$, $\quad r=\overline{1,N+1}$.

Further we consider the Cauchy problems for ordinary differential equations on subintervals

$$\frac{dz}{dt}=A(t)z+P(t), \quad z(\theta_{r-1})=0, \quad t\in[\theta_{r-1},\theta_r], \quad r=\overline{1,N+1}, \tag{18}$$

where $P(t)$ is either $(n \times n)$-matrix, or n-vector, both continuous on $[\theta_{r-1}, \theta_r], r = \overline{1, N+1}$. Consequently, solution to problem (18) is a square matrix or a vector of dimension n. Denote by $a(P, t)$ the solution to the Cauchy problem (18). Obviously,

$$a(P,t) = X_r(t) \int\limits_{\theta_{r-1}}^{t} X_r^{-1}(\tau)P(\tau)d\tau, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N+1},$$

where $X_r(t)$ is a fundamental matrix of differential equation (18) on the rth interval.

3 Numerical implementation

We offer the following numerical implementation of algorithm based on the Runge-Kutta method of the 4-th order.

1. Suppose we have a partition: $0 = \theta_0 < \theta_1 < ... < \theta_N < \theta_{N+1} = T$. Divide each r-th interval $[\theta_{r-1}, \theta_r]$, $r = \overline{1, N+1}$, into N_r parts with step $h_r = (\theta_r - \theta_{r-1})/N_r$. Assume on each interval $[\theta_{r-1}, \theta_r]$, $r = \overline{1, N+1}$, the variable $\widehat{\theta}$ takes its discrete values: $\widehat{\theta} = \theta_{r-1}$, $\widehat{\theta} = \theta_{r-1} + h_r$, ..., $\widehat{\theta} = \theta_{r-1} + (N_r - 1)h_r$, $\widehat{\theta} = \theta_r$, and denote by $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1, N+1}$, the set of such points.

2. Solving the following Cauchy problems for ordinary differential equations

$$\frac{dz}{dt} = A_0(t)z + A_0(t), \qquad z(\theta_{r-1}) = 0, \qquad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N+1},$$

$$\frac{dz}{dt} = A_0(t)z + B_j(t), \qquad z(\theta_{r-1}) = 0, \qquad t \in [\theta_{r-1}, \theta_r], \quad j = \overline{1, N},$$

$$\frac{dz}{dt} = A_0(t)z + F(t), \qquad z(\theta_{r-1}) = 0, \qquad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N+1},$$

by using the Runge-Kutta method of the 4-th order, we find the values of $(n \times n)$-matrices $a_r(A_0, \widehat{\theta})$, $a_r(B_j, \widehat{\theta})$, $j = \overline{1, N}$, and n-vector $a_r(F, \widehat{\theta})$ on $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1, N+1}$.

3. Construct the system of linear algebraic equations with respect to parameters

$$Q_*^{\widetilde{h}}(\theta)\lambda = F_*^{\widetilde{h}}(\theta), \quad \lambda \in R^{n(N+1)}. \tag{19}$$

Solving the system (19), we find $\lambda^{\widetilde{h}}$. As noted above, the elements of $\lambda^{\widetilde{h}} = (\lambda_1^{\widetilde{h}}, \lambda_2^{\widetilde{h}}, ..., \lambda_{N+1}^{\widetilde{h}})$ are the values of approximate solution to problem (5), (6) in the starting points of subintervals: $x^{\widetilde{h}_r}(\theta_{r-1}) = \lambda_r^{\widetilde{h}}$, $r = \overline{1, N+1}$.

4. To define the values of approximate solution at the remaining points of the set $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1, N+1}$, we solve the Cauchy problems

$$\frac{dx}{dt} = A_0(t)x + \sum_{j=1}^{N} B_j(t)\lambda_{j+1}^{\widetilde{h}} + F(t),$$

$$x(\theta_{r-1}) = \lambda_r^{\widetilde{h}}, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N+1}.$$

And the solutions to Cauchy problems are found by the Runge-Kutta method of the 4-th order. Thus, the algorithm allows us to find the numerical solution to the problem (5), (6). We can see that the solution of boundary value problem (5), (6) is also a solution of boundary value problem (1), (2), when the matrix $G(\theta)$ is invertible.

To illustrate the proposed approach for the numerical solving linear multi-point boundary value problem for essentially loaded differential equations (1), (2) on the basis of parameterization method, let us consider the following example.

4 Example

We consider a linear multi-point boundary value problem for essentially loaded differential equations

$$\frac{dx}{dt} = A_0(t)x + \sum_{j=1}^{3} A_j(t)\dot{x}(\theta_j) + f(t), \quad t \in (0,1), \tag{20}$$

$$\sum_{i=0}^{4} C_i x(\theta_i) = d, \quad d \in R^2, \quad x \in R^2. \tag{21}$$

Here $\theta_0 = 0$, $\theta_1 = \frac{1}{4}$, $\theta_2 = \frac{1}{2}$, $\theta_3 = \frac{3}{4}$, $\theta_4 = T = 1$,

$$A_0(t) = \begin{pmatrix} t^3 & 3t \\ 8 & t-4 \end{pmatrix}, \quad A_1(t) = \begin{pmatrix} 1 & t+1 \\ 2 & t-3 \end{pmatrix}, \quad A_2(t) = \begin{pmatrix} t & t^2+1 \\ 3t & 0 \end{pmatrix},$$

$$A_3(t) = \begin{pmatrix} 0 & t+1 \\ t^3 & 5t \end{pmatrix}, \quad C_0 = \begin{pmatrix} 2 & -4 \\ 6 & -1 \end{pmatrix}, \; C_1 = \begin{pmatrix} 3 & 1 \\ 0 & -3 \end{pmatrix}, \; C_2 = \begin{pmatrix} 4 & -6 \\ 9 & 3 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 2 & 1 \\ 5 & -3 \end{pmatrix}, \quad C_4 = \begin{pmatrix} -6 & 2 \\ 8 & 3 \end{pmatrix}, \quad d = \begin{pmatrix} 3e + 4e^2 + 2e^3 - 6e^4 - \frac{45}{8} \\ 9e^2 + 5e^3 + 8e^4 + \frac{701}{8} \end{pmatrix},$$

$$f(t) = \begin{pmatrix} 4e^{4t} - 5t - 4e - 4te^2 + 3t^2 - 3t^3 - 5t^4 - t^3e^{4t} \\ 5t^2 - 8e^{4t} - 8e - 12te^2 - 59t - 6t^3 - 4t^3e^3 - \frac{25}{2} \end{pmatrix}.$$

We use the proposed numerical implementation of algorithm for solving (20), (21). Accuracy of solution depends on the accuracy of solving the Cauchy problem on subintervals. We provide the results of the numerical implementation of algorithm by partitioning the subintervals $[0, 0.25]$, $[0.25, 0.5]$, $[0.5, 0.75]$, $[0.75, 1]$ with step $h = 0.025$.

The results of calculations of numerical solutions at the partition points are presented in Table 1.

The exact solution of the problem (20), (21) is $x^*(t) = \begin{pmatrix} e^{4t} + 5t \\ t^2 - t \end{pmatrix}$.

For the difference of the corresponding values of the exact and constructed solutions of the problem the following estimate is true:

$$\max_{j=\overline{0,40}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.0002.$$

Table 1. Results received by using MathCad15.

t	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$	t	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$
0	1.000053121	0.000016266	0.5	9.889091935	-0.249985381
0.025	1.230222697	-0.024353848	0.525	10.791205222	-0.249364093
0.05	1.471453221	-0.047474822	0.55	11.775048257	-0.247493163
0.075	1.724907995	-0.069346569	0.575	12.849216621	-0.244372621
0.1	1.99187266	-0.089969009	0.6	14.023209899	-0.240002504
0.125	2.273768066	-0.109342072	0.625	15.307526754	-0.234382854
0.15	2.572164491	-0.127465695	0.65	16.713770002	-0.227513725
0.175	2.888797358	-0.144339823	0.675	18.254762737	-0.219395173
0.2	3.225584605	-0.159964406	0.7	19.944676666	-0.210027267
0.225	3.584645878	-0.174339402	0.725	21.799173942	-0.199410088
0.25	3.968323749	-0.187464777	0.75	23.835563921	-0.187543725
0.275	4.379207157	-0.199340499	0.775	26.072976392	-0.174428284
0.3	4.820157324	-0.209966545	0.8	28.532553035	-0.160063889
0.325	5.294336387	-0.219342897	0.825	31.237659013	-0.144450679
0.35	5.80523905	-0.22746954	0.85	34.214116814	-0.127588821
0.375	6.356727557	-0.234346466	0.875	37.49046469	-0.109478506
0.4	6.953070347	-0.239973672	0.9	41.098242268	-0.090119958
0.425	7.598984775	-0.244351158	0.925	45.072306212	-0.069513439
0.45	8.299684327	-0.247478931	0.95	49.451179069	-0.047659257
0.475	9.060930792	-0.249357	0.975	54.277434806	-0.024557777
0.5	9.889091935	-0.249985381	1	59.598124901	-0.000209426

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Қадырбаева Ж.М., Қаракенова С.Г. ЕЛЕУЛІ ТҮРДЕ ЖҮКТЕЛГЕН ЖӘЙ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ҮШІН КӨП НҮКТЕЛІ ШЕТТІК ЕСЕПТІҢ САНДЫҚ ШЕШІМІ

Елеулі түрде жүктелген жәй дифференциалдық теңдеулер үшін сызықтық көп нүктелі шеттік есеп зерттеледі. Жүктелген жәй дифференциалдық теңдеудің қасиеттерін пайдалана отырып және жүктеу нүктелеріндегі ізделінді функцияның туындысының мәндерінің коэффициенттері бойынша құрастырылған матрицаның қайтымдылығын ескере отырып, қарастырып отырған есепті жүктелген жәй дифференциалдық теңдеулер үшін көп нүктелі шеттік есептке келтіреміз. Бұл есепті шешу үшін параметрлеу әдісі қолданылады. Қарастырылып отырған есептің шешімін табудың сандық әдісі ұсынылған.

Кілттік сөздер. Елеулі түрде жүктелген дифференциалдық теңдеу, көп нүктелі шарт, параметрлеу әдісі, алгоритм.

Кадирбаева Ж.М., Каракенова С.Г. ЧИСЛЕННОЕ РЕШЕНИЕ МНОГОТОЧЕЧНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ СУЩЕСТВЕННО НАГРУЖЕННЫХ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Исследуется линейная многоточечная краевая задача для существенно нагруженных обыкновенных дифференциальных уравнений. Используя свойства существенно нагруженного обыкновенного дифференциального уравнения и допуская обратимость матрицы, составленной по коэффициентам при значениях производной искомой функции в точках нагрузки, мы сводим рассматриваемую задачу к многоточечной краевой задаче для нагруженного обыкновенного дифференциального уравнения. Для решения этой проблемы используется метод параметризации. Предложен численный метод нахождения решения рассматриваемой задачи.

Ключевые слова. Существенно нагруженное дифференциальное уравнение, многоточечное условие, метод параметризации, алгоритм.

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On conditions of solvability of a nonlinear boundary value problem for a system of differential equations with a delay argument

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Abstract. In this paper, we consider a boundary value problem with essentially nonlinear two-point boundary conditions for a system of linear differential equations with a delay argument. To study this problem, we use the parametrization method of D.S. Dzhumabaev with a modified algorithm. The application of the parametrization method leads to a nonlinear operator equation, which is solved using the amplification of the sharper version of the local Hadamard theorem. A theorem on sufficient conditions for the existence of an isolated solution of the problem is proved.

Keywords. Boundary value problems, equation with delay argument, isolated solution.

Dedicated to the bright memory of an outstanding scientist,
Doctor of Physical and Mathematical Sciences, Professor,
our scientific supervisor Dzhumabaev Dulat Syzdykbekovich

1. Introduction

Due to applications in physics, biology, epidemiology, and so on, much of the literature on lagged differential equations has focused on the existence of a periodic solution, oscillation, and so on. In [1], using the parametrization method [2], necessary and sufficient conditions for the existence of an isolated solution of a periodic boundary value problem for a system of nonlinear differential equations with a delay argument are established in terms of the initial

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data. In recent years, there has been a growing interest in boundary value problems for differential equations with delay, see, for example, [3–11].

The importance and variety of applications served to increase interest in the theory of boundary value problems for differential equations with a delay argument, and the development of computer technology and its comprehensive application in applied problems presented new requirements for the developed methods, paying special attention to their constructability and feasibility.

One of the constructive methods widely used for the study of boundary value problems for differential equations is the parametrization method. In this paper, on the basis of one modification of the parametrization method algorithms, sufficient conditions are obtained for the existence of an isolated solution of a boundary value problem for a system of linear differential equations with a delay argument that satisfies essentially nonlinear boundary conditions.

2. Material and methods

We consider a nonlinear boundary value problem for a system of differential equations with delay argument

$$\frac{dx}{dt} = A(t)x(t) + B(t)x(t-\tau) + f(t), \quad x \in R^n, \quad t \in (0, N\tau), \quad \tau > 0, \tag{1}$$

$$x(t) = \mathrm{diag}[x(0)] \cdot \varphi(t), \quad t \in [-\tau, 0], \tag{2}$$

$$g(x(0), x(N\tau)) = 0, \tag{3}$$

where $(n \times n)$-matrices $A(t)$, $B(t)$ and the function $f(t)$ are continuous on $[0, N\tau]$, $\varphi : [-\tau, 0] \to R^n$ is a continuously differentiable function such that $\varphi_i(0) = 1$, $i = 1 : n$, τ is a constant delay, $\|A(t)\| \leq \alpha$, $\|B(t)\| \leq \beta$, where α, β are constant.

The solution of the boundary value problem (1)-(3) is a continuous on $[-\tau, N\tau]$, continuously differentiable on $[-\tau, 0) \bigcup (0, N\tau]$ vector function $x(t)$ that satisfies the differential equation (1) and has values $x(0)$, $x(N\tau)$, for which equalities (2), (3) are valid.

Introduce notation:

Δ_τ is a partition of interval $[-\tau, N\tau) = \bigcup\limits_{r=0}^{N} [t_{r-1}, t_r)$ by points $t_s = s\tau$, $s = \overline{-1, N}$;

$C([-\tau, N\tau], R^n)$ is a space of continuous on $[-\tau, N\tau]$ functions $x : [-\tau, N\tau] \to R^n$ with the norm $\|x\|_1 = \max\limits_{t \in [-\tau, N\tau]} \|x(t)\|$;

$C([0, N\tau], \tau, R^{nN})$ is a space of function systems $x[t] = (x_1(t), x_2(t), \ldots, x_N(t))$, where the functions $x_r(t) \in C[t_{r-1}, t_r)$ have a finite limit $\lim\limits_{t \to t_r - 0} x_r(t)$ for all $r = 1 : (N-1)$ with the norm $\|x[\cdot]\|_2 = \max\limits_{r=1:N} \sup\limits_{t \in [t_{r-1}, t_r)} \|x_r(t)\|$;

$C([-\tau, N\tau], \tau, R^{n(N+1)})$ is a space of function systems $x[t] = (x_0(t), x_1(t), x_2(t), \dots, x_N(t))$, where the functions $x_r(t) \in C[t_{r-1}, t_r)$ have a finite limit $\lim\limits_{t \to t_r - 0} x_r(t)$ for all $r = 0 : N$ with the norm $\|x[\cdot]\|_3 = \max\limits_{r=0:N} \sup\limits_{t \in [t_{r-1}, t_r)} \|x_r(t)\|$;

$C([-\tau, (N-1)\tau], \tau, R^{nN})$ is a space of function systems $x[t] = (x_0(t-\tau), x_1(t-\tau), \dots, x_{N-1}(t-\tau))$, where the functions $x_r(t) \in C[t_{r-1}, t_r)$ have a finite limit $\lim\limits_{t \to t_r - 0} x_r(t)$ for all $r = 0 : (N-1)$ with the norm $\|x[\cdot]\|_4 = \max\limits_{r=1:N} \sup\limits_{t \in [t_{r-1}, t_r)} \|x_{-1+r}(t-\tau)\|$.

We denote the restriction of function $x(t)$ to $[t_{r-1}, t_r)$ by $x_r(t)$, $r = 0 : N$, and reduce problem (1) - (3) to the equivalent multipoint boundary value problem

$$\frac{dx_r(t)}{dt} = A(t)x_r(t) + B(t)x_{-1+r}(t-\tau) + f(t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : N, \tag{4}$$

$$x_0(t) = diag[x_1(0)] \cdot \varphi(t), \quad t \in [t_{-1}, t_0], \tag{5}$$

$$g\left(x_1(0), \lim_{t \to t_N - 0} x_N(t)\right) = 0, \tag{6}$$

$$\lim_{t \to t_s - 0} x_s(t) = x_{s+1}(t_s), \quad s = 1 : (N-1), \tag{7}$$

where (7) are the conditions for matching the solution at the interior points of partition of interval $[-\tau, N\tau]$.

Solution to problem (4)-(7) is a function system

$$x^*[t] = (x_0^*(t), x_1^*(t), x_2^*(t), \dots, x_N^*(t)) \in C([-\tau, N\tau], \tau, R^{n(N+1)}),$$

with continuously differentiable on $[t_{r-1}, t_r)$ functions $x_r^*(t)$, $r = 0 : N$, that satisfy the system of differential equations with delay argument (4) and conditions (5)-(7).

Boundary value problem (4)-(7) is equivalent to the multipoint boundary value problem with parameters

$$\frac{du_r(t)}{dt} = A(t)(\lambda_r + u_r(t)) + B(t)(\lambda_{-1+r} + u_{-1+r}(t-\tau)) + f(t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : N, \tag{8}$$

$$u_r(t_{r-1}) = 0, \quad r = 1 : N, \tag{9}$$

$$\lambda_0 + u_0(t) = \Phi(t) \cdot \lambda_1, \quad t \in [t_{-1}, t_0], \quad u_0(t_{-1}) = 0, \tag{10}$$

$$g(\lambda_1, \lambda_{N+1}) = 0, \tag{11}$$

$$\lambda_r + \lim_{t \to t_r - 0} u_r(t) = \lambda_{r+1}, \quad r = 1 : N, \tag{12}$$

where $\lambda_r = x_r(t_{r-1})$, $r = 0 : N$, $\lambda_{N+1} = \lim\limits_{t \to t_N - 0} x_N(t)$, $u_r(t) = x_r(t) - \lambda_r$ at $t \in [t_{r-1}, t_r)$, $r = 0 : N$, $\Phi(t) = diag[\varphi(t)]$, $t \in [t_{-1}, t_0]$.

Let us introduce into consideration the linear operator [12, p. 145]

$$X_r(t) = I + \int_{t_{r-1}}^{t} A(\tau_1)d\tau_1 + \sum_{j=2}^{\infty} \int_{t_{r-1}}^{t} A(\tau_1) \int_{t_{r-1}}^{\tau_1} A(\tau_2) \ldots \int_{t_{r-1}}^{\tau_{j-1}} A(\tau_j)d\tau_j \ldots d\tau_2 d\tau_1,$$

$$t \in [t_{r-1}, t_r), \quad r = 1 : N,$$

where I is the identity matrix of dimension $(n \times n)$. Operator $X_r(t)$ satisfies the problem

$$\frac{dX_r}{dt} = A(t)X_r, \qquad X_r(t_{r-1}) = I, \qquad t \in [t_{r-1}, t_r), \quad r = 1 : N.$$

It is easy to check that $X_r^{-1}(t)$ exists and

$$\frac{dX_r^{-1}(t)}{dt} = -X_r^{-1}(t)A(t), \qquad X_r^{-1}(t_{r-1}) = I, \quad t \in [t_{r-1}, t_r), \quad r = 1 : N.$$

For fixed values of parameter λ_r, using the notation

$$a_r(P, t) = X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\xi)P(\xi)d\xi,$$

$$b_r(u_{-1+r}(\cdot), \tau, t) = X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\xi)B(\xi)u_{-1+r}(\xi - \tau)d\xi,$$

we write down the unique solution to the Cauchy problem (8), (9) as

$$u_r(t) = a_r(A, t)\lambda_r + a_r(B, t)\lambda_{-1+r} + b_r(u_{-1+r}(\cdot), \tau, t) + a_r(f, t), \ t \in [t_{r-1}, t_r), \ r = 1 : N, \quad (13)$$

and compose a system of functions $v[t] = (u_0(t), u_1(t), u_2(t), \ldots, u_N(t))$.

From (13) we define $\lim\limits_{t \to t_r - 0} u_r(t)$, $r = 1 : N$. Based on (10)-(12), we write down the system of nonlinear equations with respect to unknown parameters

$$Q_{*,\Delta_\tau}(\mu, v) = 0, \quad \mu = (\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_N, \lambda_{N+1}) \in R^{n(N+2)}, \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in R^{nN},$$

where the operator $Q_{*,\Delta_\tau}(\mu, v)$ has the form:

$$Q_{*,\Delta_\tau}(\mu, v) = \begin{pmatrix} \tau \cdot (-\lambda_0 + \Phi(t_{-1}) \cdot \lambda_1) \\ \tau \cdot g(\lambda_1, \lambda_{N+1}) \\ a_1(B, t_1)\lambda_0 + (I + a_1(A, t_1))\lambda_1 - \lambda_2 + b_1(u_0(\cdot), \tau, t_1) + a_1(f, t_1) \\ a_2(B, t_2)\lambda_1 + (I + a_2(A, t_2))\lambda_2 - \lambda_3 + b_2(u_1(\cdot), \tau, t_2) + a_2(f, t_2) \\ \ldots \\ a_N(B, t_N)\lambda_{N-1} + (I + a_N(A, t_N))\lambda_N - \lambda_{N+1} + b_N(u_{N-1}(\cdot), \tau, t_N) + a_N(f, t_N) \end{pmatrix}.$$

Condition A_τ**.** Suppose that
(i) system of nonlinear equations $Q_{*,\Delta_\tau}(\mu,0)=0$ has solution

$$\mu^{(0)}=\big(\lambda_0^{(0)},\lambda_1^{(0)},\ldots,\lambda_{N+1}^{(0)}\big)\in R^{n(N+2)};$$

(ii) $u_0^{(0)}(t)=-\lambda_0^{(0)}+\Phi(t)\cdot\lambda_1^{(0)}$, $t\in[t_{-1},t_0]$;
(iii) the Cauchy problem

$$\frac{du_r(t)}{dt}=A(t)(\lambda_r^{(0)}+u_r(t))+B(t)\lambda_{-1+r}^{(0)}+f(t),\quad t\in[t_{r-1},t_r),\quad r=1:N,$$

$$u_r(t_{r-1})=0,\quad r=1:N,$$

has solution $u_r^{(0)}(t)=a_r(A,t)\lambda_r^{(0)}+a_r(B,t)\lambda_{-1+r}^{(0)}+a_r(f,t)$, $t\in[t_{r-1},t_r)$, $r=1:N$;
(iv) $v^{(0)}[t]=\big(u_0^{(0)}(t),u_1^{(0)}(t),u_2^{(0)}(t),\ldots,u_N^{(0)}(t)\big)\in C([-\tau,N\tau],\tau,R^{n(N+1)})$.

By $\mu^{(0)}=\big(\lambda_0^{(0)},\lambda_1^{(0)},\ldots,\lambda_{N+1}^{(0)}\big)\in R^{n(N+2)}$ and $v^{(0)}[t]=\big(u_0^{(0)}(t),u_1^{(0)}(t),\ldots,u_N^{(0)}(t)\big)\in C([-\tau,N\tau],\tau,R^{n(N+1)})$ we define the piecewise contionuous on $[-\tau,N\tau]$ function

$$x^{(0)}(t)=\begin{cases}\lambda_r^{(0)}+u_r^{(0)}(t) & \text{at}\quad t\in[t_{r-1},t_r),\quad r=0:N,\\ \lambda_{N+1}^{(0)} & \text{at}\quad t=t_N.\end{cases}$$

Choosing the numbers $\rho_\lambda>0$, $\rho_u>0$, $\rho_x>0$, we compose the sets:

$$S(\mu^{(0)},\rho_\lambda)=\{\mu=(\lambda_0,\lambda_1,\ldots,\lambda_{N+1})\in R^{n(N+2)}:\ \|\mu-\mu^{(0)}\|=\max_{r=0:(N+1)}\|\lambda_r-\lambda_r^{(0)}\|<\rho_\lambda\},$$

$$S(v^{(0)}[t],\rho_u)=\{v(t)\in C([-\tau,N\tau],\tau,R^{n(N+1)}):\ \|(v-v^{(0)})[\cdot]\|_3<\rho_u\},$$

$$S(x^{(0)}(t),\rho_x)=\{x(t)\in C([-\tau,N\tau],R^n):\ \|x-x^{(0)}\|_0<\rho_x\},$$

$$G_0(\rho_\lambda,\rho_x)=\{(w_1,w_2)\in R^{2n}:\ \|w_1-x^{(0)}(0)\|<\rho_\lambda,\ \|w_2-x^{(0)}(N\tau)\|<\rho_x\}.$$

Condition B**.** Suppose that function $g(w_1,w_2)$ is continuous in $G_0(\rho_\lambda,\rho_x)$, and has uniformly continuous partial derivatives $g'_{w_1}(w_1,w_2)$, $g'_{w_2}(w_1,w_2)$ and inequalities

$$\sup_{(w_1,w_2)\in G_0(\rho_\lambda,\rho_x)}\|g'_{w_1}(w_1,w_2)\|\le L_1,\qquad \sup_{(w_1,w_2)\in G_0(\rho_\lambda,\rho_x)}\|g'_{w_2}(w_1,w_2)\|\le L_2,$$

where L_1, L_2 are constant, hold.

Let us assume that condition A_τ is satisfied. By $\mu^{(0)}=\big(\lambda_0^{(0)},\lambda_1^{(0)},\ldots,\lambda_{N+1}^{(0)}\big)\in\mathbb{R}^{n(N+2)}$ and $v^{(0)}[t]=\big(u_0^{(0)}(t),u_1^{(0)}(t),\ldots,u_N^{(0)}(t)\big)\in C([-\tau,N\tau],\tau,R^{n(N+1)})$, we define the sequence $(\mu^{(k)},v^{(k)}[t])$, $k=1,2,\ldots$, according to the following algorithm:

Step 1.

a) From equation $Q_{*,\Delta_\tau}(\mu, v^{(0)}) = 0$ we find

$$\mu^{(1)} = (\lambda_0^{(1)}, \lambda_1^{(1)}, \ldots, \lambda_{N+1}^{(1)}) \in \mathbb{R}^{n(N+2)};$$

b_1) evaluate $u_0^{(1)}(t) = -\lambda_0^{(1)} + \Phi(t) \cdot \lambda_1^{(1)}$, $t \in [t_{-1}, t_0]$;

b_2) from the Cauchy problem

$$\frac{du_r(t)}{dt} = A(t)(\lambda_r^{(1)} + u_r(t)) + B(t)(\lambda_{-1+r}^{(1)} + u_{-1+r}^{(0)}(t-\tau)) + f(t), \quad t \in [t_{r-1}, t_r), \quad u_r(t_{r-1}) = 0,$$

we find function $u_r^{(1)}(t)$, $r = 1 : N$;

b_3) compose the function systems

$$v^{(1)}[t] = \big(u_0^{(1)}(t), u_1^{(1)}(t), \ldots, u_N^{(1)}(t)\big) \in C([-\tau, N\tau], \tau, R^{n(N+1)}),$$

$$u^{(1)}[t] = \big(u_0^{(1)}(t-\tau), u_1^{(1)}(t-\tau), \ldots, u_{N-1}^{(1)}(t-\tau)\big) \in C([-\tau, (N-1)\tau], \tau, R^{nN}).$$

Step 2.

(a) From equation $Q_{*,\Delta_\tau}(\mu, v^{(1)}) = 0$ we find

$$\mu^{(2)} = (\lambda_0^{(2)}, \lambda_1^{(2)}, \ldots, \lambda_{N+1}^{(2)}) \in \mathbb{R}^{n(N+2)};$$

b_1) evaluate $u_0^{(2)}(t) = -\lambda_0^{(2)} + \Phi(t) \cdot \lambda_1^{(2)}$, $t \in [t_{-1}, t_0]$;

b_2) from the Cauchy problem

$$\frac{du_r(t)}{dt} = A(t)(\lambda_r^{(2)} + u_r(t)) + B(t)(\lambda_{-1+r}^{(2)} + u_{-1+r}^{(1)}(t-\tau)) + f(t), \quad t \in [t_{r-1}, t_r), \quad u_r(t_{r-1}) = 0,$$

we find the function $u_r^{(2)}(t)$, $r = 1 : N$;

b_3) compose the function systems

$$v^{(2)}[t] = \big(u_0^{(2)}(t), u_1^{(2)}(t), \ldots, u_N^{(2)}(t)\big) \in C([-\tau, N\tau], \tau, R^{n(N+1)}),$$

$$u^{(2)}[t] = \big(u_0^{(2)}(t-\tau), u_1^{(2)}(t-\tau), \ldots, u_{N-1}^{(2)}(t-\tau)\big) \in C([-\tau, (N-1)\tau], \tau, R^{nN}).$$

And so on.

3. Main results

The sufficient conditions for the feasibility and convergence of the proposed algorithm, which simultaneously ensure the existence of an isolated solution to the multipoint boundary value problem with parameters for the system of differential equations with delay argument (8)-(12), are established by

Theorem 1. *Suppose that under partition Δ_τ and some numbers $\rho_\lambda > 0$, $\rho_u > 0$, $\rho_x > 0$, the conditions A_τ, B are satisfied, Jacobi matrix $\dfrac{\partial Q_{*,\Delta_\tau}(\mu, v)}{\partial \mu} : \mathbb{R}^{n(N+2)} \to \mathbb{R}^{n(N+2)}$ is invertible for all $(\mu, v[t])$ ($\mu \in S(\mu^{(0)}, \rho_\lambda)$, $v[t] \in S(v^{(0)}[t], \rho_u)$ and the following inequalities are valid:*
1) $\Big\| \Big(\dfrac{\partial Q_{*,\Delta_\tau}(\mu, v)}{\partial \mu} \Big)^{-1} \Big\| \leq \gamma_*(\Delta_\tau)$,
2) $q_*(\Delta_\tau) = \gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau} \max\Big\{1, \max\limits_{t\in[t_{-1},t_0]} \|\Phi(t)\|, 1/\gamma_*(\Delta_\tau), e^{\alpha\tau} - 1, \beta\tau e^{\alpha\tau}\Big\} < 1$,
3) $\dfrac{\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}{1 - q_*(\Delta_\tau)} \|u^{(0)}[\cdot]\|_4 < \rho_\lambda$,
4) $\dfrac{q_*(\Delta_\tau)}{1 - q_*(\Delta_\tau)} \|u^{(0)}[\cdot]\|_4 < \rho_u$,
5) $\rho_\lambda + \rho_u < \rho_x$.

Then the sequence $(\mu^{(k)}, v^{(k)}[t])$, $k \in \mathbb{N}$, with $\mu^{(k)} \in S(\mu^{(0)}, \rho_\lambda)$ and $v^{(k)} \in S(v^{(0)}[t], \rho_u)$, determined by the algorithm, converges to $(\mu^*, v^*[t])$, the isolated solution to problem (8)-(12), *where $\mu^* \in S(\mu^{(0)}, \rho_\lambda)$, $v^*[t] \in S(v^{(0)}[t], \rho_u)$ and the following estimates are valid:*

$$\|(v^* - v^{(k)})[\cdot]\|_3 \leqslant \frac{q_*(\Delta_\tau)}{1 - q_*(\Delta_\tau)} \|(u^{(k)} - u^{(k-1)})[\cdot]\|_4, \tag{14}$$

$$\|\mu^* - \mu^{(k)}\| \leq \frac{\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}{1 - \gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}} \|(u^{(k)} - u^{(k-1)})[\cdot]\|_4. \tag{15}$$

Proof. By the partition Δ_τ, we pass from problem (1)-(3) to the equivalent multipoint boundary value problem with parameters (8)-(12). We take any $(\mu, v[t])$, where $\mu \in S(\mu^{(0)}, \rho_\lambda)$ and $v[t] \in S(v^{(0)}[t], \rho_u)$, then

$$\|\lambda_r - \lambda_r^{(0)} + u_r(t) - u_r^{(0)}(t)\| \leq \|\lambda_r - \lambda_r^{(0)}\| + \|u_r(t) - u_r^{(0)}(t)\| < \rho_\lambda + \rho_u < \rho_x,$$

$$t \in [t_{r-1}, t_r), \qquad r = 0 : N.$$

The solution to problem (8)-(12) will be sought according to the proposed algorithm.

Suppose the condition A_τ holds. Let us set $(\mu^{(0)}, v^{(0)}[t])$ as the initial approximation to the solution of problem (8)-(12). The next approximation by parameter is found from the equation

$$Q_{*,\Delta_\tau}(\mu, v^{(0)}) = 0, \qquad \mu \in R^{n(N+2)}.$$

By the conditions of the theorem, the operator $Q_{*,\Delta_\tau}(\mu, v^{(0)})$ in $S(\mu^{(0)}, \rho_\lambda)$ satisfies all the assumptions of Theorem 1 [13]. Choosing the number $\varepsilon_0 > 0$ satisfying the inequalities

$$\varepsilon_0\gamma_*(\Delta_\tau) \le \frac{1}{2}, \qquad \frac{\gamma_*(\Delta_\tau)}{1-\varepsilon_0\gamma_*(\Delta_\tau)}\|Q_{*,\Delta_\tau}(\mu^{(0)}, v^{(0)})\| < \rho_\lambda$$

and using the uniform continuity of the Jacobi matrix $\dfrac{\partial Q_{*,\Delta_\tau}(\mu, v^{(0)})}{\partial\lambda}$ in $S(\mu^{(0)}, \rho_\lambda)$, we find $\delta_0 \in (0, 0.5\rho_\lambda]$ such that

$$\Big\|\frac{\partial Q_{*,\Delta_\tau}(\mu, v^{(0)})}{\partial\mu} - \frac{\partial Q_{*,\Delta_\tau}(\tilde\mu, v^{(0)})}{\partial\mu}\Big\| < \varepsilon_0$$

for any $\mu, \tilde\mu \in S(\mu^{(0)}, \rho_\lambda)$ at $\|\mu - \tilde\mu\| < \delta_0$.

Choosing $\alpha_1 \geqslant \alpha_0 = \max\{1, \gamma_*(\Delta_\tau)\|Q_{*,\Delta_\tau}(\mu^{(0)}, v^{(0)})\|/\delta_0\}$, we construct an iterative process:

$$\begin{aligned} \mu^{(1,0)} &= \mu^{(0)}, \\ \mu^{(1,m+1)} &= \mu^{(1,m)} - \tfrac{1}{\alpha_1}\Big(\tfrac{\partial Q_{*,\Delta_\tau}(\mu^{(1,m)}, v^{(0)})}{\partial\mu}\Big)^{-1} Q_{*,\Delta_\tau}(\mu^{(1,m)}, v^{(0)}), \quad m = 0, 1, 2, \ldots. \end{aligned} \tag{16}$$

According to Theorem 1 [13], the iterative process (16) converges to $\mu^{(1)} \in S(\mu^{(0)}, \rho_\lambda)$, the isolated solution to equation $Q_{*,\Delta_\tau}(\mu, v^{(0)}) = 0$ and

$$\begin{aligned} &\|\mu^{(1)} - \mu^{(0)}\| \leqslant \gamma_*(\Delta_\tau)\|Q_{*,\Delta_\tau}(\mu^{(0)}, v^{(0)})\| \le \gamma_*(\Delta_\tau)\max_{r=1:N}\|b_r(u^{(0)}_{-1+r}(\cdot), \tau, t_r)\| \\ &\le \gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}\max_{r=1:N}\sup_{t\in[t_{r-1},t_r)}\|u^{(0)}_{-1+r}(t-\tau)\| = \gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}\|u^{(0)}[\cdot]\|_4 < \rho_\lambda. \end{aligned} \tag{17}$$

Note that the components of the function system $v^{(0)}[t] = (u^{(0)}_0(t), u^{(0)}_1(t), \ldots, u^{(0)}_N(t))$ satisfy the inequalities

$$\begin{aligned} \|u^{(0)}_0(t)\| &\le \|\lambda^{(0)}_0\| + \|\Phi(t)\|\cdot\|\lambda^{(0)}_1\|, \quad t \in [t_{-1}, t_0], \\ \|u^{(0)}_r(t)\| &\le \max\{e^{\alpha\tau} - 1, \beta\tau e^{\alpha\tau}\}\|\mu^{(0)}\| + \tau e^{\alpha\tau}\sup_{t\in[t_{r-1},t_r)}\|f(t)\|, \quad t \in [t_{r-1}, t_r), \quad r = 1:N. \end{aligned}$$

Under our assumptions, the Cauchy problem

$$\frac{du_r(t)}{dt} = A(t)(\lambda^{(1)}_r + u_r(t)) + B(t)(\lambda^{(1)}_{-1+r} + u^{(0)}_{-1+r}(t-\tau)) + f(t), \quad t \in [t_{r-1}, t_r), \quad r = 1:N,$$

$$u_r(t_{r-1}) = 0, \quad r = 1:N,$$

has the unique solution $u_r^{(1)}(t) = a_r(A,t)\lambda_r^{(1)} + a_r(B,t)\lambda_{-1+r}^{(1)} + b_r(u_{-1+r}^{(0)}(\cdot),\tau,t) + a_r(f,t)$, $t \in [t_{r-1},t_r)$, $r = 1:N$, and $u_0^{(1)}(t) = -\lambda_0^{(1)} + \Phi(t)\cdot\lambda_1^{(1)}$ at $t \in [t_{-1},t_0]$. Taking into account (17), we establish that

$$\|u_0^{(1)}(t) - u_0^{(0)}(t)\| \le \|\lambda_0^{(1)} - \lambda_0^{(0)}\| + \|\Phi(t)\|\cdot\|\lambda_1^{(1)} - \lambda_1^{(0)}\|$$

$$\le \max\left\{1, \max_{t\in[t_{-1},t_0]}\|\Phi(t)\|\right\}\|\mu^{(1)} - \mu^{(0)}\|$$

$$\le \max\left\{1, \max_{t\in[t_{-1},t_0]}\|\Phi(t)\|\right\}\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}\|u^{(0)}[\cdot]\|_4, \quad t\in[t_{-1},t_0],$$

$$\|u_r^{(1)}(t) - u_r^{(0)}(t)\| \le \|a_r(A,t)\|\cdot\|\lambda_r^{(1)} - \lambda_r^{(0)}\| + \|a_r(B,t)\|\cdot\|\lambda_{-1+r}^{(1)} - \lambda_{-1+r}^{(0)}\|$$

$$+\|b_r(u_{-1+r}^{(0)}(\cdot),\tau,t)\| \le \max\{e^{\alpha\tau}-1, \beta\tau e^{\alpha\tau}\}\cdot\|\mu^{(1)} - \mu^{(0)}\|$$

$$+\beta\tau e^{\alpha\tau}\max_{r=1:N}\sup_{t\in[t_{r-1},t_r)}\|u_{-1+r}^{(0)}(t-\tau)\|$$

$$\le \beta\tau e^{\alpha\tau}\max\{1, \gamma_*(\Delta_\tau)\max\{e^{\alpha\tau}-1, \beta\tau e^{\alpha\tau}\}\}\|u^{(0)}[\cdot]\|_4$$

$$\le \gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}\max\{1/\gamma_*(\Delta_\tau), e^{\alpha\tau}-1, \beta\tau e^{\alpha\tau}\}\|u^{(0)}[\cdot]\|_4, \quad t\in[t_{r-1},t_r), \quad r=1:N. \quad (18)$$

Thus, for the difference between function systems $v^{(1)}[t] = \big(u_0^{(1)}(t), u_1^{(1)}(t), \ldots, u_N^{(1)}(t)\big)$ and $v^{(0)}[t] = \big(u_0^{(0)}(t), u_1^{(0)}(t), \ldots, u_N^{(0)}(t)\big)$, we have the estimate:

$$\|(v^{(1)} - v^{(0)})[\cdot]\|_3 \le \max\left\{\max_{t\in[t_{-1},t_0]}\|u_0^{(1)}(t) - u_0^{(0)}(t)\|, \max_{r=1:N}\sup_{t\in[t_{r-1},t_r)}\|u_r^{(1)}(t) - u_r^{(0)}(t)\|\right\}$$
$$\le q_*(\Delta_\tau)\|u^{(0)}[\cdot]\|_4 < \rho_u,$$

i.e. $v^{(1)}[t] \in S(v^{(0)}[t], \rho_u)$.

It follows from the structure of the operator $Q_{*,\Delta_\tau}(\mu, v)$ and the equality $Q_{*,\Delta_\tau}(\mu^{(1)}, v^{(0)}) = 0$ that

$$\begin{aligned}\gamma_*(\Delta_\tau)\|Q_{*,\Delta_\tau}(\mu^{(1)}, v^{(1)})\| &= \gamma_*(\Delta_\tau)\|Q_{*,\Delta_\tau}(\mu^{(1)}, v^{(1)}) - Q_{*,\Delta_\tau}(\mu^{(1)}, v^{(0)})\| \\ &\le \gamma_*(\Delta_\tau)\max_{r=1:N}\|b_r((u_{-1+r}^{(1)} - u_{-1+r}^{(0)})(\cdot),\tau,t_r)\| \\ &\le \gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}\max_{r=1:N}\sup_{t\in[t_{r-1},t_r)}\|u_{-1+r}^{(1)}(t-\tau) - u_{-1+r}^{(0)}(t-\tau)\| \\ &\le \gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}\|(u^{(1)} - u^{(0)})[\cdot]\|_4,\end{aligned} \quad (19)$$

where $u^{(1)}[t] \in C([-\tau,(N-1)\tau],\tau,R^{nN})$.

If $\mu \in S(\mu^{(1)}, \rho_1)$, where $\rho_1 = \gamma_*(\Delta_\tau)\|Q_{*,\Delta_\tau}(\mu^{(1)}, v^{(1)})\|$, then by virtue of inequalities 2), 3) of the theorem and (17), (18), we get the estimate

$$\|\mu - \mu^{(0)}\| \leqslant \|\mu - \mu^{(1)}\| + \|\mu^{(1)} - \mu^{(0)}\| < \gamma_*(\Delta_\tau)\|Q_{*,\Delta_\tau}(\mu^{(1)}, u^{(1)})\| + \|\mu^{(1)} - \mu^{(0)}\|$$

$$\leqslant \gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}(q_*(\Delta_\tau) + 1)\|u^{(0)}[\cdot]\|_4 < \frac{\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}{1 - q_*(\Delta_\tau)}\|u^{(0)}[\cdot]\|_4 < \rho_\lambda,$$

i.e. $S(\mu^{(1)}, \rho_1) \subset S(\mu^{(0)}, \rho_\lambda)$.

The operator $Q_{*,\Delta_\tau}(\mu, v^{(1)})$ in $S(\lambda^{(1)}, \rho_1)$ satisfies all the conditions of Theorem 1 [13]. Therefore, the iterative process

$$\begin{aligned} \mu^{(2,0)} &= \mu^{(1)}, \\ \mu^{(2,m+1)} &= \mu^{(2,m)} - \tfrac{1}{\alpha}\Big(\tfrac{\partial Q_{*,\Delta_\tau}(\mu^{(2,m)}, v^{(1)})}{\partial \mu}\Big)^{-1} Q_{*,\Delta_\tau}(\mu^{(2,m)}, v^{(1)}), \quad m = 0, 1, 2, \dots, \end{aligned}$$

converges to $\mu^{(2)} = (\lambda_0^{(2)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \dots, \lambda_N^{(2)}, \lambda_{N+1}^{(2)}) \in S(\mu^{(1)}, \rho_1)$, the isolated solution of the equation $Q_{*,\Delta_\tau}(\mu, v^{(1)}) = 0$, and

$$\|\mu^{(2)} - \mu^{(1)}\| \leq \gamma_*(\Delta_\tau)\|Q_{*,\Delta_\tau}(\mu^{(1)}, v^{(1)})\|.$$

This and (19) imply the inequalities

$$\|\mu^{(2)} - \mu^{(1)}\| \leqslant \gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}\|(u^{(1)} - u^{(0)})[\cdot]\|_4,$$

$$\|v^{(2)} - v^{(1)}\|_3 \leqslant q_*(\Delta_\tau)\|(u^{(1)} - u^{(0)})[\cdot]\|_4 \leq q_*(\Delta_\tau)\|(v^{(1)} - v^{(0)})[\cdot]\|_3 \leq q_*^2(\Delta_\tau)\|u^{(0)}[\cdot]\|_4,$$

$$\|v^{(2)} - v^{(0)}\|_3 \leqslant \|v^{(2)} - v^{(1)}\|_3 + \|v^{(1)} - v^{(0)}\|_3$$

$$\leq (q_*^2(\Delta_\tau) + q_*(\Delta_\tau))\|u^{(0)}[\cdot]\|_4 < \frac{q_*(\Delta_\tau)}{1 - q_*(\Delta_\tau)}\|u^{(0)}[\cdot]\|_4 < \rho_u,$$

i.e. $v^{(2)}[t] \in S(v^{(0)}[t], \rho_u)$.

Assuming that $(\mu^{(k-1)}, v^{(k-1)}[t])$ is determined here $\mu^{(k-1)} \in S(\mu^{(0)}, \rho_\lambda)$, $v^{(k-1)}[t] \in C([-\tau, (N-1)\tau], \tau, R^{nN}))$, and the following estimates are set:

$$\|\mu^{(k-1)} - \mu^{(k-2)}\| \leq \gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}\|(u^{(k-2)} - u^{(k-3)})[\cdot]\|_4, \tag{20}$$

$$\|v^{(k-1)} - v^{(k-2)}\|_3 \leqslant q_*(\Delta_\tau)\|(u^{(k-2)} - u^{(k-3)})[\cdot]\|_4. \tag{21}$$

The k-th approximation in the parameter $\lambda^{(k)}$ is found from the equation $Q_{*,\Delta_\tau}(\mu, v^{(k-1)}) = 0$. Using (20), (21) and equality $Q_{*,\Delta_\tau}(\mu^{(k-1)}, v^{(k-2)}) = 0$, similarly to (18), we establish the inequalities

$$\gamma_*(\Delta_\tau)\|Q_{*,\Delta_\tau}(\mu^{(k-1)}, v^{(k-1)})\| \leqslant \gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}\|(u^{(k-1)} - u^{(k-2)})[\cdot]\|_4. \tag{22}$$

Let us take $\rho_{k-1} = \gamma_*(\Delta_\tau)\|Q_{*,\Delta_\tau}(\mu^{(k-1)}, v^{(k-1)})\|$ and show that $S(\mu^{(k-1)}, \rho_{k-1}) \subset S(\mu^{(0)}, \rho_\lambda)$. Indeed, in view of (20)-(22) and condition 3) of the theorem, we have:

$$\|\mu - \mu^{(0)}\| \leqslant \|\mu - \mu^{(k-1)}\| + \|\mu^{(k-1)} - \mu^{(k-2)}\| + \|\mu^{(k-2)} - \mu^{(k-3)}\| + \ldots + \|\mu^{(1)} - \mu^{(0)}\|$$

$$< \rho_{k-1} + (\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau})^{k-2}\|\mu^{(1)} - \mu^{(0)}\| + (\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau})^{k-3}\|\mu^{(1)} - \mu^{(0)}\| + \ldots + \|\mu^{(1)} - \mu^{(0)}\|$$

$$\leqslant ((\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau})^{k-1} + (\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau})^{k-2} + (\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau})^{k-3} + \ldots + 1)\|\mu^{(1)} - \mu^{(0)}\|$$

$$< \frac{\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}{1 - \gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}\|u^{(0)}[\cdot]\|_4 \leq \frac{\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}{1 - q_*(\Delta_\tau)}\|u^{(0)}[\cdot]\|_4 < \rho_\lambda.$$

Since $Q_{*,\Delta_\tau}(\mu, v^{(k-1)})$ in $S(\mu^{(k-1)}, \rho_{k-1})$ satisfies all the conditions of Theorem 1 [13], then there exists $\mu^{(k)} \in S(\mu^{(k-1)}, \rho_{k-1})$, the solution of the equation $Q_{*,\Delta_\tau}(\mu, v^{(k-1)}) = 0$, and the estimate

$$\|\mu^{(k)} - \mu^{(k-1)}\| \leqslant \gamma_*(\Delta_\tau)\|Q_{*,\Delta_\tau}(\mu^{(k-1)}, v^{(k-1)})\| \tag{23}$$

holds.

The Cauchy problem

$$\frac{du_r(t)}{dt} = A(t)(\lambda_r^{(k)} + u_r(t)) + B(t)(\lambda_{-1+r}^{(k)} + u_{-1+r}^{(k-1)}(t-\tau)) + f(t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : N,$$

$$u_r(t_{r-1}) = 0, \quad r = 1 : N,$$

has a unique solution $u_r^{(k)}(t) = a_r(A,t)\lambda_r^{(k)} + a_r(B,t)\lambda_{-1+r}^{(k)} + b_r(u_{-1+r}^{(k-1)}(\cdot), \tau, t) + a_r(f,t)$, $t \in [t_{r-1}, t_r)$, $r = 1 : N$, and $u_0^{(k)}(t) = -\lambda_0^{(k)} + \Phi(t) \cdot \lambda_1^{(k)}$ at $t \in [t_{-1}, t_0]$. If $\rho_k = \gamma_*(\Delta_\tau)\|Q_{*,\Delta_\tau}(\mu^{(k)}, v^{(k)})\| = 0$, then $Q_{*,\Delta_\tau}(\mu^{(k)}, v^{(k)}) = 0$. From this we obtain the equalities:

$$\lambda_0^{(k)} + u_0^{(k)}(t) = \Phi(t) \cdot \lambda_1^{(k)}, \quad t \in [t_{-1}, t_0],$$

$$g\Big(\lambda_1^{(k)}, \lambda_{N+1}^{(k)}\Big) = 0,$$

$$\lambda_r^{(k)} + \lim_{t \to t_t - 0} u_r^{(k)}(t) = \lambda_{r+1}^{(k)}, \qquad r = 1 : N,$$

i.e. $(\mu^{(k)}, v^{(k)}[t])$ is the solution to problem (8)-(12).

Using (22) and (23), we establish the estimates:

$$\|\mu^{(k)} - \mu^{(k-1)}\| \leq \gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}\|(u^{(k-1)} - u^{(k-2)})[\cdot]\|_4, \tag{24}$$

$$\|v^{(k)} - v^{(k-1)}\|_3 \leqslant q_*(\Delta_\tau)\|(u^{(k-1)} - u^{(k-2)})[\cdot]\|_4 \leqslant q_*(\Delta_\tau)\|(v^{(k-1)} - v^{(k-2)})[\cdot]\|_3, \tag{25}$$

$$\|v^{(k)} - v^{(0)}\|_3 \leqslant \|v^{(k)} - v^{(k-1)}\|_3 + \ldots + \|v^{(1)} - v^{(0)}\|_3$$

$$\leq (q_*^k(\Delta_\tau) + \ldots + q_*(\Delta_\tau))\|u^{(0)}[\cdot]\|_4 < \frac{q_*(\Delta_\tau)}{1 - q_*(\Delta_\tau)}\|u^{(0)}[\cdot]\|_4 < \rho_u, \quad v^{(k)}[t] \in S(v^{(0)}[t], \rho_u).$$

It follows from inequalities (24), (25) and $q_*(\Delta_\tau) < 1$ that the sequence of pairs $(\mu^{(k)}, v^{(k)}[t])$ converges to $(\mu^*, v^*[t])$, the solution of problem (8)-(12), as $k \to \infty$. Moreover, by virtue of inequalities 3), 4) of the theorem, $\mu^{(k)}, \mu^* \in S(\mu^{(0)}, \rho_\lambda)$, $k \in \mathbb{N}$, and $v^{(k)}[t], v^*[t] \in S(v^{(0)}[t], \rho_u)$. In inequalities

$$\|\mu^{(k+p)} - \mu^{(k)}\| < \frac{\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}{1-\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}} \left(1 - (\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau})^p\right) \|(u^{(k)} - u^{(k-1)})[\cdot]\|_4,$$

$$\|(v^{(k+p)} - v^{(k)})[\cdot]\|_3 \leqslant \frac{q_*(\Delta_\tau)}{1-q_*(\Delta_\tau)}(1 - q_*^p(\Delta_\tau))\|(u^{(k)} - u^{(k-1)})[\cdot]\|_4,$$

passing to the limit as $p \to \infty$, we obtain estimates (14), (15).

Let us show that the solution is isolated. Let $(\widetilde{\mu}, \widetilde{v}[t])$ be a solution to problem (8)-(12), where $\widetilde{\mu} \in S(\mu^{(0)}, \rho_\lambda)$, $\widetilde{v}[t] \in S(v^{(0)}[t], \rho_u)$. Then there are numbers $\widetilde{\delta}_1 > 0$, $\widetilde{\delta}_2 > 0$ such that $\|\widetilde{\mu} - \mu^{(0)}\| + \widetilde{\delta}_1 < \rho_\lambda$, $\|(\widetilde{v} - v^{(0)})[\cdot]\|_3 + \widetilde{\delta}_2 < \rho_u$. If $\mu \in S(\widetilde{\mu}, \widetilde{\delta}_1)$, $v[t] \in S(\widetilde{v}[t], \widetilde{\delta}_2)$, then by virtue of the inequalities

$$\|\mu - \mu^{(0)}\| \leqslant \|\mu - \widetilde{\mu}\| + \|\widetilde{\mu} - \mu^{(0)}\| \leqslant \widetilde{\delta}_1 + \|\widetilde{\mu} - \mu^{(0)}\| < \rho_\mu,$$

$$\|(v - v^{(0)})[\cdot]\|_3 \leqslant \|(v - \widetilde{v})[\cdot]\|_3 + \|(\widetilde{v} - v^{(0)})[\cdot]\|_3 \leqslant \widetilde{\delta}_2 + \|(\widetilde{v} - v^{(0)})[\cdot]\|_3 < \rho_u,$$

$\mu \in S(\mu^{(0)}, \rho_\lambda)$, $v[t] \in S(v^{(0)}[t], \rho_u)$, i.e. $S(\widetilde{\mu}, \widetilde{\delta}_1) \subset S(\mu^{(0)}, \rho_\lambda)$, $S(\widetilde{v}[t], \widetilde{\delta}_2) \subset S(v^{(0)}[t], \rho_u)$.

Take a number $\varepsilon > 0$ such that

$$\varepsilon\gamma_*(\Delta_\tau) < 1, \qquad \frac{\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}{1-\varepsilon\gamma_*(\Delta_\tau)} \max\left\{1, \max_{t\in[t_{-1},t_0]} \|\Phi(t)\|, \frac{1-\varepsilon\gamma_*(\Delta_\tau)}{\gamma_*(\Delta_\tau)}, e^{\alpha\tau} - 1, e^{\alpha\tau}\beta\tau\right\} < 1.$$

The condition B and structure of the Jacobi matrix $\dfrac{\partial Q_{*,\Delta_\tau}(\mu, v)}{\partial\mu}$ imply its uniform continuity for all $\mu \in S(\widetilde{\mu}, \widetilde{\delta}_1)$, $v[t] \in S(\widetilde{v}[t], \widetilde{\delta}_2)$. Therefore, there exists $\delta \in (0, \min\{\widetilde{\delta}_1, \widetilde{\delta}_2\}]$, for which

$$\left\|\frac{\partial Q_{*,\Delta_\tau}(\mu, v)}{\partial\mu} - \frac{\partial Q_{*,\Delta_\tau}(\widetilde{\mu}, \widetilde{v})}{\partial\mu}\right\| < \varepsilon, \quad \mu \in S(\widetilde{\mu}, \delta), \quad v[t] \in S(\widetilde{v}[t], \delta).$$

Note that if $(\widetilde{\mu}, \widetilde{v}[t])$ is a solution to problem (8)-(12), then the equality $Q_{*,\Delta_\tau}(\widetilde{\mu}, \widetilde{v}) = 0$ holds.

Let $(\widehat{\mu}, \widehat{v}[t])$ be another solution to problem (8)-(12), where $\widehat{\mu} \in S(\widetilde{\mu}, \delta)$, $\widehat{v}[t] \in S(\widetilde{v}[t], \delta)$. Since $Q_{*,\Delta_\tau}(\widetilde{\mu}, \widetilde{v}) = 0$ and $Q_{*,\Delta_\tau}(\widehat{\mu}, \widehat{v}) = 0$, then from the equalities

$$\widetilde{\mu} = \widetilde{\mu} - \left(\frac{\partial Q_{*,\Delta_\tau}(\widetilde{\mu}, \widetilde{v})}{\partial\mu}\right)^{-1} Q_{*,\Delta_\tau}(\widetilde{\mu}, \widetilde{v}), \qquad \widehat{\mu} = \widehat{\mu} - \left(\frac{\partial Q_{*,\Delta_\tau}(\widetilde{\mu}, \widetilde{v})}{\partial\mu}\right)^{-1} Q_{*,\Delta_\tau}(\widehat{\mu}, \widehat{v})$$

it follows that

$$\widetilde{\mu} - \widehat{\mu} = -\left(\frac{\partial Q_{*,\Delta_\tau}(\widetilde{\mu}, \widetilde{v})}{\partial\mu}\right)^{-1} \int_0^1 \left(\frac{\partial Q_{*,\Delta_\tau}(\widehat{\mu} + \theta(\widetilde{\mu} - \widehat{\mu}), \widetilde{v})}{\partial\mu} - \frac{\partial Q_{*,\Delta_\tau}(\widetilde{\mu}, \widetilde{v})}{\partial\mu}\right) d\theta \cdot (\widetilde{\mu} - \widehat{\mu})$$

$$-\Big(\frac{\partial Q_{*,\Delta_\tau}(\widetilde{\mu},\widetilde{v})}{\partial \mu}\Big)^{-1}(Q_{*,\Delta_\tau}(\widehat{\mu},\widetilde{v})-Q_{*,\Delta_\tau}(\widehat{\mu},\widehat{v})),$$

from where

$$\|\widetilde{\mu}-\widehat{\mu}\| \le \frac{\gamma_*(\Delta_\tau)}{1-\varepsilon\gamma_*(\Delta_\tau)}\|Q_{*,\Delta_\tau}(\widehat{\mu},\widetilde{v})-Q_{*,\Delta_\tau}(\widehat{\mu},\widehat{v})\|$$

$$\le \frac{\gamma_*(\Delta_\tau)}{1-\varepsilon\gamma_*(\Delta_\tau)}\max_{r=1:N}\Big\{\|b_r(\widetilde{u}_{-1+r}-\widehat{u}_{-1+r},\tau,t_r)\|\Big\}$$

$$\le \frac{\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}{1-\varepsilon\gamma_*(\Delta_\tau)}\|(\widetilde{u}-\widehat{u})[\cdot]\|_4 \le \frac{\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}{1-\varepsilon\gamma_*(\Delta_\tau)}\|(\widetilde{v}-\widehat{v})[\cdot]\|_3. \tag{26}$$

Since the components of function systems $\widetilde{v}[t]=(\widetilde{u}_0(t),\widetilde{u}_1(t),\ldots,\widetilde{u}_N(t))$ and $\widehat{v}[t]=(\widehat{u}_0(t),\widehat{u}_1(t),\ldots,\widehat{u}_N(t))$ satisfy the equalities

$$\widetilde{u}_0(t)=-\widetilde{\lambda}_0+\Phi(t)\cdot\widetilde{\lambda}_1,\quad t\in[t_{-1},t_0],\quad \widetilde{u}_0(t_{-1})=0,$$

$$\begin{aligned}&\frac{d\widetilde{u}_r(t)}{dt}=A(t)(\widetilde{\lambda}_r+\widetilde{u}_r(t))+B(t)(\widetilde{\lambda}_{-1+r}+\widetilde{u}_{-1+r}(t-\tau))+f(t),\quad t\in[t_{r-1},t_r),\quad r=1:N,\\ &\widetilde{u}_r(t_{r-1})=0,\quad r=1:N,\end{aligned} \tag{27}$$

and

$$\widehat{u}_0(t)=-\widehat{\lambda}_0+\Phi(t)\cdot\widehat{\lambda}_1,\quad t\in[t_{-1},t_0],\quad \widehat{u}_0(t_{-1})=0,$$

$$\begin{aligned}&\frac{d\widehat{u}_r(t)}{dt}=A(t)(\widehat{\lambda}_r+\widehat{u}_r(t))+B(t)(\widehat{\lambda}_{-1+r}+\widehat{u}_{-1+r}(t-\tau))+f(t),\quad t\in[t_{r-1},t_r),\quad r=1:N,\\ &\widehat{u}_r(t_{r-1})=0,\quad r=1:N,\end{aligned} \tag{28}$$

we get:

$$\|\widetilde{u}_0(t)-\widehat{u}_0(t)\| \le \|\widetilde{\lambda}_0-\widehat{\lambda}_0\|+\max_{t\in[t_{-1},t_0]}\|\Phi(t)\|\cdot\|\widetilde{\lambda}_1-\widehat{\lambda}_1\| \le \max\Big\{1,\max_{t\in[t_{-1},t_0]}\|\Phi(t)\|\Big\}\|\widetilde{\mu}-\widehat{\mu}\|$$

$$\le \frac{\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}{1-\varepsilon\gamma_*(\Delta_\tau)}\max\Big\{1,\max_{t\in[t_{-1},t_0]}\|\Phi(t)\|\Big\}\|(\widetilde{v}-\widehat{v})[\cdot]\|_3,\quad t\in[t_{-1},t_0]. \tag{29}$$

It follows from the Cauchy problems (27) and (28) that

$$\|\widetilde{u}_r(t)-\widehat{u}_r(t)\| \le \int_{t_{r-1}}^{t}\alpha\Big(\|\widetilde{\lambda}_r-\widehat{\lambda}_r\|+\|\widetilde{u}_r(\xi)-\widehat{u}_r(\xi)\|\Big)d\xi$$

$$+\int_{t_{r-1}}^{t}\beta\Big(\|\widetilde{\lambda}_{-1+r}-\widehat{\lambda}_{-1+r}\|+\|\widetilde{u}_{-1+r}(\xi-\tau)-\widehat{u}_{-1+r}(\xi-\tau)\|\Big)d\xi,\quad t\in[t_{r-1},t_r),\ r=1:N,$$

whence, by the Gronwall-Bellman lemma, we have the inequality

$$\|\widetilde{u}_r(t)-\widehat{u}_r(t)\|+\|\widetilde{\lambda}_r-\widehat{\lambda}_r\|\le\Big(\|\widetilde{\lambda}_r-\widehat{\lambda}_r\|$$

$$+\int_{t_{r-1}}^{t}\beta\Big(\|\widetilde{\lambda}_{-1+r}-\widehat{\lambda}_{-1+r}\|+\|\widetilde{u}_{-1+r}(\xi-\tau)-\widehat{u}_{-1+r}(\xi-\tau)\|\Big)d\xi\Big)\,e^{\alpha\tau},$$

$t\in[t_{r-1},t_r),\quad r=1:N$, i.e.

$$\|\widetilde{u}_r(t)-\widehat{u}_r(t)\|\le(e^{\alpha\tau}-1)\,\|\widetilde{\lambda}_r-\widehat{\lambda}_r\|+e^{\alpha\tau}\beta\tau\|\widetilde{\lambda}_{-1+r}-\widehat{\lambda}_{-1+r}\|$$

$$+e^{\alpha\tau}\beta\tau\sup_{t\in[t_{r-1},t_r)}\|\widetilde{u}_{-1+r}(t-\tau)-\widehat{u}_{-1+r}(t-\tau)\|$$

$$\le\max\{e^{\alpha\tau}-1,e^{\alpha\tau}\beta\tau\}\|\widetilde{\mu}-\widehat{\mu}\|+e^{\alpha\tau}\beta\|(\widetilde{u}-\widehat{u})[\cdot]\|_4$$

$$\le\max\{e^{\alpha\tau}-1,e^{\alpha\tau}\beta\tau\}\frac{\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}{1-\varepsilon\gamma_*(\Delta_\tau)}\|(\widetilde{v}-\widehat{v})[\cdot]\|_3+e^{\alpha\tau}\beta\|(\widetilde{v}-\widehat{v})[\cdot]\|_3$$

$$\le\frac{\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}{1-\varepsilon\gamma_*(\Delta_\tau)}\max\left\{\frac{1-\varepsilon\gamma_*(\Delta_\tau)}{\gamma_*(\Delta_\tau)},e^{\alpha\tau}-1,e^{\alpha\tau}\beta\tau\right\}\|(\widetilde{v}-\widehat{v})[\cdot]\|_3,\tag{30}$$

$t\in[t_{r-1},t_r),\quad r=1:N.$

From the fact that $\|(\widetilde{v}-\widehat{v})[\cdot]\|_3=\max\limits_{r=0:N}\sup\limits_{t\in[t_{r-1},t_r)}\|\widetilde{u}_r(t)-\widehat{u}_r(t)\|$, and inequalities (29) and (30) hold, the next estimate

$$\|(\widetilde{v}-\widehat{v})[\cdot]\|_3$$

$$\le\frac{\gamma_*(\Delta_\tau)\beta\tau e^{\alpha\tau}}{1-\varepsilon\gamma_*(\Delta_\tau)}\max\left\{1,\max_{t\in[t_{-1},t_0]}\|\Phi(t)\|,\frac{1-\varepsilon\gamma_*(\Delta_\tau)}{\gamma_*(\Delta_\tau)},e^{\alpha\tau}-1,e^{\alpha\tau}\beta\tau\right\}\|(\widetilde{v}-\widehat{v})[\cdot]\|_3$$

follows. From the last relation, by virtue of the choice of $\varepsilon>0$, it follows that $\widetilde{v}[t]=\widehat{v}[t]$. Then (26) implies $\widetilde{\mu}=\widehat{\mu}$. Theorem 1 is proved.

By using $\mu^{(k)}=\big(\lambda_0^{(k)},\lambda_1^{(k)},\dots,\lambda_{N+1}^{(k)}\big)\in R^{n(N+2)}$ and $v^{(k)}[t]=\big(u_0^{(k)}(t),u_1^{(k)}(t),\dots,u_N^{(k)}(t)\big)\in S(v^{(0)}[t],\rho_u)$ $(k=1,2,\dots)$ we define a piecewise continuous on $[-\tau,N\tau]$ function

$$x^{(k)}(t)=\begin{cases}\lambda_r^{(k)}+u_r^{(k)}(t) & \text{at}\quad t\in[t_{r-1},t_r),\quad r=0:N,\\ \lambda_{N+1}^{(k)} & \text{at}\quad t=t_N.\end{cases}$$

In view of the equivalence of problems (1)-(3) and (8)-(12) Theorem 1 implies

Theorem 2. *Suppose that under partition* Δ_τ *and some numbers* $\rho_\lambda > 0$, $\rho_u > 0$, $\rho_x > 0$, *the conditions* A_τ, B *are satisfied, Jacobi matrix* $\dfrac{\partial Q_{*,\Delta_\tau}(\mu, v)}{\partial \mu} : \mathbb{R}^{n(N+2)} \to \mathbb{R}^{n(N+2)}$ *is invertible for all* $(\mu, v[t])$ ($\mu \in S(\mu^{(0)}, \rho_\lambda)$, $v[t] \in S(v^{(0)}[t], \rho_u)$ *and inequalities 1)-5) of Theorem 1 take place. Then the sequence of functions* $(x^{(k)}(t))$, $k \in \mathbb{N}$, *is contained in* $S(x^{(0)}(t), \rho_x)$, *converges to* $x^*(t) \in S(x^{(0)}(t), \rho_x)$, *the isolated solution of problem* (1)-(3), *and the next inequality holds:*

$$\max_{t\in[0,T]} \|x^*(t) - x^{(k)}(t)\| \leqslant \frac{2q_*(\Delta_\tau)}{1 - q_*(\Delta_\tau)} \|(v^{(k)} - v^{(k-1)})[\cdot]\|_3.$$

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Искакова Н.Б., Темешева С.М., Абильдаева А.Д. КЕШІГУЛІ АРГУМЕНТІ БАР ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІ ҮШІН СЫЗЫҚТЫ ЕМЕС ШЕТТІК ЕСЕПТІҢ ШЕШІЛІМДІЛІК ШАРТТАРЫ ТУРАЛЫ

Бұл жұмыста кешігулі аргументі бар сызықтық дифференциалдық теңдеулер жүйесі үшін елеулі сызықтық емес екі нүктелі шеттік шарттары бар шеттік есеп қарастырылады. Бұл есепті зерттеу үшін модификацияланған алгоритмі бар Д.С. Джумабаевтың параметрлеу әдісі қолданылады. Параметрлеу әдісін қолдану сызықтық емес операторлық теңдеуге әкеледі, ол Адамар теоремасының локалды нұсқасының күшейтілуі арқылы шешіледі. Есептің оқшауланған шешімі бар болуының жеткілікті шарттары туралы теорема дәлелденді.

Кілттік сөздер. Шеттік есептер, кешігулі аргументі бар теңдеу, оқшауланған шешім.

Искакова Н.Б., Темешева С.М., Абильдаева А.Д. ОБ УСЛОВИЯХ РАЗРЕШИМОСТИ НЕЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ СИСТЕМЫ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ЗАПАЗДЫВАЮЩИМ АРГУМЕНТОМ

В данной работе рассматривается краевая задача с существенно нелинейными двухточечными граничными условиями для системы линейных дифференциальных уравнений с запаздывающим аргументом. Для исследования этой задачи используется метод параметризации Д.С. Джумабаева с модифицированным алгоритмом. Применение метода параметризации приводит к нелинейному операторному уравнению, которое решается с помощью усиления более резкого варианта локальной теоремы Адамара. Доказана теорема о достаточных условиях существования изолированного решения задачи.

Ключевые слова. Краевые задачи, уравнение с запаздывающим аргументом, изолированное решение.

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Parameterization method to solve problem with non-separated multipoint-integral conditions for high-order differential equations

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Abstract. Problem with non-separated multipoint-integral conditions for high-order differential equations is considered. Algorithms of the parametrization method are constructed and their convergence is proved. Sufficient conditions for the unique solvability of considered problem are set.

Keywords. Non-separated multipoint-integral conditions, high-order differential equations, parameterization method, algorithm, solvability.

Dedicated to the bright memory of an outstanding scientist,
Doctor of Physical and Mathematical Sciences, Professor,
our scientific supervisor Dzhumabaev Dulat Syzdykbekovich

1 Introduction

On $[0,T]$ consider boundary value problem with non-separated multipoint and integral conditions for higher order ordinary differential equations

$$\frac{d^n u}{dt^n} = a_1(t)\frac{d^{n-1}u}{dt^{n-1}} + a_2(t)\frac{d^{n-2}u}{dt^{n-2}} + a_3(t)\frac{d^{n-3}u}{dt^{n-3}} + ... + a_{n-1}(t)\frac{du}{dt} + a_n(t)u + f(t), \qquad (1)$$

$$\sum_{j=0}^{n-1}\sum_{s=0}^{m} b_{j,s}^k \frac{d^j u(t)}{dt^j}\Big|_{t=t_s} + \int_0^T \sum_{j=0}^{n-1} c_j^k(\tau)\frac{d^j u(\tau)}{d\tau^j}d\tau = d_k, \qquad k = 1,2,...,n, \qquad (2)$$

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where $u(t)$ is unknown function, the functions $a_i(t)$, $i = \overline{1,n}$, and $f(t)$ are continuous on $[0,T]$; $b^k_{j,s}$ are constants, the functions $c^k_j(t)$ are continuous on $[0,T]$, d_k are constants, where $k = \overline{1,n}$, $j = \overline{0,n-1}$, $s = \overline{0,m}$; $0 = t_0 < t_1 < ... < t_{m-1} < t_m = T$.

Let $C([0,T],\mathbb{R})$ be the space of continuous functions $u : [0,T] \to \mathbb{R}$ with norm

$||u||_0 = \max_{t\in[0,T]} |u(t)|$.

A solution to problem (1), (2) is a function $u(t) \in C([0,T],\mathbb{R})$ with derivatives $\frac{d^l u(t)}{dt^l} \in C([0,T],\mathbb{R})$, $l = \overline{1,n}$, it satisfies the differential equation (1) for all $t \in [0,T]$ and multipoint-integral conditions (2).

Mathematical modeling of many processes of the theory of oscillations, the theory of impulse systems, and the theory of multi-support beams lead to problems with multipoint-integral conditions for differential equations of various orders [1-19]. Until recently, problems with multipoint-integral conditions for systems of differential equations still remained less explored, since intermediate points of boundary conditions pose a number of serious difficulties, such as violation of the smoothness of the Green's function, the absence of a conjugate problem, etc. [1-19]. Therefore, the development of effective methods for solving the problem with multipoint and integral conditions for systems of differential equations without using the fundamental matrix and Green's function is of particular importance.

In [20], Professor D. S. Dzhumabaev proposed a parameterization method for solving the two-point boundary value problems for systems of ordinary differential equations. This method, in addition to proving the unique solvability of the problem under study, also offers an algorithm for constructing an approximate solution that converges to its exact solution. Moreover, the solvability criteria for boundary value problems for systems of differential equations are formulated in terms of the initial data, without using a fundamental matrix. In [21, 22], the parameterization method was developed for multipoint boundary value problems for systems of ordinary differential equations. The coefficient criteria for the well-posedness of a linear multipoint problem for a system of ordinary differential equations are obtained. These results were extended to nonlinear multipoint boundary value problems [23], and were also applied to the Cauchy-Nicoletti problem [24], to multipoint problems for second-order differential equations [25], to the Valle-Poussin [26] and the multipoint-integral problem for third-order differential equations [27]. Based on the equivalence of the well-posedness of the multipoint problem for systems of hyperbolic equations with mixed derivatives and families of multipoint problems for systems of ordinary differential equations, criteria for the well-posedness of the original problem are established [28]. The results are extended to quasilinear systems [29].

The goal of the present paper is to develop the parameterization method for investigating and solving the problem with non-separated multipoint and integral conditions for higher order ordinary differential equations (1), (2). Algorithms of the parameterization method are constructed for finding solutions to this problem and their convergence is proved. Conditions for the unique solvability of problems with non-separated multipoint-integral conditions for higher order ordinary differential equations are established in terms of initial data.

Let's give a scheme of method without partitioning of interval $[0, T]$.

2 Scheme of the parameterization method

Introduce the following notations:

$$u(0) = \lambda_1, \qquad \frac{du(t)}{dt}\Big|_{t=0} = \lambda_2, \qquad \frac{d^2u(t)}{dt^2}\Big|_{t=0} = \lambda_3, \qquad ..., \qquad \frac{d^{n-1}u(t)}{dt^{n-1}}\Big|_{t=0} = \lambda_n,$$

where λ_r are unknown parameters, $r = 1, 2, ..., n$.

And make the substitution

$$u(t) = \widetilde{u}(t) + \lambda_1, \ \frac{du(t)}{dt} = \frac{d\widetilde{u}(t)}{dt} + \lambda_2, \ \frac{d^2u(t)}{dt^2} = \frac{d^2\widetilde{u}(t)}{dt^2} + \lambda_3, \ ..., \ \frac{d^{n-1}u(t)}{dt^{n-1}} = \frac{d^{n-1}\widetilde{u}(t)}{dt^{n-1}} + \lambda_n,$$

where $\widetilde{u}(t)$ is new unknown function.

Then, problem (1), (2) is reduced to the following problem with parameters:

$$\frac{d^n\widetilde{u}}{dt^n} = \sum_{i=1}^{n} a_i(t)\frac{d^{n-i}\widetilde{u}}{dt^{n-i}} + \sum_{i=1}^{n} a_i(t)\lambda_{n+1-i} + f(t), \tag{3}$$

$$\widetilde{u}(0) = \lambda_1, \qquad \frac{d\widetilde{u}(t)}{dt}\Big|_{t=0} = \lambda_2, \qquad \frac{d^2\widetilde{u}(t)}{dt^2}\Big|_{t=0} = \lambda_3, \quad ..., \quad \frac{d^{n-1}\widetilde{u}(t)}{dt^{n-1}}\Big|_{t=0} = \lambda_n, \tag{4}$$

$$\sum_{j=0}^{n-1}\sum_{s=1}^{m} b_{j,s}^k \frac{d^j\widetilde{u}(t)}{dt^j}\Big|_{t=t_s} + \sum_{j=0}^{n-1}\sum_{s=0}^{m} b_{j,s}^k\lambda_{j+1} + \int_0^T \sum_{j=0}^{n-1} c_j^k(\tau)\frac{d^j\widetilde{u}(\tau)}{d\tau^j}d\tau$$

$$+\sum_{j=0}^{n-1}\int_0^T c_j^k(\tau)d\tau\lambda_{j+1} = d_k, \qquad k = 1, 2, ..., n. \tag{5}$$

So, we obtain boundary value problem with parameters for higher order differential equation (3)-(5).

A solution to problem (3)-(5) is a pair $(\widetilde{u}(t), \lambda)$, where the function $\widetilde{u}(t) \in C([0, T], \mathbb{R})$ with derivatives $\frac{d^l\widetilde{u}(t)}{dt^l} \in C([0, T], \mathbb{R})$, $l = \overline{1, n}$, parameter $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)'$, that satisfies the higher order differential equation with parameters (3) for all $t \in [0, T]$, initial conditions (4) and multipoint-integral conditions (5).

At fixed values of parameters λ_r, $r = 1, 2, ..., n$, the problem (3), (4) is a Cauchy problem for higher order ordinary differential equation. For determining unknown parameters λ_r, $r = 1, 2, ..., n$, we have equalities (5) consisting of multipoint-integral conditions.

The problems (1), (2) and (3)-(5) are equivalent in the following sense. If the function $u(t)$ is the solution of the problem (1), (2), then the pair $(\widetilde{u}(t), \lambda)$, where

$$\widetilde{u}(t) = u(t) - u(0), \quad \frac{d\widetilde{u}(t)}{dt} = \frac{du(t)}{dt} - \frac{du(t)}{dt}\Big|_{t=0}, \quad \frac{d^2\widetilde{u}(t)}{dt^2} = \frac{d^2u(t)}{dt^2} - \frac{d^2u(t)}{dt^2}\Big|_{t=0}, \ ...,$$

$$\frac{d^{n-1}\widetilde{u}(t)}{dt^{n-1}} = \frac{d^{n-1}u(t)}{dt^{n-1}} - \frac{d^{n-1}u(t)}{dt^{n-1}}\Big|_{t=0}, \qquad t \in [0, T],$$

$$\lambda_1 = u(0), \quad \lambda_2 = \frac{du(t)}{dt}\Big|_{t=0}, \quad \lambda_3 = \frac{d^2u(t)}{dt^2}\Big|_{t=0}, \ ..., \ \lambda_n = \frac{d^{n-1}u(t)}{dt^{n-1}}\Big|_{t=0},$$

will be a solution to problem (3)-(5).

Conversely, if the pair $(\widetilde{u}(t), \lambda)$, where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)'$ is the solution to problem (3)-(5), then the function $u(t)$ defined by the equalities:

$$u(t) = \widetilde{u}(t) + \lambda_1, \ \frac{du(t)}{dt} = \frac{d\widetilde{u}(t)}{dt} + \lambda_2, \ \frac{d^2u(t)}{dt^2} = \frac{d^2\widetilde{u}(t)}{dt^2} + \lambda_3, \ ..., \ \frac{d^{n-1}u(t)}{dt^{n-1}} = \frac{d^{n-1}\widetilde{u}(t)}{dt^{n-1}} + \lambda_n,$$

will be a solution of original problem (1), (2).

Let's consider the Cauchy problem (3), (4).

Assume $\dfrac{d^n\widetilde{u}}{dt^n}$ as unknown function and put

$$\frac{d^n\widetilde{u}}{dt^n} = \varphi(t), \qquad t \in [0, T]. \tag{6}$$

Taking into account initial conditions (4), we get

$$\widetilde{u}(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\varphi(s)ds + \frac{t^{n-1}}{(n-1)!}\lambda_n + \frac{t^{n-2}}{(n-2)!}\lambda_{n-1} + ... + \frac{t}{1!}\lambda_2 + \lambda_1. \tag{7}$$

Differentiating it l times by t, we obtain

$$\frac{d^l\widetilde{u}(t)}{dt^l} = \frac{1}{(n-l-1)!}\int_0^t (t-s)^{n-l-1}\varphi(s)ds + \frac{t^{n-l-1}}{(n-l-1)!}\lambda_n + \frac{t^{n-l-2}}{(n-l-2)!}\lambda_{n-1} + ... + \lambda_{l+1}, \tag{8}$$

$l = 1, 2, ..., n-1$.

Substituting obtained expressions in (3), we get Volterra integral equation of second kind for $\varphi(t)$:

$$\varphi(t) = \int_0^t K(t,s)\varphi(s)ds + f(t) + F(t, \lambda), \qquad t \in [0, T], \tag{9}$$

where the kernel $K(t,s)$ and function $F(t,\lambda)$ are defined by expressions

$$K(t,s) = a_1(t) + a_2(t)\frac{(t-s)}{1!} + a_3(t)\frac{(t-s)^2}{2!} + ... + a_n(t)\frac{(t-s)^{n-1}}{(n-1)!}, \qquad t,s \in [0,T], \quad (10)$$

$$F(t,\lambda) = \sum_{l=0}^{n-1} a_{n-l}(t)\Big[\frac{t^{n-l-1}}{(n-l-1)!}\lambda_n + \frac{t^{n-l-2}}{(n-l-2)!}\lambda_{n-1} + ... + \lambda_{l+1}\Big] + \sum_{i=1}^{n} a_i(t)\lambda_{n+1-i}, \quad (11)$$

$t \in [0,T]$, respectively.

The function $F(t,\lambda)$ is continuous on $[0,T]$, also depends on the coefficients of equation (3) and given unknown parameters λ_r, $r = 1,2,...,n$.

Substituting the corresponding values of the function $\widetilde{u}(t)$ and its derivatives from the expressions (7), (8) when $t = t_s$, $s = \overline{1,m}$, $t = \tau$, into (5), we get the following system of n equations with respect to unknown parameters λ_r:

$$\sum_{j=0}^{n-1}\sum_{s=1}^{m} b_{j,s}^k\Big\{\frac{t_s^{n-j-1}}{(n-j-1)!}\lambda_n + \frac{t_s^{n-j-2}}{(n-j-2)!}\lambda_{n-1} + ... + \lambda_{j+1}\Big\} + \sum_{j=0}^{n-1}\sum_{s=0}^{m} b_{j,s}^k\lambda_{j+1}$$

$$+ \int_0^T \sum_{j=0}^{n-1} c_j^k(\tau)\Big\{\frac{\tau^{n-j-1}}{(n-j-1)!}\lambda_n + \frac{\tau^{n-j-2}}{(n-j-2)!}\lambda_{n-1} + ... + \lambda_{j+1}\Big\}d\tau + \sum_{j=0}^{n-1}\int_0^T c_j^k(\tau)d\tau\lambda_{j+1}$$

$$= d_k - \sum_{j=0}^{n-1}\sum_{s=1}^{m} b_{j,s}^k \frac{1}{(n-j-1)!}\int_0^{t_s}(t_s - s_s)^{n-j-1}\varphi(s_1)ds_1$$

$$-\int_0^T \sum_{j=0}^{n-1} c_j^k(\tau)\frac{1}{(n-j-1)!}\int_0^{\tau}(\tau - s)^{n-l-1}\varphi(s)dsd\tau, \qquad k = 1,2,...,n. \quad (12)$$

Grouping the coefficients corresponding to the same parameters λ_r, $r = 1,2,...,n$, in the left-hand sides of the algebraic equations (12), we shall compile $(n \times n)$ matrix $Q(T)$ and write down the system of algebraic equations (12) in the following form

$$Q(T)\lambda = d - \Phi(\varphi,T), \quad (13)$$

where $\lambda = (\lambda_1,\lambda_2,...,\lambda_n)'$, $\Phi(\varphi,T) = (\Phi_1(\varphi,T),\Phi_2(\varphi,T),...,\Phi_n(\varphi,T))'$,

$$\Phi_k(\varphi,T) = \sum_{j=0}^{n-1}\sum_{s=1}^{m} b_{j,s}^k \frac{1}{(n-j-1)!}\int_0^{t_s}(t_s - s_s)^{n-j-1}\varphi(s_1)ds_1$$

$$+\int_0^T \sum_{j=0}^{n-1} c_j^k(\tau)\frac{1}{(n-j-1)!}\int_0^\tau (\tau-s)^{n-l-1}\varphi(s)dsd\tau, \qquad k=1,2,...,n.$$

The system of equations (13) allows us to define an unknown vector whose components are the parameters λ_r, $r=1,2,...,n$, through the integrals of the function φ. At fixed φ necessary and sufficient condition for existing unique solution to system of algebraic equations (13) is the invertibility of the matrix $Q(T)$.

Let's suppose that the matrix $Q(T)$ is invertible. Then the vector λ is determined from the system of algebraic equations (13) as follows:

$$\lambda = [Q(T)]^{-1}\{d-\Phi(\varphi,T)\}.$$

An existence of the invertible matrix $[Q(T)]^{-1}$ ensures compatibility of conditions (4) and (5).

This allows us to consider the problem (3), (4), (5) as a problem with initial and non-separated multipoint-integral conditions for the high-order ordinary differential equation, containing additional parameters.

3 Algorithm and conditions for its convergence

If the parameters are known λ_r, $r=1,2,...,n$, from Volterra integral equation (9) we get the function $\varphi(t)$ for all $t\in[0,T]$. Then, substituting of the parameters λ_r, $r=1,2,...,n$, and found function $\varphi(t)$ in the representations (7) and (8), we define the function $\widetilde{u}(t)$ and its derivatives for all $t\in[0,T]$. Further, taking into account the following equalities

$$u(t)=\widetilde{u}(t)+\lambda_1,\ \frac{du(t)}{dt}=\frac{d\widetilde{u}(t)}{dt}+\lambda_2,\ \frac{d^2u(t)}{dt^2}=\frac{d^2\widetilde{u}(t)}{dt^2}+\lambda_3,\ ...,\ \frac{d^{n-1}u(t)}{dt^{n-1}}=\frac{d^{n-1}\widetilde{u}(t)}{dt^{n-1}}+\lambda_n,$$

we find the solution of original problem (1), (2).

If a function $\varphi(t)$ is known, from the system of algebraic equations (13) we define the parameters λ_r, $r=1,2,...,n$. Then, substituting the function $\varphi(t)$ and founded parameters λ_r, $r=1,2,...,n$, into (7), (8), we determine the function $\widetilde{u}(t)$ and its derivatives for all $t\in[0,T]$. So, summing up

$$u(t)=\widetilde{u}(t)+\lambda_1,\ \frac{du(t)}{dt}=\frac{d\widetilde{u}(t)}{dt}+\lambda_2,\ \frac{d^2u(t)}{dt^2}=\frac{d^2\widetilde{u}(t)}{dt^2}+\lambda_3,\ ...,\ \frac{d^{n-1}u(t)}{dt^{n-1}}=\frac{d^{n-1}\widetilde{u}(t)}{dt^{n-1}}+\lambda_n,$$

we get the solution of original problem (1), (2).

Since, functions $\varphi(t)$, $\widetilde{u}(t)$ and parameters λ_r are unknown, we use an iterative process for finding solution of problem (3)-(5). The sequential approximations of the pairs $(\widetilde{u}^{(p)}(t),\lambda^{(p)})$ and $\varphi^{(p)}(t)$ are defined from the following algorithm.

Step 0. 1) Solving the system of algebraic equations (13) for $\varphi(t)=f(t)$, we get an initial

approximation $\lambda^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, ..., \lambda_n^{(0)})'$. 2) Solving the integral equation (9) for $\lambda_r = \lambda_r^{(0)}$, $r = 1, 2, ..., n$, we find $\varphi^{(0)}(t)$ for all $t \in [0, T]$. 3) In (7) and (8) substituting $\varphi^{(0)}(t)$ and $\lambda_r^{(0)}$ instead of $\varphi(t)$ and λ_r, $r = 1, 2, ..., n$, respectively, we determine the initial approximation $\widetilde{u}^{(0)}(t)$ and its derivatives $\frac{d^l\widetilde{u}^{(0)}(t)}{dt^l}$, $l = 1, 2, ..., n-1$, for all $t \in [0, T]$.
Step 1. 1) Solving the system of algebraic equations (13) for $\varphi(t) = \varphi^{(0)}(t)$, we get a first approximation $\lambda^{(1)} = (\lambda_1^{(1)}, \lambda_2^{(1)}, ..., \lambda_n^{(1)})'$. 2) Solving the integral equation (9) for $\lambda_r = \lambda_r^{(1)}$, $r = 1, 2, ..., n$, we find $\varphi^{(1)}(t)$ for all $t \in [0, T]$. 3) In (7) and (8) substituting $\varphi^{(1)}(t)$ and $\lambda_r^{(1)}$ instead of $\varphi(t)$ and λ_r, $r = 1, 2, ..., n$, respectively, we determine the first approximation $\widetilde{u}^{(1)}(t)$ and its derivatives $\frac{d^l\widetilde{u}^{(1)}(t)}{dt^l}$, $l = 1, 2, ..., n-1$, for all $t \in [0, T]$.

And so on.

Step p. 1) Solving the system of algebraic equations (13) for $\varphi(t) = \varphi^{(p-1)}(t)$, we get pth approximation $\lambda^{(p)} = (\lambda_1^{(p)}, \lambda_2^{(p)}, ..., \lambda_n^{(p)})'$. 2) Solving the integral equation (9) for $\lambda_r = \lambda_r^{(p)}$, $r = 1, 2, ..., n$, we find $\varphi^{(p)}(t)$ for all $t \in [0, T]$. 3) In (7) and (8) substituting $\varphi^{(p)}(t)$ and $\lambda_r^{(p)}$ instead of $\varphi(t)$ and λ_r, $r = 1, 2, ..., n$, respectively, we determine the pth approximation $\widetilde{u}^{(p)}(t)$ and its derivatives $\frac{d^l\widetilde{u}^{(p)}(t)}{dt^l}$, $l = 1, 2, ..., n-1$ for all $t \in [0, T]$.

The condition for realizability of the algorithm is invertibility of matrix $Q(T)$. Now, it is important to find out the conditions of convergence of the proposed algorithm, which ensure uniform convergence sequence of pairs $(\widetilde{u}^{(p)}(t), \lambda^{(p)})$ (and functions $\varphi^{(p)}(t)$) to pair $(\widetilde{u}^*(t), \lambda^*)$ is a solution of problem (3)-(5) (and $\varphi^*(t)$ is a solution of integral equation (9) for $\lambda_r = \lambda_r^*$) as $p \to \infty$ for all $t \in [0, T]$. Here $p = 1, 2, ...$

Consider problems (1), (2) and (3)-(5).

Let $\alpha = \max\limits_{(t,s)\in[0,T]\times[0,T]} |K(t, s)|, \quad \beta_j^k = \max\limits_{t\in[0,T]} |c_j^k(t)|, \ j = \overline{0, n-1}, \ k = \overline{1, n}.$

The following statement gives the conditions for convergence of algorithm proposed above, which at the same time guarantee the existence of a unique solution to problem (3)-(5).

Theorem 1. *Let*

1) the functions $a_i(t)$, $f(t)$ and $c_j^k(t)$ be continuous on $[0, T]$, where $i, k = \overline{1, n}$, $j = \overline{0, n-1}$, $s = \overline{0, m}$;

2) the $(n \times n)$ matrix $Q(T)$ be invertible and the following equalities are valid:
a) $||[Q(T)]^{-1}|| \le \gamma(T)$, where $\gamma(T)$ is a positive constant;
b)
$$q(T) = \gamma(T) \cdot \Bigg(\max_{k=\overline{1,n}} \sum_{j=0}^{n-1} \sum_{s=1}^{m} |b_{j,s}^k| \frac{1}{(n-j-1)!} \Big[e^{\alpha t_s} - 1 - \alpha t_s - ... - \frac{(\alpha t_s)^{n-j-1}}{(n-j-1)!} \Big] + \max_{k=\overline{1,n}} T \sum_{j=0}^{n-1} \beta_j^k \Big[e^{\alpha T} - 1 - \alpha T - ... - \frac{(\alpha T)^{n-j-1}}{(n-j-1)!} \Big] \Bigg) < 1.$$

Then, the sequential approximations $\widetilde{u}^{(p)}(t)$, $\lambda^{(p)}$ and $\varphi^{(p)}(t)$, determined by the algorithm, converge uniformly to $\widetilde{u}^*(t)$, λ^* and $\varphi^*(t)$, respectively, as $p \to \infty$ for all $t \in [0, T]$. Moreover, the pair $(\widetilde{u}^*(t), \lambda^*)$ is a unique solution to problem (3)-(5).

Proof of Theorem 1 is carried out according to the above algorithm.

From the equivalence of problems (3)-(5) and (1), (2) it follows that the function

$$u^*(t) = \widetilde{u}^*(t) + \lambda_1^*, \qquad t \in [0,T]$$

will be a solution of problem (1), (2).

Theorem 2. *Let*

1) the functions $a_i(t)$*,* $f(t)$ *and* $c_j^k(t)$ *are continuous on* $[0,T]$*, where* $i,k = \overline{1,n}$*,* $j = \overline{0,n-1}$*,* $s = \overline{0,m}$*;*

2) the $(n \times n)$ *matrix* $Q(T)$ *be invertible and the inequalities a), b) of Theorem 1. hold. Then the function* $u^*(t)$*, defined as the sum of functions* $\widetilde{u}^*(t) + \lambda_1^*$ *for all* $t \in [0,T]$ *is a unique solution to problem (1), (2).*

4 Main results

Consider problem (3)-(5) again. Let's put

$$K_1(t,s) = K(t,s), \qquad K_p(t,s) = \int\limits_s^t K(t,\eta)K_{p-1}(\eta,s)d\eta, \qquad p = 2,3,....$$

The function $K_p(t,s)$ is called mth iteration of the kernel $K(t,s)$. If we denote by M the norm of function $|K(t,s)|$ on $[0,T]\times[0,T]$, then the following estimate is true:

$$|K_p(t,s)| \leq M\frac{|t-s|^{p-1}}{(p-1)!}, \qquad (t,s) \in [0,T]\times[0,T].$$

Then the series $\Gamma(t,s) = \sum\limits_{p=1}^{\infty} K_p(t,s)$ converges uniformly on $[0,T]\times[0,T]$ and integral equation (9) has a unique solution in the following form

$$\varphi(t) = \int\limits_0^t \Gamma(t,s)\{f(s) + F(s,\lambda)\}ds + f(t) + F(t,\lambda), \qquad t \in [0,T]. \tag{14}$$

Here $\Gamma(t,s)$ is a resolvent of integral equation (9).

Substituting the expression (14) instead of $\varphi(t)$ into (7), we get a representation of desired function $\widetilde{u}(t)$ in the following form

$$\widetilde{u}(t) = \frac{1}{(n-1)!}\int\limits_0^t (t-s)^{n-1}\int\limits_0^s \Gamma(s,s_1)f(s_1)ds_1 ds + \frac{1}{(n-1)!}\int\limits_0^t (t-s)^{n-1}f(s)ds$$

$$+\frac{1}{(n-1)!}\int\limits_0^t (t-s)^{n-1}\int\limits_0^s \Gamma(s,s_1)\Big\{\sum_{l=0}^{n-1} a_{n-l}(s_1)\Big[\frac{s_1^{n-l-1}}{(n-l-1)!}\lambda_n+\frac{s_1^{n-l-2}}{(n-l-2)!}\lambda_{n-1}+...+\lambda_{l+1}\Big]$$

$$+\sum_{i=1}^{n} a_i(s_1)\lambda_{n+1-i}\Big\}ds_1ds+\frac{1}{(n-1)!}\int\limits_0^t (t-s)^{n-1}\Big\{\sum_{l=0}^{n-1} a_{n-l}(s)\Big[\frac{s^{n-l-1}}{(n-l-1)!}\lambda_n$$

$$+\frac{s^{n-l-2}}{(n-l-2)!}\lambda_{n-1}+...+\lambda_{l+1}\Big]+\sum_{i=1}^{n} a_i(s)\lambda_{n+1-i}\Big\}ds$$

$$+\frac{t^{n-1}}{(n-1)!}\lambda_n+\frac{t^{n-2}}{(n-2)!}\lambda_{n-1}+...+\frac{t}{1!}\lambda_2+\lambda_1. \qquad (15)$$

Similarly, let's define its derivatives by t:

$$\frac{d^l\widetilde{u}(t)}{dt^l}=\frac{1}{(n-1)!}\int\limits_0^t (t-s)^{n-l-1}\int\limits_0^s \Gamma(s,s_1)f(s_1)ds_1ds+\frac{1}{(n-1)!}\int\limits_0^t (t-s)^{n-l-1}f(s)ds$$

$$+\frac{1}{(n-1)!}\int\limits_0^t (t-s)^{n-l-1}\int\limits_0^s \Gamma(s,s_1)\Big\{\sum_{l=0}^{n-1} a_{n-l}(s_1)\Big[\frac{s_1^{n-l-1}}{(n-l-1)!}\lambda_n+\frac{s_1^{n-l-2}}{(n-l-2)!}\lambda_{n-1}+...+\lambda_{l+1}\Big]$$

$$+\sum_{i=1}^{n} a_i(s_1)\lambda_{n+1-i}\Big\}ds_1ds+\frac{1}{(n-1)!}\int\limits_0^t (t-s)^{n-l-1}\Big\{\sum_{l=0}^{n-1} a_{n-l}(s)\Big[\frac{s^{n-l-1}}{(n-l-1)!}\lambda_n$$

$$+\frac{s^{n-l-2}}{(n-l-2)!}\lambda_{n-1}+...+\lambda_{l+1}\Big]+\sum_{i=1}^{n} a_i(s)\lambda_{n+1-i}\Big\}ds$$

$$+\frac{t^{n-l-1}}{(n-l-1)!}\lambda_n+\frac{t^{n-l-2}}{(n-l-2)!}\lambda_{n-1}+...+\lambda_{l+1}, \qquad l=\overline{1,n-1}. \qquad (16)$$

Thus, the desired function $\widetilde{u}(t)$ and its derivatives are clearly expressed through the entered unknown parameters λ_r, $r=1,2,...,n$. Substituting the corresponding values of the function $\widetilde{u}(t)$ and its derivatives from the expressions (15), (16), when $t=t_s$, $s=\overline{1,m}$, $t=\tau$, into (5), we obtain the following system of n equations with respect to unknown parameters λ_r:

$$Q_*(T)\lambda=\widetilde{F}_*(f,T). \qquad (17)$$

Lemma 1. *Let the functions $a_i(t)$, $f(t)$ and $c_j^k(t)$ be continuous on $[0,T]$, where $i,k=\overline{1,n}$, $j=\overline{0,n-1}$, $s=\overline{0,m}$.*

Then the following assertions hold:

(i) the vector $\lambda^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_n^*)'$, *consisting of the values of the solution* $u^*(t)$ *and its derivatives* $\frac{d^l u^*(t)}{dt^l}$, $l = \overline{1, n-1}$, *to problem (1), (2) at the point* $t = 0$*:*

$$\lambda_1^* = u^*(0),\ \lambda_2^* = \frac{du^*(t)}{dt}\Big|_{t=0}, \lambda_3^* = \frac{d^2u^*(t)}{dt^2}\Big|_{t=0}, ..., \lambda_{n-1}^* = \frac{d^{n-2}u^*(t)}{dt^{n-2}}\Big|_{t=0}, \lambda_n^* = \frac{d^{n-1}u^*(t)}{dt^{n-1}}\Big|_{t=0},$$

satisfies system (17);

(ii) the function $u^{**}(t)$ *and its derivatives* $\frac{d^l u^{**}(t)}{dt^l}$, $l = \overline{1, n-1}$, *defined by the equalities:*

$$u^{**}(t) = \lambda_1^{**} + \widetilde{u}^{**}(t), \qquad \frac{du^{**}(t)}{dt} = \lambda_2^{**} + \frac{d\widetilde{u}^{**}(t)}{dt}, \qquad \frac{d^2u^{**}(t)}{dt^2} = \lambda_3^{**} + \frac{d^2\widetilde{u}^{**}(t)}{dt^2},$$

$$..., \ \frac{d^{n-2}u^{**}(t)}{dt^{n-1}} = \lambda_{n-1}^{**} + \frac{d^{n-2}\widetilde{u}^{**}(t)}{dt^{n-2}}, \qquad \frac{d^{n-1}u^{**}(t)}{dt^{n-1}} = \lambda_n^{**} + \frac{d^{n-1}\widetilde{u}^{**}(t)}{dt^{n-1}}, \qquad t \in [0, T],$$

where $\lambda^{**} = (\lambda_1^{**}, \lambda_2^{**}, ..., \lambda_n^{**})'$ *solves system (17) and the function* $u^{**}(t)$ *solves the Cauchy problem (3), (4) for* $\lambda_r = \lambda_r^{**}$, *and* $r = \overline{1, n}$, *is a solution to problem (1), (2).*

The existence of invertible matrix $[Q_*(T)]^{-1}$ ensures compatibility of conditions (4) and (5). It allows us to consider problem (3)-(5) as problem with non-separated multipoint-integral conditions for high-order ordinary differential equation, containing additional parameters. By substituting the components λ_r, $r = 1, 2, ..., n$, of obtained vector λ into the expression (2), we get representation of the desired solution of the problem (1), (2). From the above statement, it follows the next statement, which is setting sufficient conditions for the existence of a unique solution of problem with non-separated multipoint-integral conditions for high-order ordinary differential equation (1), (2).

Theorem 3. *Let*

1) the functions $a_i(t)$, $f(t)$ *and* $c_j^k(t)$ *be continuous on* $[0, T]$, *where* $i, k = \overline{1, n}$, $j = \overline{0, n-1}$, $s = \overline{0, m}$*;*

2) the $(n \times n)$ *matrix* $Q_*(T)$ *be invertible.*

Then the problem with non-separated multipoint-integral conditions for high-order ordinary differential equation (1), (2) has a unique solution.

Thus, the conditions for the unique solvability of problem (1), (2) are given in the terms of the matrix $Q_*(T)$, which is constructed by using of a resolvent $\Gamma(t, s)$ of Volterra integral equation of second kind (9). The kernel and right-hand side of the integral equation (9) are defined by the coefficients of the differential equation (1) and the non-separated multipoint-integral condition (2). It allows us to state that conditions for the solvability of the problem (1), (2) are set in the terms of the initial data. The solution of the integral equation (9) is an important issue and the construction of its approximate solutions relies on the calculation of iterative kernel $K_p(t, s)$.

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Иманчиев А.Е., Ермек А.А. ЖОҒАРЫ РЕТТІ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ҮШІН БӨЛІНБЕГЕН КӨПНҮКТЕЛІ-ИНТЕГРАЛДЫҚ ШАРТТАРЫ БАР ЕСЕПТЕРДІ ШЕШУДІҢ ПАРАМЕТРЛЕУ ӘДІСІ

Жоғары ретті дифференциалдық теңдеулер үшін бөлінбеген көпнүктелі-интегралдық шарттары бар есеп қарастырылды. Параметрлеу әдісінің алгоритмдері құрылды және олардың жинақтылығы дәлелденді. Зерттеліп отырған есептің бірмәнді шешілімділігінің жеткілікті шарттары орнатылды.

Кілттік сөздер. Бөлінбеген көпнүктелі-интегралдық шарттар, жоғары ретті дифференциалдық теңдеулер, параметрлеу әдісі, алгоритм, шешілімділік.

Иманчиев А.Е., Ермек А.А. МЕТОД ПАРАМЕТРИЗАЦИИ РЕШЕНИЯ ЗАДАЧ С НЕРАЗДЕЛЕННЫМИ МНОГОТОЧЕЧНО-ИНТЕГРАЛЬНЫМИ УСЛОВИЯМИ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВЫСОКОГО ПОРЯДКА

Рассмотрена задача с неразделенными многоточечно-интегральными условиями для дифференциальных уравнений высокого порядка. Построены алгоритмы метода параметризации и доказана их сходимость. Установлены достаточные условия однозначной разрешимости исследуемой задачи.

Ключевые слова. Неразделенные многоточечно-интегральные условия, дифференциальные уравнения высокого порядка, метод параметризации, алгоритм, разрешимость.

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New general solution of second order differential equation with piecewise-constant argument of generalized type and its application for solving boundary value problems

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Abstract. We consider second order differential equation with piecewise-constant argument of generalized type. An interval is divided into N parts, the values of a solution at the interior points of the subintervals are considered as additional parameters, and an ordinary differential equation with piecewise-constant argument of generalized type is reduced to the Cauchy problems on the subintervals for second order linear differential equations with parameters. Using the solutions to these problems, new general solutions to second order differential equations with piecewise-constant argument of generalized type are introduced and their properties are established. Based on the general solution, boundary condition, and continuity conditions of a solution at the interior points of the partition, the system of linear algebraic equations with respect to parameters is composed. Its coefficients and right hand sides are found by solving the Cauchy problems for linear ordinary differential equations on the subintervals. It is shown that the solvability of boundary value problems is equivalent to the solvability of systems composed. Methods for solving boundary value problems are proposed, which are based on the construction and solving of these systems.

Keywords. Differential equations with piecewise-constant argument of generalized type, Δ_N general solution, two-point boundary value problem, solvability criteria, algorithms of parameterization method.

Dedicated to the memory of Professor Dulat Dzhumabaev

1 Introduction

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On $[0,T]$ consider two-point boundary value problem for second order differential equation with piecewise-constant argument of generalized type

$$\ddot{x} = a_1(t)\dot{x}(t) + a_2(t)x(t) + a_3(t)\dot{x}(\gamma(t)) + a_4(t)x(\gamma(t)) + f(t), \tag{1}$$

$$b_{11}\dot{x}(0) + b_{21}x(0) + c_{11}\dot{x}(T) + c_{21}x(T) = d_1, \tag{2}$$

$$b_{12}\dot{x}(0) + b_{22}x(0) + c_{12}\dot{x}(T) + c_{22}x(T) = d_2, \tag{3}$$

where $x(t)$ is unknown function, the functions $a_i(t)$, $i = \overline{1,4}$, and $f(t)$ are continuous on $[0,T]$; $\gamma(t) = \zeta_j$ if $t \in [\theta_j, \theta_{j+1})$, $j = \overline{0, N-1}$; $\theta_j \le \zeta_j \le \theta_{j+1}$ for all $j = 0, 1, ..., N-1$; $0 = \theta_0 < \theta_1 < ... < \theta_{N-1} < \theta_N = T$, b_{sp}, c_{sp} and d_s are constants, where $s,p = 1,2$; $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

A function $x(t)$ is a solution to problem (1)-(3) if:

(i) $x(t)$ is continuously differentiable on $[0,T]$;

(ii) the second derivative $\ddot{x}(t)$ exists at each point $t \in [0,T]$ with the possible exception of the points θ_j, $j = \overline{0, N-1}$, where the one-sided derivatives exist; (iii) equation (1) is satisfied for $x(t)$ on each interval (θ_j, θ_{j+1}), $j = \overline{0, N-1}$, and it holds for the right second derivative of $x(t)$ at the points θ_j, $j = \overline{0, N-1}$;

(iv) boundary conditions (2), (3) are satisfied for $x(t)$ and $\dot{x}(t)$ at the points $t = 0$, $t = T$.

Differential equations with piecewise-constant argument of generalized type (DEPCAG) are introduced in the works [1-3].

Examples of the applications of these equations to the various problems have been under intensive investigation for the last decades.

Along with the study of various properties of differential equations with piecewise-constant argument, a number of authors investigated the questions of solvability and construction of solutions to boundary value problems for these equations on a finite interval. Particular attention was paid to periodic and multipoint boundary value problems for second order differential equations with piecewise-constant argument due to their wide application in natural sciences and engineering [4-15].

The aim of the present paper is to develop a constructive method for investigating and solving the boundary value problem, including an algorithm for finding a solution to problem (1), (2), (3) as well.

To this end, we use a new concept of general solution and parametrization's method [16]. This concept of general solution has been introduced for the linear Fredholm integro-differential equation in [17] and for the linear loaded differential equation and a family of such equations in [18, 19]. New general solutions are also introduced to ordinary differential equations and their properties are established in [20]. Results are developed to nonlinear Fredholm integro-differential equations [21]. Based on the general solution methods for solving boundary value problems are proposed.

The paper is organized as follows.

The interval $[0,T]$ is divided into N parts according to the partition Δ_N : $\theta_0 = 0 < \theta_1 < \theta_2 < ... < \theta_N = T$, and the Δ_N general solution to a linear second order differential equation with piecewise-constant argument of generalized type is introduced. The Δ_N general solution, denoted by $x(\Delta_N, t, \lambda)$, contains an arbitrary vectors $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N) \in \mathbb{R}^{2N}$. Using $x(\Delta_N, t, \lambda)$, we establish solvability criteria of considered problem and propose an algorithm for finding its solution.

At first, we reduce the problem for second order ordinary differential equation with piecewise-constant argument of generalized type (1)-(3) to a problem for system of two differential equations with piecewise-constant argument of generalized type.

For this we introduce a new functions $u_{(1)}(t) = x(t), \quad u_{(2)}(t) = \dot{x}(t), \quad t \in [0,T]$, and rewrite problem (1)-(3) in the following form

$$\frac{du}{dt} = A(t)u(t) + A_0(t)u(\gamma(t)) + g(t), \qquad t \in [0,T], \tag{4}$$

$$Bu(0) + Cu(T) = d, \tag{5}$$

where $u(t) = col(u_{(1)}(t), u_{(2)}(t))$ is unknown 2 dimensional vector function,

$$A(t) = \begin{pmatrix} 0 & 1 \\ a_2(t) & a_1(t) \end{pmatrix}, \qquad A_0(t) = \begin{pmatrix} 0 & 0 \\ a_4(t) & a_3(t) \end{pmatrix}, \qquad g(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix},$$

$$B = \begin{pmatrix} b_{21} & b_{11} \\ b_{22} & b_{12} \end{pmatrix}, \quad C = \begin{pmatrix} c_{21} & c_{11} \\ c_{22} & c_{12} \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$

A vector function $u(t) = col(u_{(1)}(t), u_{(2)}(t))$ is a solution to problem (4), (5) if:

(i) $u(t)$ is continuous on $[0,T]$;

(ii) the derivative $\dot{x}(t)$ exists at each point $t \in [0,T]$ with the possible exception of the points θ_j, $j = \overline{0, N-1}$, where the one-sided derivatives exist;

(iii) equation (4) is satisfied for $u(t)$ on each interval (θ_j, θ_{j+1}), $j = \overline{0, N-1}$, and it holds for the right derivative of $u(t)$ at the points θ_j, $j = \overline{0, N-1}$;

(iv) boundary condition (5) is satisfied for $u(t)$ at the points $t = 0$, $t = T$.

2 Scheme of the method and properties of new general solution

Denote by Δ_N a partition of the interval $[0,T)$:

$$[0,T) = \bigcup_{r=1}^{N} [\theta_{r-1}, \theta_r) \text{ by lines } \ t = \theta_j, \ \ j = \overline{1, N-1}.$$

Let

$C([0,T], \mathbb{R}^2)$ be the space of continuous functions $z : [0,T] \to \mathbb{R}^2$ with norm $\|z\|_1 = \max\limits_{t \in [0,T]} ||z(t)|| = \max\limits_{t \in [0,T]} \max\limits_{i=1,2} |z_i(t)|$;

$C([0,T], \Delta_N, \mathbb{R}^{2N})$ be the space of functions systems $z[t] = (z_1(t), z_2(t), \ldots, z_N(t))'$, where

$z_r : [\theta_{r-1}, \theta_r) \to \mathbb{R}^2$ are continuous and have finite left-hand side limits $\lim\limits_{t\to\theta_r-0} z_r(t)$ for all $r = \overline{1,N}$ with norm $\|z[\cdot]\|_2 = \max\limits_{r=\overline{1,N}} \sup\limits_{t\in[\theta_{r-1},\theta_r)} |z_r(t)|$.

Denote by $u_r(t)$ a restriction of function $u(t)$ on r-th interval $[\theta_{r-1}, \theta_r)$, i.e.

$u_r(t) = u(t)$ for $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1,N}$.

Then the function system $u[t] = (u_1(t), u_2(t), \ldots, u_N(t))$ belongs to $C([0,T], \Delta_N, \mathbb{R}^{2N})$, and its elements $u_r(t)$, $r = \overline{1,N}$, satisfy the following system of two ordinary differential equations with piecewise-constant argument of generalized type

$$\frac{du_r}{dt} = A(t)u_r(t) + A_0(t)u_r(\zeta_{r-1}) + g(t), \qquad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1,N}. \tag{6}$$

In (6) we take into account that $\gamma(t) = \zeta_j$ if $t \in [\theta_j, \theta_{j+1})$, $j = \overline{0, N-1}$.

Introduce an additional parameters $\lambda_r = u_r(\zeta_{r-1})$ for all $r = \overline{1,N}$. Making the substitution $z_r(t) = u_r(t) - \lambda_r$ on every r-th interval $[\theta_{r-1}, \theta_r)$, we obtain the system of two ordinary differential equations with parameters

$$\frac{dz_r}{dt} = A(t)(z_r(t) + \lambda_r) + A_0(t)\lambda_r + g(t), \qquad t \in [\theta_{r-1}, \theta_r), \qquad r = \overline{1,N}, \tag{7}$$

and initial conditions

$$z_r(\zeta_{r-1}) = 0, \qquad r = \overline{1,N}. \tag{8}$$

Problems (7), (8) are Cauchy problems for system of two ordinary differential equations with parameters on the intervals $[\theta_{r-1}, \theta_r)$, $r = \overline{1,N}$. For any fixed $\lambda_r \in \mathbb{R}^2$ and r, the Cauchy problem (7), (8) has a unique solution $z_r(t, \lambda_r)$, and the function system $z[t, \lambda] = (z_1(t, \lambda_1), z_2(t, \lambda_2), \ldots, z_N(t, \lambda_N))$ belongs to $C([0,T], \Delta_N, \mathbb{R}^{2N})$.

The function system $z[t, \lambda]$ is referred to as a solution to the Cauchy problems with parameters (7), (8). If a function system $\widetilde{u}[t] = (\widetilde{u}_1(t), \widetilde{u}_2(t), ..., \widetilde{u}_N(t))$ belongs to $C([0,T], \Delta_N, \mathbb{R}^{2N})$, and the functions $\widetilde{u}_r(t)$, $r = \overline{1,N}$, satisfy equations (6), then the function system $z[t, \widetilde{\lambda}] = (z_1(t, \widetilde{\lambda}_1), z_2(t, \widetilde{\lambda}_2), ..., z_N(t, \widetilde{\lambda}_N))$ with the elements $z_r(t, \widetilde{\lambda}_r) = \widetilde{u}_r(t) - \widetilde{\lambda}_r$, $\widetilde{\lambda}_r = \widetilde{u}_r(\zeta_{r-1})$, $r = \overline{1,N}$, is a solution to the Cauchy problems with parameters (7), (8) for $\lambda_r = \widetilde{\lambda}_r$, $r = \overline{1,N}$. Conversely, if a function system $z[t, \lambda^*] = (z_1(t, \lambda_1^*), z_2(t, \lambda_2^*), \ldots, z_N(t, \lambda_N^*))$ is a solution to problems (7), (8) for $\lambda_r = \lambda_r^*$, $r = \overline{1,N}$, then the function system $u^*[t] = (u_1^*(t), u_2^*(t), \ldots, u_N^*(t))$ with $u_r^*(t) = \lambda_r^* + z_r(t, \lambda_r^*)$, $r = \overline{1,N}$, belongs to $C([0,T], \Delta_N, \mathbb{R}^{2N})$, and the functions $u_r^*(t)$, $r = \overline{1,N}$, satisfy equations (6).

Let us now introduce a new general solution to the system of two ordinary differential equations with piecewise-constant argument of generalized type (4).

Definition 1. *Let* $z[t, \lambda] = (z_1(t, \lambda_1), z_2(t, \lambda_2), \ldots, z_N(t, \lambda_N))$ *be the solution to the Cauchy problems (7), (8) for the parameters* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N) \in \mathbb{R}^{2N}$. *Then the function* $u(\Delta_N, t, \lambda)$, *given by the equalities*

$u(\Delta_N, t, \lambda) = \lambda_r + z_r(t, \lambda_r)$, *for* $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$, *and*
$u(\Delta_N, T, \lambda) = \lambda_N + \lim\limits_{t \to T-0} z_N(t, \lambda_N)$,
is called the Δ_N *general solution to equation (4).*

As follows from Definition 1, the Δ_N general solution depends on N arbitrary vectors $\lambda_r \in \mathbb{R}^2$ and satisfies equation (4) for all $t \in (0, T) \backslash \{\theta_p, p = \overline{1, N-1}\}$.

Note that the first component of function $u(\Delta_N, t, \lambda)$: function $u_{(1)}(\Delta_N, t, \lambda)$ equals to $x(\Delta_N, t, \lambda)$. And function $x(\Delta_N, t, \lambda)$ is a Δ_N general solution to equation (1) and depends on $2N$ arbitrary constants $\lambda_{(1),r}, \lambda_{(2),r} \in \mathbb{R}$, where $\lambda_r = (\lambda_{(1),r}, \lambda_{(2),r}) \in \mathbb{R}^2$, $r = \overline{1, N}$.

Take $X_r(t)$, a fundamental matrix of the ordinary differential equation

$$\frac{dz_r}{dt} = A(t) z_r(t), \qquad t \in [\theta_{r-1}, \theta_r], \qquad r = \overline{1, N},$$

and write down the solutions to the Cauchy problems with parameters (7), (8) in the form:

$$z_r(t) = X_r(t) \int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)[A(\tau) + A_0(\tau)] d\tau \lambda_r + X_r(t) \int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau) g(\tau) d\tau,$$

$$t \in [\theta_{r-1}, \theta_r), \qquad r = \overline{1, N}.$$

Consider the Cauchy problems on the subintervals

$$\frac{du}{dt} = A(t) u + P(t), \qquad u(\zeta_{r-1}) = 0, \qquad t \in [\theta_{r-1}, \theta_r], \qquad r = \overline{1, N}, \tag{9}$$

where $P(t)$ is a square matrix or a vector of dimension 2, continuous on $[0, T]$, $\theta_{r-1} \leq \zeta_{r-1} \leq \theta_r$ for all $r = 1, 2, ..., N$. Denote by $A_r(P, t)$ a unique solution to the Cauchy problem (9) on each r-th interval. The uniqueness of the solution to the Cauchy problem for linear ordinary differential equations yields

$$A_r(P, t) = X_r(t) \int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau) P(\tau) d\tau, \qquad t \in [\theta_{r-1}, \theta_r], \qquad r = \overline{1, N}.$$

Therefore, we can represent the Δ_N general solution to equation (4) in the form:

$$u(\Delta_N, t, \lambda) = \lambda_p + A_p(A + A_0, t)\lambda_p + A_p(g, t), \qquad t \in [\theta_{p-1}, \theta_p), \quad p = \overline{1, N-1}, \tag{10}$$

$$u(\Delta_N, t, \lambda) = \lambda_N + A_N(A + A_0, t)\lambda_N + A_N(g, t), \qquad t \in [\theta_{N-1}, \theta_N]. \tag{11}$$

The following statement justifies the function $u(\Delta_N, t, \lambda)$ as a "general solution".

Theorem 1. *Let a piecewise continuous on $[0,T]$ function $\widetilde{u}(t)$ with the possible discontinuity points $t=\theta_p$, $p=\overline{1,N-1}$, be given, and $u(\Delta_N,t,\lambda)$ be the Δ_N general solution to equation (4). Suppose that the function $\widetilde{u}(t)$ has a continuous derivative and satisfies equation (4) for all $t\in(0,T)\backslash\{\theta_p,p=\overline{1,N-1}\}$. Then there exists a unique $\widetilde{\lambda}=(\widetilde{\lambda}_1,\widetilde{\lambda}_2,...,\widetilde{\lambda}_N)\in\mathbb{R}^{2N}$ such that the equality $u(\Delta_N,t,\widetilde{\lambda})=\widetilde{u}(t)$ holds for all $t\in[0,T]$.*

We omit the proof of this theorem which is quite straightforward.

Corollary 1. *Let $u^*(t)$ be a solution to equation (4) and $u(\Delta_N,t,\lambda)$ be the Δ_N general solution to equation (4). Then there exists a unique $\lambda^*=(\lambda_1^*,\lambda_2^*,\dots,\lambda_N^*)\in\mathbb{R}^{2N}$ such that the equality $u(\Delta_N,t,\lambda^*)=u^*(t)$ holds for all $t\in[0,T]$.*

If $u(t)$ is a solution to equation (4), and $u[t]=(u_1(t),u_2(t),...,u_N(t))$ is a function system composed of its restrictions to the subintervals $[\theta_{r-1},\theta_r)$, $r=\overline{1,N}$, then the equations

$$\lim_{t\to\theta_p-0}u_p(t)=u_{p+1}(\theta_p),\qquad p=\overline{1,N-1},\tag{12}$$

hold. These equations are the continuity conditions for the solution and its first derivative to equation (4) at the interior points of the partition Δ_N.

Theorem 2. *Let a function system $u[t]=(u_1(t),u_2(t),...,u_N(t))$ belong to $C([0,T],\Delta_N,\mathbb{R}^{2N})$. Assume that the functions $u_r(t)$, $r=\overline{1,N}$, satisfy equations (6) and continuity conditions (12). Then the function $u^*(t)$, given by the equalities $u^*(t)=u_r(t)$ for $t\in[\theta_{r-1},\theta_r)$, $r=\overline{1,N}$, and $u^*(T)=\lim\limits_{t\to T-0}u_N(t)$, is continuous on $[0,T]$, continuously differentiable on $(0,T)$ and satisfies equation (4).*

Proof. Equations (12), the equality $u^*(T)=\lim\limits_{t\to T-0}u_N(t)$, and belonging of $u[t]=(u_1(t),u_2(t),\ ...,u_N(t))$ to $C([0,T],\Delta_N,\mathbb{R}^{2N})$ provide continuity of the function $u^*(t)$ on the interval $[0,T]$. Since the functions $u_r(t)$, $r=\overline{1,N}$, satisfy equations (6), the function $u^*(t)$ has continuous derivative and satisfies equation (4) for all $t\in[0,T]\backslash\{\theta_p,p=\overline{1,N-1}\}$. The existence and continuity of the derivative of the function $u^*(t)$ at the points $t=\theta_p$, $p=\overline{1,N-1}$, follow from the relations:

$$\lim_{t\to\theta_p-0}\dot{u}^*(t)=A(\theta_p)u^*(\theta_p)+A_0(\theta_p)u^*(\zeta_{p-1})+g(\theta_p)=\lim_{t\to\theta_p+0}\dot{u}^*(t),\qquad p=\overline{1,N-1}.$$

Hence the function $u^*(t)$ satisfies equation (4) at the interior points of the partition Δ_N as well. Theorem 2 is proved.

3 Main results

The Δ_N general solution allows us to reduce the solvability of a boundary value problem to the solvability of a system of linear algebraic equations with respect to arbitrary vectors $\lambda_r\in\mathbb{R}^2$, $r=\overline{1,N}$.

Substituting the suitable expressions of Δ_N general solution (10), (11) into the boundary condition (5) and continuity conditions (12), we obtain the system of linear algebraic equations

$$B\lambda_1 + BA_1(A+A_0,\theta_0)\lambda_1 + C\lambda_N + CA_N(A+A_0,T)\lambda_N = d - BA_1(g,\theta_0) - CA_N(g,T), \quad (13)$$

$$\lambda_p + A_p(A,\theta_p)\lambda_p - \lambda_{p+1} - A_{p+1}(A+A_0,\theta_p)\lambda_{p+1} = -A_p(g,\theta_p) + A_{p+1}(g,\theta_p), \quad p = \overline{1, N-1}. \quad (14)$$

Denote by $Q_*(\Delta_N)$ $2N \times 2N$ matrix corresponding to the left-hand side of system (12), (13) and write the system as

$$Q_*(\Delta_N)\lambda = -F_*(\Delta_N), \qquad \lambda \in R^{2N}, \quad (15)$$

where $F_*(\Delta_N) = \big(-d + BA_1(g,\theta_0) + CA_N(g,T), A_1(g,\theta_1) - A_2(g,\theta_1), A_2(g,\theta_2) + A_3(g,\theta_2), ..., A_{N-1}(g,\theta_{N-1}) + A_N(g,\theta_{N-1})\big) \in \mathbb{R}^{2N}$.

For any partition Δ_N, Theorems 1 and 2 provide the validity of the following statement.

Lemma 1. *If $u^*(t)$ is a solution to problem (4), (5) and $\lambda_r^* = u^*(\zeta_{r-1})$, $r = \overline{1, N}$, then the vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_N^*) \in \mathbb{R}^{2N}$ is a solution to system (15). Conversely, if $\tilde\lambda = (\tilde\lambda_1, \tilde\lambda_2, \dots, \tilde\lambda_N) \in \mathbb{R}^{2N}$ is a solution to system (15) and $z[t,\tilde\lambda] = (z_1(t,\tilde\lambda_1), z_2(t,\tilde\lambda_2), \dots, z_N(t,\tilde\lambda_N))$ is the solution to Cauchy problems (7), (8) for the parameter $\tilde\lambda \in \mathbb{R}^{2N}$, then the function $\tilde u(t)$ given by the equalities $\tilde u(t) = \tilde\lambda_r + z_r(t,\tilde\lambda_r)$, $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$, and $\tilde u(T) = \tilde\lambda_N + \lim\limits_{t \to T-0} z_N(t,\tilde\lambda_N)$, is a solution to problem (4), (5).*

Definition 2. *The boundary value problem (4), (5) is called uniquely solvable if for any pair $(g(t), d)$, with $g(t) \in C([0,T], \mathbb{R}^2)$ and $d \in \mathbb{R}^2$, it has a unique solution.*

Lemma 1 and well known theorems of linear algebra imply the following two assertions.

Theorem 3. *The boundary value problem (4), (5) is solvable if and only if the vector $F_*(\Delta_N)$ is orthogonal to the kernel of the transposed matrix $(Q_*(\Delta_N))'$, i.e. iff the equality*

$$(F_*(\Delta_N), \eta) = 0$$

is valid for all $\eta \in Ker(Q_*(\Delta_N))'$, where $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^{2N}$.

Theorem 4. *The boundary value problem (4), (5) is uniquely solvable if and only if $2N \times 2N$ matrix $Q_*(\Delta_N)$ is invertible.*

Based on the results of Section 2, we propose the following algorithm for finding a solution to the linear boundary value problem (4), (5).

Step 1. Solve the Cauchy problems on the subintervals

$$\frac{dz}{dt} = A(t)z + A(t) + A_0(t), \qquad z(\zeta_{r-1}) = 0, \qquad t \in [\theta_{r-1}, \theta_r],$$

$$\frac{dz}{dt} = A(t)z + g(t), \qquad z(\zeta_{r-1}) = 0, \qquad t \in [\theta_{r-1}, \theta_r],$$

and find $A_r(A + A_0, \theta_r)$ and $A_r(g, \theta_r)$, $r = \overline{1, N}$. Here $\theta_{r-1} \le \zeta_{r-1} \le \theta_r$ for all $r = 1, 2, ..., N$.

Step 2. Using found matrices and vectors compose the system of linear algebraic equations (15).

Step 3. Solve the composed system and find $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*) \in \mathbb{R}^{2N}$. Note that the elements of λ^* are the values of the solution to problem (4), (5) at the interior points of the subintervals: $\lambda_r^* = u^*(\zeta_{r-1})$, $r = \overline{1, N}$.

Step 4. Solve the Cauchy problems

$$\frac{dz}{dt} = A(t)z + g(t), \qquad z(\zeta_{r-1}) = \lambda_r^*, \qquad t \in [\theta_{r-1}, \theta_r),$$

and define the values of the solution $u^*(t)$ at the remaining points of the subintervals.

First component of the function $u^*(t)$: function $u^*_{(1)}(t)$ equals to $x^*(t)$ and is a solution to original problem (1)-(3).

As it follows from Lemma 1, any solution to system (15) determines the values of the solution to problem (4), (5) at the beginning points of the subintervals.

The accuracy of the algorithm proposed depends on the accuracy of computing the coefficients and right-hand sides of system (15).

The Cauchy problem for ordinary differential equation is the main auxiliary problem in the algorithm proposed. By choosing an approximate method for solving that problem, we get an approximate method for solving the boundary value problem (4), (5). Solving the Cauchy problems by numerical methods leads to the numerical methods for solving problem (4), (5).

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Асанова А.Т. ЖАЛПЫЛАНҒАН ТҮРДЕГІ БӨЛІКТІ-ТҰРАҚТЫ АРГУМЕНТІ БАР ЕКІНШІ РКТТІ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУДІҢ ЖАҢА ЖАЛПЫ ШЕШІМІ ЖӘНЕ ОНЫҢ ШЕТТІК ЕСЕПТЕРДІ ШЕШУГЕ ҚОЛДАНЫСЫ

Жалпыланған түрдегі бөлікті-тұрақты аргументі бар екінші ретті дифференциалдық теңдеу қарастырылады. Аралық N бөлікке бөлінеді, шешімнің ішкіаралықтардың ішкі нүктелеріндегі мәндері қосымша параметрлер ретінде қарастырылады, ал жалпыланған түрдегі бөлікті-тұрақты аргументі бар жәй дифференциалдық теңдеу ішкіаралықтардағы параметрлері бар екінші ретті сызықты дифференциалдық теңдеулер үшін Коши есептерине келтіріледі. Осы есептердің шешімдерін пайдалана отырып жалпыланған түрдегі бөлікті-тұрақты аргументі бар екінші ретті дифференциалдық теңдеулердің жаңа жалпы шешімдері енгізіледі және олардың қасиеттері орнатылады. Жалпы шешімнің, шекаралық шарттың және шешімнің ішкі бөлу нүктелеріндегі үзіліссіздік шарттарының негізінде параметрлерге қатысты сызықты алгебралық теңдеулер жүйесі құрастырылады. Жүйенің коэффициенттері мен оң жақтары ішкіаралықтарда сызықты жәй дифференциалдық теңдеулер үшін Коши есептерін шешу арқылы табылады. Шеттік есептердің шешілімділігі құрастырылған жүйенің шешілімділігіне пара-пар екені көрсетілді. Осы жүйелерді құруға және шешуге негізделген шеттік есептерді шешудің әдістері ұсынылады.

Кілттік сөздер. Жалпыланған түрдегі бөлікті-тұрақты аргументі бар дифференциалдық теңдеулер, Δ_N жалпы шешім, қосnүктелі шеттік есеп, шешілімділік критерийлері, параметрлеу әдісінің алгоритмдері.

Асанова А.Т. НОВОЕ ОБЩЕЕ РЕШЕНИЕ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА С КУСОЧНО-ПОСТОЯННЫМ АРГУМЕНТОМ ОБОБЩЕННОГО ТИПА И ЕГО ПРИЛОЖЕНИЕ К РЕШЕНИЮ КРАЕВЫХ ЗАДАЧ

Рассматривается дифференциальное уравнение второго порядка с кусочно-постоянным аргументом обобщенного типа. Интервал разбивается на N частей, значения решения во внутренних точках подынтервалов рассматриваются как дополнительные параметры, а обыкновенное дифференциальное уравнение с кусочно-постоянным аргументом обобщенного типа сводится к задачам Коши на подынтервалах для линейных дифференциальных уравнений второго порядка с параметрами. Используя решения этих задач, вводятся новые общие решения дифференциальных уравнений второго порядка с кусочно-постоянным аргументом обобщенного типа и устанавливаются их свойства. На основе общего решения, граничного условия и условия непрерывности решения во внутренних точках разбиения составляется система линейных алгебраических уравнения относительно параметров. Коэффициенты и правые части системы находятся путем решения задач Коши для линейных обыкновенных дифференциальных уравнений на подынтервалах. Показано, что разрешимость краевых задач равносильна разрешимости составленных систем. Предлагаются методы решения краевых задач, основанные на построении и решении этих систем.

Ключевые слова. Дифференциальные уравнения с кусочно-постоянным аргументом обобщенного типа, Δ_N общее решение, двухточечная краевая задача, критерии разрешимости, алгоритмы метода параметризации.

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A numerical algorithm for solving a linear boundary value problem with a parameter

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Abstract. We consider a linear boundary value problem for an ordinary differential equation with a parameter. Based on the Dzhumabaev parametrization method, a numerical algorithm for solving the problem under consideration is proposed.

Keywords. Boundary value problem with parameter, numerical algorithm, general solution, parametrization method

In memory of Professor
Dulat Syzdykbekovich Dzhumabaev
with great gratefulness

1 Introduction

The paper deals with a linear boundary value problem with a parameter

$$\frac{dx}{dt} = A(t)x + B(t)\mu + f(t), \quad t \in (0,T), \quad x \in R^n, \quad \mu \in R^m, \tag{1}$$

$$C_0\mu + C_1x(0) + C_2x(T) = d, \quad d \in R^{m+n}, \tag{2}$$

where $n \times n$ matrix $A(t)$, $n \times m$ matrix $B(t)$ and $n-$dimensional vector $f(t)$ are continuous on $[0,T]$; $(n+m)\times m$ matrix C_0 and $(n+m)\times n$ matrices C_1 and C_2 are constant; $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

By a solution of problem (1), (2) we mean a pair $(\mu^*, x^*(t))$, where $\mu^* \in R^m$ and $x^*(t)$ is a function that is continuous on $[0,T]$ and continuously differentiable on $(0,T)$ and satisfies (1) and (2) with $\mu = \mu^*$.

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Boundary value problems with parameters for various classes of differential equations have been extensively studied by many authors (see [1-11] and references cited therein).

When a fundamental matrix $\Phi(t)$ of the differential equation $\frac{dx}{dt} = A(t)x$ is known, the general solution to Eq.(1) can be written as

$$x(t,\mu) = \Phi(t)c + \Phi(t)\int\limits_0^t \Phi^{-1}(\tau)[B(\tau)\mu + f(\tau)]d\tau, \tag{3}$$

where c is a n-dimensional constant vector. Substituting (3) into the boundary condition (2), we obtain the following equation for determining unknown parameters $c \in R^n$ and $\mu \in R^m$:

$$[C_0 + C_2\Phi(T)\int\limits_0^T \Phi^{-1}(\tau)B(\tau)d\tau]\mu + [C_1\Phi(0) + C_2\Phi(T)]c = d - C_2\Phi(T)\int\limits_0^T \Phi^{-1}(\tau)f(\tau)d\tau.$$

Hence, the existence of a unique solution to the boundary value problem (1),(2) is equivalent to the invertibility of the $(n+m)\times(n+m)$ matrix

$$D = (C_0 + C_2\Phi(T)\int\limits_0^T \Phi^{-1}(\tau)B(\tau)d\tau, \quad C_1\Phi(0) + C_2\Phi(T)).$$

In fact, it is usually impossible to explicitly find $\Phi(t)$, and hence the general solution, for a linear differential equation with variable coefficients. In this paper, we use a different approach to the concept of the general solution proposed by D.S.Dzhumabaev on the basis of the parametrization method [12]. He originally introduced new general solutions for linear Fredholm integro-differential equations [13] and later extended this concept to linear loaded differential equations and families of such equations [14], [15], nonlinear ordinary differential equations [16], and nonlinear Fredholm integro-differential equations [17]. The introduction of the new general solution provides a basis for new numerical and approximate methods for solving various classes of boundary value problems.

The aim of this paper is to apply the concept of the new general solution to Eq.(1) and develop a numerical algorithm for solving the boundary value problem with a parameter (1),(2).

2 The Δ_N-general solution to a linear ordinary differential equation with a parameter

Let Δ_N be a partition of $[0,T]$ into N subintervals by points $t_0 = 0 < t_1 < \ldots < t_{N-1} < t_N = T$. We will denote by $C([0,T],\Delta_N,R^{nN})$ the space of function systems $x[t] =$

$(x_1(t), x_2(t), \dots, x_N(t))$, where functions $x_r(t) : [t_{r-1}, t_r) \to R^n$ are continuous and have left-hand limits $\lim\limits_{t\to t_r-0} x_r(t)$ for all $r = \overline{1,N}$, with the norm $\|x[\cdot]\| = \max\limits_{r=\overline{1,N}} \sup\limits_{t\in[t_{r-1},t_r)} \|x_r(t)\|$.

Suppose that a pair $(\mu, x(t))$ is a solution to Eq.(1) and $x_r(t)$ is the restriction of $x(t)$ to the rth subinterval of Δ_N; that is, $x_r(t) = x(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$. Then the functions $x_r(t)$ satisfy the following system of equations:

$$\frac{dx_r}{dt} = A(t)x_r + B(t)\mu + f(t), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N}. \tag{4}$$

A solution to Eqs.(4) is a pair $(\hat{\mu}, \hat{x}[t])$ with $\hat{\mu} \in R^m$ and $\hat{x}[t] = (\hat{x}_1(t), \hat{x}_2(t), \dots, \hat{x}_N(t)) \in C([0,T], \Delta_N, R^{nN})$ satisfying Eqs. (4)

If $(\mu, x(t))$ is a solution to Eq.(1) and $x[t] = (x_1(t), x_2(t), \dots, x_N(t))$ is the system of the restrictions of $x(t)$ to the partition subintervals, then the components of this system must satisfy the continuity conditions at the interior points of Δ_N:

$$\lim_{t\to t_p-0} x_p(t) = x_{p+1}(t_p), \quad p = \overline{1, N-1}. \tag{5}$$

By introducing additional parameters $\lambda_r = x_r(t_{r-1})$, $r = \overline{1,N}$, and by substituting $u_r(t) = x_r(t) - \lambda_r$ on every subinterval $[t_{r-1}, t_r)$, from (4) we obtain the system of differential equations with parameters μ and λ_r

$$\frac{du_r}{dt} = A(t)(u_r + \lambda_r) + B(t)\mu + f(t), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N}, \tag{6}$$

subject to the initial conditions

$$u_r(t_{r-1}) = 0, \quad r = \overline{1,N}. \tag{7}$$

For fixed r, λ_r, and μ, each of Cauchy problems (6),(7) has a unique solution $u_r(t, \mu, \lambda_r)$. Let λ denote the vector of additional parameters λ_r: $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N) \in R^{nN}$. It is easily seen that the function system $u[t, \mu, \lambda] = (u_1(t, \mu, \lambda_1), u_2(t, \mu, \lambda_2), \dots, u_N(t, \mu, \lambda_N))$ belongs to $C([0,T], \Delta_N, R^{nN})$. We will refer to $u[t, \mu, \lambda]$ as a solution to the Cauchy problem with parameters (6),(7).

If a pair $(\tilde{\mu}, \tilde{x}[t])$ is a solution to Eq.(4), then the function system $u[t, \tilde{\mu}, \tilde{\lambda}]$ with elements $u_r(t, \tilde{\mu}, \tilde{\lambda}_r) = \tilde{x}_r(t) - \tilde{\lambda}_r$, $t \in [t_{r-1}, t_r)$, $\tilde{\lambda}_r = \tilde{x}_r(t_{r-1})$, $r = \overline{1,N}$, is a solution to the Cauchy problem with parameters (6),(7). Conversely, if a function system $u[t, \mu^*, \lambda^*]$ is a solution to the Cauchy problem (6),(7), then the pair $(\mu^*, x^*[t])$ with $x^*[t]$ composed of $x^*_r(t) = \lambda^*_r + u_r(t, \mu^*, \lambda^*_r)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, is a solution to Eq.(4).

Definition 1. *Let a function system $u[t, \mu, \lambda]$ be a solution to the Cauchy problem (6),(7) for some $\mu \in R^m$ and $\lambda \in R^{nN}$. Then the function $x(\Delta_N, t, \mu, \lambda)$ defined as*

$$x(\Delta_N, t, \mu, \lambda) = \lambda_r + u_r(t, \mu, \lambda_r), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N},$$

$$x(\Delta_N, T, \mu, \lambda) = \lambda_N + \lim_{t \to T-0} u_N(t, \mu, \lambda_N),$$

is called the Δ_N-general solution to Eq.(1).

It follows from the definition that the Δ_N-general solution depends on $m + nN$ arbitrary constants and satisfies Eq.(1) for all $t \in (0, T) \setminus \{t_p, p = \overline{1, N-1}\}$.

The following statement justifies the fact that the function $x(\Delta_N, t, \mu, \lambda)$ can be considered as a general solution to Eq.(1).

Theorem 1. *Suppose a function $\tilde{x}(t)$ satisfies the following conditions:*
(a) $\tilde{x}(t)$ is piecewise continuous on $[0, T]$ with possible discontinuities at the interior points $t = t_p$, $p = \overline{1, N-1}$, of the partition Δ_N;
(b) for all $t \in (0, T) \setminus \{t_p, p = \overline{1, N-1}\}$, the function $\tilde{x}(t)$ is continuously differentiable and satisfies Eq.(1) with some $\mu = \tilde{\mu} \in R^m$.
Then there exists a unique $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_N) \in R^{nN}$ such that equality $\tilde{x}(t) = x(\Delta_N, t, \tilde{\mu}, \tilde{\xi})$ holds for all $t \in [0, T]$.

Proof. Let $\tilde{x}(t)$ satisfy the conditions of theorem and its restrictions to the subintervals of Δ_N constitute the function system $\tilde{x}[t] = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N)$. Then the pair $(\tilde{\mu}, \tilde{x}[t])$ is a solution to Eq.(4). Setting $\tilde{\lambda}_r = \tilde{x}_r(t_{r-1})$, $r = \overline{1, N}$, and solving the Cauchy problems (6),(7) with $\mu = \tilde{\mu}$ and $\lambda_r = \tilde{\lambda}_r$, we get the functions $u_r(t, \tilde{\mu}, \tilde{\lambda}_r)$. Taking into account the relationship between the solutions to Eqs.(4) and those to the Cauchy problems (6),(7), we obtain

$$\tilde{x}(t) = \tilde{x}_r(t) = \tilde{\lambda}_r + u_r(t, \tilde{\mu}, \tilde{\lambda}_r) = x(\Delta_N, t, \tilde{\mu}, \tilde{\lambda}), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1, N},$$

$$\tilde{x}(T) = \tilde{\lambda}_N + \lim_{t \to T-0} u_N(t, \tilde{\mu}, \tilde{\lambda}_N) = x(\Delta_N, T, \tilde{\mu}, \tilde{\lambda}).$$

In order to prove the uniqueness of $\tilde{\lambda}$, assume that there is another parameter $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*) \in R^{nN}$ such that $\tilde{x}(t) = x(\Delta_N, t, \mu^*, \lambda^*)$ for all $t \in [0, T]$. Hence, by Definition 1, $\tilde{x}(t) = \tilde{x}_r(t) = \lambda_r^* + u_r(t, \mu^*, \lambda_r^*)$ for $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$, and $\tilde{x}(T) = \lambda_N^* + \lim\limits_{t \to T-0} u_N(t, \mu^*, \lambda_N^*)$, where $u_r(t, \mu^*, \lambda_r^*)$ are the solutions to the Cauchy problems (6),(7) with $\mu = \mu^*$ and $\lambda = \lambda^*$. Then, taking into account the initial conditions (7), we conclude that

$$\tilde{\lambda}_r = \tilde{x}_r(t_{r-1}) = \lambda_r^* + u_r(t, \mu^*, \lambda_r^*) = \lambda_r^*, \quad r = \overline{1, N}.$$

Theorem 1 is proved.

Corollary 1. *Let a pair $(\mu^*, x^*(t))$ be a solution to Eq.(1) and $x(\Delta_N, t, \mu, \lambda)$ be the Δ_N-general solution to Eq.(1). Then there exists a unique $\lambda^* \in R^{nN}$ such that the equality $x^*(t) = x(\Delta_N, t, \mu^*, \lambda^*)$ holds for all $t \in [0, T]$.*

Let $\Phi_r(t)$ be a fundamental matrix of the differential equation

$$\frac{dx}{dt} = A(t)x, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1, N}.$$

Then we can represent the solutions to the Cauchy problems (6),(7) in the form

$$u_r(t, \mu, \lambda_r) = \Phi_r(t) \int_{t_r}^{t} \Phi_r^{-1}(\tau)\{A(\tau)\lambda_r + B(\tau)\mu + f(\tau)\}d\tau, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1, N}. \quad (8)$$

Let us introduce the auxiliary Cauchy problems

$$\frac{dz}{dt} = A(t)z + P(t), \quad z(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1, N}, \quad (9)$$

where $P(t)$ is a square matrix or a vector continuous on $[0, T]$. Each of these problems has a unique solution $a_r(P, t)$ which can be written using $\Phi_r(t)$ as follows:

$$a_r(P, t) = \Phi_r(t) \int_{t_r}^{t} \Phi_r^{-1}(\tau)P(\tau)d\tau, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1, N}. \quad (10)$$

Thus the Δ_N-general solution to Eq.(1) can be represented in terms of solutions to auxiliary Cauchy problems in the following way:

$$x(\Delta_N, t, \mu, \lambda) = \lambda_r + a_r(A, t)\lambda_r + a_r(B, t)\mu + a_r(f, t), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1, N}, \quad (11)$$

$$x(\Delta_N, T, \mu, \lambda) = \lambda_N + a_N(A, T)\lambda_N + a_N(B, T)\mu + a_N(f, T). \quad (12)$$

Let us now turn to the boundary value problem with a parameter (1),(2) and make use of the Δ_N-general solution to Eq.(1) in solving this problem. As mentioned above, the Δ_N-general solution depends on $m+nN$ arbitrary constants that are components of vector-valued parameters $\mu \in R^m$ and $\lambda \in R^{nN}$. Substituting $x(\Delta, t, \mu, \lambda)$ into the boundary conditions (2) and the continuity conditions (5), we obtain the following system of linear algebraic equations in unknown parameters μ and λ:

$$[C_0 + C_2 a_N(B, T)]\mu + C_1\lambda_1 + [C_2 + C_2 a_N(A, T)]\lambda_N = d - C_2 a_N(f, T), \quad (13)$$

$$a_p(B, t_p)\mu + \lambda_p + a_p(A, t_p)\lambda_p - \lambda_{p+1} = -a_p(f, t_p), \quad p = \overline{1, N-1}. \quad (14)$$

Let $Q^*(\Delta_N)$ denote the square matrix of order $m + nN$ corresponding to the left-hand side of system (13),(14). Setting $\xi = (\mu, \lambda)$, we rewrite system (13),(14) in the matrix form

$$Q^*(\Delta_N)\xi = -F^*(\Delta_N), \quad \xi \in R^{m+nN}, \quad (15)$$

where $F^*(\Delta_N) = (-d + C_2 a_N(f, T), a_1(f, t_1), a_2(f, t_2), \ldots, a_{N-1}(f, t_{N-1})) \in R^{m+nN}$.

Theorem 2. *The boundary value problem (1),(2) is uniquely solvable if and only if the matrix $Q^*(\Delta_N)$ is invertible.*

3 An algorithm for solving problem (1),(2) and its numerical implementation

Using the results obtained in the previous section, we propose the following algorithm for finding a solution to the linear boundary value problem with a parameter (1),(2).

1. Choose a partition Δ_N of the interval $[0,T]$ and construct the Δ_N-general solution (11),(12).

2. Construct the system of linear algebraic equations (15) by substituting the Δ_N-general solution into the boundary conditions (2) and continuity conditions (5). Find the solution $\xi^* \in R^{m+nN}$ of system (15).

3. Substitute the components $\mu^* \in R^m$ and $\lambda^* \in R^{nN}$ of ξ^* into the Δ_N-general solution to get the solution $(\mu^*, x^*(t))$ to problem (1),(2).

The algorithm proposed relies on the Δ_N-general solution whose components, being solutions to Cauchy problems, are determined via fundamental matrices $\Phi_r(t)$. As we have mentioned above, for linear systems with variable coefficients there is no general way of getting a fundamental matrix. We therefore present a numerical version of the algorithm that involves numerical solution of Cauchy problems.

Step 1. Choose a partition Δ_N of the interval $[0,T]$. Solve the auxiliary Cauchy problems on the partition subintervals:

$$\frac{dz}{dt} = A(t)z + A(t), \quad z(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r),$$

$$\frac{dz}{dt} = A(t)z + B(t), \quad z(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r),$$

$$\frac{dz}{dt} = A(t)z + f(t), \quad z(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r),$$

and find $a_r(A, t_r)$, $a_r(B, t_r)$, and $a_r(f, t_r)$, $r = \overline{1, N}$.

Step 2. Construct the system of linear algebraic equations (15) and find its solution $\xi^* \in R^{m+nN}$. Note that the first component of $\xi^* = (\mu^*, \lambda_1^*, \ldots, \lambda_N^*)$ is the value of the parameter μ of problem (1),(2). The rest components are the values of the solution to problem (1),(2) at the left endpoints of the partition subintervals: $\lambda_r^* = x_r^*(t_{r-1})$, $r = \overline{1, N}$.

Step 3. Solve the Cauchy problems

$$\frac{dx}{dt} = A(t)x + f(t), \quad x(t_{r-1}) = \lambda_r^*, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1, N},$$

and determine the values of the solution $x^*(t)$ to problem (1),(2) at the remaining points of the partition Δ_N.

4 A numerical example

On the interval $[0, 4]$, we consider the following boundary value problem with a parameter

$$\frac{dx}{dt} = A(t)x + B(t)\mu + f(t), \quad x \in R^3, \quad \mu \in R^2, \tag{16}$$

$$x(0) = col(1, -3, 2), \quad x_1(T) = 1, \quad x_2(T) = 13, \tag{17}$$

where

$$A(t) = \begin{pmatrix} 1 & t & \sin 0.5\pi t \\ t^2 & 1-t & 0 \\ 0 & t^3 & \cos 0.5\pi t \end{pmatrix}, \quad B(t) = \begin{pmatrix} 2t & 3-t \\ 1 & t^2 \\ t-1 & t^2+1 \end{pmatrix},$$

$$f(t) = \begin{pmatrix} 0.25\pi\cos 0.25\pi t - \sin 0.25\pi t - 2\sin 0.5\pi t \cos 0.25\pi t - t^3 - 15t - 13 \\ -t^2 \sin 0.25\pi t + t^3 - 6t^2 - t - 8 \\ -0.5\pi \sin 0.25\pi t - 2\cos 0.5\pi t \cos 0.25\pi t - t^5 + 3t^3 - 4t^2 - 11t + 7 \end{pmatrix}.$$

To solve problem (16),(17), we used the proposed numerical algorithm with the Δ_5 partition of the interval $[0, 4]$ into 5 subintervals.

By solving the system of linear equations (15), we obtained the following values of the parameters μ and λ_r, $r = \overline{1,5}$.

$$\lambda_1 = \begin{pmatrix} 0.9999 \\ -2.9999 \\ 2.0000 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 1.5878 \\ -2.3600 \\ 1.6180 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1.9511 \\ -0.4399 \\ 0.6180 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 1.9511 \\ 2.7600 \\ -0.6180 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 1.5878 \\ 7.2400 \\ -1.6180 \end{pmatrix}, \mu = \begin{pmatrix} 10.9999 \\ 4.0000 \end{pmatrix}.$$

The exact solution to problem (16),(17) is the pair $(\mu^*, x^*(t))$ with

$$\mu^* = \begin{pmatrix} 11 \\ 4 \end{pmatrix} \quad \text{and} \quad x^*(t) = \begin{pmatrix} 1 + \sin 0.25\pi t \\ t^2 - 3 \\ 2\cos 0.25\pi t \end{pmatrix}, \ t \in [0, 4].$$

The maximum approximation error for obtained μ and λ_r, $r = \overline{1,5}$, is $5 \cdot 10^{-9}$.

The differences between the exact and approximate solutions to problem (16), (17) are provided in the following table:

Table 1. Absolute errors of the numerical solution

t	$\lvert x^*_{(1)}(t) - x_{(1)}(t)\rvert$	$\lvert x^*_{(2)}(t) - x_{(2)}(t)\rvert$	$\lvert x^*_{(3)}(t) - x_{(3)}(t)\rvert$
0	$0.1221245\cdot 10^{-14}$	$0.1332267\cdot 10^{-14}$	0
0.4	$0.0221004\cdot 10^{-8}$	$0.1224830\cdot 10^{-8}$	$0.1617531\cdot 10^{-8}$
0.8	$0.1587150\cdot 10^{-9}$	$0.0750204\cdot 10^{-9}$	$0.1350530\cdot 10^{-9}$
1.2	$0.0548728\cdot 10^{-7}$	$0.0563190\cdot 10^{-7}$	$0.1163317\cdot 10^{-7}$
1.6	$0.1903193\cdot 10^{-9}$	$0.2443955\cdot 10^{-9}$	$0.3542951\cdot 10^{-9}$
2.0	$0.0611203\cdot 10^{-6}$	$0.0950316\cdot 10^{-6}$	$0.3560913\cdot 10^{-6}$
2.4	$0.0993818\cdot 10^{-8}$	$0.1380603\cdot 10^{-8}$	$0.3227025\cdot 10^{-8}$
2.8	$0.0440663\cdot 10^{-5}$	$0.0964392\cdot 10^{-5}$	$0.1947860\cdot 10^{-5}$
3.2	$0.4853852\cdot 10^{-8}$	$0.2599317\cdot 10^{-8}$	$0.1151605\cdot 10^{-8}$
3.6	$0.0885035\cdot 10^{-5}$	$0.1912879\cdot 10^{-5}$	$0.5995351\cdot 10^{-5}$
4	$0.0143556\cdot 10^{-4}$	$0.0456637\cdot 10^{-4}$	$0.1030083\cdot 10^{-4}$

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Утешова Р.Е., Мурсалиев Д.Е. ПАРАМЕТРІ БАР СЫЗЫҚТЫ ШЕТТІК ЕСЕПТІ ШЕШУДІҢ САНДЫҚ АЛГОРИТМІ

Жай дифференциалдық теңдеу үшін параметрі бар сызықты шеттік есеп қарастырылады. Жұмабаев параметрлеу әдісі негізінде есепті шешудің сандық алгоритмі ұсынылады.

Кілттік сөздер. Параметрі бар шеттік есеп, сандық алгоритм, жалпы шешім, параметрлеу әдісі.

Утешова Р.Е., Мурсалиев Д.Е. ЧИСЛЕННЫЙ АЛГОРИТМ РЕШЕНИЯ ЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ С ПАРАМЕТРОМ

Рассматривается линейная краевая задача с параметром для обыкновенного дифференциального уравнения. Предлагается численный алгоритм решения рассматриваемой задачи на основе метода параметризации Джумабаева.

Ключевые слова. Краевая задача с параметром, численный алгоритм, общее решение, метод параметризации.

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An algorithm for solving multipoint boundary value problem for loaded differential and Fredholm integro-differential equations

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Abstract. A multipoint boundary value problem for the loaded differential and Fredholm integro-differential equations is considered. This problem is investigated by parameterization method. The interval is divided into parts, values of desired function at the initial points of subintervals are considered as additional parameters and the original equation is reduced to a system of integro-differential equations with parameters, where unknown functions satisfy the initial conditions on the subintervals. At the fixed values of parameters we get the special Cauchy problem for the system of linear integro-differential equations. The solution of the special Cauchy problem is constructed using the fundamental matrix of the differential equation. The system of linear algebraic equations with respect to the parameters are composed by substituting the values of the corresponding points in the boundary condition and the continuity conditions. Numerical method for solving the problem is suggested, based on the solution of the constructed system and the Bulirsch-Stoer method for solving the Cauchy problem on the subintervals.

Keywords. Loaded differential equation, integro-differential equation, multipoint problem, algorithm, parametrization method.

Dedicated to the memory of Professor Dulat Dzhumabaev

1 Introduction

Loaded differential equations are often used in applied mathematics. In particular, in [1, 2] these equations are used in solving the problems of long-term forecasting and regulating

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the level of groundwater and soil moisture. Note that the loaded differential equations in the literature are also called boundary differential equations [3]. Various problems for loaded differential equations and methods for finding their solutions are considered in [4-7]. Replacing the integral term of integro-differential equation by the quadrature formula, we obtain the loaded differential equation. Therefore, numerical and approximate methods for solving the boundary value problems for loaded differential equations are also used in solving the boundary value problems for integro-differential equations.

Integro-differential equations are often encountered in applications as mathematical models of various processes in natural sciences. Their role in the study of processes with after effects was noted in the monograph [8], and the overview of early works devoted to the initial and boundary value problems for integro-differential equations was provided as well. The solvability of various problems for the Fredholm integro-differential equations and approximate methods for finding their solutions are studied by many authors [9–21].

Statement of problem. Consider multipoint boundary value problems for the loaded differential and Fredholm integro-differential equations

$$\frac{dx}{dt} = A_0(t)\,x + \sum_{k=1}^{m} \int_0^T \varphi_k(t)\,\psi_k(s)x(s)ds + \sum_{i=1}^{N} A_i(t)x(\theta_i) + f(t), \quad t \in (0,T), \tag{1}$$

$$\sum_{p=0}^{N} B_p x(\theta_p) = d, \quad d \in R^n, \qquad x \in R^n, \tag{2}$$

where the matrises $A_j(t)$, $j = \overline{0,N}$, $\varphi_k(t)$ and $\psi_k(\tau)$, $k = \overline{1,m}$, and the vector $f(t)$ are continuous on $[0,T]$; B_p, $p = \overline{0,N}$, are constant matrices.

Let $C([0,T],\ R^n)$ denote the space of continuous on $[0,T]$ functions $x(t)$ with the norm $\|x\|_1 = \max_{t\in[0,T]} \|x(t)\|$.

Solution to problem (1), (2) is a continuously differentiable on $(0,T)$ function $x(t) \in C([0,T],\ R^n)$ satisfying the system of the loaded differential and Fredholm integro-differential equations (1) and multipoint boundary condition (2).

2 Scheme of parametrization method

The interval $[0,T)$ is divided into $N+1$ parts by the points $\theta_0 = 0 < \theta_1 < \cdots < \theta_N < \theta_{N+1} = T$ and partition $[0,T) = \bigcup_{r=1}^{N+1} [\theta_{r-1},\ \theta_r)$ is denoted by $\triangle_N$.

$C([0,T],\ \triangle_N,\ R^{n(N+1)})$ is the space of function systems $x[t] = (x_1(t), x_2(t), \ldots, x_{N+1}(t))$, where $x_r : [\theta_{r-1},\ \theta_r) \to R^n$ are continuous on $[\theta_{r-1},\ \theta_r)$ and have finite left-sided limits $\lim\limits_{t\to\theta_r-0} x_r(t)$ for all $r = \overline{1,N+1}$ with the norm $\|x[\cdot]\|_2 = \max\limits_{r=\overline{1,N+1}} \sup\limits_{t\in[\theta_{r-1},\ \theta_r)} \|x_r(t)\|$.

The restriction of the function $x(t)$ to the r-th interval $[\theta_{r-1},\ \theta_r)$ is denoted by $x_r(t)$, i.e. $x_r(t) = x(t)$ for $t \in [\theta_{r-1},\ \theta_r)$, $r = \overline{1,N+1}$. Then we introduce additional parameters $\lambda_r = x_r(\theta_{r-1})$, $r = \overline{1,N+1}$, and make the substitution $x_r(t) = u_r(t) + \lambda_r$ on each r-th interval $[\theta_{r-1},\ \theta_r)$, $r=\overline{1,N+1}$, we obtain the multipoint boundary value problem with parameters

$$\frac{du_r}{dt}=A_0(t)\left[u_r+\lambda_r\right]+\sum_{j=1}^{N+1}\sum_{k=1}^{m}\int_{\theta_{j-1}}^{\theta_j}\varphi_k(t)\psi_k(s)\left[u_j+\lambda_j\right]ds+\sum_{i=1}^{N}A_i\left(t\right)\lambda_{i+1}+f(t),\qquad(3)$$

$$t\in[\theta_{r-1},\ \theta_r),\qquad r=\overline{1,N+1},$$

$$u_r(\theta_{r-1})=0,\qquad r=\overline{1,N+1},\qquad(4)$$

$$\sum_{p=0}^{N-1}B_p\lambda_{p+1}+B_N\lambda_{N+1}+B_N\lim_{t\to T-0}u_{N+1}(t)=d,\qquad(5)$$

$$\lambda_s+\lim_{t\to\theta_s-0}u_s(t)\ =\lambda_{s+1},\qquad s=\overline{1,N},\qquad(6)$$

where (6) are conditions for matching the solution at the interior points of the partition $\triangle_N$.

Problems (1), (2) and (3)-(6) are equivalent. If $x^*(t)$ is a solution to multipoint problem (1), (2), then the pair $(u^*\left[t\right],\ \lambda^*)$, where $u^*[t]=(x^*(t)-x^*(\theta_0),x^*(t)-x^*(\theta_1),\ldots,x^*(t)-x^*(\theta_N))$, and $\lambda^*=(x^*(\theta_0),x^*(\theta_1),\ldots,x^*(\theta_N))$, is a solution to the problem (3)-(6). Conversely, if a pair $\left(\tilde{u}\left[t\right],\ \widetilde{\lambda}\right)$ with elements $\tilde{u}\left[t\right]=\left(\tilde{u}_1\left(t\right),\tilde{u}_2\left(t\right),\ldots,\tilde{u}_{N+1}(t)\right)\in C([0,T],\ \triangle_N,\ R^{n(N+1)})$, $\widetilde{\lambda}=(\widetilde{\lambda}_1,\widetilde{\lambda}_2,\ldots,\widetilde{\lambda}_{N+1})\in R^{n(N+1)}$, is a solution to problem (3)-(6), then the function $\tilde{x}\left(t\right)$ defined by the equalities $\tilde{x}\left(t\right)=\tilde{u}_r\left(t\right)+\widetilde{\lambda}_r$, $t\in[\theta_{r-1},\ \theta_r)$, $r=\overline{1,N+1}$, $\tilde{x}\left(T\right)=\lim\limits_{t\to T-0}\tilde{u}_{N+1}\left(t\right)\ +\widetilde{\lambda}_{N+1}$, is a solution to the original problem (1), (2).

For fixed λ_j problem (3), (4) is a special Cauchy problem for the system of Fredholm integro-differential equations. We have $N+1$ Cauchy problems on the intervals $[\theta_{r-1},\ \theta_r)$, r $=\overline{1,N+1}$, and the system of integro-differential equations includes the sum of integrals of all $N+1$ functions $u_r\left(t\right)$ with degenerate kernels on the segments $[\theta_{r-1},\ \theta_r]$.

If $X_r(t)$ is a fundamental matrix of the differential equation $\frac{dx_r}{dt}=A(t)x_r$ on $[\theta_{r-1},\ \theta_r]$, then the special Cauchy problem for the system of integro-differential equations with parameters (3), (4) is reduced to the equivalent system of integral equations

$$u_r(t)=X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)A_0(\tau)d\tau\lambda_r+X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)\sum_{i=1}^{N}A_i(t)\lambda_{i+1}d\tau$$

$$+X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)\sum_{j=1}^{N+1}\sum_{k=1}^{m}\int_{\theta_{j-1}}^{\theta_j}\varphi_k(\tau)\psi_k(s)[u_j(s)+\lambda_j]dsd\tau$$

$$+X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)f(\tau)d\tau,\quad t\in[\theta_{r-1},\theta_r),\qquad r=\overline{1,N+1}.\qquad(7)$$

Let $\mu_k = \sum_{j=1}^{N+1} \int_{\theta_{j-1}}^{\theta_j} \psi_k(s) u_j(s) ds,\ k = \overline{1,m}$, and rewrite (7) in the following form

$$u_r(t) = \sum_{k=1}^{m} X_r(t) \int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\, \varphi_k(\tau)\, d\tau \mu_k$$

$$+X_r(t) \int_{\theta_{r-1}}^{t} X_r^{-1}(\tau) A_0(\tau) d\tau \lambda_r + X_r(t) \int_{\theta_{r-1}}^{t} X_r^{-1}(\tau) \sum_{i=1}^{N} A_i(t) \lambda_{i+1} d\tau$$

$$+X_r(t) \int_{\theta_{r-1}}^{t} X_r^{-1}(\tau) \sum_{k=1}^{m} \varphi_k(\tau) \sum_{j=1}^{N+1} \int_{\theta_{j-1}}^{\theta_j} \psi_k(s) \lambda_j ds d\tau$$

$$+X_r(t) \int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\, f(\tau)\, d\tau, \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, N+1}. \tag{8}$$

Multiplying both sides of (8) by $\psi_p(t)$, integrating on the interval $[\theta_{r-1},\ \theta_r]$ and summing up with respect to r, we get the following system of linear algebraic equations in $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in R^{nm}$:

$$\mu_p = \sum_{k=1}^{m} G_{p,k}(\triangle_N)\mu_k + \sum_{r=1}^{N+1} V_{p,r}(\triangle_N)\lambda_r + \sum_{j=1}^{N} W_{p,j}(\triangle_N)\lambda_{j+1} + g_p(f, \triangle_N), \quad p = \overline{1,m}, \tag{9}$$

with the $(n \times n)$ matrices

$$G_{p,k}(\triangle_N) = \sum_{r=1}^{N+1} \int_{\theta_{r-1}}^{\theta_r} \psi_p(\tau)\, X_r(\tau) \int_{\theta_{r-1}}^{\tau} X_r^{-1}(s)\, \varphi_k(s)\, ds d\tau, \quad k = \overline{1,m},$$

$$V_{p,r}(\triangle_N) = \int_{\theta_{r-1}}^{\theta_r} \psi_p(\tau)\, X_r(\tau) \int_{\theta_{r-1}}^{\tau} X_r^{-1}(s)\, A_0(s)\, ds d\tau + \sum_{j=1}^{N+1} \sum_{k=1}^{m} \int_{\theta_{j-1}}^{\theta_j} \psi_p(\tau)$$

$$\times X_j(\tau) \int_{\theta_{j-1}}^{\tau} X_j^{-1}(\tau_1)\, \varphi_k(\tau_1)\, d\tau_1 d\tau \int_{\theta_{r-1}}^{\theta_r} \psi_k(s)\, ds, \quad r = \overline{1, N+1},$$

$$W_{p,j}(\triangle_N) = \sum_{r=1}^{N+1} \int_{\theta_{r-1}}^{\theta_r} \psi_p(\tau)\, X_r(\tau) \int_{\theta_{r-1}}^{\tau} X_r^{-1}(s)\, A_j(s)\, ds d\tau, \quad j = \overline{1,N},$$

and vectors of dimension n

$$g_p(f, \triangle_N) = \sum_{r=1}^{N+1} \int_{\theta_{r-1}}^{\theta_r} \psi_p(\tau)\, X_r(\tau) \int_{\theta_{r-1}}^{\tau} X_r^{-1}(s)\, f(s)\, ds d\tau, \quad p = \overline{1,m},$$

Using the matrices $G_{p,k}(\triangle_N)$, $V_{p,r}(\triangle_N)$, $W_{p,j}(\triangle_N)$, we construct matrices $G(\triangle_N) = (G_{p,k}(\triangle_N))$, $p,k=\overline{1,m}$, and $V(\triangle_N) = (V_{p,r}(\triangle_N))$, $p=\overline{1,m}$, $r=\overline{1,N+1}$, and $W(\triangle_N) = (W_{p,j}(\triangle_N))$, $p=\overline{1,m}$, $j=\overline{1,N}$. Then, system (9) can be rewritten in the form

$$[I - G(\triangle_N)]\,\mu = V(\triangle_N)\,\lambda + W(\triangle_N)\,\xi + g(f,\triangle_N), \tag{10}$$

where I is the identity matrix of dimension nm, and

$$g(f,\triangle_N) = (g_1(f,\triangle_N), g_2(f,\triangle_N), \ldots, g_m(f,\triangle_N)) \in R^{nm}.$$

Definition 1. *Partition $\triangle_N$ is called regular if the matrix $I - G(\triangle_N)$ is invertible.*

Let $\sigma(m,[0,T])$ denote the set of regular partitions $\triangle_N$ of $[0,T]$ for the equation (1).

Definition 2. *The special Cauchy problem (3), (4) is called uniquely solvable, if for any $\lambda \in R^{n(N+1)}$, $f(t) \in C([0,T],R^n)$ it has a unique solution.*

The special Cauchy problem (3), (4) is equivalent to the system of integral equations (7). This system by virtue of the kernel degeneracy is equivalent to the system of algebraic equations (9) with respect to $\mu = (\mu_1,\mu_2,\ldots,\mu_m) \in R^{nm}$. Therefore, the special Cauchy problem is uniquely solvable if and only if the partition $\triangle_N$, generating this problem, is regular.

Since the special Cauchy problem is uniquely solvable for sufficiently small partition step $h>0$, the set $\sigma(m,[0,T])$ is not empty.

Take $\triangle_N \in \sigma(m,[0,T])$ and present $[I-G(\triangle_N)]^{-1}$ in the form

$$[I-G(\triangle_N)]^{-1} = (M_{k,p}(\triangle_N)),\, k,p=\overline{1,m},$$

where $M_{k,p}(\triangle_N)$ are the square matrices of dimension n.

Then taking into account (10), we can determine elements of the vector $\mu \in R^{nm}$ from the equalities

$$\mu_k = \sum_{j=1}^{N+1}\Big(\sum_{p=1}^{m} M_{k,p}(\triangle_N)V_{p,j}(\triangle_N\Big)\lambda_j + \sum_{j=1}^{N}\Big(\sum_{p=1}^{m} M_{k,p}(\triangle_N)W_{p,j}(\triangle_N)\Big)\lambda_{j+1}$$
$$+\sum_{p=1}^{m} M_{k,p}(\triangle_N)g_p(f,\triangle_N),\, k=\overline{1,m}, \tag{11}$$

In (8), substituting the right-hand side of (11) instead of μ_k, we get the representation of functions $u_r(t)$ through λ_j, $\;j=\overline{1,N+1}$:

$$u_r(t) = \sum_{j=1}^{N+1}\left\{\sum_{k=1}^{m} X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\,\varphi_k(\tau)\,d\tau \sum_{p=1}^{m} M_{k,p}(\triangle_N)\,V_{p,j}(\triangle_N)\right\}\lambda_j$$

$$+\sum_{j=1}^{N+1}\left\{\sum_{k=1}^{m}X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)\varphi_k(\tau)d\tau\int_{\theta_{j-1}}^{\theta_j}\psi_k(s)ds\right\}\lambda_j$$

$$+\sum_{j=1}^{N}\left\{\sum_{k=1}^{m}X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)\varphi_k(\tau)d\tau\sum_{p=1}^{m}M_{k,p}(\triangle_N)W_{p,j}(\triangle_N)\right\}\lambda_{j+1}$$

$$+X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)A_0(\tau)d\tau\lambda_r+X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)\sum_{i=1}^{N}A_i(t)\lambda_{i+1}d\tau$$

$$+X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)\left[\sum_{k=1}^{m}\varphi_k(\tau)\sum_{p=1}^{m}M_{k,p}(\triangle_N)g_p(f,\triangle_N)+f(\tau)\right]d\tau, \quad r=\overline{1,N+1}. \quad (12)$$

Introduce the notations:

$$D_{r,j}(\triangle_N)=\sum_{k=1}^{m}X_r(\theta_r)\int_{\theta_{r-1}}^{\theta_r}X_r^{-1}(\tau)\varphi_k(\tau)d\tau\left[\sum_{p=1}^{m}M_{k,p}(\triangle_N)V_{p,j}(\triangle_N)\right.$$

$$\left.+\int_{\theta_{j-1}}^{\theta_j}\psi_k(s)ds\right], \quad j\neq r, \quad r,j=\overline{1,N+1},$$

$$D_{r,r}(\triangle_N)=\sum_{k=1}^{m}X_r(\theta_r)\int_{\theta_{r-1}}^{\theta_r}X_r^{-1}(\tau)\varphi_k(\tau)d\tau\left[\sum_{p=1}^{m}M_{k,p}(\triangle_N)V_{p,r}(\triangle_N)\right.$$

$$\left.+\int_{\theta_{r-1}}^{\theta_r}\psi_k(s)ds\right]+X_r(\theta_r)\int_{\theta_{r-1}}^{\theta_r}X_r^{-1}(\tau)A_0(\tau)d\tau, r=\overline{1,N+1},$$

$$E_{r,j}(\triangle_N)=\sum_{k=1}^{m}X_r(\theta_r)\int_{\theta_{r-1}}^{\theta_r}X_r^{-1}(\tau)\varphi_k(\tau)d\tau\sum_{p=1}^{m}M_{k,p}(\triangle_N)W_{p,j}(\triangle_N)$$

$$+X_r(\theta_r)\int_{\theta_{r-1}}^{\theta_r}X_r^{-1}(\tau)A_j(t)d\tau, \ j=\overline{1,N},$$

$$F_r(\triangle_N)=\sum_{k=1}^{m}X_r(\theta_r)\int_{\theta_{r-1}}^{\theta_r}X_r^{-1}(\tau)\left[\varphi_k(\tau)\sum_{p=1}^{m}M_{k,p}(\triangle_N)g_p(f,\triangle_N)+f(\tau)\right]d\tau,$$

Then from (12) we get

$$\lim_{t\to\theta_r-0}u_r(t)=\sum_{j=1}^{N+1}D_{r,j}(\triangle_N)\lambda_j+\sum_{j=1}^{N}E_{r,j}(\triangle_N)\lambda_{j+1}+F_r(\triangle_N). \quad (13)$$

Substituting the right-hand side of (13) into condition (5) and conditions of matching solution (6), we have the following system of linear algebraic equations with respect to parameters λ_r, $r=\overline{1,N+1}$:

$$[B_0+B_N D_{N+1,1}(\triangle_N)]\,\lambda_1+\sum_{p=1}^{N-1}B_p\lambda_{p+1}+\sum_{j=2}^{N}B_N D_{N+1,j}(\triangle_N)\,\lambda_j$$

$$+B_N\,[I+D_{N+1,N+1}(\triangle_N)]\,\lambda_{N+1}+\sum_{j=1}^{N}B_N E_{N+1,j}(\triangle_N)\,\lambda_{j+1}=d-B_N F_{N+1}(\triangle_N)\,, \quad (14)$$

$$[I+D_{s,s}(\triangle_N)]\,\lambda_s-[I-D_{s,s+1}(\triangle_N)]\,\lambda_{s+1}$$

$$+\sum_{j=1}^{N}D_{s,j}(\triangle_N)\,\lambda_{j+1}=-F_s(\triangle_N)\,,\quad s=\overline{1,N}. \quad (15)$$

Denoting by $Q_*(\triangle_N)$ the matrix corresponding to the left-hand side of the system of equations (14), (15), we get

$$Q_*(\triangle_N)\,\lambda=-F_*(\triangle_N)\,,\quad \lambda\in R^{n(N+1)}\,, \quad (16)$$

where $F_*(\triangle_N)=(-d+B_N F_{N+1}(\triangle_N)\,,F_1(\triangle_N)\,,\ldots,F_N(\triangle_N))$.

It is not difficult to establish that the solvability of the boundary value problem (6), (7) is equivalent to the solvability of the system (16). The solution of the system (16) is a vector $\lambda^*=(\lambda_1^*,\lambda_2^*,\ldots,\lambda_{N+1}^*)\in R^{n(N+1)}$ consisting of the values of the solutions of the original problem (6), (7) in the initial points of subintervals, i.e. $\lambda_r^*=x^*(\theta_{r-1})\,,\ \ \mathrm{r}=\overline{1,N+1}$.

Further we consider the Cauchy problems for ordinary differential equations on subintervals

$$\frac{dz}{dt}=A(t)\,z+P(t)\,,\quad z(\theta_{r-1})=0,\quad t\in[\theta_{r-1},\ \theta_r]\,,\quad \mathrm{r}=\overline{1,N+1}, \quad (17)$$

where $P(t)$ is $(n\times n)$ matrix or n vector that is continuous on $[\theta_{r-1},\ \theta_r]$, $\mathrm{r}=\overline{1,N+1}$. Consequently, the solution to problem (17) is a square matrix or a vector of dimension n. Let $a(P,t)$ denote a solution to the Cauchy problem (17). Clearly,

$$a(P,t)=X_r(t)\int_{\theta_{r-1}}^{t}X_r^{-1}(\tau)\,\mathrm{P}(\tau)\,d\tau,\quad t\in[\theta_{r-1},\ \theta_r],$$

where $X_r(t)$ is a fundamental matrix of differential equation (17) on the r-th interval.

3 Numerical implementation of the parametrization method

We offer the following numerical implementation of the parametrization method. The algorithm is based on the Bulirsch-Stoer method to solve the Cauchy problems for ordinary differential equations and it is based on Simpson's method for the estimation of definite integrals.

1. We divide $[0,T]$ into N + 1 parts by the points $0=\theta_0<\theta_1<\cdots<\theta_N<\theta_{N+1}=T$, involved in the multipoint condition. Divide each r-th interval $[\theta_{r-1},\ \theta_r]$, r $=\overline{1,N+1}$, into N_r parts with step $h_r=(\theta_r-\theta_{r-1})/N_r$. Assume that on each interval $[\theta_{r-1},\ \theta_r]$ the variable $\widehat{\theta}$ takes its discrete values: $\widehat{\theta}=\theta_{r-1}$, $\widehat{\theta}=\theta_{r-1}+h_r,\ \ldots,\ \widehat{\theta}=\theta_{r-1}+(N_r-1)h_r,\ \widehat{\theta}=\theta_r$, and denote by $\{\theta_{r-1},\ \theta_r\}$ the set of such points.

2. Using the Bulirsch-Stoer method, we find numerical solutions to Cauchy problems (17) and define values of $(n\times n)$ matrices $a_r^{h_r}\left(\varphi_k,\widehat{\theta}\right)$ on the set $\{\theta_{r-1},\ \theta_r\}$, $r=\overline{1,N+1}$, $k=\overline{1,m}$.

3. Using the values of $(n\times n)$ matrices $\psi_k(s)$ and $a_r^{h_r}\left(\varphi_k,\widehat{\theta}\right)$ on $\{\theta_{r-1},\ \theta_r\}$ and Simpson's method, we calculate the $(n\times n)$ matrices

$$\widehat{\psi}_{p,r}^{h_r}(\varphi_k)=\int_{\theta_{r-1}}^{\theta_r}\psi_p(\tau)\,a_r^{h_r}(\varphi_k,\tau)\,d\tau,\qquad p,k=\overline{1,m},\qquad r=\overline{1,N+1}.$$

Summing up the matrices $\widehat{\psi}_{p,r}^{h_r}(\varphi_k)$ over r, we find $(n\times n)$ matrices $G_{p,k}^{\tilde{h}}(\triangle_N)=\sum_{r=1}^{N+1}\widehat{\psi}_{p,r}^{h_r}(\varphi_k)$, where $\tilde{h}=(h_1,h_2,\ldots,h_{N+1})\in R^n$. Using them, we compose the $nm\times nm$ matrix $G^{\tilde{h}}(\triangle_N)=\left(G_{p,k}^{\tilde{h}}(\triangle_N)\right)$, $p,k=\overline{1,m}$. Check the invertibility of matrix $\left[I-G^{\tilde{h}}(\triangle_N)\right]:R^{nm}\to R^{nm}$.

If this matrix is invertible, we find $\left[I-G^{\tilde{h}}(\triangle_N)\right]^{-1}=\left(M_{p,k}^{\tilde{h}}(\triangle_N)\right)$, $p,k=\overline{1,m}$. If it has no the inverse, then we take a new partition. In particular, each subinterval can be divided into two.

4. Solving the Cauchy problems for ordinary differential equations

$$\frac{dz}{dt}=A_0(t)\,z+A_i(t),\quad z(\theta_{r-1})=0,\quad t\in[\theta_{r-1},\ \theta_r],\quad \mathrm{i}=\overline{0,N},$$

$$\frac{dz}{dt}=A_0(t)\,z+f(t),\quad z(\theta_{r-1})=0,\quad t\in[\theta_{r-1},\ \theta_r],\quad \mathrm{r}=\overline{1,N+1},$$

by using again the Bulirsch-Stoer method, we find the values of $(n\times n)$ matrices $a_r\left(A_0,\widehat{\theta}\right)$, $a_r\left(A_i,\widehat{\theta}\right)$, i $=\overline{1,N}$, and n vector $a_r\left(f,\widehat{\theta}\right)$ on $\{\theta_{r-1},\ \theta_r\}$, r $=\overline{1,N+1}$.

5. Applying Simpson's method on the set $\{\theta_{r-1},\ \theta_r\}$, we evaluate the definite integrals

$$\widehat{\psi}_{p,r}^{h_r}=\int_{\theta_{r-1}}^{\theta_r}\psi_p(s)\,ds,\ \widehat{\psi}_{p,r}^{h_r}(A_i)=\int_{\theta_{r-1}}^{\theta_r}\psi_p(\tau)\,a_r^{h_r}(A_i,\tau)\,d\tau,\quad \mathrm{i}=\overline{0,N},$$

$$\widehat{\psi}_{p,r}^{h_r}(f)=\int_{\theta_{r-1}}^{\theta_r}\psi_p(\tau)\,a_r^{h_r}(f,\tau)\,d\tau,\qquad p=\overline{1,m},\qquad r=\overline{1,N+1}.$$

By the equalities

$$V_{p,r}^{\tilde h}(\triangle_N)=\widehat{\psi}_{p,r}^{h_r}(A_0)+\sum_{j=1}^{N+1}\sum_{k=1}^{m}\widehat{\psi}_{p,j}^{h_j}(\varphi_k)\cdot\widehat{\psi}_{p,r}^{h_r},r=\overline{1,N+1},$$

$$W_p^{\tilde h}(A_i,\triangle_N)=\sum_{r=1}^{N+1}\widehat{\psi}_{p,r}^{h_r}(A_i),\text{i}=\overline{1,N},g_p^{\tilde h}(f,\triangle_N)=\sum_{r=1}^{N+1}\widehat{\psi}_{p,r}^{h_r}(f),p=\overline{1,m},$$

we define the $(n\times n)$ matrices $V_{p,r}^{\tilde h}(\triangle_N)$, $r=\overline{1,N+1}$, $W_p^{\tilde h}(A_i,\triangle_N)$, i $=\overline{1,N}$, and n vectors $g_p^{\tilde h}(f,\triangle_N)$, respectively, $p=\overline{1,m}$.

6. Construct the system of linear algebraic equations with respect to parameters

$$Q_*^{\tilde h}(\triangle_N)\,\lambda=-F_*^{\tilde h}(\triangle_N),\quad \lambda\in R^{n(N+1)}. \tag{18}$$

The elements of the matrix $Q_*^{\tilde h}(\triangle_N)$ and the vector $F_*^{\tilde h}(\triangle_N)=\Big(-d+CF_{N+1}^{\tilde h}(\triangle_N),F_1^{\tilde h}(\triangle_N),\ldots,\ F_N^{\tilde h}(\triangle_N)\Big)$ are defined by the equalities

$$D_{r,j}(\triangle_N)=\sum_{k=1}^{m}a_r^{h_r}(\varphi_k,\theta_r)\left[\sum_{p=1}^{m}M_{k,p}^{\tilde h}(\triangle_N)\,V_{p,j}^{\tilde h}(\triangle_N)+\widehat{\psi}_{p,j}^{h_j}\right],$$

$$j\neq r,\quad r,j=\overline{1,N+1},$$

$$D_{r,r}^{\tilde h}(\triangle_N)=\sum_{k=1}^{m}a_r^{h_r}(\varphi_k,\theta_r)\left[\sum_{p=1}^{m}M_{k,p}^{\tilde h}(\triangle_N)\,V_{p,r}^{\tilde h}(\triangle_N)+\widehat{\psi}_{p,r}^{h_r}\right]+a_r^{h_r}(A_0,\theta_r),$$

$$E_{r,j}^{\tilde h}(\triangle_N)=\sum_{k=1}^{m}a_r^{h_r}(\varphi_k,\theta_r)\sum_{p=1}^{m}M_{k,p}^{\tilde h}(\triangle_N)\,W_{p,j}^{\tilde h}(\triangle_N)+a_r^{h_r}(A_j,\theta_r),\ \ j=\overline{1,N},$$

$$F_r^{\tilde h}(\triangle_N)=\sum_{k=1}^{m}a_r^{h_r}(\varphi_k,\theta_r)\sum_{p=1}^{m}M_{k,p}^{\tilde h}(\triangle_N)\,g_p^{\tilde h}(\triangle_N)+a_r^{h_r}(f,\theta_r),r=\overline{1,N+1}.$$

Solving the system (18), we find $\lambda^{\tilde h}$. As noted above, the elements of $\lambda^{\tilde h}=(\lambda_1^{\tilde h},\lambda_2^{\tilde h},\ldots,\lambda_{N+1}^{\tilde h})$ are the values of the approximate solution to problem (1), (2) in the initial points of subintervals: $x^{\tilde h_r}(\theta_{r-1})=\lambda_r^{\tilde h}$, r $=\overline{1,N+1}$.

7. To define the values of approximate solution at the remaining points of set $\{\theta_{r-1}, \theta_r\}$, we first find

$$\mu_k^{\tilde{h}} = \sum_{j=1}^{N+1} \left(\sum_{p=1}^{m} M_{k,p}^{\tilde{h}} (\triangle_N) V_{p,j}^{\tilde{h}} (\triangle_N) \right) \lambda_j^{\tilde{h}} + \sum_{j=1}^{N} \left(\sum_{p=1}^{m} M_{k,p}^{\tilde{h}} (\triangle_N) W_{p,j}^{\tilde{h}} (\triangle_N) \right) \lambda_{j+1}^{\tilde{h}}$$

$$+ \sum_{p=1}^{m} M_{k,p}^{\tilde{h}} (\triangle_N) g_p^{\tilde{h}} (f, \triangle_N), \quad k = \overline{1, m},$$

and then solve the Cauchy problems

$$\frac{dx}{dt} = A(t) x + \mathcal{F}^{\tilde{h}}(t), \quad x(\theta_{r-1}) = \lambda_r^{\tilde{h}}, \quad t \in [\theta_{r-1}, \theta_r], \quad \text{r} = \overline{1, N+1},$$

where $\mathcal{F}^{\tilde{h}}(t) = \sum_{k=1}^{m} \varphi_k(t) \left[\mu_k^{\tilde{h}} + \sum_{j=1}^{N+1} \widehat{\psi}_{p,j}^{h_j} \lambda_j^{\tilde{h}} \right] + \sum_{i=1}^{N} A_i(t) \lambda_{i+1}^{\tilde{h}} + f(t)$.

And the solutions to Cauchy problems are found by the Bulirsch-Stoer method. Thus, the algorithm allows us to find the numerical solution to the problem (1), (2).

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Бакирова Э.А., Минглибаева Б.Б., Қасымова А.Б. ЖҮКТЕЛГЕН ДИФФЕРЕНЦИАЛДЫҚ ЖӘНЕ ФРЕДГОЛЬМ ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕРІ ҮШІН КӨП НҮКТЕЛІ ШЕТТІК ЕСЕПТІ ШЕШУДІҢ АЛГОРИТМІ

Жүктелген дифференциалдық және Фредгольм интегралдық-дифференциалдық теңдеулері үшін көп нүктелі шеттік есеп қарастырылады. Бұл есеп параметрлеу әдісімен зерттеледі. Аралық бөліктерге бөлініп, ішкі аралықтың бастапқы нүктелеріндегі берілген функцияның мәндері қосымша параметрлер ретінде қарастырылады, ал бастапқы теңдеу белгісіз функциялар ішкі аралықтағы бастапқы шарттарды қанағаттандыратын параметрлері бар интегралдық-дифференциалдық теңдеулер жүйесіне келтіріледі. Параметрлердің бекітілген мәндері үшін сызықтық интегралдық-дифференциалдық теңдеулер жүйесі үшін арнайы Коши есебін аламыз. Кошидің арнайы есебінің шешімі дифференциалдық теңдеудің іргелі матрицасының көмегімен құрылады. Параметрлерге қатысты сызықтық алгебралық теңдеулер жүйесі сәйкес мәндерін шеттік шартқа және үзіліссіздік шартына қою арқылы құрылады. Құрылған жүйені шешуге және ішкі аралықтардағы Коши есебін шешудің Булирш-Штоер әдісіне негізделген сандық әдісі ұсынылған.

Кілттік сөздер. Жүктелген дифференциалдық теңдеу, интегралдық-дифференциалдық теңдеу, көп нүктелі есеп, алгоритм, параметрлеу әдісі

Бакирова Э.А., Минглибаева Б.Б., Касымова А.Б. АЛГОРИТМ НАХОЖДЕНИЯ РЕШЕНИЯ МНОГОТОЧЕЧНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ НАГРУЖЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ И ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ФРЕДГОЛЬМА

Рассматривается многоточечная краевая задача для нагруженных дифференциальных и интегро-дифференциальных уравнений Фредгольма. Эта задача исследуется методом параметризации. Интервал разбивается на части, значения искомой функции в начальных точках подинтервалов рассматриваются как дополнительные параметры, а исходное уравнение сводится к системе интегро-дифференциальных уравнений с параметрами, где неизвестные функции удовлетворяют начальным условиям на подинтервалах. При фиксированных значениях параметров получаем специальную задачу Коши для системы линейных интегро-дифференциальных уравнений. Решение специальной задачи Коши строится с использованием фундаментальной матрицы дифференциального уравнения. Система линейных алгебраических уравнений относительно параметров составляется путем подстановки соответствующих значений в краевое условие и условия непрерывности. Предлагается численный метод решения задачи, основанный на решении построенной системы и методе Булирша-Штоера решения задачи Коши на подинтервалах

Ключевые слова. Нагруженное дифференциальное уравнение, интегро-дифференциальное уравнение, многоточечная задача, алгоритм, метод параметризации

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On a solution of a nonlinear semi-periodic boundary value problem for a third-order pseudoparabolic equation

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Abstract. A nonlinear semi-periodic boundary value problem for an evolutionary equation of the pseudoparabolic type is studied. Introducing new variables, this third-order semi-periodic problem is reduced to a periodic boundary-value problem for a family of systems of first-order ordinary differential equations and functional relations. An algorithm for finding an approximate solution to the problem under study is proposed.

Keywords. differential equations, nonlinear problem, third-order equation, boundary value problem, algorithm.

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with great gratefulness

1 Introduction

On $\Omega = [0, \omega] \times [0, T]$ the boundary value problem is considered

$$\frac{\partial^3 u}{\partial x^2 \partial t} = f\left(x, t, u, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right), \qquad (x,t) \in \overline{\Omega}, \qquad u \in R^n, \tag{1}$$

$$u(x,0) = u(x,T), \quad x \in [0,\omega], \tag{2}$$

$$u(0,t) = \varphi(t), \quad t \in [0,T], \tag{3}$$

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$$\frac{\partial u(0,t)}{\partial x} = \psi(t), \quad t \in [0,T], \tag{4}$$

where $f : \overline{\Omega} \times R^n \times R^n \to R^n$ is continuous, n-vector function $\varphi(t)$, $\psi(t)$ is continuously differentiable by $[0,T]$ and satisfies the condition $\psi(0) = \psi(T)$.

Nonlocal problems for third-order partial differential equations have been studied by many authors [1-4]. A certain interest in the study of these problems is caused in connection with their applied values. Such problems include highly porous media with complex topologies, and first of all, soil and ground. Also, such equations can describe long waves in dispersed systems. To solve this problem, we introduce new functions and apply the method of parametrization [5]. Then the nonlinear semi-periodic boundary value problem for a third-order pseudoparabolic equation is reduced to a periodic boundary value problem for a family of systems of ordinary differential equations [6-19].

Function $u(x,t) \in C(\Omega, R^n)$, having partial derivatives

$$\frac{\partial u(x,t)}{\partial x} \in C(\Omega, R^n), \quad \frac{\partial u(x,t)}{\partial t} \in C(\Omega, R^n), \quad \frac{\partial^2 u(x,t)}{\partial x^2} \in C(\Omega, R^n), \quad \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \in C(\Omega, R^n),$$

is called a classical solution to problem (1)-(4) if it satisfies system (1) for all $(x,t) \in \Omega$, and boundary conditions (2)-(4).

To find a solution, we introduce the functions $z(x,t) = \frac{\partial u(x,t)}{\partial x}$, $w(x,t) = \frac{\partial u(x,t)}{\partial t}$ and problem (1)-(4) can be written as

$$\frac{\partial^2 z}{\partial x \partial t} = f\left(x, t, u, \frac{\partial z(x,t)}{\partial x}, z(x,t), w(x,t)\right), \quad (x,t) \in \overline{\Omega}, \quad u \in R^n, \tag{5}$$

$$z(x,0) = z(x,T), \quad x \in [0,\omega], \tag{6}$$

$$z(0,t) = \psi(t), \quad t \in [0,T], \tag{7}$$

$$w(x,t) = \dot{\varphi}(t) + \int_0^x \frac{\partial z(\xi,t)}{\partial t} d\xi, \tag{8}$$

$$u(x,t) = \varphi(t) + \int_0^x z(\xi,t) d\xi. \tag{9}$$

For fixed $u(x,t)$ and $w(x,t)$ problem (5)-(7) is a semi-periodic boundary value problem for a system of second-order hyperbolic equations. Reintroduce the notation $v(x,t) = \frac{\partial z(x,t)}{\partial x}$, and problem (5)-(9) is reduced to a family of periodic boundary value problems for a system of ordinary differential equations of the form

$$\frac{\partial v}{\partial t} = f\left(x, t, u, v(x,t), z(x,t), w(x,t)\right), \quad (x,t) \in \Omega,$$

$$v(x,0) = v(x,T), \quad x \in [0,\omega],$$

functional relationships

$$z(x,t) = \psi(t) + \int_0^x v(\xi,t)d\xi, \quad (x,t) \in \Omega,$$

and (8), (9).

Thus we have reduced a semiperiodic boundary value problem for a system of hyperbolic equations to a family of periodic boundary value problems for ordinary differential equations and functional relations.

At the step $h > 0 : Nh = T$ we perform the partition

$$[0,T) = \bigcup_{r=1}^{N} [(r-1)h, rh), \quad N = 1, 2, \dots.$$

Then the domain Ω is divided into N parts. By $v_r(x,t)$, $u_r(x,t)$, $w_r(x,t)$ denote, respectively, the restriction of the functions $v(x,t)$, $u(x,t)$, $w(x,t)$ to $\Omega_r = [0,\omega] \times [(r-1)h, rh)$, $r = \overline{1,N}$. We introduce the notation $\lambda_r(x) = v_r(x,(r-1)h)$ and make a replacement $\widetilde{v}_r(x,t) = v_r(x,t) - \lambda_r(x)$, $r = \overline{1,N}$. We obtain an equivalent boundary value problem with unknown functions $\lambda_r(x)$:

$$\frac{\partial \widetilde{v}_r}{\partial t} = f(x,t,u_r(x,t),\widetilde{v}_r + \lambda_r(x), z_r(x,t), w_r(x,t)), \quad (x,t) \in \Omega_r, \tag{10}$$

$$\widetilde{v}_r(x,(r-1)h) = 0, \quad x \in [0,\omega], \quad r = \overline{1,N}, \tag{11}$$

$$\lambda_1(x) - \lambda_N(x) - \lim_{t \to T-0} \widetilde{v}_N(x,t) = 0, \quad x \in [0,\omega], \tag{12}$$

$$\lambda_s(x) + \lim_{t \to sh-0} \widetilde{v}_s(x,t) - \lambda_{s+1}(x) = 0, \quad x \in [0,\omega], \quad s = \overline{1,N-1}. \tag{13}$$

$$z_r(x,t) = \psi(t) + \int_0^x \widetilde{v}_r(\xi,t)d\xi + \int_0^x \lambda_r(\xi)d\xi, \quad (x,t) \in \Omega_r, \quad r = \overline{1,N}, \tag{14}$$

$$w_r(x,t) = \dot{\varphi}(t) + \dot{\psi}(t)x + \int_0^x \int_0^\xi \frac{\partial \widetilde{v}_r(\xi_1,t)}{\partial t} d\xi_1 d\xi, \quad (x,t) \in \Omega_r, \quad r = \overline{1,N}, \tag{15}$$

$$u_r(x,t) = \varphi(t) + \psi(t)x + \int_0^x \int_0^\xi \widetilde{v}_r(\xi_1,t)d\xi_1 d\xi + \int_0^x \int_0^\xi \lambda_r(\xi_1)d\xi_1 d\xi, (x,t) \in \Omega_r, r = \overline{1,N}, \tag{16}$$

where (13) is the condition for the continuity of the solution in the inner lines of the partition.

Problem (10),(16) for fixed $\lambda_r(x), z_r(x,t), w_r(x,t), u_r(x,t)$, is a one-parameter family of Cauchy problems for systems of ordinary differential equations, where $x \in [0,\omega]$, and is equivalent to the nonlinear integral equation

$$\widetilde{v}_r(x,t) = \int\limits_{(r-1)h}^{t} f(x,\tau,u_r(x,\tau),\widetilde{v}_r(x,\tau)+\lambda_r(x),z_r(x,\tau),w_r(x,\tau))d\tau. \tag{17}$$

Instead of $\widetilde{v}_r(x,\tau)$ substitute the corresponding right-hand side of (17) and repeat this process ν times ($\nu = 1,2,\dots$) as soon as we get

$$\widetilde{v}_r(x,t) = \int\limits_{(r-1)h}^{t} f\Bigg(x,\tau_1,u_r(x,\tau_1), \int\limits_{(r-1)h}^{\tau_1} f\Bigg(x,\tau_2,u_r(x,\tau_2),\dots$$

$$\dots\int\limits_{(r-1)h}^{\tau_{\nu-1}} f\Bigg(x,\tau_\nu,u_r(x,\tau_\nu),\widetilde{v}_r(x,\tau_\nu)+\lambda_r(x),z_r(x,\tau_\nu),w_r(x,\tau_\nu)\Bigg)d\tau_\nu+$$

$$+\dots+\lambda_r(x),z_r(x,\tau_2),w_r(x,\tau_2)\Bigg)d\tau_2+\lambda_r(x),z_r(x,\tau_1),w_r(x,\tau_1)\Bigg)d\tau_1. \tag{18}$$

Hence, defining $\lim\limits_{t\to rh-0}\widetilde{v}_r(x,t)$, substituting them into (12), (13), we obtain a system of nonlinear equations for $\lambda_r(x)$:

$$\lambda_1(x)-\lambda_N(x)-\int\limits_{(N-1)h}^{Nh} f\Bigg(x,\tau_1,u_N(x,\tau_1), \int\limits_{(N-1)h}^{\tau_1} f\Bigg(x,\tau_2,u_N(x,\tau_2),\dots$$

$$\dots\int\limits_{(N-1)h}^{\tau_{\nu-1}} f\Bigg(x,\tau_\nu,u_N(x,\tau_\nu),\widetilde{v}_N(x,\tau_\nu)+\lambda_N(x),z_N(x,\tau_\nu),w_N(x,\tau_\nu)\Bigg)d\tau_\nu+\dots$$

$$\dots+\lambda_N(x),z_N(x,\tau_2),w_N(x,\tau_2)\Bigg)d\tau_2+\lambda_N(x),z_N(x,\tau_1),w_N(x,\tau_1)\Bigg)d\tau_1=0, \qquad x\in[0,\omega],$$

$$\lambda_s(x)+\int\limits_{(s-1)h}^{sh} f\Bigg(x,\tau_1,u_s(x,\tau_1), \int\limits_{(s-1)h}^{\tau_1} f\Bigg(x,\tau_2,u_s(x,\tau_2),\dots$$

$$\dots \int\limits_{(s-1)h}^{\tau_{\nu-1}} f\bigg(x, \tau_\nu, u_s(x,\tau_\nu), \widetilde{v}_s(x,\tau_\nu) + \lambda_s(x), z_s(x,\tau_\nu), w_s(x,\tau_\nu)\bigg) d\tau_\nu + \dots$$

$$\dots + \lambda_s(x), z_s(x,\tau_2), w_s(x,\tau_2)\bigg) d\tau_2 + \lambda_s(x), z_s(x,\tau_1), w_s(x,\tau_1)\bigg) d\tau_1 - \lambda_{s+1}(x) = 0,$$

$$x \in [0,\omega], \quad s = \overline{1, N-1}.$$

which we write in the form

$$Q_{\nu,h}(x, \lambda, \widetilde{v}, z, w, u) = 0. \tag{19}$$

In the absence of partition $(N = 1, h = T)$ the system of equations (19) has the form

$$\int\limits_0^T f\bigg(x, \tau_1, u(x,\tau_1), \int\limits_0^{\tau_1} f\bigg(x, \tau_2, u(x,\tau_2), \dots \int\limits_0^{\tau_{\nu-1}} f\bigg(x, \tau_\nu, u(x,\tau_\nu), \widetilde{v}(x,\tau_\nu)$$

$$+\lambda(x), z(x,\tau_\nu), w(x,\tau_\nu)\bigg) d\tau_\nu + \dots + \lambda(x), z(x,\tau_2), w(x,\tau_2)\bigg) d\tau_2$$

$$+\lambda(x), z(x,\tau_1), w(x,\tau_1)\bigg) d\tau_1 = 0, \quad x \in [0,\omega].$$

To find a system of functions $\{\lambda_r(x), \widetilde{v}_r(x,t), z_r(x,t), w_r(x,t), u_r(x,t)\}$, $r = \overline{1,N}$, we have a closed system consisting of equations (19), (18), (14), (15) and (16), defined in terms of the function f, the partition step $h > 0$ and the number of substitutions ν.

Let us choose a step $h > 0 : Nh = T$ $(N = 1, 2, \dots)$, vector function

$$\lambda^{(0)}(x) = (\lambda_1^{(0)}(x), \lambda_2^{(0)}(x), \dots, \lambda_N^{(0)}(x))' \in C([0,\omega], R^{Nn}),$$

and assume that problem (10)–(16) for $\lambda_r(x) = \lambda_r^{(0)}(x), r = \overline{1,N}$, has a solution $\widetilde{v}_r^{(0)}(x,t) \in \widetilde{C}(\Omega_r, R^n)$, $z_r^{(0)}(x,t) \in \widetilde{C}(\Omega_r, R^n)$, $w_r^{(0)}(x,t) \in \widetilde{C}(\Omega_r, R^n)$, $u_r^{(0)}(x,t) \in \widetilde{C}(\Omega_r, R^n)$, $r = \overline{1,N}$. Many such $\lambda^{(0)}(x) \in C([0,\omega], R^{nN})$ denote $G_0(f, x, h)$, and the corresponding $\lambda^{(0)}(x)$ have the system of solutions to problems (10)–(16) in terms of

$$\widetilde{v}^{(0)}(x,[t]) = (\widetilde{v}_1^{(0)}(x,t), \widetilde{v}_2^{(0)}(x,t), \dots, \widetilde{v}_N^{(0)}(x,t))',$$

$$z^{(0)}(x,[t]) = (z_1^{(0)}(x,t), z_2^{(0)}(x,t), \dots, z_N^{(0)}(x,t))',$$

$$w^{(0)}(x,[t]) = (w_1^{(0)}(x,t), w_2^{(0)}(x,t), \dots, w_N^{(0)}(x,t))',$$

$$u^{(0)}(x,[t]) = (u_1^{(0)}(x,t), u_2^{(0)}(x,t), \dots, u_N^{(0)}(x,t))'.$$

By taking $\lambda^{(0)}(x) \in G_0(f,x,h)$, $\widetilde{v}^{(0)}(x,[t])$, $u^{(0)}(x,[t])$, $z^{(0)}(x,[t])$, $w^{(0)}(x,[t])$, continuous functions on $[0,\omega]$ $\rho(x)>0$, $\theta(x)>0$, $\phi(x)>0$ build sets:

$$S(\lambda^{(0)}(x),\rho(x)) = \big\{(\lambda_1(x),\lambda_2(x),\ldots,\lambda_N(x))' \quad \in \quad C([0,\omega],R^{nN}):$$

$$\|\lambda_r(x)-\lambda_r^{(0)}(x)\| < \rho(x), r=\overline{1,N}\big\},$$

$$S(\widetilde{v}^{(0)}(x,[t]),\theta(x)) = \big\{(\widetilde{v}_1(x,t),\widetilde{v}_2(x,t),\ldots,\widetilde{v}_N(x,t))', \widetilde{v}_r(x,t)\in \widetilde{C}(\Omega_r,R^n):$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r(x,t)-\widetilde{v}_r^{(0)}(x,t)\| < \theta(x), r=\overline{1,N}\big\},$$

$$S(z^{(0)}(x,[t]),\phi(x)) = \{(z_1(x,t),z_2(x,t),\ldots,z_N(x,t))', z_r(x,t)\in \widetilde{C}(\Omega_r,R^n):$$

$$\|z_r(x,t)-z_r^{(0)}(x,t)\| < \phi(x), (x,t)\in\Omega_r, r=\overline{1,N}\},$$

$$S(w^{(0)}(x,[t]),\phi(x)) = \{(w_1(x,t),w_2(x,t),\ldots,w_N(x,t))', w_r(x,t)\in \widetilde{C}(\Omega_r,R^n):$$

$$\|w_r(x,t)-w_r^{(0)}(x,t)\| < \phi(x), (x,t)\in\Omega_r, r=\overline{1,N}\},$$

$$S(u^{(0)}(x,[t]),\phi(x)) = \{(u_1(x,t),u_2(x,t),\ldots,u_N(x,t))', u_r(x,t)\in \widetilde{C}(\Omega_r,R^n):$$

$$\|u_r(x,t)-u_r^{(0)}(x,t)\| < \phi(x), (x,t)\in\Omega_r, r=\overline{1,N}\},$$

$$G_1^0(\rho(x),\theta(x),\phi(x)) = \{(x,t,v,z,w,u): \quad (x,t)\in\overline{\Omega},$$

$$\|v-\lambda_r^{(0)}(x)-\widetilde{v}_r^{(0)}(x,t)\| < \rho(x)+\theta(x), \quad (x,t)\in\Omega_r, \quad r=\overline{1,N},$$

$$\|v-\lambda_N^{(0)}(x)-\lim_{t\to T-0}\widetilde{v}_N^{(0)}(x,t)\| < \rho(x)+\theta(x), \quad t=T,$$

$$\|z-z_r^{(0)}(x,t)\| < \phi(x), \quad (x,t)\in\Omega_r, \quad r=\overline{1,N},$$

$$\|z-\lim_{t\to Nh-0} z_N^{(0)}(x,t)\| < \phi(x), \quad t=T,$$

$$\|w-w_r^{(0)}(x,t)\| < \phi(x), \quad (x,t)\in\Omega_r, \quad r=\overline{1,N},$$

$$\|w-\lim_{t\to Nh-0} w_N^{(0)}(x,t)\| < \phi(x), t=T, \quad \|u-u_r^{(0)}(x,t)\| < \phi(x), (x,t)\in\Omega_r, r=\overline{1,N},$$

$$\|u-\lim_{t\to Nh-0} u_N^{(0)}(x,t)\| < \phi(x), \quad t=T\}.$$

By $U_0(f,L_1(x),L_2(x),L_3(x),L_4(x),x,h)$ we denote the collection

$$(\lambda^{(0)}(x),\widetilde{v}^{(0)}(x,[t]),z^{(0)}(x,[t]),w^{(0)}(x,[t]),u^{(0)}(x,[t]),\rho(x),\theta(x),\phi(x))$$

for which the function $f(x,t,v,w,u)$ in $G_1^0(\rho(x),\theta(x),\phi(x))$ has continuous partial derivatives $f'_v(x,t,v,z,w,u)$, $f'_z(x,t,v,z,w,u)$, $f'_w(x,t,v,z,w,u)$, $f'_u(x,t,v,z,w,u)$ and

$$\|f'_v(x,t,v,z,w,u)\| \le L_1(x), \|f'_z(x,t,v,z,w,u)\| \le L_2(x),$$

$$\|f'_w(x,t,v,z,w,u)\| \le L_3(x), \|f'_u(x,t,v,z,w,u)\| \le L_4(x).$$

By system $\{\lambda_r(x), \widetilde{v}_r(x,t), z_r(x,t), w_r(x,t), u_r(x,t)\}, r = \overline{1,N}$, let us make the top five $\{\lambda(x), \widetilde{v}(x,[t]), z(x,[t]), w(x,[t]), u(x,[t])\}$, where $\lambda(x) = (\lambda_1(x), \lambda_2(x), \dots, \lambda_N(x))'$,

$$\widetilde{v}(x,[t]) = (\widetilde{v}_1(x,t), \widetilde{v}_2(x,t), \dots, \widetilde{v}_N(x,t))',$$

$$z(x,[t]) = (z_1(x,t), z_2(x,t), \dots, z_N(x,t))',$$

$$w(x,[t]) = (w_1(x,t), w_2(x,t), \dots, w_N(x,t))',$$

$$u(x,[t]) = (u_1(x,t), u_2(x,t), \dots, u_N(x,t))'.$$

Assuming the existence of $\lambda^{(0)}(x) \in G_0(f,x,h)$, for the initial approximation of problem (10)-(16) we take the functions $\{\lambda^{(0)}(x), \widetilde{v}^{(0)}(x,[t]), z^{(0)}(x,[t]), w^{(0)}(x,[t]), u^{(0)}(x,[t])\}, \quad r = \overline{1,N}$, and successive approximations are built according to the following algorithm:

Step 1. A) Assuming that

$$z_r(x,t) = z_r^{(0)}(x,t), \quad w_r(x,t) = w_r^{(0)}(x,t), \quad u_r(x,t) = u_r^{(0)}(x,t), r = \overline{1,N},$$

we find first approximations of $\lambda_r(x), \widetilde{v}_r(x,t)$ by solving problem (10)–(13). Taking

$$\lambda^{(1,0)}(x) = \lambda^{(0)}(x), \quad \widetilde{v}_r^{(1,0)}(x,t) = \widetilde{v}_r^{(0)}(x,t),$$

we find the system of couples $\{\lambda_r^{(1)}(x),\ \widetilde{v}_r^{(1)}(x,t)\}, \quad r = \overline{1,N}$, the limit of the sequence $\lambda_r^{(1,m)}(x),\ \widetilde{v}_r^{(1,m)}(x,t)$, determined by the following algorithm:

Step 1.1. a) Substituting $\widetilde{v}_r^{(1,0)}(x,t), r = \overline{1,N}$, in (19), from the system of functional equations

$$Q_{\nu,h}(x,\lambda,\widetilde{v}^{(1,0)}, z^{(0)}, w^{(0)}, u^{(0)}) = 0$$

define $\lambda_r^{(1,1)}(x)$, $r = \overline{1,N}$.

b) In the right-hand side of (18), substituting instead of $\widetilde{v}_r(x,t)$, $\lambda_r(x)$, respectively, $\widetilde{v}_r^{(1,0)}(x,t)$, $\lambda_r^{(1,1)}(x)$, $r = \overline{1,N}$, define $\{\widetilde{v}_r^{(1,1)}(x,t)\}, \quad r = \overline{1,N}$.

Step 1.2. a) Substituting $\widetilde{v}_r^{(1,1)}(x,t), r = \overline{1,N}$, in (19), from the system of functional equations

$$Q_{\nu,h}(x,\lambda,\widetilde{v}^{(1,1)}, z^{(0)}, w^{(0)}, u^{(0)}) = 0$$

define $\lambda_r^{(1,2)}(x), \quad r = \overline{1,N}$.

b) In the right-hand side of (18), substituting instead of $\widetilde{v}_r(x,t)$, $\lambda_r(x)$, respectively, $\widetilde{v}_r^{(1,1)}(x,t)$, $\lambda_r^{(1,2)}(x)$, $r = \overline{1,N}$, define $\{\widetilde{v}_r^{(1,2)}(x,t)\}$, $r = \overline{1,N}$. On $(1,m)$-th step we get a system of pairs $\{\lambda_r^{(1,m)}(x),\ \widetilde{v}_r^{(1,m)}(x,t)\}$, $r = \overline{1,N}$. Assume that a sequence of systems of couples $\{\lambda_r^{(1,m)}(x),\ \widetilde{v}_r^{(1,m)}(x,t)\}$ at $m \to \infty$ converges to $\{\lambda_r^{(1)}(x),\ \widetilde{v}_r^{(1)}(x,t)\}$, $r = \overline{1,N}$.

B) Functions $u_r^{(1)}(x,t), z_r^{(1)}(x,t), w_r^{(1)}(x,t), r=\overline{1,N}$, are determined from the relations

$$z_r^{(1)}(x,t)=\psi(t)+\int_0^x \widetilde{v}_r^{(1)}(\xi,t)d\xi+\int_0^x \lambda_r^{(1)}(\xi)d\xi, \quad (x,t)\in\Omega_r, \quad r=\overline{1,N},$$

$$w_r^{(1)}(x,t)=\dot{\varphi}(t)+\dot{\psi}(t)x+\int_0^x\int_0^\xi \frac{\partial\widetilde{v}_r^{(1)}(\xi_1,t)}{\partial t}d\xi_1 d\xi, (x,t)\in\Omega_r, r=\overline{1,N},$$

$$u_r^{(1)}(x,t)=\varphi(t)+\psi(t)x+\int_0^x\int_0^\xi \widetilde{v}_r^{(1)}(\xi_1,t)d\xi_1 d\xi+\int_0^x\int_0^\xi \lambda_r^{(1)}(\xi_1)d\xi_1 d\xi, (x,t)\in\Omega_r, r=\overline{1,N}.$$

Step 2. A) Assuming that

$$z_r(x,t)=z_r^{(1)}(x,t), \quad w_r(x,t)=w_r^{(1)}(x,t), \quad u_r(x,t)=u_r^{(1)}(x,t), \quad r=\overline{1,N},$$

we find second approximations of $\lambda_r(x), \widetilde{v}_r(x,t)$ by solving problem (10)–(13). Taking

$$\lambda^{(2,0)}(x)=\lambda^{(1)}(x), \quad \widetilde{v}_r^{(2,0)}(x,t)=\widetilde{v}_r^{(1)}(x,t),$$

we find the system of couples $\{\lambda_r^{(2)}(x),\ \widetilde{v}_r^{(2)}(x,t)\}, r=\overline{1,N}$, as the limit of the sequence $\lambda_r^{(2,m)}(x), \quad \widetilde{v}_r^{(2,m)}(x,t)$, determined by the following algorithm:

Step 2.1. a) Substituting $\widetilde{v}_r^{(2,0)}(x,t), r=\overline{1,N}$, in (19), from the system of functional equations

$$Q_{\nu,h}(x,\lambda,\widetilde{v}^{(2,0)},z^{(1)},w^{(1)},u^{(1)})=0$$

define $\lambda_r^{(2,1)}(x)$, $r=\overline{1,N}$.

b) In the right-hand side of (18), substituting instead of $\widetilde{v}_r(x,t)$, $\lambda_r(x)$, respectively, $\widetilde{v}_r^{(2,0)}(x,t)$, $\lambda_r^{(2,1)}(x)$, $r=\overline{1,N}$, define $\{\widetilde{v}_r^{(2,1)}(x,t)\}$, $r=\overline{1,N}$.

Step 2.2. a) Substituting $\widetilde{v}_r^{(2,1)}(x,t), r=\overline{1,N}$, in (19), from the system of functional equations

$$Q_{\nu,h}(x,\lambda,\widetilde{v}^{(2,1)},z^{(1)},w^{(1)},u^{(1)})=0$$

define $\lambda_r^{(2,2)}(x), \quad r=\overline{1,N}$.

b) In the right-hand side of (18), substituting instead of $\widetilde{v}_r(x,t)$, $\lambda_r(x)$, respectively, $\widetilde{v}_r^{(2,1)}(x,t)$, $\lambda_r^{(2,2)}(x)$, $r=\overline{1,N}$, define $\{\widetilde{v}_r^{(2,2)}(x,t)\}$, $r=\overline{1,N}$. At the $(2,m)$-th step we obtain a system of couples $\{\lambda_r^{(2,m)}(x),\ \widetilde{v}_r^{(2,m)}(x,t)\}$, $r=\overline{1,N}$. Assume that a sequence of systems of couples $\{\lambda_r^{(2,m)}(x),\ \widetilde{v}_r^{(2,m)}(x,t)\}$ at $m\to\infty$ converges to $\{\lambda_r^{(2)}(x),\ \widetilde{v}_r^{(2)}(x,t)\}$, $r=\overline{1,N}$.

B) Functions $u_r^{(2)}(x,t), z_r^{(2)}(x,t), w_r^{(2)}(x,t), r=\overline{1,N}$, are determined from the relations

$$z_r^{(2)}(x,t)=\psi(t)+\int\limits_0^x \widetilde{v}_r^{(2)}(x_1,t)dx_1+\int\limits_0^x \lambda_r^{(2)}(x_1)dx_1, \quad (x,t)\in\Omega_r, \quad r=\overline{1,N},$$

$$w_r^{(2)}(x,t)=\dot{\varphi}(t)+\dot{\psi}(t)x+\int\limits_0^x\int\limits_0^{x_1}\frac{\partial\widetilde{v}_r^{(2)}(x_2,t)}{\partial t}dx_2dx_1, (x,t)\in\Omega_r, r=\overline{1,N},$$

$$u_r^{(2)}(x,t)=\varphi(t)+\psi(t)x+\int\limits_0^x\int\limits_0^{x_1}\widetilde{v}_r^{(2)}(x_2,t)dx_2dx_1+\int\limits_0^x\int\limits_0^{x_1}\lambda_r^{(2)}(x_2)dx_2dx_1, (x,t)\in\Omega_r, r=\overline{1,N}.$$

Continuing the process, at the k–th step we obtain the system

$$\{\lambda_r^{(k)}(x), \widetilde{v}_r^{(k)}(x,t), z_r^{(k)}(x,t), w_r^{(k)}(x,t), u_r^{(k)}(x,t)\}, \quad r=\overline{1,N}.$$

Sufficient conditions for the feasibility, convergence of the algorithm and the existence of a solution to a multi-characteristic boundary value problem with functional parameters (10)–(16) are established by

Theorem 1. *Suppose that there is* $h>0: Nh=T, (N=1,2,\ldots)$, $\nu\in\mathbb{N}$, $(\lambda^{(0)}(x), \widetilde{v}^{(0)}(x,[t]), z^{(0)}(x,[t]), w^{(0)}(x,[t]), u^{(0)}(x,[t]), \rho(x), \theta(x), \phi(x))\in U_0(f, L_1(x), L_2(x), L_3(x), L_4(x), x, h)$, *for which the Jacobi matrix* $\dfrac{\partial Q_{\nu,h}(x,\lambda,\widetilde{v},z,w,u)}{\partial\lambda}$ *is reversible for all* $(x,\lambda(x), \widetilde{v}(x,[\cdot]), z(x,[\cdot]), w(x,[\cdot])), u(x,[\cdot])$, *where* $x\in[0,\omega]$, $(\lambda(x), \widetilde{v}(x,[t]), z(x,[t]), w(x,[t]), u(x,[t]))\in S(\lambda^{(0)}(x),\rho(x))\times S(\widetilde{v}^{(0)}(x,[t]),\theta(x))\times S(z^{(0)}(x,[t]),\phi(x))\times S(w^{(0)}(x,[t]),\phi(x))\times S(u^{(0)}(x,[t]),\phi(x))$ *and the following inequalities hold:*

1) $\left\|\left[\dfrac{\partial Q_{\nu,h}(x,\lambda,\widetilde{v},z,w,u)}{\partial\lambda}\right]^{-1}\right\|\le\gamma_\nu(x,h)$,

2) $q_\nu(x,h)=\dfrac{(L_1(x)h)^\nu}{\nu!}\left[1+\gamma_\nu(x,h)\sum\limits_{j=1}^{\nu}\dfrac{(L_1(x)h)^j}{j!}\right]\le\mu<1$,

3) $[c_0(x)+1]c_2(x)[L_2(x)+L_3(x)+L_4(x)]\int\limits_0^x c(\xi)exp\left(\int\limits_0^\xi c(\xi_1)d\xi_1\right)$

$\times\int\limits_0^\xi \max\{1,L_1(\xi_1)\}[c_0(\xi_1)c_1(\xi_1)+c_0(\xi_1)+1]\gamma_\nu(\xi_1,h)\|Q_{\nu,h}(\xi_1,\lambda^{(0)},\widetilde{v}^{(0)},z^{(0)},w^{(0)},u^{(0)})\|d\xi_1d\xi$

$+c_0(x)\gamma_\nu(x,h)\|Q_{\nu,h}(x,\lambda^{(0)}(x),\widetilde{v}^{(0)},z^{(0)},w^{(0)},u^{(0)})\|<\theta(x)$,

4) $[c_0(x)c_1(x)+c_0(x)+1]c_2(x)[L_2(x)+L_3(x)+L_4(x)]\int\limits_0^x c(\xi)exp\left(\int\limits_0^\xi c(\xi_1)d\xi_1\right)\int\limits_0^\xi\max\{1,L_1(\xi_1)\}$

$\times[c_0(\xi_1)c_1(\xi_1)+c_0(\xi_1)+1]\gamma_\nu(\xi_1,h)\|Q_{\nu,h}(\xi_1,\lambda^{(0)},\widetilde{v}^{(0)},z^{(0)},w^{(0)},u^{(0)})\|d\xi_1d\xi$

$+[c_0(x)c_1(x)+1]\gamma_\nu(x,h)\|Q_{\nu,h}(x,\lambda^{(0)},\widetilde{v}^{(0)},z^{(0)},w^{(0)},u^{(0)})\|<\rho(x),$

5) $c(x)\sum\limits_{j=0}^{k-1}\dfrac{1}{j!}\Bigg(\int\limits_0^x c(\xi)d\xi\Bigg)^j$

$\times\int\limits_0^x \max\{1,L_1(\xi)\}[c_0(\xi)c_1(\xi)+c_0(\xi)+1]\gamma_\nu(\xi,h)\|Q_{\nu,h}(\xi,\lambda^{(0)},\widetilde{v}^{(0)},z^{(0)},w^{(0)},u^{(0)})\|d\xi$

$+\max\{1,L_1(x)[c_0(x)c_1(x)+c_0(x)+1]\}\gamma_\nu(x,h)\|Q_{\nu,h}(x,\lambda^{(0)},\widetilde{v}^{(0)},z^{(0)},w^{(0)},u^{(0)})\|<\phi(x),$

where $c(x)=\max\Big\{[c_0(x)c_1(x)+2c_0(x)+2]c_2(x),L_1(x)[c_0(x)c_1(x)+2c_0(x)+2]c_2(x)+1\Big\}$

$\times[L_2(x)+L_3(x)+L_4(x)],\ \ c_0(x)=\dfrac{1}{1-q_\nu(x,h)}\sum\limits_{j=1}^{\nu}\dfrac{(L_1(x)h)^j}{j!},\ \ c_1(x)=\gamma_\nu(x,h)\dfrac{(L_1(x)h)^\nu}{\nu!},$

$c_2(x)=\dfrac{1}{1-q_\nu(x,h)}\sum\limits_{j=0}^{\nu}\dfrac{(L_1(x)h)^j}{j!}h.$

Then the sequence of functions determined by the algorithm $(\lambda^{(k)}(x),\widetilde{v}^{(k)}(x,[t]),$ $z^{(k)}(x,[t]),\ w^{(k)}(x,[t])),u^{(k)}(x,[t]),\ k=1,2,\ldots,$ *contained in* $S(\lambda^{(0)}(x),\rho(x))\times$ $S(\widetilde{v}^{(0)}(x,[t]),\theta(x))\times S(z^{(0)}(x,[t]),\phi(x))\times S(w^{(0)}(x,[t]),\phi(x))\times S(u^{(0)}(x,[t]),\phi(x))$, *converges to* $(\lambda^*(x),\widetilde{v}^*(x,[t]),z^*(x,[t]),w^*(x,[t]),u^*(x,[t]))$ *which is the solution of problem (10)-(16) and the following estimates are valid:*

a) $\max\Bigg\{\max\limits_{r=\overline{1,N}}\|\lambda_r^*(x)-\lambda_r^{(k)}(x)\|+\max\limits_{r=\overline{1,N}}\sup\limits_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{(k)}(x,t)\|,$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^*(x,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{(k)}(x,t)}{\partial t}\Big\|,$$

$$\int\limits_0^x\max_{r=\overline{1,N}}\|\lambda_r^*(\xi)-\lambda_r^{(k)}(\xi)\|d\xi+\int\limits_0^x\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(\xi,t)-\widetilde{v}_r^{(k)}(\xi,t)\|d\xi\Bigg\}$$

$$\leq\frac{c(x)}{(k-1)!}\Bigg(\int\limits_0^x c(\xi)d\xi\Bigg)^{(k-1)}e^{\int\limits_0^x c(\xi)d\xi}\int\limits_0^x\max\{1,L_1(\xi)\}[c_0(\xi)c_1(\xi)+c_0(\xi)+1]$$

$$\times\gamma_\nu(\xi,h)\|Q_{\nu,h}(\xi,\lambda^{(0)},\widetilde{v}^{(0)},z^{(0)},w^{(0)},u^{(0)})\|d\xi,$$

b) $\max\Big\{\max\limits_{r=\overline{1,N}}\sup\limits_{t\in[(r-1)h,rh)}\|z_r^*(x,t)-z_r^{(k)}(x,t)\|,\max\limits_{r=\overline{1,N}}\sup\limits_{t\in[(r-1)h,rh)}\|w_r^*(x,t)-w_r^{(k)}(x,t)\|,$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^*(x,t)-u_r^{(k)}(x,t)\|\Big\}$$

$$\leq \int_0^x \max\Big\{ \max_{r=\overline{1,N}} \|\lambda_r^*(\xi) - \lambda_r^{(k)}(\xi)\| + \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^*(\xi,t) - \widetilde{v}_r^{(k)}(\xi,t)\|,$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \Big\| \frac{\partial \widetilde{v}_r^*(\xi,t)}{\partial t} - \frac{\partial \widetilde{v}_r^{(k)}(\xi,t)}{\partial t} \Big\|,$$

$$\int_0^\xi \max_{r=\overline{1,N}} \|\lambda_r^*(\xi_1) - \lambda_r^{(k)}(\xi_1)\| d\xi + \int_0^\xi \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^*(\xi_1,t) - \widetilde{v}_r^{(k)}(\xi_1,t)\| d\xi_1 \Big\} d\xi.$$

Furthermore any solution $(\lambda(x), \widetilde{v}(x,[t]), z(x,[t]), w(x,[t]), u(x,[t]))$ *to problems (10)-(16) in* $S(\lambda^{(0)}(x), \rho(x)) \times S(\widetilde{v}^{(0)}(x,[t]), \theta(x)) \times S(z^{(0)}(x,[t]), \phi(x)) \times S(w^{(0)}(x,[t]), \phi(x)) \times S(u^{(0)}(x,[t]), \phi(x))$ *is isolated.*

The proof is given on the basis of the above algorithm, similar to the scheme of the proof of Theorem 1 of [14].

Functions $v_r^{(k)}(x,t), z_r^{(k)}(x,t), w_r^{(k)}(x,t), u_r^{(k)}(x,t)$, $k = 1, 2, \ldots$, are defined by equalities:

$$v_r^{(k)}(x,t) = \begin{cases} \lambda_r^{(k)}(x) + \widetilde{v}_r^{(k)}(x,t), & \text{at } (x,t) \in \Omega_r, \quad r = \overline{1,N}, \\ \lambda_N^{(k)}(x) + \lim\limits_{t\to T-0} \widetilde{v}_N^{(k)}(x,t), & \text{at } t = Nh, \end{cases}$$

$$z_r^{(k)}(x,t) = \psi(t) + \int_0^x v_r^{(k)}(\xi,t) d\xi,$$

$$w_r^{(k)}(x,t) = \dot{\varphi}(t) + \dot{\psi}(t) + \int_0^x \int_0^\xi \frac{\partial v_r^{(k)}(\xi_1,t)}{\partial t} d\xi_1 d\xi,$$

$$u_r^{(k)}(x,t) = \varphi(t) + \psi(t)x + \int_0^x \int_0^\xi v_r^{(k)}(\xi_1,t) d\xi_1 d\xi$$

and by $G(u^{(0)}(x,[t]), \phi(x))$ denote lots of piecewise-continuously differentiable to x, t functions $u : \overline{\Omega} \to R^n$, satisfying the inequalities $\|v(x,t) - \lambda^{(0)}(x) - \widetilde{v}^{(0)}(x,t)\| < \phi(x)$, $\|v(x,T) - \lambda^{(0)}(x) - \widetilde{v}^{(0)}(x,T)\| < \phi(x)$, $\|z(x,t) - z^{(0)}(x,t)\| < \phi(x)$, $\|z(x,T) - z^{(0)}(x,T)\| < \phi(x)$, $\|w(x,t) - w^{(0)}(x,t)\| < \phi(x)$, $\|w(x,T) - w^{(0)}(x,T)\| < \phi(x)$, $\|u(x,t) - u^{(0)}(x,t)\| < \phi(x)$, $\|u(x,T) - u^{(0)}(x,T)\| < \phi(x)$, where $v(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$, $z(x,t) = \frac{\partial u(x,t)}{\partial t}$, $w(x,t) = \frac{\partial u(x,t)}{\partial t}$.

In view of the equivalence of problems (1)-(4) and (10)-(16), Theorem 1 implies

Theorem 2. *If the conditions of Theorem 1 are satisfied, then the sequence of functions* $\{u^{(k)}(x,t)\}$, $k=1,2,\dots,$ *contained in* $G(u^{(0)}(x,[t]),\phi(x))$, *converges to* $u^*(x,t)$ *which is the solution to problem (1)-(4) in* $G(u^{(0)}(x,[t]),\phi(x))$ *and this inequality is valid*

$$\|u^*(x,t)-u^{(k)}(x,t)\|\le\int\limits_0^x c(\xi)\sum_{j=k-1}^{\infty}\frac{1}{j!}\Big(\int\limits_0^{\xi}c(\xi_1)d\xi_1\Big)^j$$

$$\times\int\limits_0^{\xi}\max\{1,L_1(\xi_1)\}[c_0(\xi_1)c_1(\xi_1)+c_0(\xi_1)+1]\gamma_\nu(\xi_1,h)\|Q_{\nu,h}(\xi_1,\lambda^{(0)},\widetilde{v}^{(0)},z^{(0)},w^{(0)},u^{(0)})\|d\xi_1 d\xi,$$

$(x,t)\in\overline{\Omega}$. *Moreover, any solution to problem (1)-(4) in* $G(u^{(0)}(x,[t]),\phi(x))$ *is isolated.*

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Орумбаева Н.Т., Кельдибекова А.Б. ҮШІНШІ РЕТТІ ПСЕВДОПАРАБОЛАЛЫҚ ТЕҢДЕУ ҮШІН БЕЙСЫЗЫҚТЫ ЖАРТЫЛАЙ ПЕРИОДТЫ ШЕТТІК ЕСЕПТІҢ БІР ШЕШІМІ ЖАЙЫНДА

Псевдопараболикалық типтегі эволюциялық теңдеу үшін бейсызықты жартылай периодты шеттіқ есеп зерттелуде. Жаңа айнымалыларды енгізу арқылы осы үшінші ретті жартылай периодты есеп қарапайым бірінші ретті периодты дифференциалдық теңдеулер жүйесі үйіріне және функционалдық қатынастарға келтіріледі. Зерттелетін есептің жуық шешімін табу алгоритмі ұсынылған.

Кілттік сөздер. Дифференциалдық теңдеулер, бейсызықты есеп, үшінші ретті теңдеу, шеттік есеп, алгоритм.

Орумбаева Н.Т., Кельдибекова А.Б. ОБ ОДНОМ РЕШЕНИИ НЕЛИНЕЙНОЙ ПОЛУПЕРИОДИЧЕСКОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ ПСЕВДОПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА

Исследуется нелинейная полупериодическая краевая задача для эволюционного уравнения псевдопараболического типа. Вводя новые переменные данная полупериодическая задача третьего порядка сводится к периодической краевой задаче для семейства систем обыкновенных дифференциальных уравнений первого порядка и функциональным соотношениям. Предложен алгоритм нахождения приближенного решения исследуемой задачи.

Ключевые слова. Дифференциальные уравнения, нелинейная задача, уравнение третьего порядка, краевая задача, алгоритм.

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An approximate solution to quasilinear boundary value problems for Fredholm integro-differential equation

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Abstract. An approximate method for solving quasilinear boundary value problem for Fredholm integro-differentiated equation is proposed. The method is based on approximation of the integral term by Simpson's formula and reduction of the initial problem to a quasilinear boundary value problem for a system of loaded differential equations. An algorithm for finding a numerical solution and a method for constructing approximate solution of approximating boundary value problem are proposed.

Keywords. quasilinear Fredholm integro-differential equation, Dzhumabaev's parameterization method, Newton method, Simpson's formula.

Dedicated to the bright memory of an outstanding scientist,
Doctor of Physical and Mathematical Sciences, Professor,
my scientific supervisor Dzhumabaev Dulat Syzdykbekovich

1 Introduction

In the present paper we consider the quasilinear boundary value problem (BVP) for Fredholm integro-differential equation (IDE)

$$\frac{dx}{dt} = A(t)x + \sum_{k=1}^{m} \varphi_k(t) \int_0^T \psi_k(\tau)x(\tau)d\tau + f_0(t) + \varepsilon f(t,x), \quad t \in (0,T), \quad x \in R^n, \qquad (1)$$

$$Bx(0) + Cx(T) = d, \quad d \in R^n, \qquad (2)$$

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where $\varepsilon > 0$, the $n \times n$ matrices $A(t)$, $\varphi_k(t)$, $\psi_k(\tau)$, $k = \overline{1,m}$, and the n vector $f_0(t)$ are continuous on $[0,T]$, $f : [0,T] \times R^n \to R^n$ is continuous; $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

Denote by $C\big([0,T], R^n\big)$ the space of continuous functions $x : [0,T] \to R^n$ with the norm $\|x\|_1 = \max\limits_{t\in[0,T]} \|x(t)\|$. A solution to problem (1), (2) is a continuously differentiable on $(0,T)$ function $x(t) \in C\big([0,T], R^n\big)$, which satisfies equation (1) and boundary condition (2).

The aim of the paper is to develop an approximate method for solving quasilinear BVP (1), (2). For this purpose, Dzhumabaev's parametrization method [1] and an approximation of integro–differential equation by a loaded differential equation are used.

BVPs for Fredholm IDEs are studied by many authors [2-8]. In finding approximate solutions to IDEs, loaded differential equations (LDEs) arise, which are obtained by replacing the integral terms of IDEs with a quadrature formula [9-12]. Loaded differential equations and problems for these equations are considered in [13-15].

2 Scheme of the method

We divide the interval $[0,T)$ into $2N$, $N \in \mathbb{N}$, parts:

$$[0,T) = \bigcup_{r=1}^{2N} [t_{r-1}, t_r), \quad t_r = rh, \quad h = \frac{T}{2N},$$

and replace the integral term in equation (1) with the Simpson's formula

$$\int_0^T \psi_k(\tau)x(\tau)d\tau = \frac{h}{3}\Bigg[\psi_k(0)x(0) + \psi_k(T)x(T)$$

$$+2\sum_{j=1}^{N-1}\psi_k(t_{2j})x(t_{2j}) + 4\sum_{j=1}^{N}\psi_k(t_{2j-1})x(t_{2j-1})\Bigg]. \tag{3}$$

Then we get LDE of the form:

$$\frac{dx}{dt} = A(t)x + \frac{h}{3}\sum_{k=1}^{m}\varphi_k(t)\Bigg[\psi_k(0)x(0) + \psi_k(T)x(T) + 2\sum_{j=1}^{N-1}\psi_k(t_{2j})x(t_{2j})$$

$$+4\sum_{j=1}^{N}\psi_k(t_{2j-1})x(t_{2j-1})\Bigg] + f_0(t) + \varepsilon f(t,x), \qquad t \in (0,T), \quad x \in R^n. \tag{4}$$

Let $x_r(t)$ be the restriction of the function $x(t)$ to the rth subinterval, i.e. $x_r(t) = x(t)$ for $t \in [t_{r-1}, t_r)$, $r = \overline{1,2N}$. Then, problem (4), (2) is reduced to the multipoint BVP for the system of the LDEs:

$$\frac{dx_r}{dt} = A(t)x_r + \frac{h}{3}\sum_{k=1}^{m}\varphi_k(t)\Bigg[\psi_k(0)x_1(0) + \psi_k(T)x(T) + 2\sum_{j=1}^{N-1}\psi_k(t_{2j})x_{2j+1}(t_{2j})$$

$$+4\sum_{j=1}^{N}\psi_k(t_{2j-1})x_{2j}(t_{2j-1})\Bigg] + f_0(t) + \varepsilon f(t,x_r), \quad t\in[t_{r-1},t_r), \quad r=\overline{1,2N}, \tag{5}$$

$$Bx_1(0) + Cx(t_{2N}) = d, \tag{6}$$

$$\lim_{t\to t_p-0} x_p(t) = x_{p+1}(t_p), \qquad p=\overline{1,2N-1}, \tag{7}$$

$$\lim_{t\to T-0} x_{2N}(t) = x(t_{2N}), \tag{8}$$

where (7), (8) are continuity conditions of the solution at the interior points of partition and at the point $t=T$.

We consider the value of functions $x_r(t)$ at the beginning points of the subintervals as additional parameters: $\lambda_r = x_r(t_{r-1})$, and make the substitution $u_r(t) = x_r(t) - \lambda_r$, $r=\overline{1,2N}$, on each rth interval we get BVP with parameters

$$\frac{du_r}{dt} = A(t)[u_r+\lambda_r] + \frac{h}{3}\sum_{k=1}^{m}\varphi_k(t)\Bigg[\psi_k(0)\lambda_1 + \psi_k(T)\lambda_{2N+1} + 2\sum_{j=1}^{N-1}\psi_k(t_{2j})\lambda_{2j+1}$$

$$+4\sum_{j=1}^{N}\psi_k(t_{2j-1})\lambda_{2j}\Bigg] + f_0(t) + \varepsilon f(t,u_r+\lambda_r), \quad t\in[t_{r-1},t_r), \quad r=\overline{1,2N}, \tag{9}$$

$$u_r(t_{r-1}) = 0, \quad r=\overline{1,2N}, \tag{10}$$

$$B\lambda_1 + C\lambda_{2N+1} = d, \tag{11}$$

$$\lambda_p + \lim_{t\to t_p-0} u_p(t) - \lambda_{p+1} = 0, \quad p=\overline{1,2N}. \tag{12}$$

Let $C\big([0,T],\Delta_{2N},R^{2nN}\big)$ denote the space of function systems $u[t] = \big(u_1(t),u_2(t),\ldots,u_{2N}(t)\big)$, where $u_r:[t_{r-1},t_r)\to R^n$ is continuous and has the finite left-sided limit $\lim_{t\to t_r-0} u_r(t)$ for any $r=\overline{1,2N}$, with the norm $\big\|u[\cdot]\big\|_2 = \max_{r=\overline{1,2N}}\sup_{t\in[t_{r-1},t_r)}\|u_r(t)\|$.

A solution to problem (9)-(12) is a pair $\left(\lambda^*, u^*[t]\right)$ with $\lambda^* = \left(\lambda_1^*, \lambda_2^*, \ldots, \lambda_{2N+1}^*\right) \in R^{n(2N+1)}$ and $u^*[t] = \left(u_1^*(t), u_2^*(t), \ldots, u_{2N}^*(t)\right) \in C\left([0,T], \Delta_{2N}, R^{2nN}\right)$, whose components $u_r^*(t)$, $r = \overline{1,2N}$, satisfy equations (9), (11), continuity conditions (12) for $\lambda_r = \lambda_r^*$, $r = \overline{1,2N+1}$, and initial conditions (10).

We will use the limit values of the solution to problem (9), (10) later on, when we turn to problem (4), (2). It is therefore reasonable to consider the following Cauchy problem on the closed subintervals:

$$\frac{dv_r}{dt} = A(t)[v_r + \lambda_r] + \frac{h}{3}\sum_{k=1}^{m}\varphi_k(t)\Bigg[\psi_k(0)\lambda_1 + \psi_k(T)\lambda_{2N+1} + 2\sum_{j=1}^{N-1}\psi_k(t_{2j})\lambda_{2j+1}$$

$$+4\sum_{j=1}^{N}\psi_k(t_{2j-1})\lambda_{2j}\Bigg] + f_0(t) + \varepsilon f(t, v_r + \lambda_r), \quad t \in [t_{r-1}, t_r], \quad r = \overline{1,2N}, \tag{13}$$

$$v_r(t_{r-1}) = 0, \quad r = \overline{1,2N}. \tag{14}$$

Denote by $\widetilde{C}\left([0,T], \Delta_{2N}, R^{2nN}\right)$ the space of function systems $v[t] = \big(v_1(t), v_2(t), \ldots, v_{2N}(t)\big)$, where $v_r : [t_{r-1}, t_r] \to R^n$ is continuous for any $r = \overline{1,2N}$, with the norm $\big\|v[\cdot]\big\|_3 = \max\limits_{r=\overline{1,2N}} \max\limits_{t\in[t_{r-1},t_r]} \|v_r(t)\|$.

It is clear that if function systems $u[t,\lambda^*]$ and $v[t,\lambda^*]$ are the solutions to problems (9), (10) and (13), (14), respectively, then

$$u_r(t,\lambda) = v_r(t,\lambda), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,2N},$$

$$\lim_{t\to t_r-0} u_r(t,\lambda) = v_r(t_r,\lambda), \quad r = \overline{1,2N}.$$

3 Algorithm for finding approximate solution

Denote by $PC([0,T], \Delta_{2N}, R^n)$ the space of piecewise continuous functions $x : [0,T] \to R^n$ with the possible discontinuity points of the first kind: $t = t_j$, $j = \overline{1,2N}$, with the norm $\|x\|_4 = \sup\limits_{t\in[0,T]} \|x(t)\|$.

Let us choose a vector $\lambda^{(0)} = \left(\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_{2N+1}^{(0)}\right) \in R^{n(2N+1)}$. Assume that the Cauchy problems (13), (14) with $\lambda = \lambda^{(0)}$ have the solutions $v_r^{(0)}(t)$, $r = \overline{1,2N}$, and the function system $v^{(0)}[t] = \left(v_1^{(0)}(t), v_2^{(0)}(t), \ldots, v_{2N}^{(0)}(t)\right) \in \widetilde{C}\left([0,T], \Delta_{2N}, R^{2nN}\right)$. Compose the piecewise continuous function $x_0(t)$ on $[0,T]$ by the equalities $x_0(t) = \lambda_r^{(0)} + u_r^{(0)}(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,2N}$, $x_0(T) = \lambda_{2N+1}^{(0)}$.

Given some positive numbers ρ, ρ_λ, and ρ_v, we introduce the following sets:

$$S\big(\lambda^{(0)},\rho_\lambda\big)=\Big\{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{2N+1})\in R^{n(2N+1)}:\|\lambda-\lambda^{(0)}\|=\max_{j=\overline{1,2N+1}}\|\lambda_j-\lambda_j^{(0)}\|<\rho_\lambda\Big\},$$

$$S\big(v^{(0)},\rho_v\big)=\Big\{v[t]\in\widetilde{C}\big([0,T],\Delta_{2N},R^{2nN}\big):\|v[\cdot]-v^{(0)}[\cdot]\|_3<\rho_v\Big\},$$

$$S\big(x_{(0)}(t),\rho\big)=\Big\{x(t)\in PC\big([0,T],\Delta_{2N},R^n\big):\|x(t)-x_{(0)}(t)\|_4<\rho\Big\}.$$

Condition A. Suppose that the Cauchy problems (13), (14) have unique solutions $v_r(t,\lambda)$, $r=\overline{1,2N}$, for all $\lambda\in S\big(\lambda^{(0)},\rho_\lambda\big)$. Moreover, the function system $v[t,\lambda]=(v_1(t,\lambda),$ $v_2(t,\lambda),\ldots,v_{2N}(t,\lambda))\in\widetilde{C}\big([0,T],\Delta_{2N},R^{2nN}\big)$, and $\big\|v[\cdot,\lambda]-v^{(0)}[\cdot]\big\|_3<\rho_v$.

Further we assume that Condition A is met and the inequality $\rho_v+\rho_\lambda\le\rho$ holds.

Substituting $v_r(t,\lambda)$ into continuity conditions (12) and taking into account equation (11), we get the system of $n(2N+1)$ nonlinear algebraic equation in parameters $\lambda\in R^{n(2N+1)}$:

$$B\lambda_1+C\lambda_{2N+1}=d, \tag{15}$$

$$\lambda_p+v_p(t_p,\lambda)-\lambda_{p+1}=0,\quad p=\overline{1,2N}. \tag{16}$$

Rewrite this system in the form:

$$Q_*(\Delta_{2N},\lambda)=0,\quad \lambda\in R^{n(2N+1)}. \tag{17}$$

If equation (17) has a solution $\lambda^*=\Big(\lambda_1^*,\lambda_2^*,\ldots,\lambda_{2N+1}^*\Big)\in S\big(\lambda^{(0)},\rho_\lambda\big)$, then the pair $\Big(\lambda^*,v[t,\lambda^*]\Big)$ is a solution to the problem (11)-(14). Therefore, the function $x^*(t)$ given by the equality $x^*(t)=\lambda_r^*+v_r^*(t,\lambda_r^*)$, for $t\in[t_{r-1},t_r)$, $r=\overline{1,2N}$, and $x^*(T)=\lambda_{2N+1}^*$ will be a solution to the problem (4), (2).

Thus, in order to solve BVP (4), (2) it is enough to find a solution to the system of algebraic equations (17) and solve the Cauchy problem (13), (14) for finding the values of parameters.

To solve system (17), we use Newton's method. The explicit form of $Q_*(\Delta_{2N},\widehat{\lambda})$ can be found only in exceptional cases. However, if $v[t,\widehat{\lambda}]$ is the solution to problem (13), (14) for $\lambda=\widehat{\lambda}\in S(\lambda^{(0)},\rho_\lambda)$ then

$$Q_*(\Delta_{2N};\widehat{\lambda})=\begin{pmatrix}B\widehat{\lambda}_1+C\widehat{\lambda}_{2N+1}-d\\ \widehat{\lambda}_1+v_1(t_1,\widehat{\lambda})-\widehat{\lambda}_2\\ \cdots\\ \widehat{\lambda}_{2N}+v_{2N}(t_{2N},\widehat{\lambda})-\widehat{\lambda}_{2N+1}\end{pmatrix}. \tag{18}$$

Condition B. The function $f(t,x)$ has the uniformly continuous partial derivatives $f'_x(t,x)$ in $G^0(\rho)=\{(t,x): t\in[0,T], \|x-x_{(0)}(t)\|<\rho\}$.

In order to find $\dfrac{\partial v_r(t,\lambda)}{\partial \lambda_i}$, $r=\overline{1,2N}$, $i=\overline{1,2N+1}$, we consider problem (13), (14). Differentiating both sides of equation (13) and initial condition (14) with respect to λ_i, we get

$$\frac{d}{dt}\left(\frac{\partial v_r(t,\lambda)}{\partial \lambda_i}\right)=A(t)\left[\frac{\partial v_r(t,\lambda)}{\partial \lambda_i}+\sigma_{ri}\right]+\frac{h}{3}\sum_{k=1}^{m}\varphi_k(t)[\psi_k(0)\sigma_{1i}+\psi_k(T)\sigma_{2N+1,i}$$

$$+2\psi_k(t_{2j})\sigma_{2j+1,i}+4\psi_k(t_{2j-1})\sigma_{2j,i}]+\varepsilon f'_x(t,v_r+\lambda_r)\left[\frac{\partial v_r(t,\lambda)}{\partial \lambda_i}+\sigma_{ri}\right],\quad t\in[t_{r-1},t_r],$$

$$\left.\frac{\partial v_r(t,\lambda)}{\partial \lambda_i}\right|_{t=t_{r-1}}=0,\quad r=\overline{1,2N},\quad i=\overline{1,2N+1},$$

where $j\neq N$, $j\in\mathbb{N}$,

$$\sigma_{li}=\begin{cases} I, & i=l, \quad I \text{ is the identity matrix of dimension } n,\\ O, & i\neq l, \quad O \text{ is } n\times n \text{ zero matrix }.\end{cases}$$

Hence, if we denote by $z_{ri}(t,\lambda_1,\dots,\lambda_{2N+1})$ the partial derivative $\dfrac{\partial v_r(t,\lambda_1,\dots,\lambda_{2N+1})}{\partial \lambda_i}$, $r=\overline{1,2N}$, $i=\overline{1,2N+1}$, then the function $z_{ri}(t,\lambda)$ is a solution to the linear matrix Cauchy problem

$$\frac{dz_{ri}}{dt}=A(t)\big[z_{ri}+\sigma_{ri}\big]+\frac{h}{3}\sum_{k=1}^{m}\varphi_k(t)\big[\psi_k(0)\sigma_{1i}+\psi_k(T)\sigma_{2N+1,i}+2\psi_k(t_{2j})\sigma_{2j+1,i}$$

$$+4\psi_k(t_{2j-1})\sigma_{2j,i}\big]+\varepsilon f'_x(t,v_r+\lambda_r)\big[z_{ri}+\sigma_{ri}\big],\quad t\in[t_{r-1},t_r], \tag{19}$$

$$z_{ri}(t_{r-1})=0,\quad r=\overline{1,2N},\quad i=\overline{1,2N+1}. \tag{20}$$

It is clear that by virtue of Condition B, the vector $Q_*(\Delta_{2N},\widehat{\lambda})$ of the dimension $n(2N+1)$ has uniformly continuous Jacobi matrix $\dfrac{\partial Q_*(\Delta_{2N},\widehat{\lambda})}{\partial \lambda}$ in $S(\lambda^{(0)},\rho_\lambda)$. The Jacobi matrix has the form:

$$\begin{pmatrix} B & O & \dots & O & C\\ I+z_{11}(t_1,\widehat{\lambda}) & -I+z_{12}(t_1,\widehat{\lambda}) & \dots & z_{1,2N}(t_1,\widehat{\lambda}) & z_{1,2N+1}(t_1,\widehat{\lambda})\\ z_{21}(t_2,\widehat{\lambda}) & I+z_{22}(t_2,\widehat{\lambda}) & \dots & z_{2,2N}(t_2,\widehat{\lambda}) & z_{2,2N+1}(t_2,\widehat{\lambda})\\ \dots & \dots & \dots & \dots & \\ z_{2N,1}(t_{2N},\widehat{\lambda}) & z_{2N,2}(t_{2N},\widehat{\lambda}) & \dots & I+z_{2N,2N}(t_{2N},\widehat{\lambda}) & -I+z_{2N,2N+1}(t_{2N},\widehat{\lambda}) \end{pmatrix}. \tag{21}$$

We solve BVP (4), (2) by the following Algorithm.

STEP 0. Solving the Cauchy problems (13), (14) for $\lambda = \lambda^{(0)}$, we find functions $v_r(t, \lambda^{(0)})$, $r = \overline{1, 2N}$. By the equalities

$x^{(0)}(t) = \lambda_r^{(0)} + v_r(t, \lambda^{(0)})$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, 2N}$, and $x^{(0)}(T) = \lambda_{2N+1}^{(0)}$,

we define the piecewise continuous on $[0, T]$ function $x^{(0)}(t)$.

STEP 1. A) Using formula (18), we compose vector

$$Q_*(\Delta_{2N}; \lambda^{(0)}) = \begin{pmatrix} B\lambda_1^{(0)} + C\lambda_{2N+1}^{(0)} - d \\ \lambda_1^{(0)} + v_1(t_1, \lambda^{(0)}) - \lambda_2^{(0)} \\ \dots \\ \lambda_{2N}^{(0)} + v_{2N}(t_{2N}, \lambda^{(0)}) - \lambda_{2N+1}^{(0)} \end{pmatrix}.$$

B) Solve the matrix Cauchy problems (19), (20) for $v_r = v_r(t, \lambda^{(0)})$, $\lambda = \lambda^{(0)}$ and find functions $z_{ri}(t, \lambda^{(0)})$, $r = \overline{1, 2N}$, $i = \overline{1, 2N+1}$. Using formula (21), we compose the Jacobi matrix

$$\frac{\partial Q_*(\Delta_{2N}, \lambda^{(0)})}{\partial \lambda}$$

$$= \begin{pmatrix} B & O & \dots & O & C \\ I + z_{11}(t_1, \lambda^{(0)}) & -I + z_{12}(t_1, \lambda^{(0)}) & \dots & z_{1,2N}(t_1, \lambda^{(0)}) & z_{1,2N+1}(t_1, \lambda^{(0)}) \\ z_{21}(t_2, \lambda^{(0)}) & I + z_{22}(t_2, \lambda^{(0)}) & \dots & z_{2,2N}(t_2, \lambda^{(0)}) & z_{2,2N+1}(t_2, \lambda^{(0)}) \\ \dots & \dots & \dots & \dots & \\ z_{2N,1}(t_{2N}, \lambda^{(0)}) & z_{2N,2}(t_{2N}, \lambda^{(0)}) & \dots & I + z_{2N,2N}(t_{2N}, \lambda^{(0)}) & -I + z_{2N,2N+1}(t_{2N}, \lambda^{(0)}) \end{pmatrix}.$$

C) Solving the system of linear algebraic equations

$$\frac{\partial Q_*(\Delta_{2N}, \lambda^{(0)})}{\partial \lambda} \Delta\lambda = -\frac{1}{\alpha} Q_*(\Delta_{2N}, \lambda^{(0)}), \quad \alpha \geq 0,$$

we find $\Delta\lambda^{(0)}$. Now, the vector $\lambda^{(1)}$ is defined as follows: $\lambda^{(1)} = \lambda^{(0)} + \Delta\lambda^{(0)}$.

D) Solving the Cauchy problems (13), (14) for $\lambda = \lambda^{(1)}$, we find functions $v_r(t, \lambda^{(1)})$, $r = \overline{1, 2N}$. By the equalities

$$x^{(1)}(t) = \lambda_r^{(1)} + v_r(t, \lambda^{(1)}), \qquad t \in [t_{r-1}, t_r), \quad r = \overline{1, 2N}, \quad \text{and} \quad x^{(1)}(T) = \lambda_{2N+1}^{(1)},$$

we define the piecewise continuous on $[0, T]$ function $x^{(1)}(t)$.

Continuing the process, in the kth step of the algorithm, we obtain $\lambda^{(k)} \in R^{n(2N+1)}$ and functions $v_r^{(k)}(t, \lambda^{(k)})$, $r = \overline{1, 2N}$. By the equalities

$$x^{(k)}(t) = \lambda_r^{(k)} + v_r(t, \lambda^{(k)}), \qquad t \in [t_{r-1}, t_r), \quad r = \overline{1, 2N}, \quad \text{and} \quad x^{(k)}(T) = \lambda_{2N+1}^{(k)},$$

we define the piecewise continuous on $[0,T]$ function $x^{(k)}(t)$. It is easily seen that

$$\|\lambda^{(k)} - \lambda^*\| \leq \gamma_* \|Q_*(\Delta_{2N}; \lambda^{(k)})\|, \tag{22}$$

where λ^* is the solution to equation (17). Estimate (22) allows us to measure the proximity between the approximate and exact solutions in the kth step of the algorithm.

Example. Solve the quasilinear BVP for Fredholm IDEs

$$\frac{dx}{dt} = \begin{pmatrix} t & t^2-1 \\ 1 & t^3 \end{pmatrix} + \begin{pmatrix} 1 & t \\ t^2 & 1 \end{pmatrix} \int\limits_0^T \begin{pmatrix} \tau & 0 \\ 1 & \tau^2 \end{pmatrix} x(\tau)d\tau$$

$$+ \begin{pmatrix} -\frac{t^4}{10} - 2t^3 - \frac{3t^2}{5} + \frac{73t}{12} + \frac{27}{20} \\ -t^4 - t^3 - \frac{9t^2}{20} - \frac{t}{5} + \frac{251}{60} \end{pmatrix} + \varepsilon \begin{pmatrix} x_1^2 \\ x_1 + x_2^2 \end{pmatrix}, \qquad t \in (0,T),$$

$$x(0) - x(T) = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

where $\varepsilon = \dfrac{1}{10}, \quad T = 1.$

The exact solution to this problem is $x^*(t) = \begin{pmatrix} t^2-2 \\ t+1 \end{pmatrix}$.

Let us solve the problem by the proposed algorithm. We divide the interval $[0,T]$ into 4 equal parts. Below, in Figures 1–3, we give the obtained results.

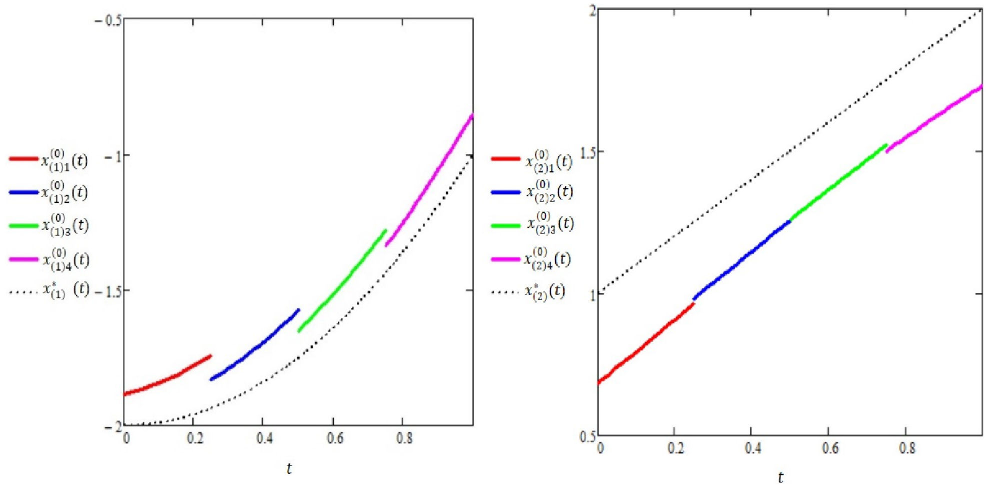

Figure 1 – Graphs of exact solution and initial approximation to the solution of the problem

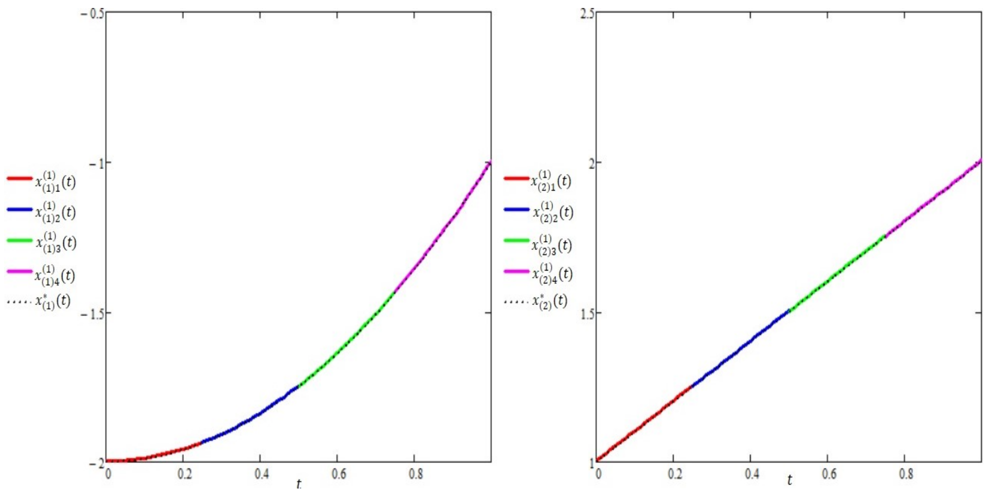

Figure 2 – Graphs of exact solution and the first approximation to the solution of the problem

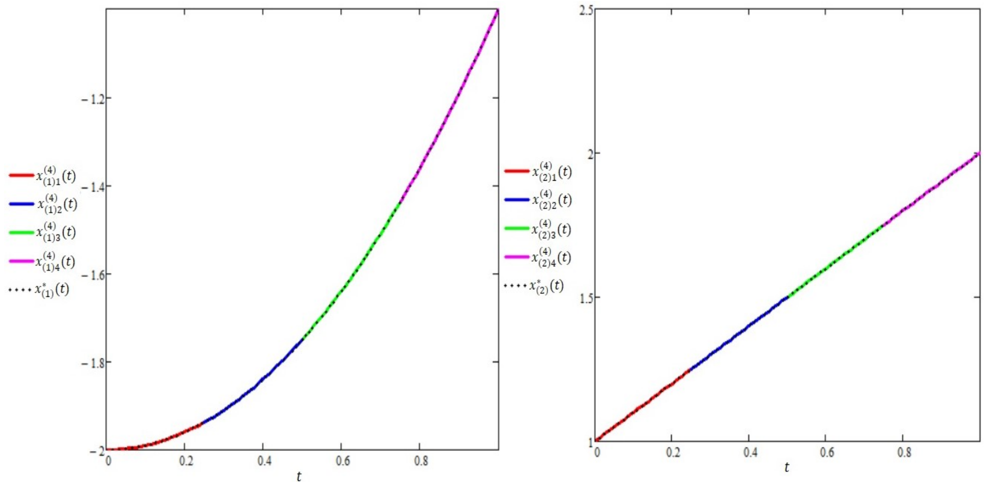

Figure 3 – Graphs of exact solution and the fourth approximation to the solution of the problem

For the difference of the exact and corresponding approximate solutions to the problem, the following estimates are true:

$$\sup_{t\in[0,T]} \|x^*(t) - x^{(0)}(t)\| < 0.2859,$$

$$\max \sup_{t\in[0,T]} \|x^*(t) - x^{(1)}(t)\| < 4.4897 \cdot 10^{-3},$$

$$\max \sup_{t\in[0,T]} \|x^*(t) - x^{(1)}(t)\| < 6.3318 \cdot 10^{-10}.$$

The offered algorithm is effective and allows us to obtain the approximate solution to the quasilinear BVP for the Fredholm IDE of higher order accuracy.

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Мынбаева С.Т. ФРЕДГОЛЬМ ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУІ ҮШІН КВАЗИСЫЗЫҚТЫ ШЕТТІК ЕСЕПТІҢ ЖУЫҚ ШЕШІМІ

Фредгольм интегралдық-дифференциалдық теңдеуі үшін квазисызықты шеттік есепті шешудің жуық әдісі ұсынылған. Бұл әдіс интегралдық мүшені Симпсон формуласымен жуықтауға және бастапқы есепті жүктелген дифференциалдық теңдеулер жүйесі үшін квазисызықты шеттік есепке келтіруге негізделген. Жуықтаушы шеттік есептің сандық шешімін табу алгоритмі және жуық шешімін құру әдісі ұсынылған.

Кілттік сөздер. квазисызықты Фредгольм интегралдық-дифференциалдық теңдеуі, Джумабаев параметрлеу әдісі, Ньютон әдісі, Симпсон формуласы.

Мынбаева С.Т. ПРИБЛИЖЕННОЕ РЕШЕНИЕ КВАЗИЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ФРЕДГОЛЬМА

Предложен приближенный метод решения квазилинейной краевой задачи для интегро-дифференцированного уравнения Фредгольма. Метод основан на аппроксимации интегрального члена формулой Симпсона и сведении исходной задачи к квазилинейной краевой задаче для системы нагруженных дифференциальных уравнений. Предложен алгоритм нахождения численного решения и метод построения приближенного решения аппроксимирующей краевой задачи.

Ключевые слова. квазилинейное интегро-дифференциальное уравнение Фредгольма, метод параметризации Джумабаева, метод Ньютона, формула Симпсона.

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Boundary value problem for systems of loaded differential equations with singularities

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Abstract. In this paper we investigate a linear two-point boundary value problem for systems of loaded differential equations with singularity. To study the task, we use the parameterization method proposed by Professor D. Dzhumabaev, that is, we introduce new parameters and, based on these parameters, we change variables. When passing to new variables, we obtain initial conditions. Using the so-called fundamental matrix of the main part and substituting the obtained solutions into the boundary value conditions, we get a system of equations for the entered parameters. According to invertibility of the matrix of this system, necessary conditions for existence of a solution to the considered problem are established.

Keywords. System of loaded differential equations, parametrization method, boundary value conditions, conformable derivative, unique solvability.

Dedicated to the memory of
Professor Dulat Dzhumabaev

1 Introduction

In [1-10], definitions and basic properties of the conformable derivative were introduced.

Definition 1. *Let a function $f : [0,\infty) \to R$. Then for all $t > 0$ the conformable derivative from the function f is defined in the form:*

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

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where $\alpha \in (0,1)$. *If* f *is differentiable up to order* α *on* $(0,a)$, $a>0$, *and* $\lim\limits_{\varepsilon\to 0+} f^{(\alpha)}(t)$ *exists, then* $f^{(\alpha)}(0) = \lim\limits_{\varepsilon\to 0+} f^{(\alpha)}(t)$.

Definition 2. *The conformable integral of the order* $\alpha \in (0,1]$ *from a function* f *is defined by the equality:*

$$I_\alpha^a(f)(t) = \int_a^t \tau^{\alpha-1} f(\tau)\, d\tau.$$

Lemma 1. *Let for* $t>0$ *functions* f *and* g *be differentiable up to order* $\alpha \in (0,1)$. *Then*

1) $T_\alpha(af+bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all* $a, b \in R$.

2) $T_\alpha(c) = 0$, *for all* $f(t) = const$.

3) $T_\alpha(fg) = fT_\alpha(f) + T_\alpha(f)g$.

4) $T_\alpha\left(\frac{f}{g}\right) = \frac{fT_\alpha(f) - T_\alpha(f)g}{g^2}$.

5) If f *is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Lemma 2. *Let* $\alpha \in (0,1]$ *and a function* f *be continuous when* $t>a$, *then*

$$T_\alpha I_\alpha^a(f)(t) = f(t).$$

In this paper, on the interval $[0,T]$, we consider the two-point boundary value problem for systems of loaded differential equations with conformable derivative

$$T_\alpha(x)(t) = A(t)x + \sum_{j=1}^{m+1} K_j(t)x(\theta_{j-1}) + f(t), \quad t \in [0,T], \tag{1}$$

$$0 = \theta_0 < \theta_1 < \ldots < \theta_m = T,$$

$$Bx(0) + Cx(T) = d, d \in R^n, \tag{2}$$

where $(n\times n)$- matrices $t^{\alpha-1}A(t)$, $K_j(t)$ and n- dimensional vector function $f(t)$ are continuous on $[0,T]$. Boundary value problem (1)-(2) is investigated by the parametrization method proposed by Professor D.S. Dzhumabaev [11-15].

2 Scheme of the method and the main assertion

We break the segment $[0,T)$: $[0,T) = \bigcup_{r=1}^{m}[\theta_{r-1},\theta_r)$, and denote restriction of the function $x(t)$ on the $r-$th interval $[\theta_{r-1},\theta_r)$, $r = \overline{1,m}$, by $x_r(t)$. Then the origin two-point boundary value problem for systems of loaded differential equations is reduced to the equivalent multi-point boundary value problem

$$T_\alpha(x_r)(t) = A(t)x_r + \sum_{j=1}^{m} K_j(t)x_j(\theta_{j-1}) + K_{m+1}(t)\lim_{t\to T-0} x_{m+1}(t) + f(t), \tag{3}$$

$$t \in [\theta_{r-1}, \theta_r), \qquad r = \overline{1, m},$$

$$Bx_1(0) + C\lim_{t\to T-0} x_m(t) = d, \tag{4}$$

$$\lim_{t\to\theta_s-0} x_s(t) = x_{s+1}(\theta_s), \qquad s = \overline{1, m-1}. \tag{5}$$

Here (5) are conditions for gluing at inner points of the partition $t = jh, \ \ j = \overline{1, N-1}$.

If a function $x(t)$ is a solution of the problem (1)-(2), then a system of its restriction $x[t] = (x_1(t), x_2(t), ..., x_m(t))'$ will be a solution of the multipoint boundary value problem (3)-(5). And in inverse, if system of vector functions $\tilde{x}[t] = (\tilde{x}_1(t), \tilde{x}_2(t), ..., \tilde{x}_m(t))'$ is a solution of the problem (3)-(5), then the function $\tilde{x}(t)$, defined by the equalities

$$\tilde{x}(t) = \tilde{x}_r(t), \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, m}, \qquad \tilde{x}(T) = \lim_{t\to T-0} \tilde{x}_m(t),$$

will be a solution of the original boundary value problem (1)-(2).

By λ_r we denote a value of the function $x_r(t)$ at the point $t = \theta_{r-1}$ and on each interval $[\theta_{r-1}, \theta_r)$, we change the variable $x_r(t) = u_r(t) + \lambda_r$, $r = \overline{1, m}$. Introduce the additional parameter $\lambda_{m+1} = \lim_{t\to T-0} x_m(t)$, then the boundary value problem (3)-(5) is reduced to the equivalent multi-point boundary value problem with parameters:

$$T_\alpha(u_r)(t) = A(t)[u_r + \lambda_r] + \sum_{j=1}^{m+1} K_j(t)\lambda_j + f(t), \tag{6}$$

$$u_r(\theta_{r-1}) = 0, \qquad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, m}\ , \tag{7}$$

$$B\lambda_1 + C\lambda_{m+1} = d, \tag{8}$$

$$\lambda_s + \lim_{t\to sh-0} u_s(t) = \lambda_{s+1}, \qquad s = \overline{1, m}. \tag{9}$$

The problems (3)-(5) and (6)-(9) are equivalent in the sense, that if the system of functions $x[t] = (x_1(t), x_2(t), ..., x_m(t))'$ is a solution of the problem (3)-(5), then the pair $(\lambda, u[t])$, where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m, \lambda_{m+1})'$, $u[t] = (u_1(t), u_2(t), ..., u_m(t))$, defined by the equalities:
$\lambda_r = x_r(\theta_{r-1})$, $r = \overline{1, m}$, $\lambda_{m+1} = \lim_{t\to T-0} x_m(t)$ and $u_r(t) = x_r(t) - x_r(\theta_{r-1})$, $r = \overline{1, m}$,

will be a solution of the problem (6)-(9). And in inverse, if the pair $(\tilde{\lambda}, \tilde{u}[t])$, where $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_{m+1})'$, $\tilde{u}[t] = (\tilde{u}_1(t), \tilde{u}_2(t), ..., \tilde{u}_m(t))'$ is a solution of the problem (6)-(9), then the system of functions $\tilde{x}[t] = (\tilde{\lambda}_1 + \tilde{u}_1(t), \tilde{\lambda}_2 + \tilde{u}_2(t), ..., \tilde{\lambda}_m + \tilde{u}_m(t))'$, $\tilde{x}(T) = \tilde{\lambda}_{m+1}$ will be a solution of the problem (3)-(5).

Appearance of the initial conditions (7) makes it possible to solve the Cauchy problem (6), (7). We use Definition 2 and results of Lemma 2, then

$$u_r(t) = \int_{\theta_{r-1}}^{t} \tau^{\alpha-1} \left(A(\tau)\left[u(\tau)+\lambda_r\right] d\tau + \sum_{j=1}^{m+1} K_j(\tau)\lambda_j + f(\tau) \right) d\tau, \tag{10}$$

$$t \in [\theta_{r-1}, \theta_r), \qquad r = \overline{1,m}.$$

From (10) defining $\lim\limits_{t\to sh-0} u_s(t)$, $s = \overline{1,m}$, and putting their corresponding expressions into the conditions (8) and (9), we get the system of linear equations concerning to the unknown parameters λ_r, $r = \overline{1, m+1:}$

$$B\lambda_1 + C\lambda_{m+1} = d, \tag{11}$$

$$\lambda_s + \int_{\theta_{s-1}}^{\theta_s} \tau^{\alpha-1} A(\tau)\lambda_s d\tau + \sum_{j=1}^{m+1} \int_{\theta_{s-1}}^{\theta_s} K_j(\tau)\lambda_j d\tau - \lambda_{s+1}$$

$$= -\int_{\theta_{s-1}}^{\theta_s} \tau^{\alpha-1} A(\tau)u(\tau)d\tau - \int_{\theta_{s-1}}^{\theta_s} \tau^{\alpha-1} f(\tau)d\tau, \tag{12}$$

combining the same parameters

$$B\lambda_1 + C\lambda_{m+1} = d, \tag{13}$$

$$\left(I + \int_{\theta_{s-1}}^{\theta_s} \tau^{\alpha-1} A(\tau)\lambda_s d\tau + \int_{\theta_{s-1}}^{\theta_s} K_s(\tau)d\tau \right)\lambda_s + \sum_{\substack{j=1 \\ j\neq s}}^{m+1} \int_{\theta_{s-1}}^{\theta_s} K_j(\tau)\lambda_j d\tau - \lambda_{s+1}$$

$$= -\int_{\theta_{s-1}}^{\theta_s} \tau^{\alpha-1} A(\tau)u(\tau)d\tau - \int_{\theta_{s-1}}^{\theta_s} \tau^{\alpha-1} f(\tau)d\tau. \tag{14}$$

A matrix of the dimension $n(m+1) \times n(m+1)$, corresponding to the left-hand side of the systems of linear equations (13), (14) is denoted by $Q(\theta)$. Then the system of linear equations (13), (14) is written in the form

$$Q(\theta)\lambda = -F(\theta) - G(u,\theta), \lambda \in R^{nN}, \tag{15}$$

where

$$F(\theta) = \left\{ d,\ \int_0^{\theta_1} \tau^{\alpha-1} f(\tau)d\tau\ , \ldots,\ \int_{T-\theta_{m-1}}^{T} \tau^{\alpha-1} f(\tau)d\tau \right\},$$

$$G(u,\theta) = \left\{ 0, \int_0^{\theta_1} \tau^{\alpha-1} A(\tau)u(\tau)d\tau\ , \ldots,\ \int_{T-\theta_{m-1}}^{T} \tau^{\alpha-1} A(\tau)u(\tau)d\tau \right\}.$$

Thus, to find unknown pairs $(\lambda, u[t])$, solutions to the problem (6)-(9), we have a closed system of equations (10), (15). We find the solution of the multipoint boundary value problem (6)-(9) as a limit of the sequence of pairs $(\lambda'^{(k)}, u^{(k)}[t])$, k =0,1,2, ..., determined by the following algorithm.

Step 0. a) Assuming, that the matrix $Q(h)$ is invertible, from the equation $Q(h)\lambda = -F(h)$ we define initial approximation by the parameter $\lambda^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, ..., \lambda_N^{(0)}) \in R^{nN}$: $\lambda^{(0)} = -[Q(h)]^{-1}F(h)$.

b) Putting the found $\lambda_r^{(0)}$, $r = \overline{1,N}$, into the right-hand side of the system of integro-differential equations (6) and solving the special Cauchy problem with conditions (7), we find $u^{(0)}[t] = (u_1^{(0)}(t), u_2^{(0)}(t), ..., u_N^{(0)}(t))'$.

Step 1. a) Putting the found values $u_r^{(0)}(t)$, $r = \overline{1,N}$, into the right side of (15), from the equation $[Q(h)]\lambda = -F(h) - G(u^{(0)}, h)$ we define $\lambda^{(1)} = (\lambda_1^{(1)}, \lambda_2^{(1)}, ..., \lambda_N^{(1)})$.

b) Putting the found $\lambda_r^{(1)}$, $r = \overline{1,N}$, into the right-hand side of the system of integro-differential equations (6) and solving the special Cauchy problem with conditions (7), we find $u^{(1)}[t] = (u_1^{(1)}(t), u_2^{(1)}(t), ..., u_N^{(1)}(t))'$.

Continuing the process, on the k-th step of the algorithm, we find the system of pairs $(\lambda^{(k)}, u^{(k)}[t])$, k =0,1,2, ... Unknown functions $u[t] = (u_1(t), u_2(t), \ldots, u_N(t))$ are determined from the special Cauchy problem for systems of integro-differential equations (6) with initial conditions (7). Unlike the Cauchy problem for ordinary differential equations, the special Cauchy problem for systems of integro-differential equations is not always solvable. Suppose that there is a so-called fundamental matrix of the principal part of the differential equation, then the following theorem is true:

Theorem [9, p. 499]. *Let $\alpha \in (0,1)$. Then a solution of the problem*

$$T_\alpha(u_r)(t) = A(t)u_r(t) + g(t,u), \quad u(t_0) = \eta \in \mathrm{R}^n$$

is determined by the formula:

$$u(t) = \eta E_\alpha(A, t - t_0) + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha(-A, s - t_0) E_\alpha(A, t - t_0) g(\tau, u(\tau)) d\tau,$$

where $E_\alpha(\lambda, s) = \exp\left(\lambda \frac{s^\alpha}{\alpha}\right)$.

According to this theorem, we obtain

$$u_r(t) = E_\alpha(A(t), t - \theta_{r-1}) \int_{\theta_{r-1}}^{t} (t-\tau)^{\alpha-1} E_\alpha(-A(\tau), \tau - \theta_{r-1}) \left(A(\tau)\lambda_r \right.$$

$$\left. + \sum_{j=1}^{m+1} K_j(\tau)\lambda_j \right) d\tau + E_\alpha(A(t), t - \theta_{r-1}) \int_{\theta_{r-1}}^{t} (t-\tau)^{\alpha-1} E_\alpha(-A(\tau), \tau - \theta_{r-1}) f(\tau) d\tau, \quad (16)$$

$$t \in [\theta_{r-1}, \theta_r), \qquad r = \overline{1,m},$$

where $E_\alpha(A(t), t-\theta_{r-1}) = \exp\left(\int_{\theta_{r-1}}^{t} (\tau-\theta_{r-1})^{\alpha-1} A(\tau)d\tau\right)$.

From (16) defining $\lim\limits_{t\to sh-0} u_s(t)$, $s = \overline{1,m}$, and putting their corresponding expressions into the conditions (8) and (9), we get the system of linear equations concerning to the unknown parameters λ_r, $r = \overline{1,m+1}$:

$$B\lambda_1 + C\lambda_{m+1} = d, \tag{17}$$

$$\lambda_s + E_\alpha(A(\theta_s), \theta_s - \theta_{s-1}) \int_{\theta_{s-1}}^{\theta_s} (t-\tau)^{\alpha-1} E_\alpha(-A(\tau), \tau-\theta_{r-1}) \left(A(\tau)\lambda_s + \sum_{j=1}^{m+1} K_j(\tau)\lambda_j\right) d\tau$$

$$-\lambda_{s+1} = -E_\alpha(A(\theta_s), \theta_s - \theta_{s-1}) \int_{\theta_{r-1}}^{\theta_s} (t-\tau)^{\alpha-1} E_\alpha(-A(\tau), \tau-\theta_{r-1}) f(\tau)d\tau\ , \tag{18}$$

combining the same parameters

$$B\lambda_1 + C\lambda_{m+1} = d, \tag{19}$$

$$\left(I + E_\alpha(A(\theta_s), \theta_s - \theta_{s-1}) \int_{\theta_{s-1}}^{\theta_s} (t-\tau)^{\alpha-1} E_\alpha(-A(\tau), \tau-\theta_{s-1}) A(\tau)d\tau\right) \lambda_s$$

$$+E_\alpha(A(\theta_s), \theta_s - \theta_{s-1}) \sum_{j=1}^{m+1} \int_{\theta_{s-1}}^{\theta_s} (t-\tau)^{\alpha-1} E_\alpha(-A(\tau), \tau-\theta_{s-1}) K_j(\tau)d\tau\lambda_j - \lambda_{s+1}$$

$$= -E_\alpha(A(\theta_s), \theta_s - \theta_{s-1}) \int_{\theta_{s-1}}^{\theta_s} (t-\tau)^{\alpha-1} E_\alpha(-A(\tau), \tau-\theta_{s-1}) f(\tau)d\tau. \tag{20}$$

A matrix of the dimension $n(m+1)\times n(m+1)$, corresponding to the left-hand side of the systems of linear equations (19), (20) is denoted by $Q(\theta)$. Then the system of linear equations (19), (20) is written in the form

$$Q_*(\theta)\lambda = -F(\theta), \qquad \lambda \in R^{nN}, \tag{21}$$

where

$$F(h) = \Bigg\{d,\ E_\alpha(A(\theta_1), \theta_1) \int_0^{\theta_1} (t-\tau)^{\alpha-1} E_\alpha(-A(\tau), \tau) f(\tau)d\tau\ , \ldots,$$

$$E_\alpha(A(T), T-\theta_{m-1}) \int_{\theta_{m-1}}^{T} (t-\tau)^{\alpha-1} E_\alpha(-A(\tau), \tau-\theta_{m-1}) f(\tau)d\tau \Bigg\}.$$

Based on the proposed algorithm on finding a solution, it follows:

Theorem 1. *For unique solvability of the boundary value problem (1), (2), it is necessary and sufficient that the matrix $Q_*(\theta)$ is invertible.*

Example. Consider

$$T_{1/2}(x)(t) = tx(0) + x\left(\frac{1}{3}\right) + x(1) + 2t, \quad t \in [0,1], \tag{22}$$

$$x(0) = x(1). \tag{23}$$

We break the interval $[0,1) = \left[0, \frac{1}{3}\right) \cup \left[\frac{1}{3}, 1\right)$, and denote restriction of the function $x(t)$ on the interval $\left[0, \frac{1}{3}\right)$ by $x_1(t)$, and on the interval $\left[\frac{1}{3}, 1\right)$ by $x_2(t)$. Then

$$T_{1/2}(x_1)(t) = tx_1(0) + x_2\left(\frac{1}{3}\right) + \lim_{t\to 1-0} x_2(t) + 2t, \quad t \in \left[0, \frac{1}{3}\right),$$

$$T_{1/2}(x_2)(t) = tx_1(0) + x_2\left(\frac{1}{3}\right) + \lim_{t\to 1-0} x_2(t) + 2t, \quad t \in \left[\frac{1}{3}, 1\right),$$

$$x_1(0) = \lim_{t\to 1-0} x_2(t),$$

$$\lim_{t\to \frac{1}{3}-0} x_1(t) = x_2\left(\frac{1}{3}\right).$$

Denote $\lambda_1 = x_1(0)$, $\lambda_2 = x_2\left(\frac{1}{3}\right)$, $\lambda_3 = \lim_{t\to 1-0} x_1(t)$, and change $x_1(t) = u_1(t) + \lambda_1$, $x_2(t) = u_2(t) + \lambda_2$. Then we obtain

$$T_{1/2}(u_1)(t) = t\lambda_1 + \lambda_2 + \lambda_3 + 2t, \quad t \in \left[0, \frac{1}{3}\right), \tag{24}$$

$$u_1(0) = 0, \tag{25}$$

$$T_{1/2}(u_2)(t) = t\lambda_1 + \lambda_2 + \lambda_3 + 2t, \quad t \in \left[\frac{1}{3}, 1\right), \tag{26}$$

$$u_2\left(\frac{1}{3}\right) = 0, \tag{27}$$

$$\lambda_1 = \lambda_3, \tag{28}$$

$$\lambda_1 + \lim_{t\to \frac{1}{3}-0} u_1(t) = \lambda_2, \tag{29}$$

$$\lambda_2 + \lim_{t\to 1-0} u_2(t) = \lambda_3. \tag{30}$$

Solutions of the Cauchy problems (24)-(25) and (26)-(27) is defined as follows

$$u_1(t) = \int_0^t \sqrt{\tau} d\tau \lambda_1 + \int_0^t \frac{d\tau}{\sqrt{\tau}} \lambda_2 + \int_0^t \frac{d\tau}{\sqrt{\tau}} \lambda_3 + 2\int_0^t \sqrt{\tau} d\tau, \quad t \in \left[0, \frac{1}{3}\right),$$

$$u_2(t) = \int_{\frac{1}{3}}^t \sqrt{\tau} d\tau \lambda_1 + \int_{\frac{1}{3}}^t \frac{d\tau}{\sqrt{\tau}} \lambda_2 + \int_{\frac{1}{3}}^t \frac{d\tau}{\sqrt{\tau}} \lambda_3 + 2\int_{\frac{1}{3}}^t \sqrt{\tau} d\tau, \quad t \in \left[\frac{1}{3}, 1\right).$$

Thus

$$u_1(t) = \sqrt{t^3}\lambda_1 + \sqrt{t}\lambda_2 + \sqrt{t}\lambda_3 + 2\sqrt{t^3}, \quad t \in \left[0, \frac{1}{3}\right), \tag{31}$$

$$u_2(t) = \left(\sqrt{t^3} - \sqrt{\frac{1}{27}}\right)\lambda_1 + \left(\sqrt{t} - \sqrt{\frac{1}{3}}\right)\lambda_2 + \left(\sqrt{t} - \sqrt{\frac{1}{3}}\right)\lambda_3 + 2\left(\sqrt{t^3} - \sqrt{\frac{1}{27}}\right), \tag{32}$$

$$t \in \left[\frac{1}{3}, 1\right).$$

From (31), (32) we determine the limits $\lim\limits_{t\to\frac{1}{3}} u_1(t)$, $\lim\limits_{t\to 1} u_2(t)$ and put into (29) and (30). Then the equations (28), (29) and (30) can be written in the form:

$$\lambda_1 - \lambda_3 = 0,$$

$$\lambda_1 + \sqrt{\frac{1}{27}}\lambda_1 + \sqrt{\frac{1}{3}}\lambda_2 + \sqrt{\frac{1}{3}}\lambda_3 - \lambda_2 = -\sqrt{\frac{4}{27}},$$

$$\lambda_2 + \left(1 - \sqrt{\frac{1}{27}}\right)\lambda_1 + \left(1 - \sqrt{\frac{1}{3}}\right)\lambda_2 + \left(1 - \sqrt{\frac{1}{3}}\right)\lambda_3 - \lambda_3 = -2\left(1 - \sqrt{\frac{1}{27}}\right).$$

Or, combining the same parameters, we get

$$\lambda_1 - \lambda_3 = 0, \tag{33}$$

$$1.19\lambda_1 - 0.42\lambda_2 + 0.58\lambda_3 = -0.38, \tag{34}$$

$$0.81\lambda_1 + 1.42\lambda_2 - 0.58\lambda_3 = -1.62. \tag{35}$$

The matrix corresponding to the left-hand side of the equations (33), (34) and (35) is denoted by $Q(\theta)$. Then

$$Q(\theta) = \begin{pmatrix} 1 & 0 & -1 \\ 1.19 & -0.42 & 1.58 \\ 0.81 & 1.42 & -0.58 \end{pmatrix}.$$

Since the matrix $Q(\theta)$ has the inverse:

$$Q^{-1}(\theta) = \begin{pmatrix} 0.5 & 0.35 & 0.1 \\ 0.49 & -0.05 & 0.69 \\ -0.51 & 0.35 & 0.1 \end{pmatrix},$$

then system of the equations (33), (34) and (35) has a unique solution and

$$\begin{cases} \lambda_1 = -0.30, \\ \lambda_2 = -1.09, \\ \lambda_3 = -0.30. \end{cases}$$

Since

$$u_1(t) = \sqrt{t^3}\lambda_1 + \sqrt{t}\lambda_2 + \sqrt{t}\lambda_3 + 2\sqrt{t^3}, \quad t \in \left[0, \frac{1}{3}\right),$$

$$u_2(t) = \left(\sqrt{t^3} - \sqrt{\frac{1}{27}}\right)\lambda_1 + \left(\sqrt{t} - \sqrt{\frac{1}{3}}\right)\lambda_2 + \left(\sqrt{t} - \sqrt{\frac{1}{3}}\right)\lambda_3 + 2\left(\sqrt{t^3} - \sqrt{\frac{1}{27}}\right),$$
$$t \in \left[\frac{1}{3}, 1\right),$$

then

$$x_1(t) = u_1(t) + \lambda_1 = \sqrt{t^3}\lambda_1 + \sqrt{t}\lambda_2 + \sqrt{t}\lambda_3 + 2\sqrt{t^3} - 0.3 \ , \quad t \in \left[0, \frac{1}{3}\right),$$

$$x_2(t) = u_2(t) + \lambda_2 = \left(\sqrt{t^3} - \sqrt{\frac{1}{27}}\right)\lambda_1 + \left(\sqrt{t} - \sqrt{\frac{1}{3}}\right)\lambda_2 + \left(\sqrt{t} - \sqrt{\frac{1}{3}}\right)\lambda_3$$
$$+2\left(\sqrt{t^3} - \sqrt{\frac{1}{27}}\right) - 1.09, \quad t \in \left[\frac{1}{3}, 1\right).$$

Therefore, the solution of the boundary value problem (22)-(23) can be written in the form

$$x(t) = \begin{cases} \sqrt{t^3}\lambda_1 + \sqrt{t}\lambda_2 + \sqrt{t}\lambda_3 + 2\sqrt{t^3} - 0.3 \ , \quad t \in \left[0, \frac{1}{3}\right), \\ \left(\sqrt{t^3} - \sqrt{\frac{1}{27}}\right)\lambda_1 + \left(\sqrt{t} - \sqrt{\frac{1}{3}}\right)\lambda_2 + \left(\sqrt{t} - \sqrt{\frac{1}{3}}\right)\lambda_3 + 2\left(\sqrt{t^3} - \sqrt{\frac{1}{27}}\right) - 1.09, \\ \qquad t \in \left[\frac{1}{3}, 1\right), \\ -0.3, \quad t = 1. \end{cases}$$

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Назарова К.Ж., Усманов Қ.Ы. ЕРЕКШЕЛІКТЕРІ БАР ЖҮКТЕЛГЕН ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕЛЕРІНЕ АРНАЛҒАН ШЕТТІК ЕСЕП ТУРАЛЫ

Бұл жұмыста ерекшелігі бар жүктелген дифференциалдық теңдеулер жүйелері үшін сызықты екі нүктелік шеттік есеп зерттеледі. Қойылған есепті зерттеу үшін профессор Д.С. Джұмабаевтың ұсынған параметрлеу әдісі қолданылады, яғни жаңа параметрлер енгізіліп, осы параметрлер негізінде жаңа айнымалыларға ауыстырылады. Жаңа айнымалыларға көшу барысында бастапқы шарттарды аламыз. Дифференциалдық теңдеудің бас бөлігінің іргелі матрицасын қолдана отырып Коши есебінің шешімін аламыз. Алынған шешімдерді шеттік шарттарға қоя отырып, біз енгізілген параметрлерге қатысты теңдеулер жүйесін аламыз. Осы жүйенің матрицасының қайтарымдылығын талап ете отырып, қарастырылып отырған есептің бірмәнді шешімділігінің қажетті шарттарын орнатамыз.

Кілттік сөздер. Жүктелген дифференциалдық теңдеулер жүйесі, параметрлеу әдісі, шекаралық шарттар, конформабельді туынды, бірмәнді шешімділік.

Назарова К.Ж., Усманов К.И. О КРАЕВОЙ ЗАДАЧЕ ДЛЯ СИСТЕМ НАГРУЖЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ОСОБЕННОСТЯМИ

В работе исследуется линейная двухточечная краевая задача для систем нагруженных дифференциальных уравнений с особенностью. Для исследования поставленной задачи используется метод параметризации, предложенный профессором Д.С. Джумабаевым, то есть вводятся новые параметры и на основе данных параметров делается замена переменных. При переходе к новым переменным получаем начальные условия. Используя фундаментальную матрицу главной части дифференциального уравнения и подставляя полученные решения в краевые условия, получаем систему уравнений относительно введённых параметров. На основе обратимости матрицы данной системы установлены необходимые условия существования решения рассматриваемой задачи.

Keywords. Система нагруженных дифференциальных уравнений, метод параметризации, краевые условия, конформабельная производная, однозначная разрешимость.

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