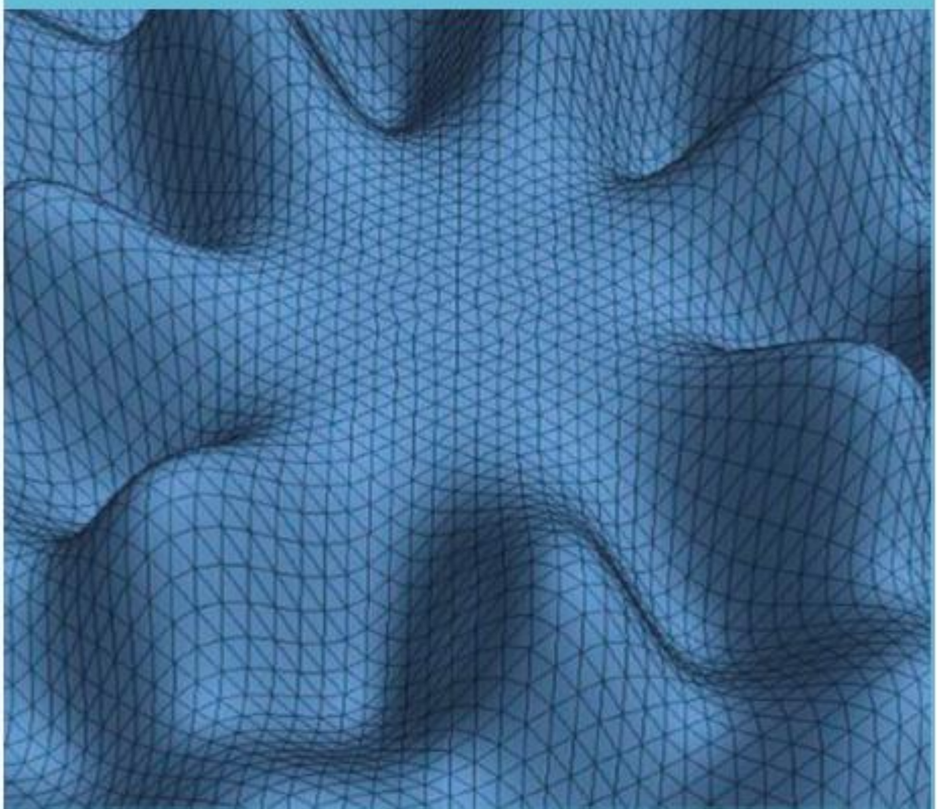
ISSN 2413-6468

KAZAKH MATHEMATICAL JOURNAL

20(3)
2020



Institute of Mathematics and Mathematical Modeling

Almaty, Kazakhstan

Institute of Mathematics and Mathematical Modeling

Vol. 20
No. 3
ISSN 2413-6468

http://kmj.math.kz/

Kazakh Mathematical Journal

(founded in 2001 as "Mathematical Journal")

Official Journal of
Institute of Mathematics and Mathematical Modeling,
Almaty, Kazakhstan

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WEB ADDRESS	http://kmj.math.kz/
PUBLICATION TYPE	Peer-reviewed open access journal Periodical Published four issues per year ISSN: 2413-6468

The Kazakh Mathematical Journal is registered by the Information Committee under Ministry of Information and Communications of the Republic of Kazakhstan № 17590-Ж certificate dated 13.03.2019.

The journal is based on the Kazakh journal "Mathematical Journal", which is publishing by the Institute of Mathematics and Mathematical Modeling since 2001 (ISSN 1682-0525).

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- Calculus of variations and optimal control
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- Mathematical modeling of heat and wave processes
- Model theory and theory of algorithms
- Numerical analysis and applications
- Operator theory
- Ordinary differential equations
- Partial differential equations
- Spectral theory
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Kazakh Mathematical Journal ISSN 2413–6468

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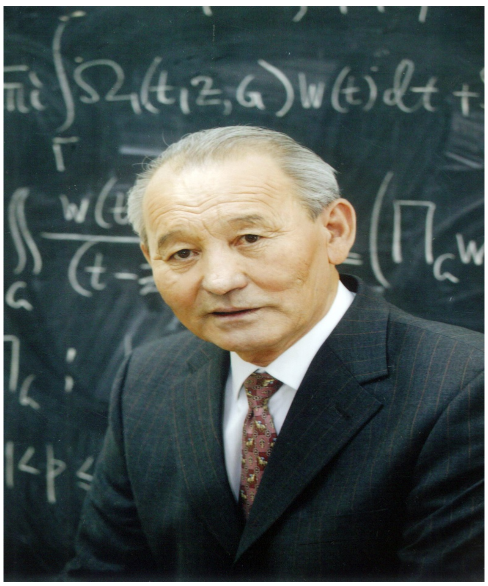
NAZARBAI KADYROVICH BLIYEV TO THE 85th ANNIVERSARY



We celebrate the 85th birth of the academician of the National Academy of Sciences of the Republic of Kazakhstan, Professor Nazarbai Kadyrovich Bliyev, the famous scientist, the specialist in the field of the theory of differential equations, mathematical physics and functional analysis who has made a great contribution to the theory of generalized analytic functions, to the theory of boundary value problems for equations of mathematical physics and to the theory of singular integral equations into functional spaces.

N.K. Bliyev was born in September 1935 in the village of Zharkamys, the Baiganinsky district of the Aktobe region, in the family of an employee. In 1952 he graduated from the Zharkamys secondary school. He dreamed of becoming a geologist but everything changed in the tenth grade. That year the Mathematics was taught by a new teacher, Zhakiya Zhusubaliev, a graduate of Ural Pedagogical Institute, who together with a famous mathematician, academician of the Academy of Sciences of the Kazakh SSR A.D. Taimanov was recommended for graduate school in Moscow but could not continue his studies for family reasons. Zh. Zhusubaliev paid attention to mathematical abilities of his student, taught him and insisted on his entering the Mathematics department of the university.

In 1952 he entered the Mathematics department of the Faculty of Physics and Mathematics of Kazakh State University named after S.M. Kirov (now al-Farabi Kazakh National University). During his university years, he listened with great interest to the lectures of Professor K.P. Persidskii and associate professors H.I. Ibrashev, M.Ya. Yataev, Sh.M. Enikeev. Under the leadership of the academician K.P. Persidskii he wrote his diploma paper on the theory of stability. In 1957 he graduated with honors from Kazakh State University named after S.M. Kirov and he was offered to go to Moscow to enter the graduate school of the Steklov Mathematical Institute (MIAS) of the USSR. But he chose to work at Guryev Pedagogical Institute named after Dosmukhamedov (now Atyrau State University named after

Dosmukhamedov) in order to have time to fulfill his filial duty to his dear grandmother Bayan Edilova who raised him and who at that time was already in old age.

In those years, Guryev Pedagogical Institute did not have enough teaching staff, N.K. Bliyev with full workload and interest worked as a teacher and then as a senior teacher till 1960, lecturing, conducting practical classes in almost all disciplines of Mathematics, that is, analytical geometry and higher algebra, the theory of functions of real and complex variables, the theory of differential equations.

In 1960 he entered the graduate school of the Steklov Mathematical Institute of the Academy of Sciences of the USSR. His first leader and mentor was the candidate and in a short time the doctor of physical and mathematical sciences Vladimir Sergeevich Vinogradov, a disciple of the academician of the Academy of Sciences of the USSR Ilya Nesterovich Vekua. Nazarbai Kadyrovich began to study the behavior of solutions of elliptic systems of differential equations in the vicinity of singular points of the coefficients. He obtained necessary and sufficient conditions for the existence of analytical solutions for degenerate first order elliptic systems in the vicinity of degeneration points. The features (degenerations) under consideration were such that it was difficult to expect the existence of any "good" solutions. Therefore, to prove the analyticity of the solution, one had to show extraordinary ingenuity and perseverance. These results laid the foundation for other studies of the possibility of the existence of continuous solutions related to questions of the theory of surfaces in the geometry. The mentioned results of N.K. Bliyev were highly appreciated by I.N. Vekua and were reported at the international conference (see I.N. Vekua. On one class of the International Conference on Analysis and Related Topics. Tokyo, April, 1969) that laid the foundation for their further close cooperation. And in his further research N.K. Bliyev followed the principle: "good" (topological) properties of solutions of elliptic equations are a consequence of the more elliptic nature of these equations, rather than the smoothness of the coefficients.

In 1965 N.K. Bliyev successfully defended his candidate thesis "On the existence of analytic solutions for degenerate elliptic systems in vicinity of a degeneration point" at the Dissertation Council of the Mathematical Institute of the Academy of Sciences of the USSR. The Institute of Mathematics of the Siberian Branch of the Academy of Sciences of the USSR gave an external review of the thesis. The official opponents were doctors of sciences K.T. Akhmedov (Baku), V. Kh. Kharasakhal (Alma-Ata). It is known from the university course that the theory of analytic functions of one complex variable is the theory of the Cauchy-Riemann system which is a special case of another elliptic system with variable coefficients, called by I.N. Vekua the generalized Cauchy-Riemann system. An attempt to construct the theory of the generalized Cauchy-Riemann system was yet undertaken by Beltrami. In the early 1930s, T. Carleman and N. Teodorescu showed that some properties of solutions of the generalized Cauchy-Riemann system are carried over to solutions of particular classes of the elliptic systems. Only in the early 50s, due to the works of I.N. Vekua, a uniform theory of general elliptic systems of two equations of the first order with two independent variables which has wide applications in different areas of analysis, geometry and mechanics was created. It is called the theory of generalized analytic functions (GAF), was constructed in Sobolev spaces W_p^l, $l \geq 0$ is an integer, $p > 2$, and it covers generalized (in some sense) solutions of the generalized Cauchy-Riemann system with coefficients belonging on the whole plane E of the complex variable z to the class $L_{p,2}$, $p > 2$, coinciding with the space L_p, $p > 2$ in the bounded domains $G \subset E$. The famous American mathematician L. Bers has a slightly different approach.

In December 1969, after another report of N.K. Bliyev at the seminar, I.N. Vekua invited him (in a convincing form) to study the problem of possibility (or impossibility) of an acceptable development of the theory of GAF onto extremely limiting cases, i.e.onto a class of coefficients of the elliptic systems summable to a power of at most two, i.e. belonging to the eigensubspaces L_p, $1 < p \leq 2$. Such a proposal by I.N.Vekua was unexpected, highly responsible and prestigious at the same time. At that time N.K. Bliyev worked at the laboratory of Professor T.I. Amanov, who was the director of the Institute of Mathematics and a disciple of Academician of the Academy of Sciences of the USSR S.M. Nikol'skii. Such connections prompted him to think: to start using the scale of Nikol'skii-Besov spaces where one can find more exact descriptions of properties of the functions. In those years these spaces were not yet adapted to study equations with variable coefficients. N.K. Bliyev was the first to succeed in proving necessary conditions for this statement about multipliers and to obtain relations between parameters of spaces into which the theory of Vekua can be extended. Here is an excerpt from the review of I.N. Vekua of these results in a letter addressed to the director of the Institute T.I. Amanov: Tbilisi, 23.06.1971 ... Today we listened to the report of N.K. Bliyev which we liked. I think that he discovered a new class of elliptic systems that admit continuous solutions. The obtained results should be unconditionally published and, in addition, it is advisable to continue further research in this direction ...". Bliyev managed to positively formulate the complete solution of the indicated problem on the scale B of Besov space, it contains the extension of the theory of Vekua known in the Sobolev spaces. We should note the success of extending the class of GAF to families of generalized solutions of general elliptic systems of differential equations on planes with coefficients from spaces with the summability exponent $p > 1$ not embedded in $L_{p,2} = L_p$. These families contain even such functions that are not summable in the usual sense but retain a number of basic topological properties of the analytic functions of a complex variable (uniqueness theorem, argument principle and others). The results of N.K. Bliyev have extraordinary consequences in various areas of mathematics. For example, he refined long-established results of a fundamental nature such as conditions for the existence of classical solutions of partial differential equations, general boundary value problems of the Riemann-Hilbert type, problems of linear connecting, quasiconformal mappings which are continuously differentiable Beltrami homeomorphisms. He established that singular integral equations are Noetherian in classes of functions which are continuous (not necessarily in Holder's sense) in terms of B-spaces and others. Thus, possibilities are enhanced and the range of applications of GAF expands.

The results on differential equations and boundary value problems in bounded domains were included in his doctoral thesis "Elliptic systems of the first order differential equations on a plane in fractional spaces and boundary value problems" which was successfully defended at the Mathematical Institute of the Academy of Sciences of the USSR in 1980. The external organization was the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of the USSR, the official opponents were Corresponding Member of the Academy of Sciences of the USSR A.V. Bitsadze, Academician of the Academy of Sciences of Ukrainian SSR I.I. Danilyuk, Doctor of Physical and Mathematical Sciences, Professor of Moscow State University, later Academician of the Academy of Sciences of Uzbekistan Sh.A. Alimov.

More detailed functional properties of GAF are presented in the monograph by Bliyev N.K. "Generalized analytic functions in fractional spaces" Alma-Ata, "Nauka", 1985. This monograph received the wide approval and the proposal of experts from far abroad to publish it in English. The results for unbounded domains are included in the monograph published in the prestigious international series "Pitman Monographs and Surveys in Pure and Applied Mathematics 86 " in English: Bliyev N. "Generalized analytic functions in fractional spaces, USA, Addison Wesley longman inc., 1997". Currently, the results by N.K. Bliyev have received the full recognition from experts and are used in foreign countries. The important results on the soliton solvability of series of nonlinear equations of mathematical physics, such as Schrodinger, Kortweg-de Vries and other equations, are due to N.K. Bliyev and his disciples.

Since 1963, the scientific activity of H.K. Bliyev is associated with the Institute (till 1965, the Sector) of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR, in which he went through all the stages of professional growth from a junior researcher to the director of the institute: since October 1963 junior researcher, since 1966 senior researcher, since 1978 the Head of Laboratory of functional analysis and theory of functions, in 1988 he was elected Director of the Institute, since 2000 Honorary Director of the Institute of Mathematics of the National Academy of Sciences of the Republic of Kazakhstan, the head of theme, since 2012 Chief Researcher (part-time) of the Institute of Mathematics and Mathematical Modeling of the Ministry of Education and Science of the Republic of Kazakhstan.

Having received the baton of the director of the Institute from Academician of the National Academy of Sciences of the Republic of Kazakhstan U.M. Sultangazin, N.K. Bliyev made his contribution to scientific and organizational activity of the Institute. Despite economic difficulties of those years of perestroika as well as the beginning of Kazakhstan's independence, he managed to organize a calm creative atmosphere, actively supporting talented young mathematicians and encouraging the scientists of the Institute to participate in various international mathematical forums. This had borne fruit. The Institute (of Theoretical and Applied Mathematics in 1992-1999) became one of the leading institutes of the Department of Physical and Mathematical Sciences of the National Academy of Sciences of the Republic of Kazakhstan, the trend of international scientific activity had increased and international contacts had been strengthened. In 1995, 11 employees received a Soros grant, three employees received an INTAS grant. There appeared scholars of different international mathematical societies, 10 employees of the institute were members of international scientific associations. That year the Institute became the winner of the INTAS grant. During those years a number of articles and monographs were published in English.

Simultaneously with the scientific activities N.K. Bliyev devoted a lot of time to teaching. Since 1964 he worked part-time at his Alma-mater, his relative Kazakh State University (now al-Farabi Kazakh National University). In September 2000, at the invitation of the rector, Nazarbai Kadyrovich completely switched to teaching and became the head of the Department of Functional Analysis and Probability Theory of al-Farabi Kazakh National University. From 2009 to the present he is Professor of the Department of Fundamental Mathematics of the Faculty of Mechanics and Mathematics of al-Farabi Kazakh National University.

N.K. Bliyev is actively involved in the scientific, organizational and social activities. He is a member of the editorial board of the journals "Izvestiya NAS RK. Seriya physiko-manematicheskaya", "Matematicheskii jurnal" (since 2019, "Kazakh Mathematical Journal"), "Vestnik KazNU im. al-Farabi". Repeatedly he was a member, vice-chairman, chairman of dissertation councils for the defense of doctoral and candidate dissertations of the Institute of Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan.

In 1999-2002 he worked part-time as academician-secretary of the Department of Physical and Mathematical Sciences of the National Academy of Sciences of the Republic of Kazakhstan. For several terms, he was a member of the Presidium of Higher Attestation Commission (SAC), chairman of the Section of Physical and Mathematical Sciences of the Terminology Committee under the Cabinet of Ministers for State Prizes of the Republic of Kazakhstan, a member of the Presidium of the National Academy of Sciences of the Republic of Kazakhstan, a deputy executive editor of "Izvestiya NAS RK. Seriya physiko-manematicheskaya", editor-in-chief of "Matematicheskii jurnal", a member of the editorial board of "Vestnik NAN RK", Encyclopedia of the Republic of Kazakhstan, Science Development Fund, dissertation councils of the Institute of Mathematics of the Academy of Sciences of Uzbekistan, Aktobe University named after K. Zhubanov. He was one of the organizers and actively participated in organizing and holding of a number of international scientific forums in Almaty, Aktobe, Semey and Karaganda.

He has published over 150 scientific papers, including one monograph, a number of articles in such highly rated mathematical publications as "Doklady AN SSSR", "Sibirskii matematicheskii jurnal", "Complex Variables and Elliptic Equations" and others. Among his direct disciples there are 18 candidates and 3 doctors of sciences who have their own schools and disciples.

The scientific achievements of N.K. Bliyev have received a worthy assessment. In 1985 he received the title of professor, in 1989 he was elected a corresponding member of the Academy of Sciences of the Kazakh SSR, in 1996 he was elected an academician of the Russian Academy of Natural Sciences, and in 2004 he became an academician of the National Academy of Sciences of the Republic of Kazakhstan. In 1998 he was awarded an honorary title "Honored Worker of Science and Technology of the Republic of Kazakhstan" , in 1999 he was awarded the international Khorezmi Prize of the first degree. The scientific and social achievements of N.K. Bliyev were marked with the certificate of honor of the Supreme Council of the Kazakh SSR, with the commemorative certificate of honor of the Central Committee of the Communistic Party of Kazakhstan, the Council of Ministers of the Kazakh SSR, the Kazakh Council of Trade Unions, the Central Committee of the youth union of Kazakhstan for the XIX party conference of the Central Committee of the Communistic Party of Soviet Union, with medals "Veteran of labour"and "10 years of the independence of the Republic of Kazakhstan".

Taking care of high-quality and professional training of the younger generation in the state language, N.K. Bliyev wrote the study guide in Kazakh "Метрикалық кеңістіктер" , Almaty: "Қазақ университеті" , 2005 and the textbook "Функционалдық анализ" , Almaty: "Universitet" , al-Farabi Kazakh National University, 2014.

N.K. Bliyev made presentations at many international scientific forums, including the International Congress of mathematicians (Poland, Warsaw, 1983), the Second European Congress of mathematicians (Hungary, Budapest, 1996), conferences of the European mathematical society (Poland, Bedlewo, 2004, 2006), with a plenary report at the International Conference "Differential equations, theory of functions and applications" (Russia, Novosibirsk, 2007), etc.

Over the years as part of various delegations he visited many countries and cities such as Delhi, Bombay, Hyderab, Madras (India), Beijin (China), Seoul (South Korea), Istanbul, Ankara, Konya (Turkey), etc. In 2015, on the occasion of his 80th birthday, an outstanding scientist-mathematician, Academician of the National Academy of Sciences, Doctor of Physical and Mathematical Sciences, Professor H.K. Bliyev was awarded with the al-Farabi silver medal and the order "Kurmet". Every year a series of books titled "Өнегелі өмір" is published at al-Farabi Kazakh Natioal University and is dedicated to those who have made a great contribution to the development of science education in Kazakhstan. In 2015, the release of this series of books was dedicated to Academician N.K. Bliyev.

Nazarbai Kadyrovich continues research in the field of generalized analytic functions in the Institute of Mathematics and Mathematical Modeling of the Ministry of Education and Science of the Republic of Kazakhstan: he was the scientific leader of the projects "Generalized analytic vectors and their applications, the solvability of soliton nonlinear equations of the dimension (1+1)" by grant funding for 2012-2014, "Boundary value problems and singular integral equations with Cauchy kernel with Carleman shift in fractional spaces " by grant funding for 2015-2017.

He is currently the scientific leader of the project "Boundedness of general $(n -$

dimensional) singular integral operators and Noetherity of corresponding singular integral equations in Besov spaces" by grant funding for 2018-2020.

Academician N.K. Bliyev is full of strength and energy to implement his new mathematical ideas.

The staff of the Institute and the Editorial Board of "Kazakh Mathematical Journal" congratulate Nazarbai Kadyrovich on his 85th jubilee and wish him good health, long life, new creative successes in his fruitful activity!

Editorial board

List of major scientific works of Bliyev Nazarbai Kadyrovich

1. Блиев Н.К. *О существовании аналитического решения одной эллиптической системы, вырождающейся в нуле*, Вестник АН КазССР, 12 (1964).

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Kazakh Mathematical Journal ISSN 2413–6468

20:3 (2020) 16–22

Unpredictable Strings

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Communicated by: Anar Assanova

Received: 02.06.2020 ⋆ Final Version: 14.09.2020 ⋆ Accepted/Published Online: 19.09.2020

Abstract. A novel notion of unpredictable strings is introduced and utilized to define unpredictable sequences on a finite number of symbols. We prove the first and second laws of large strings for random processes in discrete time. The second law of large strings is relative to the Bernoulli theorem. Theoretical and numerical backgrounds for the phenomenon are provided.

Keywords. Unpredictable strings, unpredictable sequences, Bernoulli process, Bernoulli scheme, the first law of large strings, the second law of large strings.

1 Introduction

The Bernoulli process and generally Bernoulli scheme are of most attractive and basic concepts in probability theory, statistics, random processes, dynamical processes related to chaos [1]–[7]. In the present research, we suggest to consider, beside traditional probabilistic events, a new one, which is called unpredictable string. It will provide interesting opportunities for extension of the theories as well as exploration of useful deterministic features for stochastic dynamics.

In recent papers [8], [9], new connections of deterministic chaos with random dynamics have been developed. This time, the notion of infinite sequences with unpredictable strings is introduced. This relates to the unpredictable point [10]. Numerical simulations of the Bernoulli process are performed to demonstrate that the realizations are unpredictable. They confirm that specific properties for the random dynamics are valid, namely the first and second laws of large (unpredictable) strings, which are, also, discussed theoretically. Besides, Matlab algorithm to verify sequences with inductively increasing lengths of unpredictable strings is provided.

2 Preliminaries

The notion of the realization is one of the basic in the paper. Let us provide the precise description of it not to have confusion in the comprehension. Fix natural num-

2010 Mathematics Subject Classification: 65P20, 37B10, 60G07.

bers $1, 2, \ldots, N$, and consider the set $\Omega = \{1, 2, \ldots, N\}^{\mathbb{N}}$, where $\mathbb{N}$ is the set of all natural numbers, as a sample set. Elements ω of Ω are infinite sequences $(\omega_1, \omega_2, \ldots)$ as well as finite sequences $(\omega_1, \ldots, \omega_n), n \in \mathbb{N}$, where $\omega_i, i \in \mathbb{N}$, are natural numbers from 1 to N. That is, they are members of the cylindrical sets $\Omega^n = \{1, 2, \ldots, N\}^n$. Assume, that $p(i) = 1/N$ for all $i = 1, 2, \ldots, N$. Determine a family of random variables, $X(n, \omega) : \mathbb{N} \to S$, where $S = \{s_1, \ldots, s_r\}, r \in \mathbb{N}$, is a finite set of real numbers, such that $S^{\mathbb{N}}$ is a collection of infinite and finite sequences. We shall call the sequences $\{X(k, \omega)\}, k \in \mathbb{N}$, and $\{X(k, \omega)\}_{k=1}^{n}, \omega \in \Omega$, the infinite and finite realizations of the Bernoilli scheme, respectively. Thus, the infinite realization is a sequence $\{a_k\}_k, k \in \mathbb{N}$, and the finite realization is a sequence $\{a_k\}, 1 \le k \le n, n \in \mathbb{N}$, with $a_k \in S$. They are orbits of the dynamics, which we know as the Bernoilli scheme. In the case $N = 2$, the dynamics is said to be the Bernoilli process [4], [7].

3 The unpredictable strings

In this section, we introduce the main concept of this paper, unpredictable strings and utilize it to determine unpredictable sequences.

Let a_i, $i = 0, 1, 2, \ldots$, be an infinite sequence of symbols.

Definition 1. *A finite array $(a_s, a_{s+1}, \ldots, a_{s+k})$, where s and k are positive integers, is said to be an unpredictable string of length k if $a_i = a_{s+i}$, for $i = 0, 1, 2, \ldots, k-1$, and $a_k \neq a_{s+k}$.*

The diagram in Figure 1 illustrates the definition.

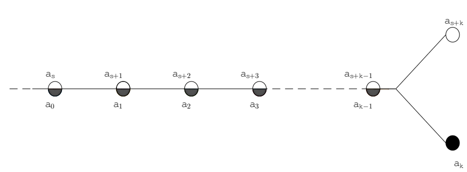

Figure 1 – The illustration of the unpredictable string of length k.

Definition 2. *The sequence a_i is unpredictable if it admits unpredictable strings with arbitrary large lengths.*

Definition 3. [10] *The sequence a_i is unpredictable if there exist sequences ζ_n, η_n of positive integers both of which diverge to infinity such that $a_{\zeta_n + l} = a_l$, $l = 0, 1, 2, \ldots, \eta_n - 1$, and $a_{\zeta_n + \eta_n} \neq a_{\eta_n}$, for each $n \in \mathbb{N}$.*

Theorem 1. *The Definitions 2 and 3 are equivalent.*

Proof. Let sequence a_i be unpredictable. Then the finite arrays $(a_{\zeta_n}, a_{\zeta_n+1}, ..., a_{\zeta_n+\eta_n})$ are unpredictable strings of length η_n, for each natural n. Thus, the sequence admits unpredictable strings with arbitrary large lengths.

Conversely, let a_i be a sequence that admits unpredictable strings of arbitrary large lengths, i.e., there is a sequence i_n, $n = 1, 2, 3, ...$, such that the finite arrays $(a_{i_n}, a_{i_n+1}, ..., a_{i_n+k})$ are unpredictable strings. By setting $\zeta_n = i_n$ and $\eta_n = i_n + k$, we deduce that the sequence a_i is unpredictable in light of Definition 3.

Fix a positive integer k and denote by S_k the sets of all indexes s such that the strings $(a_s, a_{s+1}, ..., a_{s+k})$ are unpredictable within the sequence $a_i, i = 1, 2, ...,$ which is not necessarily unpredictable.

Theorem 2. *The sets S_l and S_q do not intersect if $l < q$.*

Proof. Assume, on contrary, that sets S_l and S_q admit a common element s. Then, we have that $a_l \neq a_{s+l}$ if $s \in S_l$ and $a_l = a_{s+l}$ if $s \in S_q$. This contradiction completes the prove.

Theorem 3. *Assume that a_i is an unpredictable sequence. Then each a_j with positive j is the first element of an unpredictable string, if $a_j = a_0$.*

Proof. Assume the opposite. Then one can show that the sequence α is periodic one. That is not unpredictable sequence.

4 Numerical analysis of the Bernoulli process

We will scrutinize a realization of the Bernoulli process as a sequence consisting of the digits 1 and 0 with positive probabilities.

First, we provide an algorithm for indication of unpredictable strings in realizations of a Bernoulli scheme on finite number of complex vectors $v_1, v_2, ..., v_r$.

Let us set $a_0 = random(\{v_1, v_2, ..., v_r\})$ and $a_1 = random(\{v_1, v_2, ..., v_r\})$. Then for increasing $k = 1, 2, 3, ...,$ we define $a_{m(k)+j} = a_j$, for $j < k$, and $a_{m(k)+j} = random(\{v_1, v_2, ..., v_r\} - a_j)$, for $j = k$, where $m(k + 1) = m(k) + k$ with $m(1) = 2$.

The immediately following Algorithm 1 is for the Bernoulli process with $v_1 = 0$ and $v_2 = 1$. The sequence $(0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, ...)$ is a result of the algorithm application.

Let us introduce several characteristics that are of usage for analysis of finite realizations of the Bernoulli scheme. For fixed natural number m, consider a finite realization $a_i, i = 0, 1, ..., m$. Denote by $K(m)$ the largest length of unpredictable strings in the array. For every k between 1 and $K(m)$, denote by q_k the number of $k-$lengthy unpredictable strings within the array, by ξ_k the largest index such that $(a_{\xi_k}, a_{\xi_{k+1}}, ..., a_{\xi_k+k})$ is an unpredictable string within the array, and by $N(m)$ the number of all unpredictable strings, which have a non-empty intersection with the array.

Algorithm 1 Unpredictable sequences

$m = 2$
for $k = 1, 2, 3, ...$ **do**
  $a_0 = 0$
  $a_1 = 1$
  for $j = 0 : k$ **do**
    if $j < k$, **then**
      $a_{m+j} = a_j$
    else if $j = k$, **then**
      $a_{m+j} \neq a_j$
      $m = m + k$
    end if
  end for
end for

Now, we provide statistical results on the realization, which are obtained by Matlab simulations for the Bernoulli process with probability $p = 0.6$ and $m = 9 \times 10^5$. We have evaluated values of $K(n)$, $\xi_{K(n)}$ and $N(n)/n$, for each n from 1 to m. Ten samples of the simulations are provided in Table 1. According to the full data obtained in simulations, the realization can be considered as part of an unpredictable sequence, since there are unpredictable strings with increasing lengths. Moreover, $N(n)/n \approx p$, if n is large.

Table 1 – The values $K(n)$, $\xi_{K(n)}$ and $N(n)/n$ for the finite realization

n	$K(n)$	$\xi_{K(n)}$	$N(n)/n$
50	10	20	0.72
200	10	20	0.58
500	10	228	0.586
2000	14	1008	0.596
10000	14	3469	0.6031
20000	18	19206	0.5995
100000	21	74683	0.6014
500000	21	401088	0.6003
800000	21	663684	0.6001
900000	28	874766	0.5686

5 Laws of large strings for the Bernoulli scheme

In this section, we consider a discrete-time random process $X(n)$ with the finite state space of r different symbols $s_1, s_2, ..., s_r$.

The function admits values s_i with positive probabilities p_i, $i = 1, 2, ..., r$, which sum is equal to the unit. A realization α of the process is the sequence a_i, $i = 1, 2, ...$, and a finite realization α_m is the array a_i, $i = 1, 2, ..., m$. We claim that stochastic processes with discrete time and finite state spaces satisfy the following theorem.

Theorem 4. *(the first law of large strings). The discrete time random process $X(n)$ with the finite state space admits uncountable set of realizations, which are unpredictable sequences in the sense of Definition 2.*

Proof. Let us consider the space Σ_r of infinite sequences of finite set of symbols $s_1, s_2, ..., s_r$, with the metric

$$d(\xi, \zeta) = \Sigma_{k=0}^{\infty} \frac{|\xi_k - \zeta_k|}{2^k}, \tag{1}$$

where $\xi = (\xi_0\xi_1\xi_2...), \zeta = (\zeta_0\zeta_1\zeta_2...)$. The Bernoulli shift σ on Σ_r is defined as $\sigma(\xi_0\xi_1\xi_2...) = (\xi_1\xi_2\xi_3...)$. The map is continuous and Σ_r is a compact metric space [11].

It is clear that the set of all realizations of the random dynamics $X(n)$ coincides with the set of all sequences of the symbolic dynamics on Σ_r. According to the result in [10], the symbolic dynamics admits an unpredictable point, i^*, a sequence from the set Σ_r. There is the uncountable set of unpredictable points, which are unpredictable sequences in the sense of Definition 2.

It is important that the set of the realizations is the closure for the unpredictable orbit. The density is considered in the shift dynamics sense. The property of the metric implies that each arc of any sequence in the space coincides with some arc of the unpredictable sequence.

Let us fix an unpredictable realization of the scheme. Due to Definition 3 and Theorem 1, the following assertion is valid.

Theorem 5. *Each finite realization of the Bernoulli scheme coincides with an arc of the unpredictable realization for sure. That is, the unpredictable realization happens in each experiment of the chain, and is a certain event.*

Denote by n(m) the number of elements, which are equal to a_0 in a finite string. The limit $E[a_0] = \lim_{m\to\infty} n(m)/m$ is said to be the expected value such that $E[a_0] = p_i$, if $a_0 = s_i, i = 1, ..., r$ [4].

Theorem 3 implies the equality $N(m) = n(m)$, where $N(m)$ is the number of unpredictable strings, which intersect the array. Hence, the following proposition is correct, which can be useful for applications.

Theorem 6. *If a realization α is an unpredictable sequence, then the relation*

$$\lim_{m\to\infty} \frac{N(m)}{m} = E[a_0] \tag{2}$$

is valid.

Theorem 7. *(the second law of large strings). If the discrete time random process $X(n)$ admits a finite state space, then the relation*

$$\lim_{m\to\infty} P\left(\left|\frac{N(m)}{m} - E[a_0]\right| < \varepsilon\right) = 1 \tag{3}$$

holds for any $\varepsilon > 0$.

Proof. As it has been concluded above, Theorem 5, each finite realization of the scheme is an arc of an infinite unpredictable realization, and the relation (2) for the last one is valid. These all prove the theorem.

Example 1. *To have more impression of the unpredictable strings, let us consider the graph of the piece-wise constant function, $H(t)$, which values on intervals $[i/10,(i+1)/10), i = 0,1,\dots,199$, are assigned randomly 1 or -1 with equal probability $1/2$. The two unpredictable strings as a result of the Bernoulli process are present, in the red, in the Figure 2, (a). The second one, with length of 0.7 units, is placed between coordinates 14 and 16, shown in Figure 2, (c), while its corresponding initial arc, in Figure 2, (b). The pieces of the graph are connected with vertical lines, to improve the visibility.*

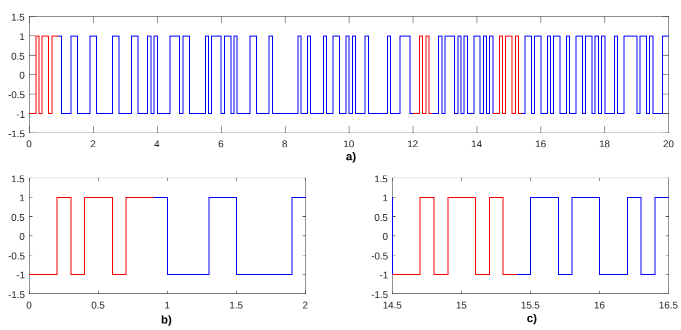

Figure 2 – The graph of the function $H(t)$, which illustrates appearance of unpredictable strings. To make the visibility better, the pieces of the graph are connected with vertical lines.

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Ахмет М., Тола А. БОЛЖАНБАЙТЫН АҚЫРЛЫ ТІЗБЕКТЕР

Таңбалардың ақырлы санында болжанбайтын тізбектерді анықтау үшін болжанбайтын ақырлы тізбектердің жаңа тұжырымдамасы енгізілді. Дискретті уақыттағы кездейсоқ процестерге арналған үлкен тізбектердің бірінші және екінші заңдарын дәлелдейміз. Үлкен тізбектердің екінші заңы Бернулли теоремасымен байланысты. Осы құбылыстың теориялық және сандық негіздері келтірілді.

Кілттік сөздер. Болжанбайтын ақырлы тізбектер, болжанбайтын тізбектер, Бернулли процесі, Бернулли схемасы, үлкен тізбектердің бірінші заңы, үлкен тізбектердің екінші заңы.

Ахмет М., Тола А. НЕПРЕДСКАЗУЕМЫЕ КОНЕЧНЫЕ ПОСЛЕДОВАТЕЛЬНОСТИ

Введено новое понятие непредсказуемых конечных последовательностей, которое используется для определения непредсказуемых последовательностей на конечном числе символов. Мы доказываем первый и второй законы больших последовательностей для случайных процессов в дискретном времени. Второй закон больших последовательностей связан с теоремой Бернулли. Приведены теоретические и численные основы этого явления.

Ключевые слова. Непредсказуемые конечные последовательности, непредсказуемые последовательности, процесс Бернулли, схема Бернулли, первый закон больших последовательностей, второй закон больших последовательностей.

Kazakh Mathematical Journal ISSN 2413–6468

20:3 (2020) 23–47

Model free boundary problems with a small parameter for the system of the parabolic equations

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Communicated by: Galina Bizhanova

Received: 01.08.2020 ⋆ Final Version: 15.09.2020 ⋆ Accepted/Published Online: 19.09.2020

Abstract. Two model problems with a small parameter in the boundary condition are studied. They were obtained by solving nonlinear problems with two free boundaries for the system of the parabolic equations. In the Hölder space there are established the uniform with respect to small parameter estimates of the solution of these problems.

Keywords. System of the parabolic equations, small parameter in the boundary condition, solution in the explicit form, uniform estimates, Hölder space.

1 Statement of the problems. Main results

In the present paper two model conjugation problems with a small parameter in the boundary condition are studied. They arise by solving the nonlinear two-phase problem with two free boundaries for the system of parabolic equations that takes into account the fluid velocity. The nonlinear two-phase problem with two free boundaries describes real physical processes, mathematical models of which contain small parameters $\varepsilon > 0$ in the condition on one and $\varkappa > 0$ in the condition on the other free boundary. Such the problem arises, for example, when extracting and transporting oil.

The problems with small parameters were investigated in [1]–[6]. J.F. Rodrigues, V.A. Solonnikov, F. Yi. [1] have investigated one-phase linear and nonlinear free boundary problems for the second order parabolic equations with a small parameter. They have established the uniform estimates with respect to small parameter of the solutions in the Hölder space. From these estimates it follows the existence of the solutions of the considered problems for a small parameter, equal to zero.

2010 Mathematics Subject Classification: 35K20, 35B25, 35C05.

Funding: This work was supported by grant AP05133898 of the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan.

The linear Stefan problem for the heat equation with a small parameter $\varkappa$ in the boundary condition with the derivative $\varkappa\partial_t\psi$, where ψ is a function describing the free boundary, was studied by G.I. Bizhanova [2]. Estimates of the solutions with constants independent on small parameter were established in the Hölder space.

In [3] the two-phase problem for the heat equation with a small parameter at the time derivative $\varepsilon\partial_t u_1$ on the boundary $x_n = 0$ was considered, in the Hölder space an estimates of the solution of the problem with the constants independent on ε were obtained. The linear one-phase problem for the heat equation with a small parameter at the time derivative $\varepsilon\partial_t u$ on the boundary $x_n = 0$ was studied in [4]. Estimates of its solution and the estimate $\varepsilon\partial_t u|_{x_n=0}$ with respect to ε are obtained.

In [5] there were constructed the solution and obtained the estimates of the Green function of the two-phase boundary value problem for the parabolic equations with two small parameters at the principal derivatives in the conjugation condition.

In [6] the linear multidimensional two-phase free boundary problem for the parabolic equations with two small parameters $\varepsilon > 0$ and $\varkappa > 0$ at the principal derivatives in the boundary condition was studied, estimates of the solution in the Hölder space are obtained.

In this article there are constructed the solutions in the explicit form of the model problems that has not been studied before. The unique solvability is proved, the uniform with respect to small parameters ε and $\varkappa$ estimates of the solutions of these problems in the Hölder space are established.

Let $D_1 := \{x : \ x' \in R^{n-1},\, x_n > 0\}$, $D_2 := \{x : \ x' \in R^{n-1},\, x_n < 0\}$, $D_{jT} = D_j \times (0,T)$, $j = 1,2$, R be hyperplane $x_n = 0$ in R^n, $R_T = R \times (0,T)$, $\varepsilon > 0$, $\varkappa > 0$ be small parameters.

Model conjugation problem I. It is required to find the unknown functions $v(x,t)$, $u_1(x,t)$, $r_1(x',t)$, satisfying the following equations and conditions

$$\partial_t v - a^2\triangle v - \alpha_1\Big(\partial_t r_1 - a^2\triangle' r_1\Big) = f(x,t) \quad \text{in} \quad D_{1T}, \tag{1}$$

$$\partial_t u_1 - a_1^2\triangle u_1 - \beta_1\Big(\partial_t r_1 - a_1^2\triangle' r_1\Big) = f_1(x,t) \quad \text{in} \quad D_{1T}, \tag{2}$$

$$v\big|_{t=0} = v_0(x), \quad u_1\big|_{t=0} = u_{01}(x) \quad \text{in} \quad D_1, \quad r_1\big|_{t=0} = 0 \quad \text{on} \quad R, \tag{3}$$

$$v\big|_{x_n=0} = \varphi_1(x',t), \quad u_1\big|_{x_n=0} = \varphi_2(x',t) \quad \text{on} \quad R_T, \tag{4}$$

$$\varepsilon\partial_t v - d_1\nabla^T v\big|_{x_n=0} - \varepsilon\alpha_3\partial_t r_1 + d_2'\nabla'^T r_1 = \varphi_3(x',t) \quad \text{on} \quad R_T, \tag{5}$$

where all coefficients are constant, a, a_1 are positive constants, $\triangle' = \partial_{x_1}^2 + \cdots + \partial_{x_{n-1}}^2$, $\nabla' = (\partial_{x_1}, \ldots, \partial_{x_{n-1}})$, $d_1 = (d_1', d_{1n})$, $d_j' = (d_{j\,1}, \ldots, d_{j\,(n-1)})$, $j = 1,2$, are vectors.

We shall investigate the problem in the Hölder space $C_{x\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega_T)$, $\alpha \in (0,1)$, of the functions $u(x,t)$ with the norm $|u|_{\Omega_T}^{(2+\alpha)}$ [7]; $\overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(\Omega_T)$ is the subspace of functions

$u(x,t)\in C_{x\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(\Omega_T)$ such that

$$\partial_t^k u\Big|_{t=0}=0,\quad k=0,1.$$

Theorem 1. *Let* $d_{1\,n}>0,\ \alpha_1>0,\ \alpha_3>0,\ 0<\varepsilon<\varepsilon_0,\ \alpha\in(0,1)$.
For every functions $f(x,t),\ f_1(x,t)\in \overset{\circ}{C}{}_{x\ t}^{\alpha,\frac{\alpha}{2}}(D_{1T}),\ \varphi_j\in \overset{\circ}{C}{}_{x'\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(R_T),\ j=1,2,$
$\varphi_3\in \overset{\circ}{C}{}_{x'\ \ t}^{1+\alpha,\frac{1+\alpha}{2}}(R_T)$*, the problem (1) - (5) has a unique solution* $v,\ u_1\in \overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(D_{1T})$,
$\varepsilon\partial_t v|_{x_n=0}\in \overset{\circ}{C}{}_{x'\ \ t}^{1+\alpha,\frac{1+\alpha}{2}}(R_T),\ r_1\in \overset{\circ}{C}{}_{x'\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(R_T),\ \varepsilon\partial_t r_1\in \overset{\circ}{C}{}_{x'\ \ t}^{1+\alpha,\frac{1+\alpha}{2}}(R_T)$*, and it satisfies the estimate*

$$|v|_{D_{1T}}^{(2+\alpha)}+|u_1|_{D_{1T}}^{(2+\alpha)}+|\varepsilon\partial_t v|_{R_T}^{(1+\alpha)}+|r_1|_{R_T}^{(2+\alpha)}+|\varepsilon\partial_t r_1|_{R_T}^{(1+\alpha)}$$
$$\le C_1\Big(|f|_{D_{1T}}^{(\alpha)}+|f_1|_{D_{1T}}^{(\alpha)}+\sum_{j=1}^{2}|\varphi_j|_{R_T}^{(2+\alpha)}+|\varphi_3|_{R_T}^{(1+\alpha)}\Big),\tag{6}$$

where the constant C_1 *is independent on* ε.

Model conjugation problem II. It is required to find the unknown functions $v(x,t)$, $u_j(x,t),\ j=1,2,\ r_2(x',t)$, satisfying the following equations and conditions

$$\begin{gathered}
\partial_t v-a^2\triangle v-\alpha_2\Big(\partial_t r_2-a^2\triangle' r_2\Big)=f(x,t)\quad\text{in}\quad D_{1T},\\
\partial_t u_1-a_1^2\triangle u_1-\beta_2\Big(\partial_t r_2-a_1^2\triangle' r_2\Big)=f_1(x,t)\quad\text{in}\quad D_{1T},\\
\partial_t u_2-a_2^2\triangle u_2-\beta_3\Big(\partial_t r_2-a_1^2\triangle' r_2\Big)=f_2(x,t)\quad\text{in}\quad D_{2T},\\
v\Big|_{t=0}=v_0(x)\quad\text{in}\quad D_1,\quad u_j\Big|_{t=0}=u_{0j}(x)\quad\text{in}\quad D_j,\quad r_2\Big|_{t=0}=0\quad\text{on}\quad R,\\
u_1\Big|_{x_n=0}=\psi_0(x',t),\quad u_2\Big|_{x_n=0}=\psi_1(x',t),\quad v\Big|_{x_n=0}=\psi_2(x',t)\quad\text{on}\quad R_T,\\
\lambda_1\partial_{x_n}u_1-\lambda_2\partial_{x_n}u_2+\varkappa\partial_t r_2+d_3'\nabla'^T r_2=\psi_3(x',t)\quad\text{on}\quad R_T,
\end{gathered}\tag{7}$$

where all coefficients are constant, $a,\ a_j,\ \lambda_j,\ j=1,2$, are positive constants and $d_3'=(d_{3\,1},\dots,d_{3\,(n-1)})$ is a vector.

Theorem 2. *Let* $\beta_j>0,\ j=2,3,\ 0<\varkappa<\varkappa_0,\ \alpha\in(0,1)$.

For every functions $f(x,t)\in \overset{\circ}{C}{}_{x\ t}^{\alpha,\frac{\alpha}{2}}(D_{1T}),\ f_j(x,t)\in \overset{\circ}{C}{}_{x\ t}^{\alpha,\frac{\alpha}{2}}(D_{jT}),\ \psi_0\in \overset{\circ}{C}{}_{x'\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(R_T)$, $\psi_j\in \overset{\circ}{C}{}_{x'\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(R_T),\ \psi_3\in \overset{\circ}{C}{}_{x'\ \ t}^{1+\alpha,\frac{1+\alpha}{2}}(R_T),\ j=1,2,$ *the problem (7) has a unique solution* $v\in \overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(D_{1T}),\ u_j\in \overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(D_{jT}),\ r_2\in \overset{\circ}{C}{}_{x'\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(R_T),\ \varkappa\partial_t r_2\in \overset{\circ}{C}{}_{x'\ \ t}^{1+\alpha,\frac{1+\alpha}{2}}(R_T)$,

$j = 1, 2,$ *and it satisfies the estimate*

$$|v|^{(2+\alpha)}_{D_{1T}} + \sum_{j=1}^{2} |u_j|^{(2+\alpha)}_{D_{jT}} + |r_2|^{(2+\alpha)}_{R_T} + |\varkappa \partial_t r_2|^{(1+\alpha)}_{R_T}$$

$$\le C_2 \Big(|f|^{(\alpha)}_{D_{1T}} + \sum_{j=1}^{2} |f_j|^{(\alpha)}_{D_{jT}} + \sum_{j=0}^{2} |\psi_j|^{(2+\alpha)}_{R_T} + |\psi_3|^{(1+\alpha)}_{R_T} \Big), \tag{8}$$

where the constant C_2 *is independent on* $\varkappa$*.*

2 Auxiliary problems. Construction of a solution to the model problem I

We reduce the problem (1)–(5) to the problem with homogeneous equations, homogeneous initial and boundary conditions (4). For this we construct the auxiliary functions $V(x,t)$, $U_1(x,t)$, as solutions of the first boundary value problems for the parabolic equations

$$\begin{gathered} \partial_t V - a^2 \triangle V = f(x,t) \quad \text{in} \quad D_{1T}, \\ V\big|_{t=0} = v_0(x) \quad \text{in} \quad D_1, \quad V\big|_{x_n=0} = \varphi_1(x',t) \quad \text{on} \quad R_T; \end{gathered} \tag{9}$$

$$\begin{gathered} \partial_t U_1 - a_1^2 \triangle U_1 = f_1(x,t) \quad \text{in} \quad D_{1T}, \\ U_1\big|_{t=0} = u_{01}(x) \quad \text{in} \quad D_1, \quad U_1\big|_{x_n=0} = \varphi_2(x',t) \quad \text{on} \quad R_T. \end{gathered} \tag{10}$$

The problems (9)–(10) have unique solutions $V(x,t),\ U_1(x,t) \in \overset{\circ}{C}{}^{2+\alpha, 1+\frac{\alpha}{2}}_{x\quad\ t}(\bar{D}_{1T})$ [7], and the following estimates for them are fulfilled

$$|V|^{(2+\alpha)}_{D_{1T}} \le C_3 \Big(|f|^{(\alpha)}_{D_{1T}} + |\varphi_1|^{(2+\alpha)}_{R_T} \Big), \tag{11}$$

$$|U_1|^{(2+\alpha)}_{D_{1T}} \le C_4 \Big(|f_1|^{(\alpha)}_{D_{1T}} + |\varphi_2|^{(2+\alpha)}_{R_T} \Big). \tag{12}$$

In the equations and conditions of the problem (1)–(5) we make the substitution

$$v(x,t) = V(x,t) + \alpha_1 r_1 + w(x,t), \quad u_1(x,t) = U_1(x,t) + \beta_1 r_1 + z_1(x,t), \tag{13}$$

where $w(x,t)$, $z_1(x,t)$ are new unknown functions.

Taking into account that the constructed functions $V(x,t),\ U_1(x,t)$ satisfy the equations and conditions of the problems (9), (10), we obtain the problem for functions $w(x,t),\ z_1(x,t)$ and $r_1(x',t)$:

$$\partial_t w - a^2 \triangle w = 0 \quad \text{in} \quad D_{1T}, \tag{14}$$

$$\partial_t z_1 - a_1^2 \triangle z_1 = 0 \quad \text{in} \quad D_{1T}, \tag{15}$$

$$w\big|_{t=0} = 0, \quad z_1\big|_{t=0} = 0 \quad \text{in} \quad D_1, \quad r_1\big|_{t=0} = 0 \quad \text{on} \quad R, \tag{16}$$

$$w\big|_{x_n=0} + \alpha_1 r_1 = 0 \quad \text{on} \quad R_T, \quad z_1\big|_{x_n=0} + \beta_1 r_1 = 0 \quad \text{on} \quad R_T, \tag{17}$$

$$\varepsilon\partial_t w - d_1\nabla^T w\big|_{x_n=0} + \varepsilon(\alpha_1 - \alpha_3)\partial_t r_1 + \big(d_2' - \alpha_1 d_1'\big)\nabla'^T r_1 = \Phi(x',t) \quad \text{on} \quad R_T, \tag{18}$$

where

$$\Phi(x',t) = \varphi_3(x',t) - \Big(\varepsilon\partial_t V - d_1\nabla^T V\Big)\Big|_{x_n=0} \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\quad t}(R_T)$$

and satisfies the estimate

$$|\Phi|^{(1+\alpha)}_{R_T} \le C_5\Big(|\varphi_3|^{(1+\alpha)}_{R_T} + (1+\varepsilon)|V|^{(2+\alpha)}_{D_{1T}}\Big). \tag{19}$$

Theorem 3. *Let* $d_{1\,n} > 0, \ \alpha_1 > 0, \ \alpha_3 > 0, \ 0 < \varepsilon < \varepsilon_0.$

For every function $\Phi(x',t) \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\quad t}(R_T)$, $\alpha \in (0,1)$, *the problem (14) - (18) has a unique solution* $w \in \overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{x\quad t}(D_{1T})$, $z_1 \in \overset{\circ}{C}{}^{2+l,1+\frac{l}{2}}_{x\quad t}(D_{1T})$, $\varepsilon\partial_t w|_{x_n=0} \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\quad t}(R_T)$, $r_1 \in \overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{x'\quad t}(R_T)$, $\varepsilon\partial_t r_1 \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\quad t}(R_T)$, *and it satisfies the estimate*

$$|w|^{(2+\alpha)}_{D_{1T}} + |z_1|^{(2+\alpha)}_{D_{1T}} + |\varepsilon\partial_t w|^{(1+\alpha)}_{R_T} + |r_1|^{(2+\alpha)}_{R_T} + |\varepsilon\partial_t r_1|^{(1+\alpha)}_{R_T} \le C_6|\Phi(x',t)|^{(1+\alpha)}_{R_T}, \tag{20}$$

where the constant C_6 *does not depend on* ε.

We apply Laplace transform with respect to the variable t and Fourier transform with respect to x' [8] to the problem (14)–(18):

$$FL[u(x,t)] = \tilde{u}(s', x_n, p) = \int_0^\infty e^{-pt}dt \int_{R^{n-1}} e^{-ix's'}dx',$$

where $s' = (s_1, \ldots, s_{n-1})$.

The solution of the problem in the domain of Laplace and Fourier images has the form

$$\tilde{w} = \frac{\alpha_1}{\varepsilon\alpha_3\zeta}\tilde{\Phi}e^{-k_1 x_n}, \quad \tilde{z}_1 = \frac{\beta_1}{\varepsilon\alpha_3\zeta}\tilde{\Phi}e^{-k_2 x_n}, \quad \tilde{r}_1 = -\frac{1}{\varepsilon\alpha_3\zeta}\tilde{\Phi},$$

where

$$\zeta = p - i\frac{b'}{\varepsilon}s' + \frac{b_n}{\varepsilon}k_1, \ b = (b', \ b_n) = \left(\frac{d_2'}{\alpha_3}, \ \frac{\alpha_1 d_{1n}}{\alpha_3}\right), \ k_1 = \frac{\sqrt{p + a^2 s'^2}}{a}, \ k_2 = \frac{\sqrt{p + a_1^2 s'^2}}{a_1}.$$

Here $Re\zeta \ge C_0 > 0$, so we can represent $\dfrac{1}{\zeta}$ as follows

$$\frac{1}{\zeta} = \int_0^\infty e^{-\zeta u}du, \tag{21}$$

and obtain

$$\tilde{w} = \frac{\alpha_1}{\varepsilon\alpha_3}\tilde{\Phi}\int\limits_0^\infty e^{-(p-i\frac{b'}{\varepsilon}s'+\frac{b_n}{\varepsilon}k_1)u-k_1x_n}du = \frac{\alpha_1}{\varepsilon\alpha_3}\tilde{\Phi}\cdot\tilde{G}_\varepsilon,$$

$$\tilde{z}_1 = \frac{\beta_1}{\varepsilon\alpha_3}\tilde{\Phi}\int\limits_0^\infty e^{-(p-i\frac{b'}{\varepsilon}s'+\frac{b_n}{\varepsilon}k_1)u-k_2x_n}du = \frac{\beta_1}{\varepsilon\alpha_3}\tilde{\Phi}\cdot\tilde{G}_{1,\varepsilon}.$$

Applying the inverse Laplace and Fourier transforms and convolution formula, we find the functions w, z_1 in the explicit forms

$$w = \frac{\alpha_1}{\varepsilon\alpha_3}\int\limits_0^t d\tau\int\limits_{R^{n-1}}\Phi(y',\tau)G_\varepsilon\big(x'-y',x_n,t-\tau\big)dy',$$

$$z_1 = \frac{\beta_1}{\varepsilon\alpha_3}\int\limits_0^t d\tau\int\limits_{R^{n-1}}\Phi(y',\tau)G_{1,\varepsilon}\big(x'-y',x_n,t-\tau\big)dy',$$

where

$$G_\varepsilon(x,t) = -2a^2\int\limits_0^t \partial_{x_n}\Gamma\Big(x'+\frac{b'u}{\varepsilon},\ x_n+\frac{b_nu}{\varepsilon},\ t-u\Big)du$$

$$= \int\limits_0^t \frac{x_n+\frac{b_nu}{\varepsilon}}{\big(2a\sqrt{\pi(t-u)}\big)^n(t-u)}e^{-\frac{(x'+\frac{b'u}{\varepsilon})^2+(x_n+\frac{b_nu}{\varepsilon})^2}{4a^2(t-u)}}\,du,$$

$$G_{1,\varepsilon}(x,t) = \int\limits_0^t \partial_{x_n}g_{1,\varepsilon}(x'+\frac{b'u}{\varepsilon},x_n,\frac{b_nu}{\varepsilon},t-u)du,$$

$$g_{1,\varepsilon}(x'+\frac{b'u}{\varepsilon},x_n,\frac{b_nu}{\varepsilon},t)$$

$$= 4a^2a_1^2\int\limits_0^t d\tau_1\int\limits_{R^{n-1}}\Gamma_1(x'-\eta'+\frac{b'u}{\varepsilon},x_n,t-\tau_1)\partial_{\eta_n}\Gamma(\eta',\eta_n+\frac{b_nu}{\varepsilon},\tau_1)\Big|_{\eta_n=0}d\eta'$$

$$= -2a_1^2\int\limits_0^t d\tau_1\int\limits_{R^{n-1}}\frac{1}{\big(2a_1\sqrt{\pi(t-\tau_1)}\big)^n}\,e^{-\frac{(x'-\eta'+\frac{b'u}{\varepsilon})^2+x_n^2}{4a_1^2(t-\tau_1)}}\,\frac{\frac{b_nu}{\varepsilon}}{\big(2a\sqrt{\pi\tau_1}\big)^n\tau_1}\,e^{-\frac{\eta'^2+(\frac{b_nu}{\varepsilon})^2}{4a^2\tau_1}}d\eta',$$

$$\Gamma_j(x,t) = \frac{1}{(2a_j\sqrt{\pi t})^n}\,e^{-\frac{x^2}{4a_j^2t}},\ j=0,1,\ \Gamma_0(x,t)\equiv\Gamma(x,t),\ a_0=a.$$

Taking into account that $r_1 = -\frac{1}{\alpha_1} w\big|_{x_n=0}$, we find the function r_1:

$$r_1 = -\frac{1}{\varepsilon\alpha_3}\int\limits_0^t d\tau \int\limits_{R^{n-1}} \Phi(y',\tau)G_\varepsilon\big(x'-y',0,t-\tau\big)dy',$$

where

$$G_\varepsilon(x',0,t) = -2a^2\int\limits_0^t \partial_{x_n}\Gamma\big(x'+\frac{b'}{\varepsilon}u,\ x_n+\frac{b_n}{\varepsilon}u,\ t-u\big)\Big|_{x_n=0}du$$

$$= \int\limits_0^t \frac{\frac{b_n}{\varepsilon}u}{\big(2a\sqrt{\pi(t-u)}\big)^n(t-u)} e^{-\frac{\left(x'+\frac{b'}{\varepsilon}u\right)^2+\left(\frac{b_n}{\varepsilon}u\right)^2}{4a^2(t-u)}}\ du.$$

The fundamental solutions $\Gamma_j(x,t)$ of the heat equations (14), (15) satisfy the estimate [7]

$$\Big|\partial_t^k\partial_x^m\Gamma_j(x,t)\Big| \le C_7\frac{1}{t^{\frac{n+2k+|m|}{2}}}\ e^{-\frac{x^2}{8a_j^2t}},\ j=0,1. \tag{22}$$

For the function g_1 and Green's function $G_{1,\varepsilon}$ the following estimates hold [3]

$$\Big|\partial_t^k\partial_x^m\partial_{x_n}g_{1,\varepsilon}(x'+\frac{b'u}{\varepsilon},x_n,\frac{b_nu}{\varepsilon},t)\Big| \le C_8\frac{1}{t^{\frac{n+2k+|m|+1}{2}}}\ e^{-\frac{q_1^2x^2+q_2^2u^2}{t}}, \tag{23}$$

$$\Big|\partial_t^k\partial_x^m G_{1,\varepsilon}(x,t)\Big| \le C_9\varepsilon\ \frac{1}{t^{\frac{n+2k+|m|}{2}}}\ e^{-\frac{q_1^2x^2}{t}} + C_{10}\frac{1}{(q_1^2x^2+q_2^2t^2)^{\frac{n+2k+|m|-1}{2}}}\ e^{-\frac{q_1^2x^2+q_2^2t^2}{4t}}, \tag{24}$$

where

$$q_1^2 = \frac{b_n^2}{16\tilde{a}^2(b'^2+b_n^2)},\quad q_2^2 = \frac{b_n^2}{16\tilde{a}^2\varepsilon^2},$$

the constants C_8 - C_{10} do not depend on ε, $\tilde{a} = max(a,a_1)$.

3 Estimates for the functions $w(x,t)|_{x_n=0}$ and $z_1(x,t)|_{x_n=0}$

Consider the functions $w(x,t)$, $z_1(x,t)$ on the plane $x_n=0$

$$w(x',0,t) = \frac{\alpha_1}{\varepsilon\alpha_3}\int\limits_0^t d\tau\int\limits_{R^{n-1}} \Phi(y',\tau)G_\varepsilon(x'-y',x_n,t-\tau)dy'|_{x_n=0}$$

$$= -\frac{2a^2\alpha_1}{\varepsilon\alpha_3}\int\limits_0^t d\tau\int\limits_{R^{n-1}}\Phi(y',\tau)dy'\int\limits_0^{t-\tau}\partial_{x_n}\Gamma\left(x'-y'+\frac{b'}{\varepsilon}u,x_n+\frac{b_n}{\varepsilon}u,t-\tau-u\right)du|_{x_n=0}$$

$$= -\frac{2a^2\alpha_1}{\varepsilon\alpha_3}\int\limits_0^t d\tau \int\limits_{R^{n-1}} dy' \int\limits_0^\tau \Phi(y',\tau-u)\partial_{x_n}\Gamma\left(x'-y'+\frac{b'u}{\varepsilon},\frac{b_n u}{\varepsilon},t-\tau\right)du := \omega_1(x',t);$$

$$z_1(x',0,t) = \frac{\beta_1}{\varepsilon\alpha_3}\int\limits_0^t d\tau \int\limits_{R^{n-1}} \Phi(y',\tau)G_{1,\varepsilon}\big(x'-y',x_n,t-\tau\big)dy'|_{x_n=0}$$

$$= \frac{\beta_1}{\varepsilon\alpha_3}\int\limits_0^t d\tau \int\limits_{R^{n-1}} \Phi(y',\tau)dy' \int\limits_0^{t-\tau} \partial_{x_n} g_{1,\varepsilon}(x'-y'+\frac{b'u}{\varepsilon},x_n,\frac{b_n u}{\varepsilon},t-\tau-u)du|_{x_n=0}$$

$$= \frac{\beta_1}{\varepsilon\alpha_3}\int\limits_0^t d\tau \int\limits_{R^{n-1}} dy' \int\limits_0^\tau \Phi(y',\tau-u)\partial_{x_n} g_{1,\varepsilon}(x'-y'+\frac{b'u}{\varepsilon},0,\frac{b_n u}{\varepsilon},t-\tau)du := \omega_2(x',t).$$

Lemma. *Let* $0<\varepsilon<\varepsilon_0$, $d_{1\,n}>0$, $\alpha_1>0$, $\alpha_3>0$, $\Phi(x',t) \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\ \ t}(R_T)$, $\alpha\in(0,1)$. *Then the function* $\omega_j(x',t) \in \overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{x'\ \ t}(R_T)$ *and satisfies the estimate*

$$|\omega_j|^{(2+\alpha)}_{R_T} \le C_{11}|\Phi(x',t)|^{(1+\alpha)}_{R_T}, \quad j=1,2, \tag{25}$$

where the constant C_{11} *is independent on* ε.

Proof. To prove the lemma we must estimate the norm of the function $\omega(x',t) := \omega_1(x',t)$ in Hölder space [7]

$$|\omega|^{(2+\alpha)}_{R_T} = \sum_{2k+|m'|\le 2} |\partial_t^k\partial_{x'}^{m'}\omega|_{R_T} + [\partial_t\omega]^{(\alpha)}_{R_T} + \sum_{\mu,\nu=1}^{n-1}[\partial^2_{x_\mu x_\nu}\omega]^{(\alpha)}_{R_T} + \sum_{\nu=1}^{n-1}[\partial_{x_\nu}\omega]^{(\frac{1+\alpha}{2})}_{t,R_T}, \tag{26}$$

where

$$|u|_{R_T} = \sup_{(x',t)\in R_T}|u(x',t)|, \quad [u]^{(\alpha)}_{R_T} := [u]^{(\alpha)}_{x',R_T} + [u]^{(\frac{\alpha}{2})}_{t,R_T},$$

$$[u]^{(\alpha)}_{x',R_T} = \sup_{(x',t),(z',t)\in R_T}\frac{|u(x',t)-u(z',t)|}{|x'-z'|^\alpha}, \quad [u]^{(\frac{\alpha}{2})}_{t,R_T} = \sup_{(x',t),(x',t_1)\in R_T}\frac{|u(x',t)-u(x',t_1)|}{|t-t_1|^\alpha}.$$

Therefore, we must obtain estimates for the Hölder constants

$$\frac{\alpha_1}{\varepsilon\alpha_3}[(\Phi*\partial_t G_\varepsilon)|_{x_n=0}]^{(\alpha)}_{R_T}, \quad \frac{\alpha_1}{\varepsilon\alpha_3}[(\Phi_{x_\nu}*\partial_{x_\mu}G_\varepsilon)|_{x_n=0}]^{(\alpha)}_{R_T},$$

$$\frac{\alpha_1}{\varepsilon\alpha_3}[(\Phi_{x_\nu}*G_\varepsilon)|_{x_n=0}]^{(\frac{1+\alpha}{2})}_{t,R_T}, \quad \nu,\mu=1,\dots,n-1.$$

We shall make use of the following notations and estimates for the function $\Phi(x',t) \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\ \ t}(R_T)$: $M_{k+1} = [\partial^k_{x_\nu}\Phi]^{(\frac{1+\alpha-k}{2})}_{t,R_T}$, $M_3 = [\Phi_{x_\nu}]^{(\alpha)}_{x',R_T}$;

$$|\partial^k_{x_\nu}\Phi(x',t)| \le M_{k+1} t^{\frac{1+\alpha-k}{2}}; \tag{27}$$

$$|\partial^k_{x_\nu}\Phi(x',t) - \partial^k_{x_\nu}\Phi(x',t_1)| \le M_{k+1}(t-t_1)^{\frac{1+\alpha-k}{2}}, \quad t_1 \le t; \tag{28}$$

$$|\Phi_{x_\nu}(x',t) - \Phi_{z_\nu}(z',t)| \le M_3|x'-z'|^{\alpha}, \quad k=0,1, \quad \nu=1,\dots,n-1. \tag{29}$$

We estimate the Hölder constants with respect to t. For that we represent the derivatives $\partial_t\omega$, $\partial^2_{x_\mu x_\nu}\omega$, $\nu,\mu = 1,\dots,n-1$, in the form

$$\partial_t\omega(x',t) = -\frac{2a^2\alpha_1}{\varepsilon\alpha_3}\Big(\int\limits_0^t d\tau \int\limits_{R^{n-1}} dy' \int\limits_0^{\tau} [\Phi(y',\tau-u) - \Phi(y',t-u)]$$

$$\times \partial_t\partial_{x_n}\Gamma(x'-y'+\frac{b'}{\varepsilon}u, \frac{b_n}{\varepsilon}u, t-\tau)du$$

$$+\int\limits_0^t du \int\limits_{R^{n-1}} \Phi(y',t-u)\partial_{x_n}\Gamma(x'-y'+\frac{b'}{\varepsilon}u, \frac{b_n}{\varepsilon}u, t-u)dy'\Big); \tag{30}$$

$$\partial^2_{x_\mu x_\nu}\omega(x',t) = -\frac{2a^2\alpha_1}{\varepsilon\alpha_3}\int\limits_0^t d\tau \int\limits_{R^{n-1}} dy' \int\limits_0^{\tau} [\Phi_{y_\nu}(y',\tau-u) - \Phi_{x_\nu}(x',\tau-u)]$$

$$\times \partial_{x_\mu}\partial_{x_n}\Gamma(x'-y'+\frac{b'}{\varepsilon}u, \frac{b_n}{\varepsilon}u, t-\tau)du. \tag{31}$$

For the definiteness we assume that $t_1 < t$ and compose the differences

$$\Delta_1 := \partial_t\omega(x',t) - \partial_{t_1}\omega(x',t_1)$$

$$= -\frac{2a^2\alpha_1}{\varepsilon\alpha_3}\Big(\int\limits_{t_1}^t d\tau \int\limits_{R^{n-1}} dy' \int\limits_0^{\tau} [\Phi(y',\tau-u) - \Phi(y',t-u)]\partial_t\partial_{x_n}\Gamma(x'-y'+\frac{b'}{\varepsilon}u, \frac{b_n}{\varepsilon}u, t-\tau)du$$

$$+\int\limits_0^{t_1} d\tau \int\limits_{R^{n-1}} dy' \int\limits_0^{\tau} [\Phi(y',\tau-u) - \Phi(y',t_1-u)]du \int\limits_{t_1}^t \partial^2_{t_2}\partial_{x_n}\Gamma(\cdot, t_2-\tau)dt_2$$

$$+\int\limits_0^{t_1} du \int\limits_{R^{n-1}} [\Phi(y',t-u) - \Phi(y',t_1-u)]\partial_{x_n}\Gamma(\cdot, t-t_1)dy'$$

$$+\int\limits_{t_1}^{t} du \int\limits_{R^{n-1}} \Phi(y', t-u)\partial_{x_n}\Gamma(\cdot, t-u)dy' + \int\limits_{0}^{t_1} du \int\limits_{R^{n-1}} \Phi(y', t_1-u)dy' \int\limits_{t_1}^{t} \partial_{t_2}\partial_{x_n}\Gamma(\cdot, t_2-u)dt_2\Big);$$

$$\Delta_2 := \partial^2_{x_\mu x_\nu}\omega(x', t) - \partial^2_{x_\mu x_\nu}\omega(x', t_1)$$

$$= -\frac{2a^2\alpha_1}{\varepsilon\alpha_3}\Big(\int\limits_{t_1}^{t} d\tau \int\limits_{R^{n-1}} dy' \int\limits_{0}^{\tau} [\Phi_{y_\nu}(y', \tau-u) - \Phi_{x_\nu}(x', \tau-u)]\partial_{x_\mu}\partial_{x_n}\Gamma(\cdot, t-\tau)du$$

$$+\int\limits_{0}^{t_1} d\tau \int\limits_{R^{n-1}} dy' \int\limits_{0}^{\tau} [\Phi_{y_\nu}(y', \tau-u) - \Phi_{x_\nu}(x', \tau-u)]du \int\limits_{t_1}^{t} \partial_{t_2}\partial_{x_\mu}\partial_{x_n}\Gamma(\cdot, t_2-\tau)dt_2\Big).$$

First, we estimate Δ_1. Applying the estimate (22) for the function Γ and the estimates (27), (28) for the function Φ, we shall have

$$|\Delta_1| \leq C_{12}\frac{M_1}{\varepsilon}\Big(\int\limits_{t_1}^{t} d\tau \int\limits_{0}^{\tau} \frac{(t-\tau)^{\frac{1+\alpha}{2}}}{(t-\tau)^{\frac{n+3}{2}}} du \int\limits_{R^{n-1}} e^{-\frac{(x'-y'+\frac{b'u}{\varepsilon})^2+(\frac{b_n u}{\varepsilon})^2}{8a^2(t-\tau)}} dy'$$

$$+\int\limits_{t_1}^{t} dt_2 \int\limits_{0}^{t_1} d\tau \int\limits_{0}^{\tau} \frac{(t_1-\tau)^{\frac{1+\alpha}{2}}}{(t_2-\tau)^{\frac{n+5}{2}}} du \int\limits_{R^{n-1}} e^{-\frac{(x'-y'+\frac{b'u}{\varepsilon})^2+(\frac{b_n u}{\varepsilon})^2}{8a^2(t_2-\tau)}} dy'$$

$$+\frac{(t-t_1)^{\frac{1+\alpha}{2}}}{(t-t_1)^{\frac{n+1}{2}}} \int\limits_{0}^{t_1} du \int\limits_{R^{n-1}} e^{-\frac{(x'-y'+\frac{b'u}{\varepsilon})^2+(\frac{b_n u}{\varepsilon})^2}{8a^2(t-t_1)}} dy' + \int\limits_{t_1}^{t} \frac{(t-u)^{\frac{1+\alpha}{2}}}{(t-u)^{\frac{n+1}{2}}} du \int\limits_{R^{n-1}} e^{-\frac{(x'-y'+\frac{b'u}{\varepsilon})^2+(\frac{b_n u}{\varepsilon})^2}{8a^2(t-u)}} dy'$$

$$+\int\limits_{t_1}^{t} dt_2 \int\limits_{0}^{t_1} \frac{(t_1-u)^{\frac{1+\alpha}{2}}}{(t_2-u)^{\frac{n+3}{2}}} du \int\limits_{R^{n-1}} e^{-\frac{(x'-y'+\frac{b'u}{\varepsilon})^2+(\frac{b_n u}{\varepsilon})^2}{8a^2(t_2-u)}} dy'\Big).$$

Integrating over y', we obtain

$$|\Delta_1| \leq C_{13}\frac{M_1}{\varepsilon}\Big(\int\limits_{t_1}^{t} \frac{1}{(t-\tau)^{\frac{3-\alpha}{2}}} d\tau \int\limits_{0}^{\tau} e^{-\frac{b_n^2 u^2}{8a^2\varepsilon^2(t-\tau)}} du$$

$$+\int\limits_{t_1}^{t} dt_2 \int\limits_{0}^{t_1} \frac{(t_1-\tau)^{\frac{1+\alpha}{2}}}{(t_2-\tau)^3} d\tau \int\limits_{0}^{\tau} e^{-\frac{b_n^2 u^2}{8a^2\varepsilon^2(t_2-\tau)}} du + (t-t_1)^{\frac{\alpha-1}{2}} \int\limits_{0}^{t_1} e^{-\frac{b_n^2 u^2}{8a^2\varepsilon^2(t-t_1)}} du$$

$$+\int\limits_{t_1}^{t}\frac{(t-u)^{1+\frac{\alpha}{2}}}{(t-u)^{\frac{3}{2}}}e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2(t-u)}}du+\int\limits_{t_1}^{t}dt_2\int\limits_{0}^{t_1}\frac{(t_1-u)^{\frac{1+\alpha}{2}}}{(t_2-u)^2}e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2(t_2-u)}}du\Big).$$

We integrate the first three integrals over u, for this in the first integral we make change $\frac{b_n u}{\sqrt{8}a\varepsilon\sqrt{t-\tau}}=\zeta$, in the second and the third integrals we introduce similar substitutions, and estimate them by Poisson integral; we make use of the inequality $t_1-\tau\le t_2-\tau$ in the second, $(t-u)^{1+\frac{\alpha}{2}}\le t(t-t_1)^{\frac{\alpha}{2}}$ and extend the integration domain up to the interval $(0,t)$ in the fourth, $\sqrt{t_1-u}\le\sqrt{t_2-u}$, $(t_1-u)^{\frac{\alpha}{2}}\le t_2^{\frac{\alpha}{2}}$ and extend the domain of integration over u from $(0,t_1)$ to $(0,t_2)$ in the last integrals, then we obtain

$$|\Delta_1|\le C_{14}\frac{M_1}{\varepsilon}\Big(\varepsilon(t-t_1)^{\frac{\alpha}{2}}+t(t-t_1)^{\frac{\alpha}{2}}\int\limits_{0}^{t}\frac{1}{(t-u)^{\frac{3}{2}}}e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2(t-u)}}du$$

$$+\int\limits_{t_1}^{t}t_2^{\frac{\alpha}{2}}dt_2\int\limits_{0}^{t_2}\frac{1}{(t_2-u)^{\frac{3}{2}}}e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2(t_2-u)}}du\Big).$$

Applying the estimate for the integral [2]

$$\int\limits_{0}^{t}\frac{1}{(t-u)^{\frac{3}{2}}}e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2(t-u)}}du\le C_{15}\frac{\varepsilon}{t}, \tag{32}$$

we shall have

$$|\partial_t\omega(x',t)-\partial_{t_1}\omega(x',t_1)|:=|\Delta_1|\le C_{16}M_1(t-t_1)^{\frac{\alpha}{2}},\quad [\partial_t\omega]_{t,R_T}^{(\frac{\alpha}{2})}\le C_{16}M_1. \tag{33}$$

Now we evaluate the difference Δ_2 with the help of the inequality (22) for the function Γ and estimate (29) for the function Φ

$$|\Delta_2|\le C_{17}\frac{M_3}{\varepsilon}\Big(\int\limits_{t_1}^{t}d\tau\int\limits_{0}^{\tau}du\int\limits_{R^{n-1}}\frac{|x'-y'|^\alpha}{(t-\tau)^{\frac{n+2}{2}}}e^{-\frac{(x'-y'+\frac{b'u}{\varepsilon})^2+(\frac{b_nu}{\varepsilon})^2}{8a^2(t-\tau)}}dy'$$

$$+\int\limits_{t_1}^{t}dt_2\int\limits_{0}^{t_1}d\tau\int\limits_{0}^{\tau}du\int\limits_{R^{n-1}}\frac{|x'-y'|^\alpha}{(t_2-\tau)^{\frac{n+4}{2}}}e^{-\frac{(x'-y'+\frac{b'u}{\varepsilon})^2+(\frac{b_nu}{\varepsilon})^2}{8a^2(t_2-\tau)}}dy'\Big).$$

We apply the inequality [2]

$$|\xi|^\alpha e^{-\xi^2}\le C_\alpha e^{-\xi^2/2},\quad \alpha\ge 0, \tag{34}$$

and integrate over y', u, then over τ and t_2

$$|\Delta_2| \le C_{18}\frac{M_3}{\varepsilon}\Big(\int\limits_{t_1}^{t}\frac{1}{(t-\tau)^{\frac{3-\alpha}{2}}}d\tau\int\limits_0^{\tau}e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2(t-\tau)}}du$$

$$+\int\limits_{t_1}^{t}dt_2\int\limits_0^{t_1}\frac{1}{(t_2-\tau)^{\frac{5-\alpha}{2}}}d\tau\int\limits_0^{\tau}e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2(t_2-\tau)}}du\Big)$$

$$\le C_{19}M_3\Big(\int\limits_{t_1}^{t}\frac{d\tau}{(t-\tau)^{1-\frac{\alpha}{2}}}+\int\limits_{t_1}^{t}dt_2\int\limits_0^{t_1}\frac{d\tau}{(t_2-\tau)^{2-\frac{\alpha}{2}}}\Big)\le C_{20}M_3(t-t_1)^{\frac{\alpha}{2}}.$$

Thus,

$$|\partial^2_{x_\mu x_\nu}\omega(x',t)-\partial^2_{x_\mu x_\nu}\omega(x',t_1)| := |\Delta_2| \le C_{20}M_3(t-t_1)^{\frac{\alpha}{2}},\quad [\partial^2_{x_\mu x_\nu}\omega(x',t)]^{(\frac{\alpha}{2})}_{t,R_T}\le C_{20}M_3. \quad (35)$$

To estimate the Hölder constant $[\partial_{x_\nu}\omega]^{(\frac{1+\alpha}{2})}_{t,R_T}$ we represent the derivative $\partial_{x_\nu}\omega$, $\nu = 1,\dots,n-1$, in the form

$$\partial_{x_\nu}\omega(x',t) = -\frac{2a^2\alpha_1}{\varepsilon\alpha_3}\Big(\int\limits_0^{t}d\tau\int\limits_{R^{n-1}}dy'\int\limits_0^{\tau}[\Phi_{y_\nu}(y',\tau-u)-\Phi_{y_\nu}(y',t-u)]$$

$$\times\partial_{x_n}\Gamma(x'-y'+\frac{b'u}{\varepsilon},\frac{b_nu}{\varepsilon},t-\tau)du+\int\limits_0^{t}du\int\limits_{R^{n-1}}\Phi_{y_\nu}(y',t-u)dy'\int\limits_0^{t-u}\partial_{x_n}\Gamma(\cdot,\tau)d\tau\Big), \quad (36)$$

compose the difference

$$\Delta_3 := \partial_{x_\nu}\omega(x',t)-\partial_{x_\nu}\omega(x',t_1)$$

$$= -\frac{2a^2\alpha_1}{\varepsilon\alpha_3}\Big(\int\limits_{t_1}^{t}d\tau\int\limits_{R^{n-1}}dy'\int\limits_0^{\tau}[\Phi_{y_\nu}(y',\tau-u)-\Phi_{y_\nu}(y',t-u)]\partial_{x_n}\Gamma(x'-y'+\frac{b'u}{\varepsilon},\frac{b_nu}{\varepsilon},t-\tau)du$$

$$+\int\limits_0^{t_1}d\tau\int\limits_{R^{n-1}}dy'\int\limits_0^{\tau}[\Phi_{y_\nu}(y',\tau-u)-\Phi_{y_\nu}(y',t_1-u)]du\int\limits_{t_1}^{t}\partial_{t_2}\partial_{x_n}\Gamma(\cdot,t_2-\tau)dt_2$$

$$+\int\limits_{t_1}^{t}du\int\limits_{R^{n-1}}\Phi_{y_\nu}(y',t-u)dy'\int\limits_0^{t-u}\partial_{x_n}\Gamma(\cdot,\tau)d\tau$$

$$+\int\limits_0^{t_1} du \int\limits_{R^{n-1}} [\Phi_{y_\nu}(y',t-u)-\Phi_{y_\nu}(y',t_1-u)]dy' \int\limits_0^{t-t_1} \partial_{x_n}\Gamma(\cdot,\tau)d\tau$$

$$+\int\limits_{t_1}^{t} d\tau \int\limits_{R^{n-1}} dy' \int\limits_0^{t_1} \Phi_{y_\nu}(y',t_1-u)\partial_{x_n}\Gamma(\cdot,\tau-u)du\Big)$$

and estimate it using the estimate (22) for the function Γ, the estimates (27), (28) for function Φ. Integrating the first over y', we shall have

$$|\Delta_3| \le C_{21}\frac{M_2}{\varepsilon}\bigg(\int\limits_{t_1}^{t} \frac{d\tau}{(t-\tau)^{1-\frac{\alpha}{2}}} \int\limits_0^{\tau} e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2(t-\tau)}}du$$

$$+\int\limits_{t_1}^{t} dt_2 \int\limits_0^{t_1} \frac{t_1-\tau)^{\frac{\alpha}{2}}}{(t_2-\tau)^2}d\tau \int\limits_0^{\tau} e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2(t_2-\tau)}}du + \int\limits_{t_1}^{t}(t-u)^{\frac{\alpha}{2}}du \int\limits_0^{t-u}\frac{1}{\tau}\, e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2\tau}}d\tau$$

$$+(t-t_1)^{\frac{\alpha}{2}} \int\limits_0^{t-t_1}\frac{d\tau}{\tau}\int\limits_0^{t_1} e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2\tau}}du + \int\limits_{t_1}^{t} d\tau \int\limits_0^{t_1} \frac{(t_1-u)^{\frac{\alpha}{2}}(\tau-u)^{\frac{1}{2}}}{(\tau-u)^{\frac{3}{2}}} e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2(\tau-u)}}du\bigg).$$

In the first, in the second and the fourth integrals we integrate over u and apply the inequality $t_1-\tau \le t_2-\tau$ in the second integral. In the third integral we make use of the inequality $(t-u)^{\frac{\alpha}{2}} \le (t-t_1)^{\frac{\alpha}{2}}$ and in the integral upper limit over τ the inequality $t-u \le t-t_1$. In the last integral we apply the inequalities $t_1-u \le t_1 \le \tau$, $\sqrt{\tau-u} \le \tau$, extend the integration domain over u from $(0,t_1)$ to $(0,\tau)$. Then we obtain

$$|\Delta_3| \le C_{22}\frac{M_2}{\varepsilon}\bigg(\varepsilon(t-t_1)^{\frac{1+\alpha}{2}} + (t-t_1)^{\frac{\alpha}{2}} \int\limits_0^{t-t_1}\frac{d\tau}{\tau}\int\limits_{t_1}^{t} e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2\tau}}du$$

$$+\int\limits_{t_1}^{t} \tau^{\frac{1+\alpha}{2}}d\tau \int\limits_0^{\tau} \frac{1}{(\tau-u)^{\frac{3}{2}}} e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2(\tau-u)}}du\bigg).$$

Integrating further over u and τ, estimating in the last term the integral over u with the help of the inequality (32), we shall have

$$|\partial_{x_\nu}\omega(x',t)-\partial_{x_\nu}\omega(x',t_1)| := |\Delta_3| \le C_{23}M_2(t-t_1)^{\frac{1+\alpha}{2}}, \quad [\partial_{x_\nu}\omega]^{(\frac{1+\alpha}{2})}_{t,R_T} \le C_{23}M_2. \qquad (37)$$

In the formulas (30), (31) in the integral over y' we carry out the change of the variable $y' - \frac{b'u}{\varepsilon} = \varsigma$, then the derivatives $\partial_t\omega,\ \partial^2_{x_\mu x_\nu}\omega, \quad \nu,\mu = 1,\dots,n-1$, may be written as

$$\partial_t\omega(x',t) = -\frac{2a^2\alpha_1}{\varepsilon\alpha_3}\Big(\int\limits_0^t d\tau \int\limits_{R^{n-1}} dy' \int\limits_0^\tau [\Phi(y' + \frac{b'}{\varepsilon}u, \tau-u) - \Phi(y' + \frac{b'}{\varepsilon}u, t-u)]$$

$$\times\partial_t\partial_{x_n}\Gamma(x'-y', \frac{b_n u}{\varepsilon}, t-\tau)du + \int\limits_0^t du \int\limits_{R^{n-1}} \Phi(y' + \frac{b'}{\varepsilon}u, t-u)\partial_{x_n}\Gamma(x'-y', \frac{b_n u}{\varepsilon}, t-u)dy'\Big);$$

$$\partial^2_{x_\mu x_\nu}\omega(x',t) = -\frac{2a^2\alpha_1}{\varepsilon\alpha_3}\int\limits_0^t d\tau \int\limits_{R^{n-1}} dy' \int\limits_0^\tau [\Phi_{y_\nu}(y' + \frac{b'}{\varepsilon}u, \tau-u) - \Phi_{x_\nu}(x' + \frac{b'}{\varepsilon}u, \tau-u)]$$

$$\times\partial_{x_\mu}\partial_{x_n}\Gamma(x'-y', \frac{b_n u}{\varepsilon}, t-\tau)du.$$

We compose the differences of these derivatives, denoting $r = |x' - z'|$,

$$\Delta_4 := \partial_t\omega(x',t) - \partial_t\omega(z',t) \tag{38}$$

$$= -\frac{2a^2\alpha_1}{\varepsilon\alpha_3}\left(\int\limits_0^t d\tau \int\limits_{|y'-z'|\le 2r} dy' \int\limits_0^\tau [\Phi(y' + \frac{b'u}{\varepsilon}, \tau-u) - \Phi(y' + \frac{b'u}{\varepsilon}, t-u)]\right.$$

$$\times\Big(\partial_t\partial_{x_n}\Gamma(x'-y', \frac{b_n u}{\varepsilon}, t-\tau) - \partial_t\partial_{z_n}\Gamma(z'-y', \frac{b_n u}{\varepsilon}, t-\tau)\Big)du$$

$$+\int\limits_0^t d\tau \int\limits_{|y'-z'|>2r} dy' \int\limits_0^\tau [\Phi(y' + \frac{b'u}{\varepsilon}, \tau-u) - \Phi(y' + \frac{b'u}{\varepsilon}, t-u)]$$

$$\times\sum_{i=1}^{n-1}(x_i - z_i)\int\limits_0^1 \partial_t\partial^2_{z_i z_n}\Gamma(z'-y'+\lambda(x'-z'), \frac{b_n u}{\varepsilon}, t-\tau)d\lambda du$$

$$+\int\limits_0^t du \int\limits_{|y'-z'|\le 2r} \Phi(y' + \frac{b'u}{\varepsilon}, t-u)\Big(\partial_{x_n}\Gamma(x'-y', \frac{b_n u}{\varepsilon}, t-u) - \partial_{z_n}\Gamma(z'-y', \frac{b_n u}{\varepsilon}, t-u)\Big)dy'$$

$$+\int\limits_0^t du \int\limits_{|y'-z'|>2r} \Phi(y' + \frac{b'u}{\varepsilon}, t-u)$$

$$\times\sum_{i=1}^{n-1}(x_i-z_i)\int_0^1 \partial_{z_i}\partial_{z_n}\Gamma(z'-y'+\lambda(x'-z'),\frac{b_n u}{\varepsilon},t-u)d\lambda dy'\Bigg);$$

$$\Delta_5 := \partial^2_{x_\mu x_\nu}\omega(x',t)-\partial^2_{z_\mu z_\nu}\omega(z',t) \tag{39}$$

$$=-\frac{2a^2\alpha_1}{\varepsilon\alpha_3}\Bigg(\int_0^t d\tau\int_{|y'-z'|\le 2r}dy'\int_0^\tau[\Phi_{y_\nu}(y'+\frac{b'u}{\varepsilon},\tau-u)-\Phi_{x_\nu}(x'+\frac{b'u}{\varepsilon},\tau-u)]$$

$$\times\partial_{x_\mu}\partial_{x_n}\Gamma(x'-y',\frac{b_n u}{\varepsilon},t-\tau)du$$

$$-\int_0^t d\tau\int_{|y'-z'|\le 2r}dy'\int_0^\tau[\Phi_{y_\nu}(y'+\frac{b'u}{\varepsilon},\tau-u)-\Phi_{z_\nu}(z'+\frac{b'u}{\varepsilon},\tau-u)]\partial_{z_\mu}\partial_{z_n}\Gamma(z'-y',\frac{b_n u}{\varepsilon},t-\tau)du$$

$$+\int_0^t d\tau\int_{|y'-z'|>2r}dy'\int_0^\tau[\Phi_{y_\nu}(y'+\frac{b'u}{\varepsilon},\tau-u)-\Phi_{x_\nu}(x'+\frac{b'u}{\varepsilon},\tau-u)]$$

$$\times\sum_{i=1}^{n-1}(x_i-z_i)\int_0^1 \partial^2_{z_i z_\mu}\partial_{z_n}\Gamma(z'-y'+\lambda(x'-z'),\frac{b_n u}{\varepsilon},t-\tau)d\lambda du$$

$$+\int_0^t d\tau\int_0^\tau[\Phi_{x_\nu}(x'+\frac{b'u}{\varepsilon},\tau-u)-\Phi_{z_\nu}(z'+\frac{b'u}{\varepsilon},\tau-u)]du$$

$$\times\int_{|y'-z'|>2r}\partial_{y_\mu}\partial_{z_n}\Gamma(z'-y',\frac{b_n u}{\varepsilon},t-\tau)dy'\Bigg).$$

We evaluate the difference Δ_4. We apply the inequalities (22) for the function Γ and (27), (28) for the function Φ; when integrating over y' we pass to the spherical coordinates assuming $\rho=|x'-y'|$ in the first and fourth integrals, $\rho=|z'-y'|$ in the second and fifth integrals, $\rho=|z'-y'+\lambda(x'-z')|$ in the third and last integrals, then we shall have

$$|\Delta_4|\le C_{24}\frac{M_1}{\varepsilon}\Bigg(\Big(\int_0^{3r}+\int_0^{2r}\Big)\rho^{n-2}d\rho\int_0^t\frac{1}{(t-\tau)^{\frac{n+2-\alpha}{2}}}d\tau\int_0^\tau e^{-\frac{\rho^2+(\frac{b_n u}{\varepsilon})^2}{8a^2(t-\tau)}}du$$

$$+r\int\limits_r^\infty \rho^{n-2}d\rho\int\limits_0^t \frac{1}{(t-\tau)^{\frac{n+3-\alpha}{2}}}d\tau\int\limits_0^\tau e^{-\frac{\rho^2+(\frac{b_nu}{\varepsilon})^2}{8a^2(t-\tau)}}du$$

$$+\Big(\int\limits_0^{3r}+\int\limits_0^{2r}\Big)\rho^{n-2}d\rho\int\limits_0^t \frac{1}{(t-u)^{\frac{n-\alpha}{2}}}e^{-\frac{\rho^2+(\frac{b_nu}{\varepsilon})^2}{8a^2(t-u)}}du+r\int\limits_r^\infty \rho^{n-2}d\rho\int\limits_0^t \frac{1}{(t-u)^{\frac{n+1-\alpha}{2}}}e^{-\frac{\rho^2+(\frac{b_nu}{\varepsilon})^2}{8a^2(t-u)}}du\Bigg).$$

We integrate the first two integrals over u, then when integrating over τ we make the change $\frac{\rho^2}{8a^2(t-\tau)}=\varsigma^2$; in the last two integrals we apply the inequality $|\xi|^\alpha e^{-\xi^2}\le C_\alpha e^{-\xi^2/2},\ \alpha\ge 0$ (34), then we obtain

$$|\Delta_4|\le C_{25}\frac{M_1}{\varepsilon}\Bigg(\varepsilon\Big(\int\limits_0^{3r}+\int\limits_0^{2r}\Big)\rho^{\alpha-1}d\rho\int\limits_0^\infty \varsigma^{n-2-\alpha}e^{-\varsigma^2}d\varsigma$$

$$+\varepsilon r\int\limits_r^\infty \rho^{\alpha-2}d\rho\int\limits_0^\infty \varsigma^{n-1-\alpha}e^{-\varsigma^2}d\varsigma$$

$$+\Bigg(\Big(\int\limits_0^{3r}+\int\limits_0^{2r}\Big)\rho^{\alpha-1}d\rho+r\int\limits_r^\infty \rho^{\alpha-2}d\rho\Bigg)\int\limits_0^t \frac{t-u}{(t-u)^{\frac{3}{2}}}e^{-\frac{b^2u^2}{8a^2\varepsilon^2(t-u)}}du\Bigg).$$

In the integral over u we use the inequality $t-u\le t$ and the estimate (32), then after integration we shall have

$$|\partial_t\omega(x',t)-\partial_t\omega(z',t)|:=|\Delta_4|\le C_{26}M_1|x'-z'|^\alpha,\quad [\partial_t\omega]^{(\alpha)}_{x',R_T}\le C_{26}M_1. \tag{40}$$

Now we evaluate the difference Δ_5 using the inequality (22) for the function Γ and estimate (29) for the function Φ. The first two integrals in Δ_5 are estimated as the first two ones in Δ_4; in the third integral when integrating over y' we pass to the spherical coordinates assuming $\rho=|z'-y'+\lambda(x'-z')|$, and make use of the inequality $|x'-y'|^\alpha\le C_{27}(\rho^\alpha+r^\alpha)\le 2C_{27}\rho^\alpha$, $r\le\rho$. In the last integral, denoting it by I_4, if $n\ge 3$ we apply the formula

$$\int\limits_{|y'-z'|>2r}\partial_{y_\mu}\partial_{z_n}\Gamma(z'-y',\frac{b_nu}{\varepsilon},t-\tau)dy'=\int\limits_{|y'-z'|=2r}\partial_{z_n}\Gamma(z'-y',\frac{b_nu}{\varepsilon},t-\tau)cos(\vec{n},y_\mu)dS_{y'},$$

where $\vec{n}$ is the normal to the sphere $|y'-z'|=2r$, then we shall have

$$|I_4|\le C_{28}\frac{M_3}{\varepsilon}r^\alpha\int\limits_0^t d\tau\int\limits_0^\tau du\Big|\int\limits_{|y'-z'|=2r}\partial_{z_n}\Gamma(z'-y',\frac{b_nu}{\varepsilon},t-\tau)cos(\vec{n},y_\mu)dS_{y'}\Big|.$$

Thus, we obtain

$$|\Delta_5| \leq C_{29}\frac{M_3}{\varepsilon}\left(\int\limits_0^t d\tau\Big(\int\limits_0^{3r}+\int\limits_0^{2r}\Big)\rho^{n-2+\alpha}d\rho\int\limits_0^{\tau}\frac{1}{(t-\tau)^{\frac{n+2}{2}}}e^{-\frac{\rho^2+(\frac{b_n u}{\varepsilon})^2}{8a^2(t-\tau)}}du\right.$$

$$\left.+r\int\limits_0^t d\tau\int\limits_r^{\infty}\rho^{n-2+\alpha}d\rho\int\limits_0^{\tau}\frac{1}{(t-\tau)^{\frac{n+3}{2}}}e^{-\frac{\rho^2+(\frac{b_n u}{\varepsilon})^2}{8a^2(t-\tau)}}du+\int\limits_0^t d\tau\int\limits_0^{\tau}\frac{r^{n-2+\alpha}}{(t-\tau)^{\frac{n+1}{2}}}e^{-\frac{4r^2+(\frac{b_n u}{\varepsilon})^2}{8a^2(t-\tau)}}du\right).$$

Further we integrate Δ_5 over u, τ and ρ

$$|\Delta_5| \leq C_{30}M_3\left(\Big(\int\limits_0^{3r}+\int\limits_0^{2r}\Big)\rho^{n-2+\alpha}d\rho\int\limits_0^{t}\frac{1}{(t-\tau)^{\frac{n+1}{2}}}e^{-\frac{\rho^2}{8a^2(t-\tau)}}d\tau\right.$$

$$\left.+r\int\limits_r^{\infty}\rho^{n-2+\alpha}d\rho\int\limits_0^{t}\frac{1}{(t-\tau)^{\frac{n+2}{2}}}e^{-\frac{\rho^2}{8a^2(t-\tau)}}d\tau+r^{n-2+\alpha}\int\limits_0^{t}\frac{1}{(t-\tau)^{\frac{n}{2}}}e^{-\frac{r^2}{2a^2(t-\tau)}}d\tau\right)$$

$$\leq C_{31}\left(\Big(\int\limits_0^{3r}+\int\limits_0^{2r}\Big)\rho^{\alpha-1}d\rho\int\limits_0^{\infty}\varsigma^{n-2}e^{-\varsigma^2}d\varsigma+r\int\limits_r^{\infty}\rho^{\alpha-2}d\rho\int\limits_0^{\infty}\varsigma^{n-1}e^{-\varsigma^2}d\varsigma\right.$$

$$\left.+r^{\alpha}\int\limits_0^{\infty}\varsigma^{n-3}e^{-\varsigma^2}d\varsigma\right)\leq C_{32}M_3|x'-z'|^{\alpha}.$$

For $n=2$ the last integral I_4 is equal to zero.

Therefore, we have obtained the required estimate for the difference (39)

$$|\partial^2_{x_\mu x_\nu}\omega(x',t)-\partial^2_{z_\mu z_\nu}\omega(z',t)| := |\Delta_5| \leq C_{32}M_3|x'-z'|^{\alpha}, \quad [\partial^2_{x_\mu x_\nu}\omega]^{(\alpha)}_{x',R_T} \leq C_{32}M_3. \tag{41}$$

We evaluate the modulus of the function $\omega(x',t)$

$$|\omega(x',t)| \leq C_{33}\frac{M_1}{\varepsilon}\int\limits_0^t d\tau\int\limits_{R^{n-1}}dy'\int\limits_0^{\tau}\frac{(\tau-u)^{\frac{1+\alpha}{2}}}{(t-\tau)^{\frac{n+1}{2}}}e^{-\frac{(x'-y'+\frac{b'u}{\varepsilon})^2+(\frac{b_n u}{\varepsilon})^2}{8a^2(t-\tau)}}du$$

$$\leq C_{34}\frac{M_1}{\varepsilon}\int\limits_0^t\frac{\tau^{\frac{1+\alpha}{2}}}{t-\tau}d\tau\int\limits_0^{\tau}e^{-\frac{b_n^2u^2}{8a^2\varepsilon^2(t-\tau)}}du \leq C_{35}M_1t^{1+\frac{\alpha}{2}}. \tag{42}$$

The modules of derivatives $\partial_t\omega,\ \partial^2_{x_\mu x_\nu}\omega,\ \partial_{x_\nu}\omega,\ \nu,\mu=1,\dots,n-1$, defined by formulas (30), (31), (36), are evaluated in the same way, then we shall have the estimates

$$|\partial_t\omega(x',t)|\le C_{36}M_1t^{\frac{\alpha}{2}},\quad |\partial^2_{x_\mu x_\nu}\omega(x',t)|\le C_{37}M_3t^{\frac{\alpha}{2}},\quad |\partial_{x_\nu}\omega(x',t)|\le C_{38}M_2t^{\frac{1+\alpha}{2}}. \tag{43}$$

Thus, we have estimated all the terms of the norm (26) of the function $\omega(x',t)$, all constants in the obtained estimates do not depend on ε.

Gathering the estimates (42), (43), (33), (35), (37), (40), (41), we obtain an inequality (25) for the function $\omega(x',t):=\omega_1(x',t)$. The estimate for $\omega_2(x',t)$ is established in the same way as for $\omega_1(x',t)$, for this we make use of the inequalities (27)–(29) for the function Φ and the estimate (23) for the function $g_{1,\varepsilon}\big(x'+\dfrac{b'u}{\varepsilon},x_n,\dfrac{b_nu}{\varepsilon},t\big)$.

4 Proofs of Theorem 3 and Theorem 1

Proof of Theorem 3. The functions $w(x,t)$, $z_1(x,t)$ satisfy the heat equations (14), (15) and, moreover, in accordance with Lemma and the estimate (25) on the plane $x_n=0$ the functions $w(x,t)|_{x_n=0}=\omega_1(x',t)$, $z_1(x,t)|_{x_n=0}=\omega_2(x',t)$ belong to the space $\overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{x'\quad t}(R_T)$ and for them the estimates hold

$$|w(x,t)|_{x_n=0}|^{(2+\alpha)}_{R_T}\le C_{39}|\Phi(x',t)|^{(1+\alpha)}_{R_T},\quad |z_1(x,t)|_{x_n=0}|^{(2+\alpha)}_{R_T}\le C_{40}|\Phi(x',t)|^{(1+\alpha)}_{R_T}, \tag{44}$$

with constants C_{39}, C_{40} independent on the small parameter ε.

The functions $w(x,t)$, $z_1(x,t)$ may be considered as solutions of the first boundary-value problems for the equations (14), (15) in D_{1T} with the trace on the plane $x_n=0$ from the space $\overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{x'\quad t}(R_T)$, but then the functions $w(x,t)$, $z_1(x,t)$ belong to the space $\overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{x\quad t}(D_{1T})$ and due to (44) satisfy the estimate [7]

$$\begin{aligned}|w(x,t)|^{(2+\alpha)}_{D_{1T}}&\le |w(x,t)|_{x_n=0}|^{(2+\alpha)}_{R_T}\le C_{41}|\Phi(x',t)|^{(1+\alpha)}_{R_T},\\ |z_1(x,t)|^{(2+\alpha)}_{D_{1T}}&\le |z_1(x,t)|_{x_n=0}|^{(2+\alpha)}_{R_T}\le C_{42}|\Phi(x',t)|^{(1+\alpha)}_{R_T},\end{aligned} \tag{45}$$

where the constants C_{41}, C_{42} do not depend on ε.

From the formula $r_1=-\dfrac{1}{\alpha_1}w\big|_{x_n=0}$ and the estimate (44) it follows that the function $r_1(x',t)$ belongs to the space $\overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{x'\quad t}(R_T)$ and satisfies the estimate

$$|r_1|^{(2+\alpha)}_{R_T}\le\frac{C_{39}}{\alpha_1}|\Phi(x',t)|^{(1+\alpha)}_{R_T}. \tag{46}$$

From the boundary condition (18) we obtain that the time derivatives $\varepsilon\partial_t w(x,t)\big|_{x_n=0}$, $\varepsilon\partial_t r_1(x',t)$ belong to the space $\overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\quad t}(R_T)$ and satisfy the estimate

$$|\varepsilon\partial_t w|^{(1+\alpha)}_{R_T} + |\varepsilon\partial_t r_1|^{(1+\alpha)}_{R_T} \le C_{43}|\Phi(x',t)|^{(1+\alpha)}_{R_T}, \tag{47}$$

where the constant C_{43} does not depend on ε.

Gathering estimates (45)–(47) for functions $w(x,t)$, $z_1(x,t)$, $r_1(x',t)$ and time derivatives $\varepsilon\partial_t w(x,t)\big|_{x_n=0}$, $\varepsilon\partial_t r_1(x',t)$ we derive the required inequality (20). Theorem 3 is proved.

Proof of Theorem 1. Remembering the change formulas (13) and applying the inequalities (10), (11), the estimate (19) for the function $\Phi(x',t)$, due to Theorem 3 and the estimate (20), we obtain the inequality (6) and the proof of Theorem 1.

Corollary 1. *The problem (1)–(5) with* $\varepsilon = 0$ *has a unique solution* $v \in \overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{x\quad t}(D_{1T})$, $u_1 \in \overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{x\quad t}(D_{1T})$, $r_1 \in \overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{x\quad t}(R_T)$, *and it satisfies the estimate*

$$|v|^{(2+\alpha)}_{D_{1T}} + |u_1|^{(2+l)}_{D_{1T}} + |r_1|^{(2+l)}_{R_T} \le C_{44}\Big(|f|^{(\alpha)}_{D_{1T}} + |f_1|^{(\alpha)}_{D_{1T}} + \sum_{j=1}^{2}|\varphi_j|^{(2+\alpha)}_{R_T} + |\varphi_3|^{(1+\alpha)}_{R_T}\Big). \tag{48}$$

5 Construction of a solution to the model problem II. Proof of Theorem 2

We construct the auxiliary functions $U_j(x,t)$, $j=1,2$, $V(x,t)$ as solutions of the first boundary value problems

$$\begin{gathered}\partial_t U_j - a_1^2\triangle U_j = f_j(x,t) \quad \text{in} \quad D_{1T},\\ U_j\big|_{t=0} = u_{0j}(x) \quad \text{in} \quad D_j, \quad U_j\big|_{x_n=0} = \psi_{j-1}(x',t) \quad \text{on} \quad R_T.\end{gathered} \tag{49}$$

$$\begin{gathered}\partial_t V - a^2\triangle V = f(x,t) \quad \text{in} \quad D_{1T},\\ V\big|_{t=0} = v_0(x) \quad \text{in} \quad D_1, \quad V\big|_{x_n=0} = \psi_2(x',t) \quad \text{on} \quad R_T.\end{gathered} \tag{50}$$

The problems (49), (50) have unique solutions $V(x,t) \in \overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{x\quad t}(\bar D_{1T})$, $U_j(x,t) \in \overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{x\quad t}(\bar D_{jT})$, $j=1,2$, which satisfy the inequalities [7]

$$|U_j|^{(2+\alpha)}_{D_{jT}} \le C_{45}\Big(|f_j|^{(\alpha)}_{D_{jT}} + |\psi_{j-1}|^{(2+\alpha)}_{R_T}\Big), \tag{51}$$

$$|V|^{(2+\alpha)}_{D_{1T}} \le C_{46}\Big(|f|^{(\alpha)}_{D_{1T}} + |\psi_2|^{(2+\alpha)}_{R_T}\Big). \tag{52}$$

In the equations and conditions of the problem (7) we make the substitution

$$v(x,t) = V(x,t) + \alpha_2 r_2 + w(x,t), \quad u_j(x,t) = U_j(x,t) + \beta_{j+1}r_2 + z_j(x,t), \tag{53}$$

where $w(x,t)$, $z_j(x,t)$ $j=1,2$, are new unknown functions, we obtain

$$\partial_t w - a^2 \triangle w = 0 \quad \text{in} \quad D_{1T}, \tag{54}$$

$$\partial_t z_j - a_j^2 \triangle z_j = 0 \quad \text{in} \quad D_{jT}, \quad j=1,2, \tag{55}$$

$$w\big|_{t=0} = 0 \quad \text{in} \quad D_1, \quad z_j\big|_{t=0} = 0 \quad \text{in} \quad D_j, \quad j=1,2, \quad r_2\big|_{t=0} = 0 \quad \text{on} \quad R, \tag{56}$$

$$w\big|_{x_n=0} + \alpha_2 r_2 = 0, \quad z_1\big|_{x_n=0} + \beta_2 r_2 = 0, \quad z_2\big|_{x_n=0} + \beta_3 r_2 = 0 \quad \text{on} \quad R_T, \tag{57}$$

$$\lambda_1 \partial_{x_n} z_1 - \lambda_2 \partial_{x_n} z_2 + \varkappa \partial_t r_2 + d_3' \nabla'^T r_2 = \Psi(x',t) \quad \text{on} \quad R_T, \tag{58}$$

where

$$\Psi(x',t) = \psi_3(x',t) - \Big(\lambda_1 \partial_{x_n} U_1 - \lambda_2 \partial_{x_n} U_2\Big)\big|_{x_n=0} \in \overset{\circ}{C}{}_{x\ \ t}^{1+\alpha,\frac{1+\alpha}{2}}(R_T)$$

and the following estimate is fulfilled

$$|\Psi|_{R_T}^{(1+\alpha)} \le C_{47}\Big(|\psi_3|_{R_T}^{(1+\alpha)} + \sum_{j=1}^{2} |U_j|_{D_{jT}}^{(2+\alpha)}\Big). \tag{59}$$

Theorem 4. *Let $\beta_j > 0$, $j=2,3$, $0 < \varkappa < \varkappa_0$.*

For every function $\Psi(x',t) \in \overset{\circ}{C}{}_{x'\ t}^{1+\alpha,\frac{1+\alpha}{2}}(R_T)$, $\alpha \in (0,1)$, the problem (54)–(58) has a unique solution $w \in \overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(D_{1T})$, $z_j \in \overset{\circ}{C}{}_{x\ \ t}^{2+l,1+\frac{l}{2}}(D_{jT})$, $j=1,2$, $r_2 \in \overset{\circ}{C}{}_{x'\ t}^{2+\alpha,1+\frac{\alpha}{2}}(R_T)$, $\varkappa\partial_t r_2 \in \overset{\circ}{C}{}_{x'\ t}^{1+\alpha,\frac{1+\alpha}{2}}(R_T)$, and it satisfies the estimate

$$|w|_{D_{1T}}^{(2+\alpha)} + \sum_{j=1}^{2} |z_j|_{D_{jT}}^{(2+\alpha)} + |r_2|_{R_T}^{(2+\alpha)} + |\varkappa \partial_t r_2|_{R_T}^{(1+\alpha)} \le C_{48} |\Psi(x',t)|_{R_T}^{(1+\alpha)}, \tag{60}$$

where the constant C_{48} does not depend on $\varkappa$.

Proof. To the problem (54)–(58) we apply Laplace transform with respect to the variable t and Fourier transform with respect to x'. From the equations (54), (55) and the initial conditions (56) of the problem we find the solution in the domain of images of Laplace and Fourier transforms

$$\tilde{w} = A e^{-k x_n}, \qquad \tilde{z}_1 = A_1 e^{-k_1 x_n}, \quad x_n > 0, \qquad \tilde{z}_2 = A_2 e^{k_2 x_n}, \quad x_n < 0, \tag{61}$$

where

$$k = \frac{1}{a}\sqrt{p + a^2 s'^2}, \qquad k_j = \frac{1}{a_j}\sqrt{p + a_j^2 s'^2}, \quad j=1,2,$$

and $A = A(s',p)$, $A_j = A_j(s',p)$, $j=1,2$, are unknown coefficients, which are determined from the boundary conditions on the hyperplane $x_n = 0$.

The boundary conditions (57)–(58) in the image domain of Laplace and Fourier transforms have the form

$$A=-\alpha_2\tilde r_2,\ A_1=-\beta_2\tilde r_2,\quad A_2=-\beta_3\tilde r_2\quad \text{on}\quad R_T,$$
$$-\lambda_1k_1A_1-\lambda_2k_2A_2+\big(\varkappa p+id_3's'\big)\tilde r_2=\tilde\Psi(s',p).$$

From these conditions we find the functions $\tilde r_2,\ A,\ A_j,\ j=1,2,$

$$\tilde r_2=\frac{1}{\varkappa\zeta_1}\,\tilde\Psi(s',p),\quad A=-\frac{\alpha_2}{\varkappa\zeta_1}\,\tilde\Psi(s',p),\quad A_j=-\frac{\beta_{j+1}}{\varkappa\zeta_1}\,\tilde\Psi(s',p),\quad j=1,2,$$

where

$$\zeta_1=p+\frac{\mu_1}{\varkappa}k_1+\frac{\mu_2}{\varkappa}k_2+i\frac{c'}{\varkappa}s',\quad \mu_1=\lambda_1\beta_2,\quad \mu_2=\lambda_2\beta_3,\quad c'=d_3'.$$

$Re\zeta_1\ge C>0$, if the conditions of the theorem $\beta_j>0,\ j=2,3,$ are fulfilled. Due to this condition we can represent $\dfrac{1}{\zeta_1}$ as an integral (21), in which $\zeta:=\zeta_1$.

Substituting the functions $A,\ A_j$ into (61) and applying the representation $\dfrac{1}{\zeta_1}$ as an integral, we write the solution of the problem in the image domain of Laplace and Fourier transforms in the form

$$\tilde z_j=-\frac{\beta_{j+1}}{\varkappa}\,\tilde\Psi(s',p)\int\limits_0^\infty e^{-\zeta_1\sigma-k_j|x_n|}d\sigma,\quad j=1,2,$$

$$\tilde r_2=\frac{1}{\varkappa}\,\tilde\Psi(s',p)\int\limits_0^\infty e^{-\zeta_1\sigma}d\sigma,\qquad \tilde w=-\frac{\alpha_2}{\varkappa}\,\tilde\Psi(s',p)\int\limits_0^\infty e^{-\zeta_1\sigma-kx_n}d\sigma.$$

With the help of inverse Laplace and Fourier transforms we obtain the solution to the problem (54) - (58) in the explicit form. First, we find the functions $z_1(x,t),\ z_2(x,t)$

$$z_j(x,t)=-\frac{\beta_{j+1}}{\varkappa}\int\limits_0^t d\tau\int\limits_{R^{n-1}}\Psi(y',\tau)G_j(x'-y',x_n,t-\tau)dy',\quad j=1,2,\tag{62}$$

where

$$G_j(x,t)=\int\limits_0^t\partial_{x_n}g_j(x'+\frac{c'u}{\varkappa},|x_n|,\frac{u}{\varkappa},t-u)du,\tag{63}$$

$$g_1(x'+\frac{c'u}{\varkappa},x_n,\frac{u}{\varkappa},t)=4a_1^2a_2^2\int\limits_0^t d\tau_1\int\limits_{R^{n-1}}\Gamma_1(x'+\frac{c'u}{\varkappa}-\eta',x_n+\frac{\mu_1u}{\varkappa}u,t-\tau_1)$$

$$\times\partial_{\eta_n}\Gamma_2(\eta',\eta_n+\frac{\mu_2 u}{\varkappa},\tau_1)\Big|_{\eta_n=0}d\eta'$$

$$=-2a_1^2\int\limits_0^t d\tau_1\int\limits_{R^{n-1}}\frac{1}{\left(2a_1\sqrt{\pi(t-\tau_1)}\right)^n}\,e^{-\frac{(x'+\frac{c'u}{\varkappa}-\eta')^2+(x_n+\frac{\mu_1 u}{\varkappa})^2}{4a_1^2(t-\tau_1)}}$$

$$\times\frac{\frac{\mu_2 u}{\varkappa}}{\left(2a_2\sqrt{\pi\tau_1}\right)^n\tau_1}\,e^{-\frac{\eta'^2+(\frac{\mu_2 u}{\varkappa})^2}{4a_2^2\tau_1}}d\eta',\quad x_n>0, \tag{64}$$

$$g_2(x'+\frac{c'u}{\varkappa},-x_n,\frac{u}{\varkappa},t)=4a_1^2a_2^2\int\limits_0^t d\tau_1\int\limits_{R^{n-1}}\Gamma_2(x'+\frac{c'u}{\varkappa}-\eta',\frac{\mu_2 u}{\varkappa}-x_n,t-\tau_1)$$

$$\times\partial_{\eta_n}\Gamma_1(\eta',\frac{\mu_1 u}{\varkappa}-\eta_n,\tau_1)\Big|_{\eta_n=0}d\eta'$$

$$=2a_2^2\int\limits_0^t d\tau_1\int\limits_{R^{n-1}}\frac{1}{\left(2a_2\sqrt{\pi(t-\tau_1)}\right)^n}\,e^{-\frac{(x'+\frac{c'u}{\varkappa}-\eta')^2+(\frac{\mu_2 u}{\varkappa}-x_n)^2}{4a_2^2(t-\tau_1)}}$$

$$\times\frac{\frac{\mu_1 u}{\varkappa}}{\left(2a_1\sqrt{\pi\tau_1}\right)^n\tau_1}\,e^{-\frac{\eta'^2+(\frac{\mu_1 u}{\varkappa})^2}{4a_1^2\tau_1}}d\eta',\quad x_n<0, \tag{65}$$

$\Gamma_j(x,t),\ j=1,2,$ is a fundamental solution to the heat equation (55).

Taking into account that $r_2=-\frac{1}{\beta_2}z_1\big|_{x_n=0}=-\frac{1}{\beta_3}z_2\big|_{x_n=0}$, then

$$r_2(x',t)=\frac{1}{\varkappa}\int\limits_0^t d\tau\int\limits_{R^{n-1}}\Psi(y',\tau)G_j(x'-y',0,t-\tau)dy' \tag{66}$$

$$=\frac{1}{\varkappa}\int\limits_0^t d\tau\int\limits_{R^{n-1}}\Psi(y',\tau)dy'\int\limits_0^{t-\tau}\partial_{x_n}g_j(x'-y'+\frac{c'u}{\varkappa},0,\frac{u}{\varkappa},t-\tau-u)du.$$

Now we find the function $w(x,t)$

$$w(x,t)=-\frac{\alpha_2}{\varkappa}\int\limits_0^t d\tau\int\limits_{R^{n-1}}\Psi(y',\tau)G_3(x'-y',x_n,t-\tau)dy', \tag{67}$$

where

$$G_3(x,t)=\int\limits_0^t \partial_{x_n} g_3(x'+\frac{c'u}{\varkappa},x_n,\frac{u}{\varkappa},t-u)du, \qquad (68)$$

$$g_3(x'+\frac{c'u}{\varkappa},x_n,\frac{u}{\varkappa},t)=-2a^2\int\limits_0^t d\tau_2\int\limits_{R^{n-1}}\Gamma(\bar\eta',x_n,\tau_2)\partial_{\bar\eta}g_1(x'+\frac{c'u}{\varkappa}-\bar\eta',\bar\eta_n,\frac{u}{\varkappa},t-\tau_2)\Big|_{\bar\eta_n=0}d\bar\eta'$$

$$=-2a^2\int\limits_0^t d\tau_2\int\limits_{R^{n-1}}\frac{1}{\left(2a\sqrt{\pi\tau_2}\right)^n}\,e^{-\frac{\bar\eta'^2+x_n^2}{4a^2\tau_2}}\cdot\partial_{\bar\eta}g_1(x'+\frac{c'u}{\varkappa}-\bar\eta',\bar\eta_n,\frac{u}{\varkappa},t-\tau_2)\Big|_{\bar\eta_n=0}d\bar\eta',\quad x_n>0, \quad (69)$$

$\Gamma(x,t)$ is a fundamental solution to the heat equation (54).

The functions $\Gamma_0(x,t)\equiv\Gamma(x,t),\ \Gamma_1(x,t),\ \Gamma_2(x,t)$ satisfy the estimate (22), where $j=0,1,2$.

For the constructed functions $g_j,\ G_j,\ \ j=1,2$, defined by formulas (64), (65), (63), and functions $g_3,\ G_3$, defined by formulas (68), (69), the following inequalities hold

$$\left|\partial_t^k\partial_x^m\partial_{x_n}g_j(x'+\frac{c'u}{\varkappa},x_n,\frac{u}{\varkappa},t)\right|\le C_{49}\frac{1}{t^{\frac{n+2k+|m|+1}{2}}}\,e^{-\frac{q_1^2x^2+q_2^2u^2}{t}},\quad j=1,2,3, \qquad (70)$$

$$\left|\partial_t^k\partial_x^m G_j(x,t)\right|\le C_{50}\varkappa\,\frac{1}{t^{\frac{n+2k+|m|}{2}}}\,e^{-\frac{q_1^2x^2}{t}}$$

$$+C_{51}\frac{1}{(q_1^2x^2+q_2^2t^2)^{\frac{n+2k+|m|-1}{2}}}\,e^{-\frac{q_1^2x^2+q_2^2t^2}{4t}},\quad j=1,2,3, \qquad (71)$$

where

$$q_1^2=\frac{\mu^2}{16\tilde a^2(c'^2+\mu_1^2+\mu_2^2)},\quad q_2^2=\frac{\mu^2}{16\tilde a^2\varkappa^2},$$

the constants C_{49} - C_{51} do not depend on $\varkappa$, $\mu=min(\mu_1,\mu_2)$, $\tilde a=max(a_1,a_2)$ for functions $g_j,\ G_j,\ \ j=1,2,\ \ \tilde a=max(a,a_1,a_2)$ for functions $g_3,\ G_3$.

Inequalities (70), (71) for functions $g_j,\ G_j,\ \ j=1,2$, were proved in [2]. The estimates for functions g, G are established in the same way.

For the norms of the function $\Psi(x',t)\in\overset{\circ}{C}{}_{x'\ \ t}^{1+\alpha,\frac{1+\alpha}{2}}(R_T)$ we introduce the following notation

$$\hat M_{k+1}=[\partial_{x_\nu}^k\Psi]_{t,R_T}^{(\frac{1+\alpha-k}{2})},\quad \hat M_3=[\Psi_{x_\nu}]_{x',R_T}^{(\alpha)}$$

and estimates

$$|\partial_{x_\nu}^k\Psi(x',t)|\le\hat M_{k+1}t^{\frac{1+\alpha-k}{2}}; \qquad (72)$$

$$|\partial^k_{x_\nu}\Psi(x',t)-\partial^k_{x_\nu}\Psi(x',t_1)|\le \hat{M}_{k+1}(t-t_1)^{\frac{1+\alpha-k}{2}},\quad t_1\le t; \tag{73}$$

$$|\Psi_{x_\nu}(x',t)-\Psi_{z_\nu}(z',t)|\le \hat{M}_3|x'-z'|^{\alpha},\quad k=0,1,\quad \nu=1,\dots,n-1. \tag{74}$$

Theorem 4 is proved as Theorem 2 and the estimate (60) is established as the estimate (20), using the inequalities (72)–(74) for the function Ψ and estimates (70), (71) for functions g_j, G_j, $j=1,2,3$.

Proof of Theorem 2. Remembering the change formulas (53) and applying the inequalities (51), (52) for functions $U_j(x,t)\in \overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(\bar{D}_{jT})$, $j=1,2$, $V(x,t)\in \overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(\bar{D}_{1T})$, the estimate (59) for the function $\Psi(x',t)$, due to Theorem 4 and the estimate (60), we obtain the estimate (8) and the proof of Theorem 2.

Corollary 2. *The problem (7) with* $\varkappa=0$ *has a unique solution* $v\in \overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(D_{1T})$, $u_j\in \overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(D_{jT})$, $j=1,2$, $r_2\in \overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\frac{\alpha}{2}}(R_T)$, *and it satisfies the estimate*

$$|v|^{(2+\alpha)}_{D_{1T}}+\sum_{j=1}^{2}|u_j|^{(2+\alpha)}_{D_{jT}}+|r_2|^{(2+\alpha)}_{R_T}\le C_{52}\Big(|f|^{(\alpha)}_{D_{1T}}+\sum_{j=1}^{2}|f_j|^{(\alpha)}_{D_{jT}}+\sum_{j=0}^{2}|\psi_j|^{(2+\alpha)}_{R_T}+|\psi_3|^{(1+\alpha)}_{R_T}\Big). \tag{75}$$

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Сарсекеева А.С. ПАРАБОЛАЛЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІ ҮШІН КІШІ ПАРАМЕТРІ БАР МОДЕЛЬДІ ЕРКІН ШЕКАРАЛЫ ЕСЕПТЕР

Шекаралық шартында кіші параметрі бар екі модельді есеп зерттелінеді. Олар екі еркін шекарасы бар параболалық теңдеулер жүйесі үшін сызықтық емес есепті шешуде пайда болады. Гельдер кеңістігінде осы есептер шешімдерінің кіші параметр бойынша бірқалыпты бағалаулары алынған.

Кілттік сөздер. Параболалық теңдеулер жүйелері, шекаралық шарттағы кіші параметр, айқын түрдегі шешім, бірқалыпты бағалаулар, Гельдер кеңістігі.

Сарсекеева А.С. МОДЕЛЬНЫЕ ЗАДАЧИ СО СВОБОДНЫМИ ГРАНИЦАМИ С МАЛЫМ ПАРАМЕТРОМ ДЛЯ СИСТЕМЫ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ

Изучены две модельные задачи с малым параметром в граничном условии. Они возникают при решении нелинейных задач с двумя свободными границами для системы параболических уравнений. В пространстве Гельдера установлены равномерные относительно малого параметра оценки решения этих задач.

Ключевые слова. Системы параболических уравнений, малый параметр в граничном условии, решение в явном виде, равномерные оценки, пространство Гельдера.

Kazakh Mathematical Journal ISSN 2413–6468

20:3 (2020) 48–56

On basis property of systems of eigenfunctions of a loaded second-order differential operator with antiperiodic boundary value conditions

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Communicated by: Abdizhakhan Sarsenbi

Received: 02.08.2020 ⋆ Final Version: 02.09.2020 ⋆ Accepted/Published Online: 25.09.2020

Abstract. We consider an eigenvalue problem for a second-order differential equation with a loaded term that contains value of derivative of the desired function at zero, with regular, but not strong regular boundary value conditions. We study the basis properties of systems of eigenfunctions of the loaded operator of multiple differentiation, with antiperiodic boundary value conditions. It is known that the system of eigenfunctions of an operator defined by formally self-adjoint differential expression with arbitrary self-adjoint boundary value conditions, providing a discrete spectrum, forms an orthonormal basis. Along with this, it is known, that in the case of non-self-adjoint ordinary differential operators, the basis properties of systems of root functions, in addition to boundary conditions, can also be influenced by coefficients of the differential operator. Moreover, the basis properties of root functions can change even at whatever pleasing small change in values of the coefficients. V.A. Il'in first noted this result in his work. A.S. Makin developed the ideas of V.A. Il'in in the case of non-self-adjoint perturbation of a self-adjoint periodic problem. In the work of A.S. Makin the operator changes due to perturbation of one of the boundary value conditions.

In this paper, the considered operator is a non-self-adjoint perturbation of the self-adjoint antiperiodic problem. In contrast to the work of A.S. Makin, here the perturbation occurs due to a change in the equation; and the boundary value conditions are antiperiodic. Characteristic determinant of the considered spectral problem is constructed, which is an entire analytical function. Theorems on stability and instability of the basis property of systems of eigenfunctions are proved.

Keywords. Characteristic determinant, Riesz basis, loaded operator, antiperiodic boundary value conditions, regular, not strong regular, perturbation.

1. Introduction

In the case of non-self-adjoint ordinary differential operators, in addition to the boundary value conditions, values of coefficients of the differential operator also influence to the basis

2010 Mathematics Subject Classification: Primary 34B05, 34L10; Secondary 34B09.

properties of systems of root functions. Knowing this fact, the basis properties of root functions can change with a small change in the values of coefficients. V.A. Il'in first noted this result in his work [1]. A.S. Makin developed the ideas of V.A. Il'in in the case of a non-self-adjoint perturbation of a self-adjoint periodic problem, where operator changes due to the perturbation of one of the boundary conditions [2].

In this paper, we consider another variant of perturbation of the self-adjoint problem, in particular, the spectral problem of the following form in the space $L_2(0,1)$:

$$Lu = -u'' + \overline{q(x)} \cdot u'(0) = \lambda u(x), 0 < x < 1, \tag{1}$$

$$U_1(u) = u(0) + u(1) = 0, U_2(u) = u'(0) + u'(1) = 0, \tag{2}$$

where $q(x) \in L_1(0,1)$.

Equations of the type (1) belong to the class of loaded differential equations, since the second term on the left-hand side of the equality (1) contains the value of derivative of the desired function at zero. The considered problem (1)–(2) is a non-self-adjoint perturbation of antiperiodic problem, and for a periodic problem they were studied in [3, 4]. In contrast to [2], here the perturbation occurs due to a change in the equation.

Questions of the basis property of root functions of loaded differential operators were studied in the works of I.S. Lomov [5, 6]. He managed to extend the method of spectral decompositions of V.A.Il'in [1] to the case of loaded differential operators. By another method, the basis properties of functional differential equations were studied in [7]. Earlier, other approaches to the study of the Samarsky-Ionkin type problems were published in our works [8,9].

2. Characteristic determinant of a spectral problem

Assuming $u'(0)$ as some independent constant, we see that the general solution of the equation (1) is representable in the form

$$u(x) = C_1 \cos\sqrt{\lambda}x + C_2 \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + u'(0) \int\limits_0^x \overline{q(\zeta)} \frac{\sin\sqrt{\lambda}(x-\zeta)}{\sqrt{\lambda}} d\zeta. \tag{3}$$

Therefore, first considering $x = 0$, and then satisfying (3) the boundary value condition (2), we get the system of the equations, which can be represented in the vector-matrix form as follows:

$$\begin{bmatrix} 0 & -1 & 1 \\ 1+\cos\sqrt{\lambda} & \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} & \frac{1}{\sqrt{\lambda}} \cdot \int\limits_0^1 \overline{q(\zeta)} \sin\sqrt{\lambda}(1-\zeta) d\zeta \\ -\sqrt{\lambda}\sin\sqrt{\lambda} & 1+\cos\lambda & -\int\limits_0^1 \overline{q(\zeta)} \cos\sqrt{\lambda}(1-\zeta) d\zeta \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ u'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4}$$

We directly obtain the characteristic determinant $\Delta(\lambda)$ from (4):

$$\Delta(\lambda) = 2(1+\cos\sqrt{\lambda}) - (1+\cos\sqrt{\lambda})\cdot\int\limits_0^1 \overline{q(\zeta)}\cos\sqrt{\lambda}(1-\zeta)d\zeta$$

$$+\sin\sqrt{\lambda}\cdot\int\limits_0^1 \overline{q(\zeta)}\sin\sqrt{\lambda}(1-\zeta)d\zeta \qquad (5)$$

In the case when $q(x)=0$, we get characteristic determinant of the antiperiodic spectral problem:

$$L_0u = -u''(x) = \lambda u(x), \qquad 0<x<1, \qquad (6)$$

$$u'(0)+u'(1)=0, \qquad u(0)+u(1)=0, \qquad (7)$$

$\Delta_0(\lambda) = 2(1+\cos\sqrt{\lambda})$. Numbers $\lambda_k^0 = ((2k-1)\pi)^2, k=1,2,3,\dots,$ are double eigenvalues, moreover

$$u_{k_0}^0 = \sqrt{2}\cos((2k-1)\pi x), \qquad u_{k_1}^0 = \sqrt{2}\sin((2k-1)\pi x)$$

are corresponding eigenfunctions, which form the complete orthonormal system in $L_2(0,1)$.

Function $q(x)$ can be represented in the form of expansion in a Fourier series by the trigonometric system $\{u_{k_0}^0,\, u_{k_1}^0\}$:

$$q(x) = \sum_{k=1}^{\infty}[a_k cos((2k-1)\pi x) + b_k sin((2k-1)\pi x)]. \qquad (8)$$

Then, after calculating the integrals from (5), we have

$$\Delta(\lambda) = \Delta_0(\lambda)\cdot A(\lambda),$$

where

$$A(\lambda) = \left[1+\sum_{k=1}^{\infty}\overline{b_k}\frac{(2k-1)\pi}{\lambda-((2k-1)\pi)^2}\right]. \qquad (9)$$

We formulate the result as the following theorem.

Theorem 1. *Characteristic determinant of the spectral problem for the loaded second-order differential equation with antiperiodic boundary conditions (1)–(2) can be represented in the form (9), where $\Delta_0(\lambda)$ is the characteristic determinant of the antiperiodic spectral problem of multiple differentiation (6)–(7); b_k are Fourier coefficients of the expansion (8) of the functions $q(x)$ by the trigonometric system of eigenfunctions of the spectral problem (6)–(7).*

Remark 1. *The function* $A(\lambda)$ *from (9) has poles of the second order at the points* $\lambda = \lambda_k^0$, *but the function* $\Delta_0(\lambda)$ *has zeros of the second order at the same points. Therefore, the function* $\Delta(\lambda)$, *represented by the formula (9), is an entire analytic function of the variable* λ.

3. The case of the basis property of root functions

The characteristic determinant (9) looks simpler when

$$q(x) = \sum_{k=1}^{N} [a_k cos((2k-1)\pi x) + b_k sin((2k-1)\pi x)].$$

That is, there exists a number N such that $a_k = b_k = 0$ for all $k > N$. In this case, formula (9) takes the form

$$\Delta_1(\lambda) = \Delta_0(\lambda) \left[1 + \sum_{k=1}^{N} \overline{b_k} \frac{(2k-1)\pi}{\lambda - ((2k-1)\pi)^2} \right]. \tag{10}$$

From this particular case of formula (9), we have the following corollary.

Corollary 1. *For any preassigned numbers (a complex* λ *and a positive integer* $\hat{m}$*) there always exists a function* $q(x)$ *such that* $\hat{\lambda}$ *will be an eigenvalue of problem (1) - (2) of multiplicity* $\hat{m}$.

From the analysis of formula (10) it is easy to see that $\Delta(\lambda_k^0) = 0$ for all $k > N$. That is, all eigenvalues λ_k^0, $k > N$, of the unperturbed antiperiodic problem are the eigenvalues of the spectral problem (1)–(2). It is also not difficult to show that the multiplicity of the eigenvalues λ_k^0, $k > N$, is also preserved. Moreover, from the condition of orthogonality of the trigonometric system it follows that in this case:

$$\int_0^1 \overline{q(x)} u_{kj}^0(x) dx = 0,\ j = \overline{0,1},\ k > N.$$

Thus, the eigenfunctions $u_{kj}^0(x)$ of the antiperiodic problem when $k > N$ satisfy the boundary value conditions (2) and, therefore, they are eigenfunctions of the spectral problem (1) - (2). Hence, in this case the system of eigenfunctions of (1)–(2) and the system of eigenfunctions of the antiperiodic problem (an orthonormal basis) differ from each other only in a finite number of the first members. Consequently, the system of eigenfunctions of (1)–(2) also forms the Riesz basis in $L_2(0,1)$. The set of functions $q(x)$, that can be represented as a finite series (8), is dense in $L_1(0,1)$. Thus, we have proved the following result.

Theorem 2. *Let $q(x) \in L_1(0,1)$. Then the system of eigenfunctions of the spectral problem (1)–(2) forms Riesz basis in the space $L_2(0,1)$, and is complete in $L_1(0,1)$.*

4. Instability of the basis property

Now we show the absence of the basis properties of eigenfunctions system of the spectral problem (1)–(2).

Theorem 3. *The set of functions $q(x) \in L_1(0,1)$, such that the system of eigenfunctions of the spectral problem (1)–(2) does not form even a normal basis in $L_2(0,1)$, is dense in $L_1(0,1)$.*

Proof. Let in (8) the coefficients $b_k \neq 0$ for all sufficiently large k. Then from (9) we note that $\lambda = \lambda_k^0$ is a simple eigenvalue of the spectral problem (1)–(2). By direct calculation we get that

$$u_k^1 = b_k \cdot \cos((2k-1)\pi)x - a_k \cdot \sin((2k-1)\pi)x$$

are eigenfunctions of (1)–(2), corresponding to $\lambda_k^0 = ((2k-1)\pi)^2$. Moreover, the eigenfunction of the dual problem [10]:

$$L^*(v) \equiv -v''(x) = \lambda v(x), \qquad 0 < x < 1,$$

$$V_1(v) \equiv v'(0) + v'(1) = 0, \qquad V_2(v) = v(0) + v(1) = \int_0^1 q(x)u(x)dx, \qquad q(x) \in L_1(0,1),$$

corresponding to the eigenvalue λ_k^0, is $v_k^1 = c_k \cdot \cos((2k-1)\pi)x$.

Since the eigenfunctions of the dual problems form a biorthogonal system, then we have the equality of the scalar product $(u_k^1, v_k^1) = 1$. Hence, it is easy to obtain $b_k\overline{c_k} = 2$. Therefore,

$$\left\|u_k^1\right\| \cdot \left\|v_k^1\right\| = \sqrt{1 + \left|\frac{a_k}{b_k}\right|^2}. \tag{11}$$

Denote by $\sigma_N(x)$ a partial sum of the Fourier series (8). It is obvious, that the set of functions, which can be represented as the infinite series

$$\overline{q(x)} = \sigma_N(x) + \sum_{k=N+1}^{\infty} \left[\overline{a_k}\cos((2k-1)\pi x) + \overline{b_k}\sin((2k-1)\pi x)\right],$$

where $\overline{a_k} = 2^{-k}$, $\overline{b_k} = \dfrac{2^{-k}}{k}$, $k > N$, is dense in $L_1(0,1)$. However, from (11) it follows that for the corresponding eigenfunctions $\overline{q(x)}$ and for the corresponding eigenfunctions systems of the direct and adjoint problems the following holds:

$$\lim_{k\to\infty} \left\|u_k^1\right\| \cdot \left\|v_k^1\right\| = \infty.$$

That is, the condition of uniform minimal property (see [11] and references in it) of the system does not hold, and therefore, it does not form even a basis in $L_2(0,1)$.

Since adjoint operators possess the Riesz basis property of the eigenfunction.

5. Conclusion

Results of this paper demonstrate stability of the basis property of eigenfunctions of a loaded operator of multiple differentiation with antiperiodic boundary conditions that are regular but not strongly regular [12–16].

6. Acknowledgments

Author thanks Professor B.E. Kanguzhin and Corresponding Member of the NAS of RK M.A. Sadybekov for their attention to the work. Author is supported by the funding from the Ministry of Education and Science of the Republic of Kazakhstan.

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Иманбаев Н.С. АНТИПЕРИОДТЫҚ ШЕТТІК ШАРТТАРМЕН БЕРІЛГЕН ЕКІНШІ РЕТТІ ДИФФЕРЕНЦИАЛДЫҚ ЖҮКТЕЛГЕН ОПЕРАТОРДЫҢ МЕНШІКТІ ФУНКЦИЯЛАР ЖҮЙЕСІНІҢ БАЗИСТІЛІГІ

Регулярлы, бірақ күшейтілмеген регулярлы шеттік шарттармен берілген, құрамында ізделінді функциядан алынған туындының нөл нүктесіндегі мәнімен қамтылған жүктелген қосылғышы бар екінші ретті дифференциалдық теңдеудің меншікті мәндерін зерттеуге арналған есеп қарастырылады. Антипериодтық шеттік шарттармен берілген екінші ретті дифференциалдық жүктелген оператордың меншікті функциялар жүйесінің базистілігі мәселесі зерттеледі. Дискретті спектрмен қамтамасыз ететін еркін түрдегі өзіне-өзі түйіндес шеттік шарттармен және өзіне-өзі түйіндес формальды дифференциалдық амалмен берілген оператордың меншікті функциялар жүйесінің ортонормаланған базис құратындығы белгілі жәй. Осымен қатар, өзіне-өзі түйіндес емес қарапайым дифференциалдық операторлар үшін де түбірлік функциялардың базистілігі не шеттік шарттардан бөлек дифференциалдық оператордың коэффициентерінің мәндері де әсер ететіндігі белгілі. Бұл жағдайда коэффициенттердің мәндері шамалы ғана өзгергенде түбірлік функциялардың базистілік қасиеттеріне әсер етеді. Мұндай нәтиже алғаш В.А. Ильиннің жұмысында аталған болатын. В.А. Ильиннің идеясы өзіне-өзі түйіндес периодтық есеп үшін өзіне-өзі түйіндес емес толқытылғандағы жағдайда А.С. Макиннің еңбегінде дамытылды. А.С. Макиннің жұмысында оператор шеттік шарттардың біреуін толқытқанда өзгерген болатын.

Бұл мақаладағы қарастырылып отырған оператор өзіне-өзі түйіндес антипериодты есептің өзіне-өзі түйіндес емес толқытуы болып табылады. Қарастырылып отырған жұмыстың А.С.Макиннің еңбегіндегі оператордан өзгешелігі, бұл жұмыста толқыту теңдеуге көшеді және шеттік шарттардың антипериодтылығында. Қарастырылып отырған спектралдық есептің характеристикалық анықтауышы құрылған және ол бүтін аналитикалық функция болып табылады. Меншікті функциялар жүйесінің базистілік қасиеттерінің орнықтылығы, орнықсыздығы туралы теоремалар дәлелденген.

Кілттік сөздер. характеристикалық анықтауыш, Рисс базистілігі, жүктелген оператор, антипериодтық шеттік шарттар, регулярлы, бірақ күшейтілген регулярлы емес, толқытылу.

Иманбаев Н.С. О БАЗИСНОСТИ СИСТЕМ СОБСТВЕННЫХ ФУНКЦИЙ НАГРУЖЕННОГО ДИФФЕРЕНЦИАЛЬНОГО ОПЕРАТОРА ВТОРОГО ПОРЯДКА С АНТИПЕРИОДИЧЕСКИМИ КРАЕВЫМИ УСЛОВИЯМИ

Рассматривается задача на собственные значения дифференциального уравнения второго порядка с нагруженным слагаемым, содержащим значение производной от искомой функции в точке нуль, с регулярными, но неусиленно регулярными краевыми условиями. Исследуется вопрос базисности систем собственных функций нагруженного оператора кратного дифференцирования с антипериодическими краевыми условиями. Известно, что система собственных функций оператора, заданного формально самосопряженным дифференциальным выражением, с произвольными самосопряженными краевыми условиями, обеспечивающими дискретный спектр, образует ортонормированный базис. Наряду с этим, известно, что в случае несамосопряженных обыкновенных дифференциальных операторов на базисность систем корневых функций, помимо краевых условий, могут влиять также значения коэффициентов дифференциального оператора. При этом базисные свойства корневых функций могут изменяться даже при сколь угодном малом изменении значений коэффициентов. Этот результат впервые отмечен в работе В.А.Ильина. Идеи В.А.Ильина были развиты А.С.Макиным на случай несамосопряженного возмущения самосопряженной периодической задачи. Оператор в работе А.С.Макина изменялся за счет возмущения одного из краевых условий.

В настоящей работе рассматриваемый оператор является несамосопряженным возмущением самосопряженной антипериодической задачи. В отличие от работы А.С.Макина здесь возмущение происходит за счет изменения уравнения и краевые условия являются антипериодическими. Построен характеристический определитель рассматриваемой спектральной задачи, который является целой аналитической функцией. Доказаны тео-

ремы об устойчивости и неустойчивости свойства базисности систем собственных функций.

Ключевые слова. характеристический определитель, базис Рисса, нагруженный оператор, антипериодические краевые условия, регулярные, неусиленно регулярные, возмущение.

Kazakh Mathematical Journal ISSN 2413–6468

20:3 (2020) 57–68

Approximation by means of Fourier series in Lebesgue spaces with variable exponent

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Communicated by: Daurenbek Bazarkhanov

Received: 11.03.2020 ⋆ Final Version: 05.09.2020 ⋆ Accepted/Published Online: 25.09.2020

Abstract. The approximation properties of means of trigonometric Fourier series in variable exponent Lebesgue spaces are studied.

Keywords. Lebesgue spaces with a variable exponent, approximation by trigonometric polynomials, Fourier series, means of Fourier series, modulus of continuity.

1. Introduction and The Main Results

Let $L_p(\mathbb{T})$, $1 \leq p < \infty$, be the Lebesgue space of all measurable 2π-periodic functions defined on $\mathbb{T} := [0, 2\pi]$ such that

$$\|f\|_p := \left(\int_{\mathbb{T}} |f(x)|^p \, dx \right) < \infty.$$

In the Lebesgue spaces $L_p(\mathbb{T})$, $1 \leq p < \infty$, we define integral modulus of continuity of f by

$$\omega_p(f, \delta) := \sup_{0 \leq |h| \leq \delta} \left\{ \frac{1}{2\pi} \int_{\mathbb{T}} |f(x+h) - f(x)|^p \, dx \right\}^{\frac{1}{p}}.$$

We define the Lipschitz class $Lip(\alpha, p)$ $(1 \leq p < \infty,\ 0 < \alpha \leq 1)$ as

$$Lip(\alpha, p) = \left\{ f \in L_p(\mathbb{T}) : \omega_p(f, \delta) = O(\delta^\alpha),\ \delta > 0 \right\}.$$

Let us denote by $\wp$ the class of Lebesgue measurable functions $p = p(x) : \mathbb{T} \longrightarrow [1, \infty)$ such that

$$1 < p_* := \underset{x \in \mathbb{T}}{ess \inf} p(x) \leq p^* := \underset{x \in \mathbb{T}}{ess \sup}\, p(x) < \infty. \tag{1}$$

2010 Mathematics Subject Classification: Primary 42A10, 41A25; Secondary 46E30.

The conjugate exponent of $p(x)$ is shown by $p'(x) := \frac{p(x)}{p(x)-1}$. For $p \in \wp$, we define a class $L^{p(\cdot)}(\mathbb{T})$ of 2π -periodic measurable functions $f : \mathbb{T} \to \mathbb{C}$ satisfying the condition

$$\int\limits_{\mathbb{T}} |f(x)|^{p(x)}\, dx < \infty.$$

This class $L^{p(\cdot)}(\mathbb{T})$ is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} := \|f\|_{L^{p(\cdot)}(\mathbb{T})} := \inf\{\ \lambda > 0 : \int\limits_{\mathbb{T}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1\}.$$

It is known that for $p(x) := p\ (0 < p \leq 1)$, the space $L^{p(x)}(\mathbb{T})$ coincides with the Lebesgue space $L^{p}(\mathbb{T})$. Note that detailed information about properties of the Lebesque spaces with variable exponent can be found in [1], [2], [3], [4], and [5].

We denote by $\mathcal{M}$ the set of all measurable 2π -periodic functions $p : \mathbb{R} \to [1, \infty)$ satisfying the conditions (1) and

$$|\ p(x) - p(y)\ | \leq \frac{c_1}{-\ln|x-y|},\ 0 < |x-y| \leq \frac{1}{2}.$$

Unfortunately the space $L^{p(\cdot)}(\mathbb{T})$ is not $p(\cdot)$-continuous and not transation invariant [3]. Note that from condition $f(x) \in L^{p(x)}(\mathbb{T})$ it does not follow $f(x+h) \in L^{p(x)}(\mathbb{T})$.

Let $p \in \mathcal{M}$, $f \in L^{p(\cdot)}(\mathbb{T})$. We define the shift operator T_h by

$$T_h(f)(x) := \frac{1}{h}\int\limits_{0}^{h} |f(x+t) - f(x)|\, dt$$

and the moduli of continuity of the function f by

$$\Omega_{p(\cdot)}(f, \delta) := \sup_{|h| \leq \delta} \|T_h(f)\|_{p(\cdot)},\ \delta > 0.$$

Note that the function $\Omega_{p(\cdot)}(f, \cdot)$ is continuous, nonnegative and satisfies

$$\lim_{\delta \to 0} \Omega_{p(\cdot)}(f, \delta) = 0,\ \Omega_{p(\cdot)}(f + f_1, \cdot) \leq \Omega_{p(\cdot)}(f, \cdot) + \Omega_{p(\cdot)}(f_1, \cdot)$$

for $f,\ f_1 \in L^{p(\cdot)}$.

Let $p \in \mathcal{M}$. For $0 < \alpha \leq 1$ we set

$$Lip\,(\alpha, p(\cdot)) = \left\{ f \in L^{p(\cdot)} : \Omega_{p(\cdot)}(f, \delta) = O(\delta^{\alpha}),\ \delta > 0 \right\}.$$

According to [6] in the Lebesgue spaces L^p the moduli of continuity $\omega_p(f,\cdot)$ and $\Omega_{p(\cdot)}(f,\cdot)$ are equivalent.

Let $\{d_n\}_0^\infty$ be a sequence of positive real numbers. If there exists a constant C, depending on the sequence $\{d_n\}_0^\infty$ only, such that, for all $n \geq m$ the inequality

$$d_n \leq C d_m \quad (p_n \geq c p_m)$$

satisfies, then sequence $\{d_n\}_0^\infty$ is called *almost monotone decreasing (increasing)*. In the paper such sequences will be denoted by $\{d_n\}_0^\infty \in$AMDS and $\{d_n\}_0^\infty \in$AMIS, respectively.

We also use the notation

$$\Delta l_n = l_n - l_{n+1},\ \Delta_m l(n,m) := l(n,m) - l(n,m+1).$$

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x,f),\ A_k(x,f) \quad : \quad = a_k(f)\cos kx + b_k(f)\sin kx \tag{2}$$

be the Fourier series of the function $f \in L_1(\mathbb{T})$, where $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function f. The *nth partial sums* of the series (2) is defined by

$$S_n(x,f) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k(x,f).$$

As in the [7] we suppose that $\mathbb{F}$ is an infinite subset of $\mathbb{N}$ and consider $\mathbb{F}$ as the range of strictly increasing sequence of positive integers, say $\mathbb{F} = \{\lambda(n)\}_1^\infty$. Following [8], the Cesáro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k,\ n = 1,2,...,$$

where $\{x_k\}$ is a sequence of a real or complex numbers. Therefore, the $C_\lambda-$ method yields a subsequence of the Cesáro method C_1, and hence it is regular for any λ. C_λ is obtained by deleting a set of rows from Cesáro matrix. We suppose that $\{d_n\}_0^\infty$ is a sequence of positive real numbers. We define the mean of the series (2), as

$$N_n^\lambda(f;x) \quad = \quad \frac{1}{D_{\lambda(n)}} \sum_{m=0}^{n} d_{\lambda(n)-m} s_m(f;x),$$

where $D_n := \sum_{m=0}^{n} d_m \neq 0 \ \ (n \geq 0)$, $\ d_{-1} = D_{-1} = 0$. Note that in the case $d_n = 1$, $n \geq 0$, $N(f;x)$ is equal to the mean

$$\sigma_n^\lambda(f;x) = \frac{1}{\lambda(n)+1}\sum_{m=0}^{\lambda(n)} S_m(f;x).$$

We consider trigonometric polynomial defined by

$$N_n^\lambda(f,x) = \frac{1}{D_{\lambda(n)}}\sum_{m=0}^{\lambda(n)} d_{\lambda(n)-m} S_m(f,x),$$

where $D_{\lambda(n)} := \sum\limits_{m=0}^{\lambda(n)} d_m \neq 0$, $d_{-1} = D_{-1} := 0$.

Note that in the paper [7] M.L. Mittal and M.V. Singh gave some conditions on the sequence $\{d_n\}_0^\infty$ and obtained results about approximation of the functions by $N_n^\lambda(f)$ in $Lip(\alpha,p)$, $0 < p \leq 1$. The problems of approximation theory in Lebesgue spaces with variable exponents have been investigated by several authors (see, for example, [9], [10], [6], [11], [12], [13], [14]).

In the present paper, the analogues of result [see [7], Theorem 5] was obtained for variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{T})$. Similar problems of the approximation of the functions in the different spaces have been studied in [6], [11]– [32].

Note that in the proof of the main results we use the methods as in the proofs of [19], [6] and [7].

Our main results are as follows.

Theorem 1. *If $p \in M$, $f \in Lip\,(\alpha, p(\cdot))$ and $\{d_n\}_0^\infty$ is a sequence of positive numbers and if one of the following conditions*

(A) $0 < \alpha < 1$, and $\{d_n\}_0^\infty \in AMDS$,

(B) $0 < \alpha < 1$, $\{d_n\}_0^\infty \in AMIS$ and $(\lambda(n)+1)\,d_{\lambda(n)} = O(D_{\lambda(n)})$ holds,

(C) $\alpha = 1$ and $\sum\limits_{k=1}^{\lambda(n)-1} k\,|\Delta d_k| = O\left(D_{\lambda(n)}\right)$,

(D) $\alpha = 1$, $\sum\limits_{k=0}^{\lambda(n)-1} |\Delta d_k| = O\left(\frac{D_{\lambda(n)}}{\lambda(n)}\right)$ and $(\lambda(n)+1)\,d_{\lambda(n)} = O(D_{\lambda(n)})$

is maintained, then for $n = 1, 2, ...$

$$\left\| f - N_n^\lambda(f) \right\|_{p(\cdot)} = O((\lambda(n))^{-\alpha})$$

holds.

In the proof of main results we need the following lemmas.

Lemma 1 [6]. *Let $p \in \mathcal{M}$ and $0 < \alpha \leq 1$. Then for every $f \in Lip\,(\alpha, p(\cdot))$ the estimate*

$$\|f - S_n(f)\|_{p(\cdot)} = O(n^{-\alpha}),\ n = 1, 2, ...,$$

holds.

Lemma 2 [6]. *Let $p \in \mathcal{M}$ and $f \in Lip\,(1, p(\cdot))$. Then for $n = 1, 2, ...$ the estimate*

$$\|S_n(f) - \sigma_n(f)\|_{p(\cdot)} = O(n^{-1})$$

holds.

Lemma 3 [7]. *Let $\{d_n\}_0^\infty \in AMDS$ or let $\{d_n\}_0^\infty \in AMIS$ and satisfy the relation $(\lambda(n)+1) = O(D_{\lambda(n)})$. Then, for $0 < \alpha < 1$, the estimate*

$$\sum_{m=1}^{\lambda(n)} m^{-\alpha} d_{\lambda(n)-m} = O((\lambda(n))^{-\alpha} D_{\lambda(n)})$$

holds.

2. Proofs of Theorems

Proof of Theorem 1. First of all we consider cases (A) and (B) together. The following relation holds:

$$N_n^\lambda(f;x) = \frac{1}{D_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} d_{\lambda(n)-m} \left\{ s_m(f;x) - f(x) \right\}.$$

Then by virtue of Lemmas 1 and 3 and condition $(\lambda(n)+1)\, d_{\lambda(n)} = O(D_{\lambda(n)})$ we reach

$$\begin{aligned}
\left\| f - N_n^\lambda(f) \right\|_{p(\cdot)} &\leq \frac{1}{D_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} d_{\lambda(n)-m} \left\| s_m(f) - f \right\|_{p(\cdot)} \\
&= \frac{1}{D_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} d_{\lambda(n)-m} \left\| s_m(f) - f \right\|_{p(\cdot)} \\
&\quad + \frac{d_{\lambda(n)}}{D_{\lambda(n)}} \left\| s_0(f) - f \right\|_{p(\cdot)} \\
&= \frac{1}{D_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} d_{\lambda(n)-m} O(m^{-\alpha}) + O\left(\frac{d_{\lambda(n)}}{D_{\lambda(n)}} \right) \\
&= O\left((\lambda(n))^{-\alpha} \right).
\end{aligned}$$

We suppose that (D) conditions hold. Using Abel's transformation, we have

$$N_n^\lambda(f;x) = \frac{1}{D_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} D_{\lambda(n)-m} u_m(f;x).$$

Then we find that

$$s_n^\lambda(f;x) - N_n^\lambda(f;x) = \frac{1}{D_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} (D_{\lambda(n)} - D_{\lambda(n)-m}) u_m(f;x). \tag{3}$$

If Abel's transformation is administered to (3), we obtain

$$\begin{aligned} s_n^\lambda(f;x) - N_n^\lambda(f;x) &= \frac{1}{D_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} \Delta_m (m^{-1}(D_{\lambda(n)} - D_{\lambda(n)-m})) \\ &\times \sum_{k=1}^{m} k u_k(f;x) + \frac{1}{(\lambda(n)+1)} \sum_{k=1}^{\lambda(n)} k u_k(f;x). \end{aligned}$$

Using (3), we have

$$\begin{aligned} \left\| s_n^\lambda(f;x) - N_n^\lambda(f;x) \right\|_{p(\cdot)} &\leq \frac{1}{D_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} \left| \Delta_m (m^{-1}(D_{\lambda(n)} - D_{\lambda(n)-m})) \right| \\ &\times \left\| \sum_{k=1}^{m} k u_k(f) \right\|_{p(\cdot)} + \frac{1}{(\lambda(n)+1)} \left\| \sum_{k=1}^{\lambda(n)} k u_k(f) \right\|_{p(\cdot)}. \end{aligned} \tag{4}$$

It is clear that

$$s_n(f;x) - \sigma_n(f;x) = \frac{1}{n+1} \sum_{k=1}^{n} k u_k(f;x). \tag{5}$$

Then from Lemma 2 and (5) we conclude that

$$\left\| \sum_{k=1}^{n} k u_k \right\|_{p(\cdot)} = (n+1) \left\| s_n(f) - \sigma_n(f) \right\|_{p(\cdot)} = O(1). \tag{6}$$

Consideration of (4) and (6) gives us

$$\begin{aligned} &\left\| s_n^\lambda(f) - N_n^\lambda(f) \right\|_{p(\cdot)} \\ = \; &O\left(\frac{1}{D_{\lambda(n)}}\right) \sum_{m=1}^{\lambda(n)} \left| \Delta_m (m^{-1}(D_{\lambda(n)} - D_{\lambda(n)-m})) \right| \\ &+ O((\lambda(n))^{-1}). \end{aligned} \tag{7}$$

The following relation holds:

$$\begin{aligned}\Delta_m\left(\frac{D_{\lambda(n)}-D_{\lambda(n)-m}}{m}\right) &= \frac{1}{m}\Delta_m\left(D_{\lambda(n)}-D_{\lambda(n)-m}\right)\\ &\quad+\frac{D_{\lambda(n)}-D_{\lambda(n)-m-1}}{m(m+1)}\\ &= \frac{D_{\lambda(n)-m-1}-D_{\lambda(n)-m}}{m}\\ &\quad+\frac{D_{\lambda(n)}-D_{\lambda(n)-m-1}}{m(m+1)}\\ &= \frac{D_{\lambda(n)}-D_{\lambda(n)-m-1}}{m(m+1)}-\frac{D_{\lambda(n)-m}}{m}\\ &= \frac{1}{m(m+1)}\left[D_{\lambda(n)}-D_{\lambda(n)-m-1}-(m+1)d_{\lambda(n)-m}\right],\end{aligned}$$

$$\begin{aligned}&\Delta_m\left(\frac{D_{\lambda(n)}-D_{\lambda(n)-m}}{m}\right)\\ =&\ \frac{1}{m(m+1)}\sum_{k=\lambda(n)-m}^{\lambda(n)}d_k-(m+1)d_{\lambda(n)-m}.\end{aligned} \tag{8}$$

We prove that the inequality

$$\begin{aligned}&\left|\sum_{k=\lambda(n)-m}^{\lambda(n)}d_k-(m+1)d_{\lambda(n)-m}.\right|\\ \leq&\ \sum_{k=1}^{m}k\left|d_{\lambda(n)-k+1}-d_{\lambda(n)-k}\right|\end{aligned} \tag{9}$$

holds. We suppose that $m=1$. Then

$$\begin{aligned}&\left|\sum_{k=\lambda(n)-m}^{\lambda(n)}d_k-2d_{\lambda(n)-1}\right|\\ =&\ \left|d_{\lambda(n)}-d_{\lambda(n)}-1\right|.\end{aligned}$$

That is the inequality (9) holds for $m=1$. We suppose that the inequality (9) is true for $m=j$. We prove the inequality (9) for $m=j+1$. For $m=j+1$ we find that

$$
\begin{aligned}
& \left| \sum_{k=\lambda(n)-(j+1)}^{\lambda(n)} d_k - (j+2) d_{\lambda(n)-(j+1)} \right| \\
= & \left| \sum_{k=\lambda(n)-j}^{\lambda(n)} d_k - (j+1) d_{\lambda(n)-(j+1)} \right| \\
= & \left| \sum_{k=\lambda(n)-j}^{\lambda(n)} d_k - (j+1) d_{\lambda(n)-j} + (j+1) d_{\lambda(n)-j} - (j+1) d_{\lambda(n)-(j+1)} \right| \\
\leq & \left| \sum_{k=\lambda(n)-j}^{\lambda(n)} d_k - (j+1) d_{\lambda(n)-j} \right| + \left| (j+1) d_{\lambda(n)-j} - (j+1) d_{\lambda(n)-(j+1)} \right| \\
\leq & \sum_{k=1}^{j} k \left| d_{\lambda(n)-k+1} - d_{\lambda(n)-k} \right| + (j+1) \left| d_{\lambda(n)-j} - d_{\lambda(n)-(j+1)} \right| \\
= & \sum_{k=1}^{j+1} k \left| d_{\lambda(n)-k+1} - d_{\lambda(n)-k} \right| .
\end{aligned}
$$

Consequently, the inequality (9) is true for any $1 \leq m \leq \lambda(n)$. Consideration of (8) and (9) gives us

$$
\begin{aligned}
& \sum_{m=1}^{\lambda(n)} \left| \Delta_m \left(\frac{D_{\lambda(n)} - D_{\lambda(n)-m}}{m} \right) \right| \\
\leq & \sum_{m=1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{m} k \left| d_{\lambda(n)-k+1} - d_{\lambda(n)-k} \right| \\
\leq & \sum_{k=1}^{\lambda(n)} k \left| d_{\lambda(n)-k+1} - d_{\lambda(n)-k} \right| \sum_{m=k}^{\infty} \frac{1}{m(m+1)} \\
= & \sum_{k=0}^{\lambda(n)-1} \left| \Delta d_k \right| = O\left(\frac{D_{\lambda(n)}}{\lambda(n)} \right).
\end{aligned}
$$

The last inequality and (7) imply that

$$
\left\| s_n^{\lambda}(f) - N_n^{\lambda}(f) \right\|_{p(\cdot)} = O\left((\lambda(n))^{-1} \right). \tag{10}
$$

Using (10) and Lemma 1, for $\alpha = 1$ we get

$$\left\| f - N_n^{\lambda}(f) \right\|_{p(\cdot)} = O\left((\lambda(n))^{-1} \right).$$

Next, we consider case (C). First of all we prove that if the condition

$$\sum_{k=1}^{\lambda(n)-1} k \left| \Delta p_k \right| = O\left(P_{\lambda(n)} \right)$$

satisfies, then the relation

$$\sum_{m=1}^{\lambda(n)} \Delta_m \left(\frac{D_{\lambda(n)} - D_{\lambda(n)-m}}{m} \right) = O\left(\frac{D_{\lambda(n)}}{\lambda(n)} \right) \tag{11}$$

holds.

We denote by r the integral part of $(\lambda(n)/2)$. Taking the relations (8) and (9) into account, we obtain

$$\begin{aligned}
&\sum_{m=1}^{\lambda(n)} \Delta_m \left(\frac{D_{\lambda(n)} - D_{\lambda(n)-m}}{m} \right) \\
\leq\ & \sum_{m=1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{m} k \left| \Delta_k d_{\lambda(n)-k} \right| \\
=\ & \sum_{m=1}^{r} \frac{1}{m(m+1)} \sum_{k=1}^{m} k \left| \Delta_k d_{\lambda(n)-k} \right| \\
& + \sum_{m=r+1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{m} k \left| \Delta_k d_{\lambda(n)-k} \right| \\
=\ & J_1 + J_2.
\end{aligned} \tag{12}$$

If we use Abel's transformation and the condition

$$\sum_{k=1}^{\lambda(n)-1} k \left| \Delta_k d_k \right| = O\left(D_{\lambda(n)} \right) \tag{13}$$

in the case (C), we find that

$$J_1 \leq \sum_{k=1}^{r} \left| \Delta_k d_{\lambda(n)-k} \right| \leq \sum_{j=r-2}^{\lambda(n)-1} \left| \Delta d_j \right| = O\left(\frac{D_{\lambda(n)}}{\lambda(n)} \right). \tag{14}$$

J_2 can be written as

$$\begin{aligned} J_2 &= \sum_{m=r+1}^{\lambda(n)} \frac{1}{m\,(m+1)} \left[\sum_{k=1}^{r} k\left|\Delta d_{\lambda(n)-k}\right| + \sum_{k=r}^{m} k\left|\Delta d_{\lambda(n)-k}\right| \right] \\ &: = J_{21} + J_{22}. \end{aligned}$$

Using the condition (13), we find that

$$J_{21} \leq \sum_{m=r}^{\lambda(n)} \frac{1}{(m+1)} \sum_{j=r-2}^{\lambda(n)-1} |\Delta d_j| = O\left(\frac{D_{\lambda(n)}}{\lambda(n)}\right), \tag{15}$$

$$\begin{aligned} J_{22} &\leq \sum_{m=r}^{\lambda(n)} \frac{1}{(m+1)} \sum_{k=r}^{m} \left|\Delta d_{\lambda(n)-k}\right| \\ &= O\left(\frac{1}{\lambda(n)}\right) \left[|\Delta d_0| + 2\,|\Delta d_1| + ... + (r+1)\,|\Delta d_{r+1}| \right] \\ &= O\left(\frac{D_{\lambda(n)}}{\lambda(n)}\right). \end{aligned} \tag{16}$$

Now combining (12), (14), (15), and (16), we obtain the relation (11). Consequently, using (11), (7) and Lemma 1 we reach

$$\left\| f - N_n^{\lambda}(f) \right\|_{p(\cdot)} = O((\lambda(n))^{-1}).$$

Thus, the proof of Theorem 1 is complete.

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Жафаров С.З. АЙНЫМАЛЫ КӨРСЕТКІШТІ ЛЕБЕГ КЕҢІСТІКТЕРІНДЕГІ ФУРЬЕ ҚАТАРЛАРЫ ОРТАШАЛАРЫ АРҚЫЛЫ АППРОКСИМАЦИЯ

Айнымалы көрсеткішті Лебег кеңістіктеріндегі орташа тригонометриялық Фурье қатарларының аппроксимациялық қасиеттері зерттеледі.

Кілттік сөздер.

Айнымалы көрсеткішті Лебег кеңістіктері, тригонометриялық полиномдармен аппроксимациялау, Фурье қатарлары, Фурье қатарларының орташалары, үзіліссіздік модулі.

Джафаров С.З. АППРОКСИМАЦИЯ СО СРЕДНИМИ РЯДОВ ФУРЬЕ В ПРОСТРАНСТВАХ ЛЕБЕГА С ПЕРЕМЕННЫМ ПОКАЗАТЕЛЕМ

Изучаются аппроксимационные свойства средних тригонометрических рядов Фурье в пространствах Лебега с переменным показателем.

Ключевые слова. Пространства Лебега с переменным показателем, аппроксимация тригонометрическими полиномами, ряды Фурье, средние рядов Фурье, модуль непрерывности.

Kazakh Mathematical Journal ISSN 2413–6468

20:3 (2020) 69-78

On the boundedness of maximal and Riesz-type potential operators in global Morrey-type spaces with variable exponent on bounded sets

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Communicated by: Mukhtarbai Otelbaev

Received: 09.08.2020 ⋆ Final Version: 19.09.2020 ⋆ Accepted/Published Online: 25.09.2020

Abstract. In this paper, we consider global Morrey-type spaces $GM_{p(\cdot),w(\cdot),\theta}(\Omega)$ with variable exponent $p(\cdot)$, where $\Omega \subset R^n$ is a bounded open set. Conditions for the boundedness of a maximal operator, fractional maximal operator, and Riesz type potential from the global Morrey-type space $GM_{p_1(\cdot),w_1(\cdot),\theta}(\Omega)$ to the global Morrey space $GM_{p_2(\cdot),w_2(\cdot),\theta}(\Omega)$ are obtained for different ratios between variable indicators $p_1(x), p_2(x)$ and between functions $w_1(x,r), w_2(x,r)$. Spanne-type and Adams-type theorems are proved.

Keywords. Morrey space, global Morrey-type spaces with variable exponent, Riesz potential, maximal function, fractional maximal operator, boundedness of an operator in spaces.

1 Introduction, definitions and auxiliary results

The classical Morrey space was introduced by Charles Morrey in 1938 [1] in connection with the study of solutions of quasilinear elliptic differential equations. In recent decades, the questions of the boundedness of various operators in general spaces of Morrey type have been actively investigated.

The questions of the boundedness of the maximal operator, the fractional maximal operator, and the Riesz potential in various function spaces have been well studied. For classical Lebesgue spaces, they are presented in detail in the monographs [2] - [4]. Then, such results were extended to general Morrey type spaces. The results of the boundedness of classical operators of function theory in general Morrey type spaces are presented in detail in the review articles by V.I. Burenkov [5], [6] and in the Adams's book [7].

2010 Mathematics Subject Classification: 35M99, 35R99, 53C35.

In this article, we will consider the questions connected with the boundedness of Hardy-Littlewood maximal operator, fractional maximal operator, potential type operator on global Morrey-type spaces with variable exponent $GM_{p(\cdot),w(\cdot),\theta}$.

Let us present the necessary definitions and notations.

Let $f \in L_{loc}(\Omega)$, where $\Omega \subset R^n$ is an open bounded set. We consider the following operators:

- Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{\tilde{B}(x,r)} |f(y)|dy,$$

where $B(x,r)$ is an open ball in R^n centered at the point $x \in R^n$ of radius r, and $|B(x,r)|$ is the volume of this ball, $\tilde{B}(x,r) = B(x,r) \cap \Omega$;

- the fractional maximal operator of variable order $\alpha(x)$

$$M^{\alpha(x)} f(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha(x)}{n}-1} \int_{\tilde{B}(x,r)} |f(y)|dy,$$

where $0 \le \alpha(x) < n$;

- Riesz potential type operator with variable order $\alpha(x)$

$$I^{\alpha(x)} f(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha(x)}} dy,$$

where $0 < \alpha(x) < n$.

When $\alpha(x) = \alpha = const$ these operators coincide, respectively, with the classical maximal fractional operator M^α and Riesz potential I^α.

For $\lambda \in R,\ 0 < p \le \infty$, Morrey space $M_p^\lambda(R^n)$ is defined as the set of all functions $f \in L_p^{loc}(R^n)$ with finite quasi-norm

$$||f||_{M_p^\lambda(R^n)} = \sup_{x \in R^n,\ r>0} r^{-\lambda} \left(\int\limits_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty.$$

If we replace here the power function $r^{-\lambda}$ by an arbitrary positive function $w(x,r)$ measurable on Ω, then we get the space $M_{p,w(\cdot)}(\Omega)$, called the generalized Morrey space. Such spaces were considered in [8] – [10].

Global Morrey-type spaces $GM_{p,w(\cdot),\theta}(\Omega)$, containing the generalized Morrey spaces, were considered in [11] – [14], in which sufficient and, in the case of some parameters, necessary conditions for the boundedness of classical operators of function theory in these spaces are obtained.

We will consider the global Morrey-type spaces $GM_{p(\cdot),w(\cdot),\theta}(\Omega)$ with variable exponent $p(\cdot)$, where $\Omega\subset R^n$ is a bounded open set.

Let $p(x)$ be a measurable function on the open bounded set $\Omega\subset R^n$ with values $(1,\infty)$. Suppose

$$1<p_-\le p(x)\le p_+<\infty, \tag{1}$$

where $p_-=p_-(\Omega)=\inf_{x\in\Omega}p(x)$, $p_+=p_+(\Omega)=\sup_{x\in\Omega}p(x)$.

We denote by $P^{\log}(\Omega)$ the class of functions defined on Ω satisfying the log-condition

$$|p(x)-p(y)|\le\frac{C}{-\ln|x-y|},\quad |x-y|\le\frac{1}{2},\quad (x,y)\in\Omega,$$

where $C=C(p)>0$ does not depend on x and y.

Denote by $L_{p(\cdot)}(\Omega)$ the Lebesque space with variable exponent [15] which is defined as the set of all measurable functions $f(x)$ on Ω such that

$$J_{p(\cdot)}(f)=\int_\Omega|f(x)|^{p(x)}\,dx<\infty,$$

where the norm is defined as follows

$$||f||_{p(\cdot)}=\inf\left\{\eta>0,\ J_{p(\cdot)}\left(\frac{f}{\eta}\right)\le1\right\}.$$

The Morrey spaces $M_{p(\cdot)}^{\lambda(\cdot)}(R^n)$ with variable exponents $p(\cdot)$ were introduced and studied in [16], [17].

Let $w(x,r)$ be positive measurable function on $\Omega\times[0,l]$, where $\Omega\subset R^n$, $l=diam\Omega$, $1\le\theta<\infty$. The generalized Morrey space $M_{p(\cdot),w(\cdot)}(\Omega)$, with variable exponent $p(\cdot)$ is defined by the norm

$$||f||_{M_{p(\cdot),w}(\Omega)}=\sup_{x\in\Omega,\ r>0}\frac{r^{-\frac{n}{p(x)}}}{w(x,r)}||f||_{L_{p(\cdot)}(B(x,r))}.$$

The generalized Morrey spaces $M_{p(\cdot),w(\cdot)}(\Omega)$ with variable exponent $p(\cdot)$, were studied in [18], [19] and boundedness conditions for the maximal operator, fractional maximal operator, and Riesz potential in this spaces were obtained.

We will define the global Morrey-type spaces $GM_{p(\cdot),w(\cdot),\theta}(\Omega)$ with variable exponent $p(\cdot)$. Throughout this work, we will assume that $\inf_{x\in\Omega,\ t>0}w_2(x,t)>0$.

Definition. *Let* $p(\cdot)\in P^{log}(\Omega)$, $w(x,r)$ *be a positive measurable function on* $\Omega\times[0,l]$, *where* $\Omega\subset R^n$, $l=diam\Omega$, $1\le\theta\le\infty$. *The global Morrey type spaces* $GM_{p(\cdot),w(\cdot),\theta}(\Omega)$ *with variable exponent* $p(\cdot)$ *is defined as the set of all functions* $f\in L_{p(\cdot)}^{log}(R^n\Omega)$ *with a finite norm*

$$\|f\|_{GM_{p(\cdot),w(\cdot),\theta}(\Omega)}=\sup_{x\in\Omega}\left\|w^{-1}(x,r)\,r^{-\frac{n}{p(x)}}||f||_{L_{p(\cdot)(B(x,r))}}\right\|_{L_{\theta(0,l)}}<\infty.$$

When $w(x,r) = r^{\frac{\lambda(x)-n}{p(x)}}$ the corresponding space is denoted by $GM^{\lambda(\cdot)}_{p(\cdot),\theta}(\Omega)$:

$$GM^{\lambda(\cdot)}_{p(\cdot),\theta}(\Omega) = GM_{p(\cdot),w(\cdot),\theta}(\Omega) w(x,r) = r^{\frac{\lambda(x)-n}{p(x)}}.$$

Note that

$$GM_{p(\cdot),w(\cdot),\theta}(\Omega) = M_{p(\cdot),w(\cdot)}(\Omega)$$

at $\theta = \infty$.

For $p(x) = const$ we have global Morrey type spaces $GM_{p,w,\theta}$ which were considered by V.I. Burenkov, V.S. Guliyev, A. Gogatishvili, R. Mustafaev and etc. [11] – [14].

In this paper, we obtain boundedness conditions for a maximal operator, fractional maximal operator, and Riesz potential type operator from one global Morrey-type space $GM_{p_1(\cdot),w_1,(\cdot)\theta}(\Omega)$ with a variable exponent $p(\cdot)$ to another space $GM_{p_2(\cdot),w_2(\cdot),\theta}(\Omega)$ at various ratios between variables exponents $p_1(x), p_2(x)$ and between functions $w_1(x,r), w_2(x,r)$.

We will denote by C and C_α positive constants that depend on the indicated parameters and, generally speaking, are different in different formulas.

Here are some auxiliary statements.

Theorem A [18]. *Let $\Omega \subset R^n$ be a bounded open set, $p(\cdot) \in P^{\log}(\Omega)$ and satisfy condition (1).*

Then

$$||Mf||_{L_{p(\cdot)}(\tilde{B}(x,t))} < Ct^{\frac{n}{p(x)}} \int_t^l r^{-\frac{n}{p(x)}-1} ||f||_{L_{p(\cdot)}(\tilde{B}(x,r))} dr, \quad 0 < t < \frac{l}{2}, \tag{2}$$

where C does not depend on f, x, t.

Theorem B [18]. *Let $\Omega \subset R^n$ be a bounded open set, $p(\cdot), \alpha(\cdot) \in P^{\log}(\Omega)$ satisfy condition (1) and*

$$\inf_{x\in\Omega} \alpha(x) > 0, \qquad \sup_{x\in\Omega} \alpha(x)p(x) < n, \tag{3}$$

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}. \tag{4}$$

Then

$$||I^{\alpha(\cdot)} f||_{L_{q(\cdot)}(\tilde{B}(x,t))} < Ct^{\frac{n}{q(x)}} \int_t^l r^{-\frac{n}{q(x)}-1} ||f||_{L_{p(\cdot)}(\tilde{B}(x,r))} dr, \quad 0 < t < \frac{l}{2},$$

where C does not depend on f, x, t.

Theorem C [18]. *Let $\Omega \subset R^n$ be a bounded open set, $p(x) \in P^{log}(\Omega)$ and satisfy condition (1), the function $\alpha(x)$ satisfy condition (3).*

Then

$$|I^{\alpha(\cdot)}f(x)| < Ct^{\alpha(x)}Mf(x) + C\int_t^l r^{\alpha(x)-\frac{n}{p(x)}-1}||f||_{L_p(B(x,r))}dr, \quad 0 < t \le \frac{l}{2}, \tag{5}$$

where the constant C *does not depend on* f, x, t.

2 The main results

In the following statements we give the conditions for boundedness of the maximal operator, the fractional maximal operator, and the potential Riesz potential type operator from space $GM_{p_1(\cdot),w_1(\cdot),\theta}(\Omega)$ to space $GM_{p_2(\cdot),w_2(\cdot),\theta}(\Omega)$.

Theorem 1. *Let* $\Omega \subset R^n$ *be a bounded open set,* $p(\cdot) \in P^{log}(\Omega)$ *and satisfy condition (1),* $1 < \theta < \infty$, $\frac{1}{\theta} + \frac{1}{\theta'} = 1$, *positive measurable functions* $w_1(x,r)$, $w_2(x,r)$ *satisfy the condition*

$$A^\theta = \sup_{x\in\Omega}\int_0^l \frac{1}{w_2^\theta(x,t)}\left\{\int_t^l \left(r^{-1}w_1(x,r)\right)^{\theta'} dr\right\}^{\frac{\theta}{\theta'}} dt < \infty. \tag{6}$$

Then the maximal operator M *is bounded from* $GM_{p(\cdot),w_1(\cdot),\theta}(\Omega)$ *to* $GM_{p(\cdot),w_2(\cdot),\theta}(\Omega)$.

Proof. According to Theorem A, we have

$$||Mf||_{GM_{p(\cdot),w_2(\cdot),\theta}(\Omega)} = \sup_{x\in\Omega}\left\|w_2^{-1}(x,t)r^{-\frac{n}{p(x)}}||Mf||_{L_{p(\cdot)}(B(x,r))}\right\|_{L_\theta(0,l)}$$

$$< C\sup_{x\in\Omega}\left\|w_2^{-1}(x,t)\int_t^l r^{-\frac{n}{p(x)}-1}||f||_{L_{p(\cdot)}(B(x,r))}\right\|_{L_\theta(0,l)}$$

$$= C\sup_{x\in\Omega}\left\{\int_0^l \left[w_2^{-1}(x,t)\int_t^l r^{-1}w_1(x,r)\frac{r^{-\frac{n}{p(x)}}}{w_1(x,r)}||f||_{L_{p(\cdot)}(B(x,r))}dr\right]^\theta dt\right\}^{\frac{1}{\theta}}$$

$$= \sup_{x\in\Omega}\left\{\int_0^l \frac{1}{w_2(x,t)}\left[\int_t^l \left(r^{-1}w_1(x,r)\right)^{\theta'} dr\right]^{\frac{\theta}{\theta'}}\left[\int_t^l\left(\frac{r^{-\frac{n}{p(x)}}}{w_1(x,r)}||f||_{L_{p(\cdot)}(B(x,r))}\right)^\theta dr\right]^{\frac{1}{\theta}} dt\right\}$$

$$\le ||f||_{GM_{p(\cdot),w_1,\theta}(\Omega)}\cdot A.$$

This implies the boundedness of the operator M from $GM_{p(\cdot),w_1(\cdot),\theta}(\Omega)$ to $GM_{p(\cdot),w_2(\cdot),\theta}(\Omega)$.
Theorem 1 is proved.

Consequence. *Let* $\Omega \subset R^n$ *be a bounded open set,*

$$\lambda(x) \ge 0,\ 1 < \theta < \infty, \qquad \lambda(x) - \mu(x) > \theta' p(x).$$

Then the operator M *is bounded from* $GM^{\lambda(\cdot)}_{p(\cdot),\theta}(\Omega)$ *to* $GM^{\mu(\cdot)}_{p(\cdot),\theta}(\Omega)$.

Theorem 2. *Let* $\Omega\subset R^n$ *be a bounded open set,* $p(\cdot)\in P^{log}(\Omega)$ *and satisfy condition (1),* $1<\theta<\infty$, $\frac{1}{\theta}+\frac{1}{\theta'}=1$, $\alpha(x)$, $q(x)$ *satisfy conditions (3) and (4), and the functions* $w_1(x,r)$, $w_2(x,r)$ *satisfy the condition*

$$B=\sup_{x\in\Omega}\left\|w_2^{-1}(x,t)\left\|r^{\alpha(x)-1}w_1(x,r)\right\|_{L_{\theta'}(t,l)}\right\|_{L_\theta(0,l)}<+\infty, \tag{7}$$

where C *does not depend on* x *and* t.

Then the operators $M^{\alpha(\cdot)}$, $I^{\alpha(\cdot)}$ *are bounded from* $GM_{p(\cdot),w_1(\cdot),\theta}(\Omega)$ *to* $GM_{q(\cdot),w_2(\cdot),\theta}(\Omega)$.

Proof. Let $f\in GM_{p(\cdot),w_1,\theta}(\Omega)$. Estimate the norm

$$||I^{\alpha(\cdot)}f||_{GM_{q(\cdot),w_2(\cdot),\theta}(\Omega)}=\sup_{x\in\Omega}\left\|w_2^{-1}(x,t)t^{-\frac{n}{q(x)}}||I^{\alpha(\cdot)}f||_{L_{q(\cdot)}(\tilde{B}(x,r))}\right\|_{L_\theta(0,l)}.$$

For this, it is enough to consider the values $t\in(0,\frac{l}{2})$ due to condition $\inf_{x\in\Omega,\ t>0}w_2(x,t)>0$.

By using Theorem B and Holder's inequality, we have

$$||I^{\alpha(\cdot)}||_{GM_{q(\cdot),w_2(\cdot),\theta}(\Omega)}\le C\sup_{x\in\Omega}\left\|w_2^{-1}(x,t)\int_t^l r^{-\frac{n}{q(x)}-1}||f||_{L_{q(\cdot)}(\tilde{B}(x,r))}dr\right\|_{L_\theta(0,l)}$$

$$=C\sup_{x\in\Omega}\left\|w_2^{-1}(x,t)\int_t^l r^{-n\left(\frac{1}{p(x)}-\frac{1}{q(x)}-1\right)}w_1(x,r)w_1^{-1}(x,r)r^{-\frac{n}{p(x)}}||f||_{L_{q(\cdot)}(\tilde{B}(x,r))}dr\right\|_{L_\theta(0,l)}$$

$$\le C\sup_{x\in\Omega}\left\|w_2^{-1}(x,t)\left\|r^{\alpha(x)-1}w_1(x,r)\right\|_{L_{\theta'}(t,l)}\cdot\left\|w_1^{-1}(x,r)r^{-\frac{n}{p(x)}}||f||_{L_{q(\cdot)}(\tilde{B}(x,r))}\right\|_{L_\theta(t,l)}\right\|_{L_\theta(0,l)}$$

$$\le C||f||_{GM_{p(\cdot),w_1(\cdot),\theta}(\Omega)}\sup_{x\in\Omega}\left\|w_2^{-1}(x,t)\left\|r^{\alpha(x)-1}w_1(x,r)\right\|_{L_{\theta'}(t,l)}\right\|_{L_\theta(0,l)}.$$

From this and condition (5) it follows that

$$||I^{\alpha(\cdot)}||_{GM_{q(\cdot),w_2(\cdot),\theta}(\Omega)}<C_1||f||_{GM_{p(\cdot),w_1(\cdot),\theta}(\Omega)},$$

that is, the operator $I^{\alpha(\cdot)}$ is bounded from $GM_{p(\cdot),w_1(\cdot),\theta}(\Omega)$ to $GM_{q(\cdot),w_2(\cdot),\theta}(\Omega)$.

The boundedness of fractional maximal operator $M^{\alpha(\cdot)}$ in these spaces follows from the following estimate

$$M^{\alpha(\cdot)}f(x)\le cI^{\alpha(\cdot)}|f|(x),\quad 0<\alpha(x)<n,$$

where c does not depend on f and x. This estimate is known for $\alpha(x)=\alpha=const$.

For variable $\alpha(x)$ we put

$$c=\sup_{x\in\Omega}\left(\frac{n}{|S^{n-1}|}\right)^{1-\frac{\alpha(x)}{n}}<\infty,$$

where S^{n-1} is a unit sphere R^{n-1}. It is clear that $|B(x,r)|=\frac{|S^{n-1}|}{n}r^n$.

Therefore

$$M^{\alpha(\cdot)}f(x)=\frac{1}{|B(x,r)|^{1-\frac{\alpha(x)}{n}}}\int_{(\tilde{B}(x,r))}|f(y)|dy$$

$$\leq\left(\frac{n}{|S^{n-1}|}\right)^{1-\frac{\alpha(x)}{n}}\int_{(\tilde{B}(x,r))}\frac{|f(y)|}{|x-y|^{n-\alpha(x)}}dy\leq cI^{\alpha(\cdot)}|f|(x).$$

Theorem 2 is proved.

Theorem 3. *Let* $\Omega\subset R^n$ *be a bounded open set,* $p(\cdot)\in P^{log}(\Omega)$ *and satisfy condition (1),* $\alpha(x)$ *satisfy condition (3),* $1<\theta<\infty$, $\frac{1}{\theta}+\frac{1}{\theta'}=1$ *and the functions* $w_1(x,t)$ *and* $w_2(x,t)$, *where* $w_2(x,t)=(w_1(x,t))^{\frac{p(\cdot)}{q(\cdot)}}$, *satisfy the conditions*

$$\sup_{x\in\Omega}\int_0^l\frac{1}{w_2^\theta(x,t)}\left\{\int_t^l\left(r^{-1}w_1(x,r)\right)^{\theta'}dr\right\}^{\frac{\theta}{\theta'}}dt<\infty, \tag{8}$$

$$\left\{\int_t^l\left(r^{\alpha(x)-1}w_1(x,r)\right)^{\theta'}dr\right\}^{\frac{\theta}{\theta'}}dt\leq Cr^{-\frac{\alpha(x)p(x)}{q(x)-p(x)}}. \tag{9}$$

Then the operators $M^{\alpha(\cdot)}$ *and* $I^{\alpha(\cdot)}$ *are bounded from* $GM_{p(\cdot),w_1(\cdot),\theta}(\Omega)$ *to* $GM_{q(\cdot),w_2(\cdot),\theta}(\Omega)$.

Proof. Let $f\in GM_{p(\cdot),w_1(\cdot)\theta}(\Omega)$. We will estimate the following norm

$$\|I^\alpha f\|_{GM_{p(\cdot),w_2(\cdot)(\Omega)},\theta}=\sup_{x\in\Omega}\left\|\frac{t^{-\frac{n}{q(x)}}}{w_2(x,t)}||I^{\alpha(\cdot)}f\chi_{(B(x,t))}||_{L^{q(\cdot)}(\Omega)}\right\|_{L_\theta(0,l)}.$$

According to Theorem C

$$\left|I^{\alpha(\cdot)}f(x)\right|<Ct^{\alpha(x)}Mf(x)+C\int_t^l r^{\alpha(x)-\frac{n}{p(x)}-1}||f||_{L_{p(\cdot)(B(x,r))}}dr=L_1(x)+L_2(x),$$

where $0<t<\frac{l}{2}$. By applying Holder's inequality and condition (8), we get

$$L_2(x)=\int_t^l r^{\alpha(x)-\frac{n}{p(x)}-1}||f||_{L_{p(\cdot)}(B(x,r))}dr=\int_t^l r^{\alpha(x)-1}w_1(x,r)\frac{r^{-\frac{n}{p(x)}}}{w_1(x,r)}||f||_{L_{p(\cdot)}(B(x,r))}dr$$

$$\leq \left\{ \int_t^l \left(\frac{r^{-\frac{n}{p(x)}}}{w_1(x,r)} ||f||_{L_{p(\cdot)}(B(x,r))} \right)^{\theta} dr \right\}^{\frac{1}{\theta}} \cdot \left\{ \int_t^l \left(r^{\alpha(x)-1} w_1(x,r) \right)^{\theta'} dr \right\}^{\frac{1}{\theta'}}$$

$$\leq ||f||_{GM_{p(\cdot),w_1(\cdot),\theta}(\Omega)} \cdot r^{-\frac{\alpha(x)p(x)}{p(x)-q(x)}}.$$

Therefore

$$\left| I^{\alpha(\cdot)} f(x) \right| < C r^{\alpha(x)} Mf(x) + ||f||_{GM_{p(\cdot),w_1(\cdot),\theta}(\Omega)} \cdot r^{-\frac{\alpha(x)p(x)}{p(x)-q(x)}}.$$

Here we choose r, so that

$$r = \left(\frac{||f||_{GM_{p(\cdot),w_1(\cdot),\theta}(\Omega)}}{Mf(x)} \right)^{-\frac{q(x)-p(x)}{\alpha(x)p(x)}}.$$

Then

$$\left| I^{\alpha(\cdot)} f(x) \right| < C(Mf(x))^{\frac{p(x)}{q(x)}} \cdot ||f||^{1-\frac{p(x)}{q(x)}}_{GM_{p(\cdot),w_1(\cdot),\theta}(\Omega)}.$$

Therefore

$$\left| I^{\alpha(\cdot)} f(x) \right|^{q(x)} < C(Mf(x))^{p(x)} \cdot ||f||^{q(x)-p(x)}_{GM_{p(\cdot),w_1(\cdot),\theta}(\Omega)}.$$

Then we have

$$||I^{\alpha(\cdot)} f||_{L_{q(\cdot)}(\tilde{B}(x,t))} \leq C ||Mf||_{L_{p(\cdot)}(\tilde{B}(x,t))}.$$

Hence,

$$\left\| \frac{r^{-\frac{n}{q(x)}}}{w_2(x,r)} ||I^{\alpha(\cdot)} f||_{L_{q(\cdot)}(\tilde{B}(x,t))} \right\|_{L_\theta(0,\infty)} \leq C \left\| \frac{r^{-\frac{n}{q(x)}}}{(w_1(x,r))^{\frac{p(x)}{q(x)}}} ||Mf||_{L_{p(\cdot)}(\tilde{B}(x,t))} \right\|_{L_\theta(0,\infty)}.$$

Hence, in view of the boundedness of the maximal operator by Theorem 1, we obtain

$$||I^{\alpha(\cdot)} f||_{GM_{q(\cdot),w_2(\cdot),\theta}(\Omega)} \leq C_1 ||Mf||_{GM_{p(\cdot),w_1(\cdot),\theta}}.$$

Hence, under the condition of the theorem, it follows that the operator I^α is bounded from $GM_{p(\cdot),w_1(\cdot),\theta}(\Omega)$ to $GM_{p(\cdot),w_2(\cdot),\theta}(\Omega)$.

Theorem 3 is proved.

Theorem 2 is a Spanne type result and Theorem 3 is an Adams type result. The similar theorems for the generalized Morrey spaces $M_{p(\cdot),w(\cdot)}(\Omega)$ with variable exponent $p(\cdot)$ were proved in [18].

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Адилханов А.Н., Бокаев Н.А., Онербек Ж.М. МАКСИМАЛДЫ ОПЕРАТОРЛАР МЕН РИСС ТИПТЕС ПОТЕНЦИАЛДЫҢ ШЕНЕЛГЕН ОБЛЫСТАРДА АНЫҚТАЛҒАН АЙНЫМАЛЫ КӨРСЕТКІШТІ МОРРИ ТИПТЕС ГЛОБАЛДЫ КЕҢІСТІКТЕРДЕ ШЕНЕЛГЕНДІГІ ТУРАЛЫ

Бұл жұмыста айнымалы $p(\cdot)$ көрсеткішті Морри типтес кеңістіктер $GM_{p(\cdot),w(\cdot),\theta}(\Omega)$ қарастырылған, мұндағы $\Omega \subset R^n$ – шенелген ашық жиын. Айнымалы $p_1(x), p_2(x)$ көрсеткіштердің және $w_1(x,r), w_2(x,r)$ функцияларының өзара сәйкес қатынастарында максималды оператордың, бөлшек-максималды оператордың және Рисс типтес потенциалдың Морри типтес $GM_{p_1(\cdot),w_1(\cdot),\theta}(\Omega)$ глобалды кеңістігінен Морри типтес $GM_{p_2(\cdot),w_2(\cdot),\theta}(\Omega)$ глобалды кеңістігіне шенелгендігінің шарттары алынған. Спейн және Адамс типтес теоремалар дәлелденген.

Кілттік сөздер. Морри кеңістігі, айнымалы көрсеткішті Морри типтес глобалды кеңістіктер, Рисс потенциалы, максималды функция, бөлшек-максималды оператор, оператордың кеңістіктерде шенелгендігі.

Адилханов А.Н., Бокаев Н.А., Онербек Ж.М. ОБ ОГРАНИЧЕННОСТИ МАКСИМАЛЬНЫХ ОПЕРАТОРОВ И ПОТЕНЦИАЛА ТИПА РИССА В ГЛОБАЛЬНЫХ ПРОСТРАНСТВАХ ТИПА МОРРИ С ПЕРЕМЕННЫМ ПОКАЗАТЕЛЕМ НА ОГРАНИЧЕННЫХ ОБЛАСТЯХ

В данной работе рассмотрены глобальные пространства типа Морри $GM_{p(\cdot),w(\cdot),\theta}(\Omega)$ с переменным показателем $p(\cdot)$, где $\Omega \subset R^n$ – ограниченное открытое множество. Получены условия ограниченности максимального оператора, дробно-максимального оператора и потенциала типа Рисса из глобального пространства типа Морри $GM_{p_1(\cdot),w_1(\cdot),\theta}(\Omega)$ в глобальное пространство типа Морри $GM_{p_2(\cdot),w_2(\cdot),\theta}(\Omega)$ при соответствующих соотношениях между переменными показателями $p_1(x), p_2(x)$ и между функциями $w_1(x,r), w_2(x,r)$. Доказаны теоремы типа Спейна и Адамса.

Ключевые слова. пространство Морри, глобальные пространства типа Морри с переменным показателем, потенциал Рисса, максимальная функция, дробно-максимальный оператор, ограниченность оператора в пространствах.

Kazakh Mathematical Journal ISSN 2413–6468

20:3 (2020) 79–91

Problem for the integro-differential equation with weakly singular kernels

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Communicated by: Abdizhahan Sarsenbi

Received: 02.09.2020 ⋆ Accepted/Published Online: 17.09.2020 ⋆ Final Version: 26.09.2020

Abstract. In the present paper, we study a linear boundary value problem for a system of integro-differential equations with weakly singular kernels. Conditions for the solvability of the considered problem are established using a method based on splitting the interval and introducing additional parameters. Necessary and sufficient conditions for the solvability of the two-point problem for the integro-differential equations with weakly singular kernels are received.

Keywords. Linear boundary value problem, integro-differential equations, kernel with weakly singularity, parameterization method, solvability.

1 Introduction

We consider a linear two-point boundary value problem for the system of Fredholm integro-differential equations with weakly singular kernels on $[0,T]$:

$$\frac{dx}{dt} = A(t)x + \int_0^T K(t,s)x(s)ds + f(t), \qquad x \in R^n, \tag{1}$$

$$Bx(0) + Cx(T) = d, \qquad d \in R^n, \tag{2}$$

where $(n\times n)$ matrix $A(t)$ and n vector $f(t)$ are continuous on $[0,T]$, $(n\times n)$ matrix $K(t,s)$ has the form $K(t,s) = \frac{1}{|t-s|^{\alpha}}H(t,s)$, and $(n\times n)$ matrix $H(t,s)$ is continuous on $[0,T]\times[0,T]$, $0<\alpha<1$, $\|x\| = \max\limits_{i=\overline{1,n}}|x_i|$.

2010 Mathematics Subject Classification: 34K05; 34K10; 45J05; 45L05.

Funding: This work was done with support of grant AP 08855726 of the Ministry of Education and Science of the Republic of Kazakhstan.

A solution to problem (1), (2) is a continuous on $[0, T]$ and continuously differentiable on $(0, T)$ vector function $x(t)$ satisfying the Fredholm integro-differential equation (1) and boundary condition (2).

Integro-differential equations and different problems for them are used as mathematical models of various physical processes [1]. Linear boundary value problems for the Fredholm integro-differential equations are investigated in [2-7] for the cases of smooth kernels. Two-point boundary value problems for linear Fredholm integro-differential equations with weakly singular or other non-smooth kernels are studied by the Galerkin and collocation methods in [8-18].

In present paper, based on the parameterization method [19] we investigate the existence and uniqueness of solution to problem (1), (2). Dividing the interval $[0, T]$ into N parts, we introduce additional parameters. While applying the method to problem (1), (2), an intermediate problem, special Cauchy problem for the system of integro-differential equations with parameters, arises. Note, the problem is always uniquely solvable for sufficiently small partition step. This property of the intermediate problem in [2] allowed us to establish necessary and sufficient conditions for solvability and unique solvability of problem (1), (2) in the case of smooth kernels. In [3-6], the smallness of interval's partition step is also required for solving the linear boundary value problems for Fredholm integro-differential equations. In [7], the arbitrary partitions of the interval are considered.

Hereby we expand the results of paper [2] to a linear two-point boundary value problem for a system of Fredholm integro-differential equations with weakly singular kernels. Algorithms of parameterization method are based on the smallness of interval's partition and solving the system of algebraic equations with respect to the additional parameters introduced. If a fundamental matrix of differential part of Eq.(1) is known and the erasing definite integrals can be evaluated, then the algorithm gives a solution to the linear two-point boundary value problem for the Fredholm integro-differential equations in the explicit form.

Let $C([0, T], R^n)$ denote the space of continuous functions $x : [0, T] \to R^n$ with the norm $||x||_1 = \max\limits_{t \in [0,T]} ||x(t)||$.

2 Scheme of the method

Given a step $h > 0 : Nh = T$ we introduce the partition $[0, T) = \bigcup\limits_{r=1}^{N} [(r-1)h, rh)$ and restrict $x(t)$ to the rth interval $[(r-1)h, rh)$, which is denoted by $x_r(t)$, i.e., $x_r(t) = x(t)$ for $t \in [(r-1)h, rh)$.

Problem (1), (2) is then reduced to the equivalent multi-point boundary value problem

$$\frac{dx_r}{dt} = A(t)x_r + \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K(t,s)x_j(s)ds + f(t), \qquad t \in [(r-1)h, rh), \qquad r = \overline{1, N}, \quad (5)$$

$$Bx_1(0) + C \lim_{t \to T-0} x_N(t) = d, \tag{6}$$

$$\lim_{t \to ph-0} x_p(t) = x_{p+1}(ph), \qquad p = \overline{1, N-1}, \tag{7}$$

where (7) are conditions for matching the solution at the interior points of the partition of $[0, T)$.

Let $C([0,T], h, R^{nN})$ denote the space of systems of functions $x[t] = (x_1(t), x_2(t), ..., x_N(t))$, where $x_r : [(r-1)h, rh) \to R^n$ are continuous and have finite left limits $\lim\limits_{t \to rh-0} x_r(t)$ for all $r = \overline{1,N}$, with the norm $||x[\cdot]||_2 = \max\limits_{r=\overline{1,N}} \sup\limits_{t \in [(r-1)h, rh)} ||x_r(t)||$.

Obviously, $C([0,T], h, R^{nN})$ is a complete space.

Introducing the parameters $\lambda_r \hat{=} x_r[(r-1)h]$ and making the substitution $u_r(t) = x_r(t) - \lambda_r$ at every rth interval, we obtain the parametric boundary value problem

$$\frac{du_r}{dt} = A(t)(u_r + \lambda_r) + \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K(t,s)[u_j(s) + \lambda_j]ds + f(t), \qquad t \in [(r-1)h, rh), \tag{8}$$

$$u_r[(r-1)h] = 0, \qquad r = \overline{1,N}, \tag{9}$$

$$B\lambda_1 + C\lambda_N + C \lim_{t \to T-0} u_N(t) = d, \tag{10}$$

$$\lambda_p + \lim_{t \to ph-0} u_p(t) - \lambda_{p+1} = 0, \qquad p = \overline{1, N-1}. \tag{11}$$

The solution of problem (8)–(11) is a pair $(\lambda^*, u^*[t])$ with elements $\lambda^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_N^*) \in R^{nN}$ and $u^*[t] = (u_1^*(t), u_2^*(t), ..., u_N^*(t)) \in C([0,T], h, R^{nN})$. If $(\lambda^*, u^*[t])$ is a solution to problem (8)–(11), then $x^*(t)$, defined by the relations: $x^*(t) = \lambda_r^* + u_r^*(t)$ for $t \in [(r-1)h, rh)$ and $r = \overline{1,N}$, and $x^*(T) = \lambda_N^* + \lim\limits_{t \to T-0} u_N^*(t)$, solves problem (1), (2). Conversely, if $\widetilde{x}(t)$ is a solution to problem (1), (2), then the pair $(\widetilde{\lambda}, \widetilde{u}[t])$ with elements $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, ..., \widetilde{\lambda}_N) \in R^{nN}$, and $\widetilde{u}[t] = (\widetilde{u}_1(t), \widetilde{u}_2(t), ..., \widetilde{u}_N(t)) \in C([0,T], h, R^{nN})$, where $\widetilde{\lambda}_r = \widetilde{x}[(r-1)h]$, $\widetilde{u}_r(t)$ is the restriction of $\widetilde{x}(t) - \widetilde{x}[(r-1)h]$ to $[(r-1)h, rh)$ for $r = \overline{1,N}$, solves problem (8)–(11). By introducing additional parameters, we obtain initial data (9) for the unknown functions $u_r(t)$, $r = \overline{1,N}$. For fixed parameter values $\lambda \in R^{nN}$ the system of functions $u[t]$ is determined from problem (8), (9), which is a special Cauchy problem for systems of integro-differential equations. Problem (8), (9) is equivalent to the system of integral equations

$$u_r(t) = X(t) \int\limits_{(r-1)h}^{t} X^{-1}(\tau_1) A(\tau_1) d\tau_1 \lambda_r$$

$$+X(t)\int\limits_{(r-1)h}^{t}X^{-1}(\tau_1)\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh}K(\tau_1,s)[u_j(s)+\lambda_j]dsd\tau_1$$

$$+X(t)\int\limits_{(r-1)h}^{t}X^{-1}(\tau_1)f(\tau_1)d\tau_1,\qquad t\in[(r-1)h,rh),\quad r=\overline{1,N}. \tag{12}$$

Solving (12), we find a representation of $u_r(t)$ in terms of $\lambda\in R^{nN}$, $r=\overline{1,N}$, and $f(t)$. Substituting them into (10) and (11) yields a system of equations for finding the unknown parameters. Thus, if the parametrization method is applied to problem (1), (2), we also have to solve an intermediate problem, namely, the special Cauchy problem (8), (9) or the equivalent system of integral equations (12). However, in contrast to the above methods, the partition step $h>0: Nh=T$ can always be chosen so that problem (8), (9) is uniquely solvable.

Consider $h_0>0$ satisfying the inequality

$$\sigma(h_0)\equiv\beta T\frac{1}{1-\alpha}h_0^{1-\alpha}e^{\alpha_0 h_0}<1, \tag{13}$$

where $\alpha_0=\max\limits_{t\in[0,T]}||A(t)||$ and $\beta=\max\limits_{(t,s)\in[0,T]\times[0,T]}||H(t,s)||$. Let us show that, for any $h\in(0,h_0]: Nh=T$ system (12) is uniquely solvable.

We use the equality

$$X(t)\int\limits_{a}^{t}X^{-1}(\tau_1)F(\tau_1)d\tau_1=\int\limits_{a}^{t}F(\tau_1)d\tau_1+\int\limits_{a}^{t}A(\tau_1)\int\limits_{a}^{\tau_1}F(\tau_2)d\tau_2d\tau_1$$

$$+\int\limits_{a}^{t}A(\tau_1)\int\limits_{a}^{\tau_1}A(\tau_2)\int\limits_{a}^{\tau_2}F(\tau_3)d\tau_3d\tau_2d\tau_1+...,\qquad a,t\in[0,T], \tag{14}$$

which holds for any function $F(t)$ that is continuous on $[0,T]$. Indeed, the functional series on the right-hand side of (14) converges uniformly on $[0,T]$ and, like the left-hand side of (14), solves the Cauchy problem

$$\frac{dx}{dt}=A(t)x+F(t),\qquad x(a)=0,\quad t\in[0,T]. \tag{15}$$

Since problem (15) is uniquely solvable, we have (14). By using (14), we obtain the estimates

$$\Big|\Big|X(t)\int\limits_{(r-1)h}^{t}X^{-1}(\tau_1)\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh}K(\tau_1,s)u_j(s)dsd\tau_1\Big|\Big|$$

$$= \Big|\Big|X(t)\int\limits_{(r-1)h}^{t}X^{-1}(\tau_1)\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh}\frac{1}{|\tau_1-s|^{\alpha}}H(\tau_1,s)u_j(s)dsd\tau_1\Big|\Big|$$

$$\leq \beta e^{\alpha_0 h}\int\limits_{(r-1)h}^{rh}\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh}\frac{1}{|\tau_1-s|^{\alpha}}dsd\tau_1\cdot||u[\cdot]||_2$$

$$\leq \beta e^{\alpha_0 h}\int\limits_{(r-1)h}^{rh}\sum_{j=1}^{N}\Big\{\int\limits_{(j-1)h}^{\tau_1}\frac{1}{(\tau_1-s)^{\alpha}}ds+\int\limits_{\tau_1}^{jh}\frac{1}{(s-\tau_1)^{\alpha}}ds\Big\}d\tau_1\cdot||u[\cdot]||_2$$

$$= \beta e^{\alpha_0 h}\int\limits_{(r-1)h}^{t}\sum_{j=1}^{N}\frac{1}{1-\alpha}\Big\{(jh-\tau_1)^{1-\alpha}-(\tau_1-(j-1)h)^{1-\alpha}\Big\}d\tau_1\cdot||u[\cdot]||_2$$

$$\leq \beta e^{\alpha_0 h}\int\limits_{(r-1)h}^{t}\sum_{j=1}^{N}\frac{1}{1-\alpha}(\tau_1-jh-(\tau_1-(j-1)h))^{1-\alpha}d\tau_1\cdot||u[\cdot]||_2$$

$$= \beta e^{\alpha_0 h}\int\limits_{(r-1)h}^{t}\sum_{j=1}^{N}\frac{1}{1-\alpha}h^{1-\alpha}d\tau_1\cdot||u[\cdot]||_2 \leq \beta e^{\alpha_0 h}T\frac{1}{1-\alpha}h^{1-\alpha}\cdot||u[\cdot]||_2$$

$$= \sigma(h_0)\cdot||u[\cdot]||_2, \qquad t\in[(r-1)h,rh), \quad r=\overline{1,N}. \tag{16}$$

Using (16) and the inequality $\sigma(h_0)<1$ and applying the contraction mapping principle, we prove the unique solvability of systems (12) for any $h\in(0,h_0]:Nh=T$.

Setting $t=\tau$ in (12) and multiplying both sides by $K(t,\tau)$, we integrate the result with respect to τ on the interval $[(r-1)h,rh]$ and sum up the left- and right-hand sides over r to obtain

$$\sum_{r=1}^{N}\int\limits_{(r-1)h}^{rh}K(t,\tau)u_r(\tau)d\tau=\sum_{r=1}^{N}\int\limits_{(r-1)h}^{rh}K(t,\tau)X(\tau)\int\limits_{(r-1)h}^{\tau}X^{-1}(\tau_1)$$

$$\times\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh}K(\tau_1,s)u_j(s)dsd\tau_1d\tau+\sum_{r=1}^{N}\int\limits_{(r-1)h}^{rh}K(t,\tau)X(\tau)\int\limits_{(r-1)h}^{\tau}X^{-1}(\tau_1)\Big\{A(\tau_1)\lambda_r$$

$$+\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh}K(\tau_1,s)ds\lambda_j+f(\tau_1)\Big\}d\tau_1d\tau, \qquad t\in[0,T]. \tag{17}$$

After introducing the notation

$$\Phi_h(t) = \sum_{j=1}^{N} \int_{(j-1)h}^{jh} K(t,s)u_j(s)ds,$$

$$M_r(h,t) = \int_{(r-1)h}^{rh} K(t,\tau)X(\tau) \int_{(r-1)h}^{\tau} X^{-1}(\tau_1)A(\tau_1)d\tau_1 d\tau$$

$$+\sum_{j=1}^{N} \int_{(j-1)h}^{jh} K(t,\tau)X(\tau) \int_{(j-1)h}^{\tau} X^{-1}(\tau_1) \int_{(r-1)h}^{rh} K(\tau_1,s)dsd\tau_1 d\tau,$$

$$F(h,t) = \sum_{j=1}^{N} \int_{(j-1)h}^{jh} K(t,\tau)X(\tau) \int_{(j-1)h}^{\tau} X^{-1}(\tau_1)f(\tau_1)d\tau_1 d\tau,$$

Eq. (17) can be written as

$$\Phi_h(t) = \sum_{j=1}^{N} \int_{(j-1)h}^{jh} K(t,\tau)X(\tau) \int_{(j-1)h}^{\tau} X^{-1}(\tau_1)\Phi_h(\tau_1)d\tau_1 d\tau$$

$$+\sum_{r=1}^{N} M_r(h,t)\lambda_r + F(h,t), \qquad t \in [0,T]. \tag{18}$$

Once again using estimates (16), we conclude that Eq. (18) is uniquely solvable for $h \in (0,h_0] : Nh = T$.

Defining sequences of matrices and vectors depending on $t \in [0,T]$ by the relations

$$M_r^{(0)}(h,t) = M_r(h,t), \quad M_r^{(k)}(h,t) = \sum_{j=1}^{N} \int_{(j-1)h}^{jh} K(t,\tau)X(\tau) \int_{(r-1)h}^{\tau} X^{-1}(\tau_1)M_r^{(k-1)}(h,\tau_1)d\tau_1 d\tau,$$

$$F^{(0)}(h,t) = F(h,t), \qquad F^{(k)}(h,t) = \sum_{j=1}^{N} \int_{(j-1)h}^{jh} K(t,\tau)X(\tau) \int_{(j-1)h}^{\tau} X^{-1}(\tau_1)F^{(k-1)}(h,\tau_1)d\tau_1 d\tau,$$

$k = 1, 2, ...$, and applying the method of successive approximations, we find the solution of Eq. (18) in the form

$$\Phi_h(t) = \sum_{r=1}^{N} D_r(h,t)\lambda_r + F_h(t), \qquad t \in [0,T], \tag{19}$$

where $D_r(h,t) = \sum\limits_{k=0}^{\infty} M_r^{(k)}(h,t)$, $F_h(t) = \sum\limits_{k=0}^{\infty} F^{(k)}(h,t)$.

Note that for $h \in (0, h_0] : Nh = T$ the functional series converge uniformly and $D_r(h,t)$, $r = \overline{1,N}$, and $F_h(t)$ are continuous on $[0,T]$. Substituting (19) into the right-hand side of (12), we express $u_r(t)$ in terms of λ_r and $f(t)$:

$$u_r(t) = X(t) \int\limits_{(r-1)h}^{t} X^{-1}(\tau)A(\tau)d\tau\lambda_r$$

$$+ \sum_{j=1}^{N} X(t) \int\limits_{(r-1)h}^{t} X^{-1}(\tau)\Big[D_j(h,\tau) + \int\limits_{(j-1)h}^{jh} K(\tau,s)ds\Big]d\tau\lambda_j$$

$$+X(t) \int\limits_{(r-1)h}^{t} X^{-1}(\tau)[f(\tau) + F_h(\tau)]d\tau, \qquad t \in [(r-1)h, rh), \quad r = \overline{1,N}. \tag{20}$$

Finding $\lim\limits_{t\to T-0} u_N(t)$ and $\lim\limits_{t\to ph-0} u_p(t)$, $p = \overline{1, N-1}$, substituting them into conditions (10) and (11), and multiplying both sides of (10) by $h > 0$, we obtain a linear system of equations for λ_r, $r = \overline{1,N}$:

$$h\Big\{B + CX(T) \int\limits_{T-h}^{T} X^{-1}(\tau)\Big[D_1(h,\tau) + \int\limits_{0}^{h} K(\tau,s)ds\Big]d\tau\Big\}\lambda_1$$

$$+hC \sum_{j=2}^{N-1} X(T) \int\limits_{T-h}^{T} X^{-1}(\tau)\Big[D_j(h,\tau) + \int\limits_{(j-1)h}^{jh} K(\tau,s)ds\Big]d\tau\lambda_j$$

$$+hC\Big\{I + X(T) \int\limits_{T-h}^{T} X^{-1}(\tau)\Big[A(\tau) + D_N(h,\tau) + \int\limits_{T-h}^{T} K(\tau,s)ds\Big]d\tau\Big\}\lambda_N$$

$$= hd - hCX(T)\int\limits_{T-h}^{T} X^{-1}(\tau)[f(\tau)+F_h(\tau)]d\tau, \tag{21}$$

$$\sum_{j=1}^{p-1} X(ph)\int\limits_{(p-1)h}^{ph} X^{-1}(\tau)\Big[D_j(h,\tau)+\int\limits_{(j-1)h}^{jh} K(\tau,s)ds\Big]d\tau\lambda_j + \Big\{I + X(ph)\int\limits_{(p-1)h}^{ph} X^{-1}(\tau)\Big[A(\tau)$$

$$+D_p(h,\tau)+\int\limits_{(p-1)h}^{ph} K(\tau,s)ds\Big]d\tau\Big\}\lambda_p - \Big\{I - X(ph)\int\limits_{(p-1)h}^{ph} X^{-1}(\tau)\Big[D_{p+1}(h,\tau)$$

$$+\int\limits_{ph}^{(p+1)h} K(\tau,s)ds\Big]d\tau\Big\}\lambda_{p+1} + \sum_{j=p+2}^{N} X(ph)\int\limits_{(p-1)h}^{ph} X^{-1}(\tau)\Big[D_j(h,\tau)+\int\limits_{(j-1)h}^{jh} K(\tau,s)ds\Big]d\tau\lambda_j$$

$$= -X(ph)\int\limits_{(p-1)h}^{ph} X^{-1}(\tau)[f(\tau)+F_h(\tau)]d\tau, \qquad p=\overline{1,N-1}. \tag{22}$$

By denoting $nN\times nN$ matrix corresponding to the left-hand side of system (21), (22) by $Q_{*,*}(h)$, this system can be written as

$$Q_{*,*}(h)\lambda = -F_{*,*}(h), \qquad \lambda\in R^{nN}, \tag{23}$$

where $F_{*,*}(h) = \Big(-hd+hCX(T)\int\limits_{T-h}^{T} X^{-1}(\tau)[f(\tau)+F_h(\tau)]d\tau, X(h)\int\limits_{0}^{h} X^{-1}(\tau)[f(\tau)+F_h(\tau)]d\tau,$

$$X(2h)\int\limits_{h}^{2h} X^{-1}(\tau)[f(\tau)+F_h(\tau)]d\tau, ..., X[(N-1)h]\int\limits_{(N-2)h}^{(N-1)h} X^{-1}(\tau)[f(\tau)+F_h(\tau)]d\tau\Big).$$

3 Main results

System (23) with $h\in(0,h_0] : Nh = T$ has the following property.

Lemma 1. *Let $h\in(0,h_0] : Nh=T$. Then the following assertions hold:*

(a) The vector $\lambda^* = (\lambda_1^*,\lambda_2^*,...,\lambda_N^*)\in R^{nN}$, consisting of the values of the solution $x^*(t)$ to problem (1), (2) at the partition points $\lambda_r^* = x^*[(r-1)h]$, $r=\overline{1,N}$, satisfies system (23);

(b) The function $\widetilde{x}(t)$, defined by the equalities: $\widetilde{x}(t) = \widetilde{\lambda}_r + \widetilde{u}_r(t)$, $t\in[(r-1)h, rh)$, $r=\overline{1,N}$, and $\widetilde{x}(T) = \widetilde{\lambda}_N + \lim\limits_{t\to T-0}\widetilde{u}_N(t)$, where $\widetilde{\lambda} = (\widetilde{\lambda}_1,\widetilde{\lambda}_2,...,\widetilde{\lambda}_N)\in R^{nN}$ solves system (23) and the system of functions $\widetilde{u}[t] = (\widetilde{u}_1(t),\widetilde{u}_2(t),...,\widetilde{u}_N(t))$ solves the special Cauchy problem (8), (9) for $\lambda_r = \widetilde{\lambda}_r$, and $r=\overline{1,N}$, and is the solution to problem (1), (2).

Proof. (a) Let $x^*(t)$ be a solution to problem (1), (2). Then the pair $[(\lambda_1^*, \lambda_2^*, ..., \lambda_N^*), (u_1^*(t), u_2^*(t), ..., u_N^*(t))]$ with elements $\lambda_r^* = x^*[(r-1)h]$, and $u_r^*(t) = x^*(t) - x^*[(r-1)h]$, $t \in [(r-1)h, rh)$, $r = \overline{1, N}$, is a solution to the equivalent parametric boundary value problem (8)-(11). Taking into account the assumption $h \in (0, h_0] : Nh = T$ and repeating the above reasoning, we see that $\lambda^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_N^*) \in R^{nN}$ satisfies system (23).

(b) Let $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, ..., \widetilde{\lambda}_N) \in R^{nN}$ be a solution to systems (23). Since $h \in (0, h_0] : Nh = T$, the special Cauchy problem (8), (9) has a unique solution for any $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N) \in R^{nN}$. Its solution for $\lambda = \widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, ..., \widetilde{\lambda}_N)$) is denoted by $\widetilde{u}[t] = (\widetilde{u}_1(t), \widetilde{u}_2(t), ..., \widetilde{u}_N(t))$. Let us show that the pair $(\widetilde{\lambda}, \widetilde{u}[t])$ solves problem (8)–(11). Indeed, (8) and (9) hold by virtue of the choice of $\widetilde{u}[t]$ from $\widetilde{\lambda}$. If $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, ..., \widetilde{\lambda}_N)$ satisfies (23), then it satisfies (21) as well; i.e.,

$$B\widetilde{\lambda}_1 + C\widetilde{\lambda}_N + C\Big\{X(T)\int_{T-h}^{T} X^{-1}(\tau_1)A(\tau_1)d\tau_1\widetilde{\lambda}_N + \sum_{j=1}^{N} X(T)\int_{T-h}^{T} X^{-1}(\tau_1)\Big[D_j(h, \tau_1)$$

$$+ \int_{(j-1)h}^{jh} K(\tau_1, s)ds\Big]d\tau_1\widetilde{\lambda}_j + X(T)\int_{T-h}^{T} X^{-1}(\tau_1)[f(\tau_1) + F_h(\tau_1)]d\tau_1\Big\} = d. \qquad (24)$$

The pair $(\widetilde{\lambda}, \widetilde{u}[t])$ satisfies (20). Therefore, the expression in curly brackets on the left-hand side of (24) is equal to $\lim\limits_{t \to T-0} \widetilde{u}_N(t)$ and boundary condition (10) is fulfilled. Similarly, using (20) and (22), we show that (11) holds. Then the function $\widetilde{x}(t)$, constructed using the pair $[(\widetilde{\lambda}_1, \widetilde{\lambda}_2, ..., \widetilde{\lambda}_N), (\widetilde{u}_1(t), \widetilde{u}_2(t), ..., \widetilde{u}_N(t))]$ is a solution to problem (1), (2). Lemma 1 is proved.

Theorem 1. *If the matrix* $Q_{*,*}(h) : R^{nN} \to R^{nN}$ *is invertible for some* $h \in (0, h_0] : Nh = T$, *then problem* (1), (2) *has the unique solution* $x^*(t)$ *satisfying the estimate*

$$||x^*||_1 \le \frac{e^{\alpha_0 h}}{1-\sigma(h)}\Big[1 + \gamma_{*,*}(h)\max\Big(1 + h||C||\frac{e^{\alpha_0 h}}{1-\sigma(h)}, \frac{e^{\alpha_0 h}}{1-\sigma(h)}\Big)\Big]h \cdot \max(||f||_1, ||d||), \quad (25)$$

where $\gamma_{*,*}(h) = ||[Q_{*,*}(h)]^{-1}||$ *and* $\sigma(h) = \beta T \frac{1}{1-\alpha} h_0^{1-\alpha} e^{\alpha_0 h_0}$.

Proof. For given $f(t)$, d, $h \in (0, h_0] : Nh = T$, we construct system (23) and, using the invertibility of $Q_{*,*}(h)$, find its unique solution

$$\lambda^* = -[Q_{*,*}(h)]^{-1}F_{*,*}(h), \qquad \lambda^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_N^*) \in R^{nN}.$$

Taking into account that $h \in (0, h_0] : Nh = T$ we solve Cauchy problem (8), (9) with the found parameter values to obtain a system of functions $u^*[t] = (u_1^*(t), u_2^*(t), ..., u_N^*(t))$. Then, according to Lemma 1, the function $x^*(t)$, defined by the equalities $x^*(t) = \lambda_r^* + u_r^*(t)$,

$t \in [(r-1)h, rh)$, $r = \overline{1,N}$, and $x^*(T) = \lambda_N^* + \lim\limits_{t \to T-0} u_N^*(t)$, solves problem (1), (2). Let us show that the solution is unique. Assume that, in addition to $x^*(t)$ problem (1), (2) has another solution $\widetilde{x}(t)$. Then problem (8)–(11), in addition to $(\lambda^*, u^*[t])$ has another solution $(\widetilde{\lambda}, \widetilde{u}[t])$. According to Lemma 1, system (23) is satisfied by both λ^* and $\widetilde{\lambda}$; i.e.,

$$Q_{*,*}(h)\lambda^* = -F_{*,*}(h), \qquad Q_{*,*}(h)\widetilde{\lambda} = -F_{*,*}(h).$$

Since $Q_{*,*}(h)$ is invertible, these relations imply $\lambda^* = \widetilde{\lambda}$. Since the special Cauchy problem (8), (9) has a unique solution, we have $u_r^*(t) = \widetilde{u}_r(t)$, $t \in [(r-1)h, rh)$, $r = \overline{1,N}$, and $\lim\limits_{t \to T-0} u_N^*(t) = \lim\limits_{t \to T-0} \widetilde{u}_N(t)$, whence $x^*(t) = \widetilde{x}(t)$ for all $t \in [0,T]$. Let us prove estimate (25).

Since $\sigma(h) \le \sigma(h_0) < 1$ for $h \in (0, h_0] : Nh = T$, it holds that

$$||F_h||_1 \le \frac{1}{1-\sigma(h)} \max_{t \in [0,T]} ||F(h,t)||.$$

Based on (14), we obtain

$$\Big|\Big| X(rh) \int\limits_{(r-1)h}^{rh} X^{-1}(\tau)[f(\tau) + F_h(\tau)]d\tau \Big|\Big|$$

$$\le e^{\alpha_0 h} \int\limits_{(r-1)h}^{rh} ||f(\tau)||d\tau + e^{\alpha_0 h} \frac{h}{1-\sigma(h)} \max_{t \in [0,T]} ||F(h,t)||$$

$$\le e^{\alpha_0 h} \cdot h \Big[||f||_1 + \sigma(h) \frac{1}{1-\sigma(h)} ||f||_1 \Big] = e^{\alpha_0 h} \cdot h \frac{1}{1-\sigma(h)} ||f||_1, \qquad r = \overline{1,N},$$

which implies the estimate

$$||\lambda^*|| \le \gamma_{*,*}(h) ||F_{*,*}(h)|| \le \gamma_{*,*}(h) \max\Big(h||d|| + h||C||$$

$$\times \Big|\Big| X(T) \int\limits_{T-h}^{T} X^{-1}(\tau)[f(\tau) + F_h(\tau)]d\tau \Big|\Big|, \max_{p=\overline{1,N-1}} \Big|\Big| X(ph) \int\limits_{(p-1)h}^{ph} X^{-1}(\tau)[f(\tau) + F_h(\tau)]d\tau \Big|\Big| \Big)$$

$$\le \gamma_{*,*}(h) \cdot \max\Big(1 + h||C|| \frac{e^{\alpha_0 h}}{1-\sigma(h)}, \frac{e^{\alpha_0 h}}{1-\sigma(h)} \Big) h \max(||f||_1, ||d||). \tag{26}$$

Since

$$||D_j(h,t)|| = \Big|\Big| \sum_{k=0}^{\infty} M_j^{(k)}(h,t) \Big|\Big| \le \frac{1}{1-\sigma(h)} \max_{r=\overline{1,N}} \max_{t \in [0,T]} ||M_r(h,t)|| \le \frac{\beta h}{1-\sigma(h)} [e^{\alpha_0 h} - 1 + \sigma(h)],$$

it follows from (20) and (26) that

$$||u^*[\cdot]||_2 \le \Big[e^{\alpha_0 h} - 1 + Te^{\alpha_0 h} \max_{j=\overline{1,N}} \max_{t\in[0,T]} ||D_j(h,t)|| + \sigma(h)\Big] ||\lambda^*||$$

$$+\frac{e^{\alpha_0 h}}{1-\sigma(h)} \cdot h \cdot ||f||_1 \le \Bigg[\frac{e^{\alpha_0 h} - 1 + \sigma(h)}{1-\sigma(h)} \gamma_{*,*}(h) \max\Big(1 + h||C|| \frac{e^{\alpha_0 h}}{1-\sigma(h)}, \frac{e^{\alpha_0 h}}{1-\sigma(h)}\Big)$$

$$+\frac{e^{\alpha_0 h}}{1-\sigma(h)}\Bigg] \cdot h \max(||f||_1, ||d||). \qquad (27)$$

Using (26), (27) and relation $||x^*||_1 \le ||\lambda^*|| + ||u^*[\cdot]||_2$, we have (25). Theorem 1 is proved.

Definition 1. *Problem (1), (2) is called uniquely solvable if for any pair $(f(t), d)$, where $f(t) \in C([0,T], R^n)$ and $d \in R^n$, it has a unique solution.*

Theorem 2. *If problem (1), (2) is uniquely solvable, then the matrix $Q_{*,*}(h)$ is invertible for all $h \in (0, h_0] : Nh = T$.*

Proof. Assume the opposite, i.e., there exists $\widetilde{h} \in (0, h_0] : \widetilde{N}\widetilde{h} = T$ such that $Q_{*,*}(\widetilde{h})$ is not invertible. Then the homogeneous system of equations

$$Q_{*,*}(\widetilde{h})\lambda = 0 \qquad (28)$$

has a nontrivial solution $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, ..., \widetilde{\lambda}_N) \in R^{n\widetilde{N}}$.

In the case of a homogeneous boundary value problem for an integro-differential equation, i.e., for problem (1), (2) with $f(t) = 0$ and $d = 0$, system (23) becomes (28). Therefore, by Lemma 1, the function defined by the relations $\widetilde{x}(t) = \widetilde{\lambda}_r + \widetilde{u}_r(t)$, $t \in [(r-1)h, rh)$, $r = \overline{1, \widetilde{N}}$ and $\widetilde{x}(T) = \widetilde{\lambda}_{\widetilde{N}} + \lim_{t\to T-0} \widetilde{u}_{\widetilde{N}}(t)$, where the system of functions $\widetilde{u}[t] = (\widetilde{u}_1(t), \widetilde{u}_2(t), ..., \widetilde{u}_{\widetilde{N}}(t))$ solves problem (8), (9) with $\lambda_r = \widetilde{\lambda}_r$, $r = \overline{1, \widetilde{N}}$ and $f(t) = 0$, is the nontrivial solution of the homogeneous boundary value problem. This contradicts the unique solvability of problem (1), (2), since, when $f(t) = 0$, $d = 0$ it has, in addition to the trivial solution, the nontrivial one $\widetilde{x}(t)$. Theorem 2 is proved.

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Асанова А.Т., Нұрмұқанбет Ш.Н. ӘЛСІЗ ЕРЕКШЕЛІКТІ ӨЗЕГІ БАР ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУ ҮШІН ЕСЕП

Әлсіз ерекшелікті өзегі бар интегралдық-дифференциалдық теңдеулер жүйесі үшін сызықты шеттік есеп қарастырылады. Қарастырылатын есептің шешілімділік шарттары аралықты бөлу мен қосымша параметрлер енгізуге негізделген әдіс көмегімен орнатылады. Зерттеліп отырған есептің шешілімділігінің қажетті және жеткілікті шарттары алынды.

Кілттік сөздер. Сызықты шеттік есеп, интегралдық-дифференциалдық теңдеулер, әлсіз ерекшелігі бар өзек, параметрлеу әдісі, шешілімділік.

Асанова А.Т., Нурмуканбет Ш.Н. ЗАДАЧА ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ СО СЛАБОЙ ОСОБЕННОСТЬЮ

Рассматривается линейная краевая задача для системы интегро-дифференциальных уравнений с ядром со слабой особенностью. Установлены условия разрешимости рассматриваемой задачи с помощью метода, основанного на разбиении интервала и введении дополнительных параметров. Получены необходимые и достаточные условия разрешимости исследуемой задачи.

Ключевые слова. Линейная краевая задача, интегро-дифференциальные уравнения, ядро со слабой особенностью, метод параметризации, разрешимость.

Kazakh Mathematical Journal ISSN 2413–6468

20:3 (2020) 92–100

Some remarks on definability of types and conservative extension

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Communicated by: Beibut Kulpeshov

Received: 15.09.2020 ⋆ Accepted/Published Online: 20.09.2020 ⋆ Final Version: 26.09.2020

Abstract. We study the question of the existence of special extensions of a set A, which are characterized by the fact that the type over A of any tuple in the extension satisfies a condition C, where C is some property of types; C can mean either that any type under consideration is definable, or that any type is locally isolated, or that any type is non-definable, etc. In particular, we study the question of the existence of a conservative extension of a model.

Keywords. Conservative extension, definability of types.

1 Introduction

Here we describe a method for constructing models using the Tarski-Vaught criterion and the theory of non-orthogonality of 1-types for constructing a conservative extension.

Theorem 1 (Tarski-Vaught)**.** *Let A be a subset of a model $\mathcal{M}$ of a complete theory T. For the set A to be an elementary submodel of the model $\mathcal{M}$, it is necessary and sufficient that for any formula of the form $\exists x\phi(x,\bar{a})$, where $\bar{a}\in A$, the following condition holds: $\mathcal{M}\models\exists x\phi(x,\bar{a})$ implies that there exists $b\in A$ such that $\mathcal{M}\models\phi(b,\bar{a})$.*

Throughout the paper, $\mathcal{N}$ is a saturated model of the theory T of large cardinality and the cardinality of all models and sets under consideration are less than the cardinality of the model $\mathcal{N}$. Assume we are going to construct an elementary submodel $\mathcal{M}$ of $\mathcal{N}$ such that the types of tuples of elements from $M\setminus A$ have some property C (C-types), which we define later. We can divide properties of types of elements or tuples of elements from $M\setminus A$ into the following kinds:

2010 Mathematics Subject Classification: 03C15.

Funding: The authors were supported by the grant AP05132688 of SC of the MES of RK.

1. isolated over finite subsets from the family of finite subsets of A,
2. isolated over A,
3. not isolated but locally isolated over A,
4. not locally isolated, non-isolated and not belonging to a fixed family of non-isolated (non) definable types over A.

The case of isolated types over a finite set from the family of finite subsets A is considered as a rule in small theories over a countable set A.

A condition C on a type over A defines a special extension B: $A \subset_c B$.

Definition 1. *We say that B is a C-extension of A and write $A \subset_C B$ for this if the type $tp(\bar{\alpha}/A)$ satisfies the condition C for any tuple $\bar{\alpha} \in B \setminus A$.*

Definition 2. *We say that a condition C satisfies the transitivity property if the following holds for any three sets:*

$$A \subset_C B \wedge B \subset_C D \Rightarrow A \subset_C D.$$

In any case, the following four conditions are necessary for constructing a model which is a C-extension of a set A:

U0$_C$. For any sets $A \subseteq B$ and any C-type $p \in S_1(A)$ there exists a C-type $q \in S_1(B)$ which extends p (the extension property).

U1$_C$. For any tuple $\bar{\alpha} \in N \setminus A$ whose type satisfies the condition C, for any formula $\phi(x, \bar{\alpha}, \bar{a})$, where $\bar{a} \in A$ and $\mathcal{N} \models \exists x(\phi(x, \bar{\alpha}, \bar{a}))$, there exists a type $p(x) \in S_1(A\bar{\alpha})$ such that $\phi(x, \bar{\alpha}, \bar{a}) \in p$ and for any $\beta \in N$ with $\beta \models p(x)$, we obtain that $tp(\beta\bar{\alpha}/A)$ satisfies the condition C.

U2$_C$. For the theory T, the condition C has the transitivity property.

U3$_C$. The theory T has the restriction property for C-types, that is if $tp(\bar{\alpha}/A)$ is a C-type, then $tp(\bar{\beta}/A)$ is a C-type for any $\bar{\beta} \subseteq \bar{\alpha}$.

Failure to meet at least one of these four conditions prevents the constructing a model with the property C. In the case when we build a model over finite subsets of A, condition U2$_C$ is bypassed by simultaneously constructing a countable family of countable models nested into each other, in a countable number of steps, in this case, the choice of a new element at each final step is carried out (to satisfy the Tarski-Vaught condition) so that the new element together with the already selected one forms a tuple of elements whose type over A is a C-type.

Definition 3. *We say that a model $\mathcal{M}$, which is a C-extension of a set A, is a C-ω-saturated extension of A if q is realized in $\mathcal{M}$ whenever q is a C-type from $S_1(A \cup \bar{\alpha})$ for some $\bar{\alpha} \in M \setminus A$.*

The existence of a C-ω-saturated extension of a set A is due to the joint extension property for C-1-types.

Definition 4. *We say a complete theory T has the joint extension property for C-1-types if for any set $A \subset N$, where $\mathcal{N} \models T$ is sufficiently saturated, for any $\bar{\alpha}, \beta, \gamma \in N \setminus A$ the following is true: if the types $q := tp(\beta/A \cup \bar{\alpha})$, $p := tp(\gamma/A \cup \bar{\alpha})$, $tp(\bar{\alpha}/A)$ are C-types over A, then the type $tp(\gamma\beta/A\bar{\alpha})$ is a C-type.*

$U4_C$. The theory T has the joint extension property for C-1-types.

Theorem 2. *Let T be a complete theory, then the following is true:*

1) For any set A, there is a model $\mathcal{M}$ with ($A \subset M \preceq N$) which is constructed using Tarski-Vaught test.

2) If the theory T satisfies the conditions $U0_C$, $U1_C$, $U2_C$, and $U3_C$, then for any set A there exists a model $\mathcal{M} \preceq \mathcal{N}$ such that $A \subseteq_C M$.

3) If the theory T satisfies the conditions $U0_C$, $U1_C$, $U2_C$, $U3_C$, and $U4_C$, then for any set A there exists a model $\mathcal{M} \preceq \mathcal{N}$ such that $A \subseteq_C M$ and $\mathcal{M}$ is a C-ω-saturated extension of A.

Proof. 1) This is well-known, nevertheless we remind the proof of this statement. To build models over any set and without conditions on types (general case), there is no need to satisfy any of Un_C. The construction of the model is as follows.

Step 1. Consider the set of all formulas with one variable and constants from A: $F_1(x, A) := \{\phi(x, \bar{a}) \mid \bar{a} \in A,\ \mathcal{N} \models \exists x\phi(x, \bar{a})\}$. The set A_1 contains all elements of realizations of formulas from $F_1(x, A)$. Obviously, $A \subseteq A_1$.

Step $n + 1 (n < \omega)$. We have $F_1^{n+1}(x, A_n) := \{\phi(x, \bar{a}) \mid \bar{a} \in A_n,\ \mathcal{N} \models \exists x\phi(x, \bar{a})\}$. The set A_{n+1} contains all elements of realizations of formulas from $F_1^{n+1}(x, A_n)$. Construction is carried out by sequential implementation of formulas from $F_1^{n+1}(x, A_n) := \{\phi(x, \bar{a}) | \bar{a} \in A_n\}$ with a fixed enumeration. Clearly, $A_n \subseteq A_{n+1}$.

The model $\mathcal{M}$ is defined as the union $M = \bigcup_{n<\omega} A_n$. Then by construction $A \subseteq M$. Tarski-Vaught criterion implies that $\mathcal{M} \preceq \mathcal{N}$.

2) The condition $U1_C$ provides the possibility of applying the Tarski-Vaught criterion for choosing a C-type, which contains $\phi_{k+1}(x, \bar{a}) \in F_1^{n+1}(x, A_n)$ and realizes this type by some element α_{k+1}, whose type over $A_n \cup \{\alpha_0, \ldots, \alpha_n\}$ is a C-type. The transitivity property $U2_C$ provides the possibility to move from A_n to A_{n+1} (on the limit steps). We show below this process in details.

We put $A^0 = A$.

Step 1. Consider the set of all formulas with one variable and constants from A^0: $F_1(x, A^0) := \{\phi_i(x, \bar{a}_i) \mid \bar{a} \in A^0,\ \mathcal{N} \models \exists x\phi_i(x, \bar{a}_i), i < \lambda\}$.

We consider $\phi_0(x, \bar{a}_0)$. By $U1_C$ there is a C-type $p(x) \in S_1(A)$ which contains $\phi_0(x, \bar{a}_0)$. Let α_0 realize p and $A^0_0 = A^0 \cup \{\alpha_0\}$. Now we assume that we have realized $\phi_i(x, \bar{a}_i)$ and constructed A^0_i. We consider $\phi_{i+1}(x, \bar{a}_{i+1})$. By $U1_C$ there is a C-type $p_{i+1}(x) \in S_1(A^0)$ which contains $\phi_{i+1}(x, \bar{a}_{i+1})$. By the extension property $U0_C$ there is an extension $q_{i+1}(x) \in S_1(A^0_i)$ of p_{i+1} which is a C-type. Now we realize q_{i+1} by some α_{i+1} and put $A^0_{i+1} = A^0_i \cup \{\alpha_{i+1}\}$. By construction, $tp(\alpha_0, \alpha_1, \ldots, \alpha_{i+1}/A)$ is a C-type. By the restriction property $U3_C$ the type of any subtuple of $(\alpha_0, \alpha_1, \ldots, \alpha_{i+1})$ is a C-type, that is why A^0_{i+1} is a C-extension of A^0_μ, where $\mu = 0$ if $i+1$ is a natural number or μ is the largest limit ordinal which is less that $i+1$. By the transitivity property $U2_C$ the set A^0_{i+1} is a C-extension of $A^0 = A$, provided that A^0_μ is a C-extension of A^0. We show the later below.

If μ is a limit ordinal we put $A^0_\mu = \bigcup_{i<\mu} A^0_i$. We show that A^0_μ is the C-extension of A^0. Let $\nu = 0$ if $\mu = \omega_0$, otherwise ν is the largest limit ordinal which is less than μ. By the induction hypothesis A^0_ν is the C-extension of A^0. So, in order to show that A^0_μ is the C-extension of A^0 it is sufficient by transitivity to show that A^0_μ is the C-extension of A^0_ν. Let $\bar\beta \in A^0_\mu \setminus A^0_\nu$. Then $\bar\beta \in A^0_{\nu+k}$ for some finite k. Since $A^0_{\nu+k}$ is the C-extension of A^0_ν, so $tp(\bar\beta/A^0_\nu)$ is the C-type by the restriction property $U3_C$.

We put $A^1 = A^0_\lambda$. It is the C-extension of A^0. The set A^1 contains all elements of realizations of formulas from $F_1(x, A^0)$.

Step $n+1 (n < \omega)$. We put $F^{n+1}_1(x, A^n) := \{\phi_i(x, \bar a_i) \mid \bar a \in A^n,\ \mathcal{N} \models \exists x \phi(x, \bar a),\ i < \lambda\}$. Up to changing the superscript 0 with n we repeat Step 1 in order to construct A^{n+1}. Clearly, $A^n \subseteq_C A^{n+1}$.

We define $\mathcal{M}$ as the union $M = \bigcup_{n<\omega} A^n$. Then by construction $A \subseteq_C M$. Tarski-Vaught criterion implies that $\mathcal{M} \preceq \mathcal{N}$.

3) Let $\mathcal{M}_1$ be a C-extension of A constructed the way which we have described in the previous item of this theorem. Assume that we have constructed $\mathcal{M}_n$, which is the C-extension of A. We enumerate all tuples $\bar\alpha$ from $M_n \setminus A$ and all C-types $p \in S_1(A\bar\alpha)$ as p_i for $i < \kappa$. Let

$$B_{n+1} = \{\beta_i : \beta_i \models p_i \text{ and } p_i \text{ is not reallized in } \mathcal{M}^n,\ i < \kappa\}.$$

The joint extension property $U4_C$ guarantees that B_{n+1} is the C-extension of A. By the previous item of this theorem there exists the C-extension $\mathcal{M}_{n+1}$ of B_{n+1} which is an elementary submodel of $\mathcal{N}$. By the transitivity property $\mathcal{M}_{n+1}$ is the C-extension of A.

We put $\mathcal{M} = \bigcup_{n<\omega} \mathcal{M}_n$. Obviously, $\mathcal{M}$ is a C-ω-saturated extension of A and an elementary submodel of $\mathcal{N}$. The theorem is proved.

2 Conditions for constructing a conservative extension with a given property of models

Let a D-property of a type over a set A be that this type is definable over the set A, that is, $p \in S(A)$ is definable, in this case we will say and write that p is a D-type.

Definition 5. *For sets $A \subset B$, we say that B is a D-extension (conservative) of A ($A \subset_D B$) if $tp(\bar\alpha/A)$ is a D-type for any $\bar\alpha \in B \setminus A$.*

An important condition for constructing a conservative extension is the transitivity property U2_C, and for our situation, U2_D.

The following is well-known.

Theorem 3. *Any complete theory satisfies the transitivity property $U2_D$ and the restriction property $U3_D$.*

Thus, taking into account this theorem for D-extensions, we can formulate the following.

Theorem 4. *If a complete theory T satisfies the conditions $U0_D$ and $U1_D$, then for any set A there exists a D-extension (conservative extension) of A.*

It is well-known, that the extension property does not hold in general. Indeed, we consider $\mathcal{M} = (\mathbb{N}, <)$ and $\mathcal{N} = (\mathbb{N} \cup \mathbb{Z}', <)$, where each $n' \in \mathbb{Z}'$ is bigger than any $k \in \mathbb{N}$. Let s stand for the successor function, which is definable in this structures. Let $A = \mathbb{N} \cup \{0'\}$, $B = \mathbb{N} \cup \mathbb{Z}'$, and let

$$p = \{n < x : x \in \mathbb{N}\} \cup \{s^m(x) < 0' : m < \omega\}.$$

Clearly, that p defines a unique complete type over A, which is definable. Moreover, p defines a unique complete type over B, but this type is not definable.

The joint extension property $U4_D$ in general does not hold.

Theorem 5 [1]**.** *There is an o-minimal theory T such that for $A \subset \mathcal{N} \models T$ and $p, q \in S_1(A)$ the following holds:*

1. *q is weakly orthogonal to p;*
2. *both q and p are locally isolated types and hence are D-types;*
3. *the unique 2-type $p(x) \cup q(y) \in S_2(A)$ is not definable.*

In this [1] example, A is just a subset. This example can be modified so that A contains an elementary submodel. But we obtain a weakly o-minimal theory.

Let $\Sigma = \{=, <, R^4, E\}$ and T be the theory of the signature Σ which consists of the following axioms.

1. $<$ is a dense linear order without endpoints;
2. E is an equivalence relation with convex infinite classes and the order induced on E-classes is a dense linear order without endpoints;
3. $R(x, y, z, t)$ implies that $y, z, t \in [x]_E$;
4. for each x, y, z there is a unique t with $R(x, y, z, t)$;
5. for each fixed z the restriction of R to $[z]_E$ is an ordered Abelian divisible group, where the addition $x+y = t$ is defined as $R(x, y, z, t)$. For example, if some E-class is $(\mathbb{Q}, <, +)$ then $R(x, y, z, t)$ is equivalent to $x + y = z + t$.

If we consider $\mathbb{Q}$ copies of $(\mathbb{Q},<,+)$, where each copy is an E-class, we obtain a prime model of T. Thus, T is complete. Let us fix a finite set $A=\{a_1<a_2<\dots a_n\}$ in a model $\mathcal{M}$ of T. We consider b_1 and b_2 such that each of them belongs to neither of $[a_i]$ and either $b_1,b_2<a_1$, or $a_i<b_1,b_2<a_{i+1}$ for some $i<n$, or $a_n<b_1,b_2$, then obviously there exists an automorphism which fixes A pointwise and moves b_1 to b_2. Now we consider $\varphi(x,\bar{a})$. Clearly, $\varphi(\mathcal{M},\bar{a})\cap[a_i]$ is a finite union of intervals and points because the restriction of $\mathcal{M}$ to a_i is o-minimal. Also we can state that each convex set of the form $(a_i,a_{i+1})\setminus([a_i]\cup[a_{i+1}])$ either belongs to $\varphi(\mathcal{M},\bar{a})$ or has an empty intersection with $\varphi(\mathcal{M},\bar{a})$. That is why this theory is weakly o-minimal.

Now we proceed as in [1]. We consider an elementary extension of $\mathcal{M}$ and consider $\alpha\notin M$ such that $[\alpha]\cap M=\emptyset$. Let $\langle a_n:n<\omega\rangle$ be an increasing sequence of rational numbers, converging to $\sqrt{2}$, and let $\langle b_n:n<\omega\rangle$ be a decreasing sequence of rational numbers, converging to $\sqrt{2}$. Let $A=M\cup\{a_n\cdot\alpha,b_n\cdot\alpha:n<\omega\}$. Let $\beta=\sqrt{2}\cdot\alpha$, $\gamma=\pi\cdot\alpha$ and $\delta=(\pi-\sqrt{2})\alpha$. Repeating reasoning form [1], we can prove that the types $tp(\gamma/A)$ and $tp(\delta/A)$ are locally isolated and weakly orthogonal to each other, but their union defines a complete type, which contains $tp(\beta/A)$ and this type is not definable.

Taking into account the last theorem we will restrict ourselves mainly to considering the definability of the union of two weakly orthogonal 1-types over the union of a model and a definable finite set, and in the case of a positive answer, it becomes possible to construct a D-ω-extension.

We consider a proof that generalizes the consideration of similar questions for the particular case of weakly o-minimal theories [1].

Definition 6. We say that a D-extension B of a set A is an ω-saturated D-extension if for any tuple $\bar{\alpha}\in M\setminus A$ each D-1-type from $S_1(A\cup\bar{\alpha})$ is realized in B. We write $A\subset_{D,\omega}B$ for this.

Now we can reformulate Theorem 2.

Theorem 5. *1) Let a complete theory T satisfy the conditions $U0_D$ and $U1_D$. Then for any set A there is a model $\mathcal{M}$, which is a D-extension of A and $\mathcal{M}\preceq\mathcal{N}$.*

2) Let a complete theory T satisfy the conditions $U0_D$, $U1_D$, and $U4_D$. Then for any set A there is a model $\mathcal{M}_\omega$, which is an ω-saturated D-extension of A and $\mathcal{M}_\omega\preceq\mathcal{N}$.

It is well-known that if A is a model, then the extension property $U0_D$ holds. Then we obtain another corollary of Theorem 2.

Theorem 6. *1) Let a complete theory T satisfy the condition $U1_D$. Then for any model A there is a model $\mathcal{M}$, which is a D-extension of A and $\mathcal{M}\preceq\mathcal{N}$. Note that if A is a model, it is not necessary that $\mathcal{M}$ is a proper extension of A, by construction $A=\mathcal{M}$ may happen.*

2) Let a complete theory T satisfy the condition $U1_D$ and $U4_D$. Then for any model A there is a model $\mathcal{M}_\omega$, which is an ω-saturated D-extension of A and $\mathcal{M}_\omega\preceq\mathcal{N}$.

3 Classes of complete theories for which the existence of a definable 1-type ensures the existence of an elementary conservative extension

Let $\mathcal{M}$ be a model of a complete theory T. Let $S_1(M)$ contain a definable non-isolated 1-type. From the construction in Theorems 2 and 6 we know that in order to construct a conservative extension, it is necessary to select a subclass of D-types such that they provide construction steps, that is, $U1_D$.

Theorem 7. *Let T be a complete theory. Assume that for any infinite formula $\phi(x,\bar{a})$ with $\bar{a}\in A$, there exists a definable 1-type $p\in S_1(A)$ such that $\phi(x,\bar{a})\in p$. Then for any model of this theory there is a proper conservative extension.*

Recall the following definition.

Definition 7. *Let A be an arbitrary set in a saturated model $\mathcal{N}$ of a complete theory and $p,q\in S(A)$. We say that p is not almost orthogonal to q if there exists $\phi(\bar{x},\bar{\alpha},\bar{a})$ such that $\phi(\mathcal{N},\bar{\alpha},\bar{a})\subset q(\mathcal{N})$, where $\bar{\alpha}\models p$.*

Proposition 1. *Let A be an arbitrary set in a saturated model of a complete theory. Let p and $q\in S(A)$ be such that p is not almost orthogonal to q. If p is definable, then so is q.*

Proof. Let $p,q\in S(A)$ be such that p is not almost orthogonal to q. This means that there exists $\phi(\bar{x},\bar{\alpha},\bar{a})$ such that $\phi(\mathcal{N},\bar{\alpha},\bar{a})\subset q(\mathcal{N})$, where $\bar{\alpha}\models p$. Since $q(\mathcal{N})=\bigcap_{H\in q}H(\mathcal{N},\bar{c})$, so $\phi(\mathcal{N},\bar{\alpha},\bar{a})\subset H(\mathcal{N},\bar{c})$. Let $K_H(\bar{\alpha},\bar{c},\bar{a})=\forall\bar{x}(\phi(\bar{x},\bar{\alpha},\bar{a})\to H(\bar{x},\bar{c}))$. Hence, $K_H(\bar{z},\bar{c},\bar{a})\in p$. Conversely, if $K_H(\bar{z},\bar{c},\bar{a})\in p$, then $H(\bar{x},\bar{c})\in q$. Thus, $K_H(\bar{z},\bar{c},\bar{a})\in p$ if and only if $H(\bar{x},\bar{c})\in q$. Let p be definable. We show that q is also definable. Suppose there exists an A-formula $\Theta(x,\bar{y})$ such that the set $B_{\Theta,q}:=\{\bar{b}\in A\mid \Theta(x,\bar{b})\in q\}$ is infinite. The definability of the type q means that the set $B_{\Theta,q}$ is definable over A for each Θ. Now we define $B_{K_\Theta,p}:=\{\bar{b}\in A\mid K_\Theta(z,\bar{b},\bar{a})\in p\}$. Since the type p is definable, the set $B_{K_\Theta,p}$ is definable, too. It follows from definition that $B_{K_\Theta,p}=B_{\Theta,q}$. The proposition is proved.

Proposition 1 can be strengthened for the class of weakly o-minimal theories, as it has been shown in [2] and [3]. The first author proved that if two one-types p and q over a set A of a model of a weakly o-minimal theory are not weakly orthogonal, that is their union $p(x)\cup q(y)$ has at least two completions over A, then p is definable if and only if q is definable. It would be interesting to investigate the question for which theories the this property holds: if two types are not weakly orthogonal, then definability of one of these two types implies definability of the other type.

The condition for existence of a definable type over an arbitrary set A containing a formula with parameters from the set A for constructing a D-extension can be weakened in the case of the existence of a conservative extension of a model. The following holds.

Theorem 8. *Let T be a complete theory and $\mathcal{M}\prec\mathcal{N}\models T$. For the existence of a conservative elementary extension $\mathcal{M}_1$ with $\mathcal{M}\prec\mathcal{M}_1\prec\mathcal{N}$ it is necessary and sufficient that for any tuple* *of elements* $\bar{\alpha} \in N \setminus M$ *such that* $tp(\bar{\alpha}/M)$ *is definable, for any* $(M\bar{\alpha})$*-formula* $\phi(x, \alpha, \bar{a})$ *such that* $\mathcal{N} \models \exists x \phi(x, \bar{\alpha}, \bar{a})$ *the following holds: if* $\phi(\mathcal{N}, \bar{\alpha}, \bar{a}) \cap M = \emptyset$*, then there is a definable 1-type* $q \in S_1(M\bar{\alpha})$ *with* $\phi(x, \bar{\alpha}, \bar{a}) \in q$.

Thus, the condition for the existence of at least one definable 1-type must be accompanied by the condition for the existence of at least one definable 1-type for a set that is the union of the universe of a model and a tuple of elements, whose type over the model is definable. The question of the existence of a saturated conservative elementary expansion requires the condition of joint expansion, but for the model.

The existence of a conservative extension of the model provided that at least one definable 1-type exists is provided by such a condition as the condition of the existence of a simple model over a set. In this case, in the proof of the theorem, we need facts about isolated types and orthogonality of types.

Theorem 9. *Let* T *be a complete theory such that there is a prime model over any set. Then for any model that has at least one definable non-isolated 1-type there is a conservative extension.*

Proof. Let $\mathcal{M} \models T$ and $p \in S_1(M)$ be definable and non-isolated. Let $\alpha \models p$ and $\mathcal{N}$ be a prime model over $M\alpha$. Then p is not almost orthogonal to any type $q \in S(M)$ which is realized in $\mathcal{N}$. By Proposition 1 the type q is definable. The theorem is proved.

4 Classes of complete theories for models of which there is a conservative extension

The class of o-minimal theories contains a complete theory in which no model has a conservative extensions, due to the fact that there are no definable 1-types over any model of this theory. Nevertheless, over an arbitrary model M of an o-minimal theory, if there is at least one definable 1-type $p \in S_1(M)$, then for an arbitrary realization $\alpha \in p(\mathcal{N})$ in a large saturated model $\mathcal{N}$, one can take a prime model over $M \cup \{\bar{\alpha}\}$, which exists by Pillay-Steinhorn theorem [4]. Then a simple model $(\mathcal{M}, \alpha)$ is a conservative extension of $\mathcal{M}$. This is explained by the fact that any tuple of elements from $\bar{\beta} \in (M, \bar{\alpha})/M$ of the type $tp(\bar{\beta}/M\alpha)$ is isolated and, therefore, is definable over $(M \cup \{\alpha\})$. Then, since the type of α over M is definable, it follows that the type $tp(\bar{\beta}/M)$ is definable, too.

In [1] B. Baizhanov proved that the class of weakly o-minimal theories satisfies the conditions of Theorem 2 and, therefore, any model of a weakly o-minimal theory with a dense order has a conservative extension. Moreover, for any model of the weakly o-minimal theory, the existence of at least one 1-definable 1-type over the model implies the existence of the conservative extension.

The class of o-stable theories contains its proper subclass of weakly o-minimal theories, which in turn contains its proper subclass of o-minimal theories. Therefore, we consider the class of o-stable and not weakly o-minimal theories. The question of the existence of a conservative extension of the model of o-stable theories is reduced to proving the condition in Theorem 5.

An interesting question is to investigate the existence of conservative extension for the class of o-stable theories [5], [6].

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Байжанов Б., Орынбасаров Д., Вербовский В. ТИПТІҢ АНЫҚТАЛЫМДЫҒЫ МЕН КОНСЕРВАТИВТІ КЕҢЕЙТУЛЕР ТУРАЛЫ КЕЙБІР ЕСКЕРТПЕЛЕР

A жиынының ерекше кеңейтулерінің бар болуын зерттейміз, олар осы кеңейтудегі кез келген кортеж A-ның үстіндегі тип C шартын қанағаттандыруымен сипатталады, мұнда C — типке қойылған қандай да бір шарт; C – кез келген, қарастырылып жатқан тип локалды оқшауланған, немесе анықталымды, немесе анықталымды емес, және т.т. болатын шарт болуы мүмкін. Атап айтқанда, моделдің консервативті кеңейтуі бар болуы мәселесін зерттейміз.

Түйінді сөздер. Консервативті кеңейтулер, типтердің анықталымдығы.

Байжанов Б., Орынбасаров Д., Вербовский В. НЕКОТОРЫЕ ЗАМЕЧАНИЯ ОБ ОПРЕДЕЛИМОСТИ ТИПОВ И КОНСЕРВАТИВНЫХ РАСШИРЕНИЯХ

Мы изучаем вопрос существования особых расширения множества A, которые характеризуются тем, что тип над A любого кортежа из данного расширения удовлетворяет условию C, где C — некоторое условие на типы; C может быть тем условием, что любой рассматриваемый тип локально изолированный, или определимый, или неопределимый, и так далее. В частности, мы изучаем вопрос существования консервативного расширения модели.

Ключевые слова. Консервативные расширения, определимость типов.

Kazakh Mathematical Journal ISSN 2413–6468

20:3 (2020) 101-110

Optimal numerical integration on classes of smooth functions in several variables

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Received: 10.09.2020 ⋆ Final Version: 22.09.2020 ⋆ Accepted/Published Online: 30.09.2020

Abstract. In the paper, we establish estimates exact in order for error of optimal cubature formulas on the Nikol'skii – Besov and Lizorkin – Triebel type spaces, $B^{s\,\mathtt{m}}_{p\,q}(\mathbb{T}^m)$ and $L^{s\,\mathtt{m}}_{p\,q}(\mathbb{T}^m)$, respectively, for a number of relations between parameters s, p, q, $\mathtt{m}$ ($s = (s_1, \ldots, s_n) \in \mathbb{R}^n_+$, $1 \le p, q \le \infty$, $\mathtt{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n, m = m_1 + \cdots + m_n$).

Keywords. Numerical integration, optimal cubature formula, lattice, Frolov's cubature formula, Nikol'skii–Besov/Lizorkin–Triebel function space/class, mixed smoothness, multidimensional torus

1 Introduction

Let Ω be a compactum in $\mathbb{R}^m\,(m \ge 2)$ (with nonempty interior), F a set (class) of complex–valued continuous functions with domain Ω. In numerical integration, for the approximation of the integral

$$\int\limits_{\Omega} f(x)dx, \quad f \in \mathrm{F},$$

expressions of the form (cubature formulas)

$$\mathscr{Q}(f, C_N, \Lambda_N) := \sum_{k=1}^{N} c(k) f(\lambda(k)), \tag{1}$$

are used; here $C_N := (c(1), \ldots, c(N)) \in \mathbb{C}^N$ are weights and $\Lambda_N := (\lambda(1), \ldots, \lambda(N)) \subset \Omega^N$ is grid of nodes of the cubature formula, and

$$\mathscr{R}(f, \Omega, C_N, \Lambda_N) := \int\limits_{\Omega} f(x)dx - \mathscr{Q}(f, C_N, \Lambda_N)$$

2010 Mathematics Subject Classification: 42B35, 41A63, 65D30.

Funding: The work was supported by the Ministry of Education and Science under grant AP05133257.

is its error on a function f. Denote

$$\mathscr{R}(\mathrm{F},\Omega,C_N,\Lambda_N) := \sup\{|\mathscr{R}(f,\Omega,C_N,\Lambda_N)| \ \mid \ f\in \mathrm{F}\}.$$

The problem of optimal numerical integration under consideration here consists in determining the exact (in N) order of the quantity

$$\mathscr{R}_N(\mathrm{F},\Omega) := \inf\{\mathscr{R}(\mathrm{F},\Omega,C_N,\Lambda_N)\,|\,C_N,\Lambda_N\} \tag{2}$$

(which is (N-th) optimal error of numerical integration) and constructing a sequence $(C_N^*,\Lambda_N^* \mid N\in\mathbb{N})$ of weights and nodes such that the errors $\mathscr{R}(\mathrm{F},\Omega,C_N^*,\Lambda_N^*)$ of the cubature formulas (1) realize the order of the optimal error (2). Cubature formulas $\mathscr{Q}(f,C_N^*,\Lambda_N^*)$ are called optimal (in order).

A lot of works are devoted to the study of different formulations of problems of optimal numerical integration for various classes of smooth functions in several variables, see, for example, monographs [1], [3, ch.6] and surveys [2], [4, ch.8] and the bibliography therein. The construction and study of optimal (or, at least, "good") cubature formulas for certain classes of (periodic) functions of mixed smoothness originates in the well-known works of N.M. Korobov [5], N.S. Bakhvalov [6], and E. Hlawka [7]. Comprehensive survey [4], monograph [3], papers [8], [9], [10] show that interest in the problem of optimal numerical integration we will study here is unabated; there is also a fairly detailed history of the issue and an extensive bibliography.

In this section, we give exact (in the sense of the order) estimates for the quantity (2) in the case when $\Omega=\mathbb{T}^m$ is m-diensional torus, F is the function class $\mathrm{B}^{s\,\mathtt{m}}_{p\,q}(\mathbb{T}^m)$ of Nikol'skii–Besov type or $\mathrm{L}^{s\,\mathtt{m}}_{p\,q}(\mathbb{T}^m)$ of Lizorkin–Triebel type, for a number of relations between the parameters of these classes.

Let us introduce the notation that we will use throughout this article. Let $k\in\mathbb{N}$, $\mathrm{z}_k=\{1,\ldots,k\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{R}_+=(0,+\infty)$. For $x=(x_1,\ldots,x_k), y=(y_1,\ldots,y_k)\in\mathbb{R}^k$, put $xy=x_1y_1+\ldots+x_ky_k,\ |x|=|x_1|+\ldots+|x_k|,\ |x|_\infty=\max(|x_\kappa|:\kappa\in\mathrm{z}_k)$; $x\le y\ (x<y)$ $\Leftrightarrow x_\kappa\le y_\kappa\ (x_\kappa<y_\kappa)$ for all $\kappa\in\mathrm{z}_k$. When $x<y$ we denote by $[x,y]$, $[x,y)$, (x,y) closed, half-open and open parallelepipeds with "lower left corner" x and "upper right corner" y in $\mathbb{R}^k$, respectively. For $\alpha=(\alpha_1,\ldots,\alpha_k)\in\mathbb{N}_0^k$, as usual, $x^\alpha=x_1^{\alpha_1}\cdots x_k^{\alpha_k}$, $\partial^\alpha:=\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k}$, where ∂_κ is partial derivative with respect to κ-th variable.

Let $\mathcal{S}:=\mathcal{S}^{(k)}:=\mathcal{S}(\mathbb{R}^k)$ and $\mathcal{S}'=\mathcal{S}'(\mathbb{R}^k)$ be the Schwartz spaces of test functions and tempered distributions, respectively; $\widehat{f}\equiv\mathcal{F}_k(f)$ and $\mathcal{F}_k^{-1}(f)$ be direct and inverse Fourier transforms of $f\in\mathcal{S}'(\mathbb{R}^k)$; in particular, for $\varphi\in\mathcal{S}^{(k)}$,

$$\widehat{\varphi}(\xi)=\mathcal{F}_k(\varphi)(\xi)=\int\limits_{\mathbb{R}^k}\varphi(x)e^{-2\pi i\,\xi x}dx,\quad \mathcal{F}_k^{-1}(\varphi)(\xi)=\int\limits_{\mathbb{R}^k}\varphi(x)e^{2\pi i\,\xi x}dx,\quad \xi\in\mathbb{R}^k.$$

Let $\mathbb{T}^k=(\mathbb{R}/\mathbb{Z})^k$ be k-dimensional torus; we denote by $\widetilde{\mathcal{S}}'\equiv\mathcal{S}'(\mathbb{T}^k)$ the space of all distributions f from $\mathcal{S}'$ 1-periodic in each variable (i.e. such that $\langle f,\varphi(\cdot+y)\rangle=\langle f,\varphi\rangle$ for all

$\varphi \in \mathcal{S}^{(k)}$ and for any $y \in \mathbb{Z}^k$) and by $\widetilde{\mathcal{S}} := \widetilde{\mathcal{S}}^{(k)} := \mathcal{S}(\mathbb{T}^k)$ the space of all infinitely differentiable functions on $\mathbb{T}^k$ endowed with the topology of uniform convergence of all derivatives. Then the space $\mathcal{S}'(\mathbb{T}^k)$ is naturally identified with the space topologically dual to $\mathcal{S}(\mathbb{T}^k)$. It is well known that $f \in \widetilde{\mathcal{S}}'$ if and only if $\operatorname{supp} \widehat{f} \subset \mathbb{Z}^k$, i.e. distribution $\widehat{f}$ vanishes on the open set $\mathbb{R}^k \setminus \mathbb{Z}^k$.

Let $f : \mathbb{R}^k \to \mathbb{C}$ be an arbitrary function, its periodization $\widetilde{f} : \mathbb{T}^k \to \mathbb{C}$ is defined as (formal) sum of the series $\sum\limits_{\xi \in \mathbb{Z}^k} f(x + \xi)$.

First we choose the function $\eta_0 := \eta_0^{(k)} \in \mathcal{S}^{(k)}$ such that

$$0 \le \widehat{\eta}_0(\xi) \le 1, \quad \xi \in \mathbb{R}^k; \quad \widehat{\eta}_0(\xi) = 1, \quad |\xi|_\infty \le 1; \quad \operatorname{supp} \widehat{\eta}_0 = \{\xi \in \mathbb{R}^k \mid |\xi|_\infty \le 2\}.$$

Put $\widehat{\eta}(\xi) := \widehat{\eta}^{(k)}(\xi) = \widehat{\eta}_0(2^{-1}\xi) - \widehat{\eta}_0(\xi)$, $\widehat{\eta}_j(\xi) := \widehat{\eta}_j(\xi) = \widehat{\eta}(2^{1-j}\xi)$, $j \in \mathbb{N}$. Then

$$\sum_{j \in \mathbb{N}_0} \widehat{\eta}_j(\xi) \equiv 1, \quad \xi \in \mathbb{R}^k,$$

i.e. $\{\widehat{\eta}_j(\xi) \,|\, j \in \mathbb{N}_0\}$ is smoth partition of unity (corresponding to "corridors") on $\mathbb{R}^k$. It is clear that

$$\eta(x) := \eta^{(k)}(x) = 2^k \eta_0(2x) - \eta_0(x), \, \eta_j(x) := \eta_j^{(k)}(x) = 2^{(j-1)k} \eta(2^{j-1}x), \, j \in \mathbb{N}. \tag{3}$$

Denote

$$\mathrm{H}^{(k)}(x) := \{\eta_j^{(k)}(x) \,|\, j \in \mathbb{N}_0\}(x \in \mathbb{R}^k), \quad \widetilde{\mathrm{H}}^{(k)}(x) = \{\widetilde{\eta}_j^{(k)}(x) \,|\, j \in \mathbb{N}_0\}(x \in \mathbb{T}^k). \tag{4}$$

Let $m, n \in \mathbb{N}$, $m \ge 2$, $0 < p, \, q \le \infty$;

$L_p(\mathbb{I}^m)$ be the space of measurable functions $f : \mathbb{I}^m \to \mathbb{C}$, which are Lebesgue integrable in p-th power (essentially bounded when $p = \infty$) over $\mathbb{I}^m$, endowed with standard quasi-norm (norm if $p \ge 1$)

$$\| \, f \,|\, L_p(\mathbb{I}^m) \, \| = \Big(\int\limits_{\mathbb{I}^m} | \, f(x) \, |^p dx \Big)^{\frac{1}{p}} \; (p < \infty), \; \| \, f \,|\, L_\infty(\mathbb{I}^m) \, \| = \operatorname{ess\,sup}(| \, f(x) \, | : x \in \mathbb{I}^m);$$

here $\mathbb{I}$ is $\mathbb{R}$ or $\mathbb{T}$; $L_p := L_p(\mathbb{R}^m)$, $\widetilde{L}_p := L_p(\mathbb{T}^m)$; sometimes we will identify $\mathbb{T}^m$ with cube $[\mathbf{0}, \mathbf{1})$ in $\mathbb{R}^m$ (we write $\mathbf{a} = (a, \ldots, a) \in \mathbb{R}^m$ for $a \in \mathbb{R}$);

$\ell_q := \ell_q^{(n)} := \ell_q(\mathbb{N}_0^n)$ be the space of (multiple complex) number sequences $(c_k) = (c_k : k \in \mathbb{N}_0^n)$ with finite standard quasi-norm (norm if $q \ge 1$) $\|(c_k) \,|\, \ell_q\|$;

$\ell_q(L_p(\mathbb{I}^m))$ (respectively, $L_p(\mathbb{I}^m; \ell_q)$) be the space of function sequences $(g_k(x)) = (g_k(x) : k \in \mathbb{N}_0^n)$ $(x \in \mathbb{I}^m)$ with finite standard quasi-norm (norm if $p, q \ge 1$)

$$\| \, (g_k(x)) \,|\, \ell_q(L_p(\mathbb{I}^m)) \, \| = \| \, (\, \| \, g_k \,|\, L_p(\mathbb{I}^m) \|) \,|\, \ell_q \, \|$$

(respectively, $\| (g_k(x)) \,|\, L_p(\mathbb{I}^m;\ell_q) \| = \| \| (g_k(\cdot)) \,|\, \ell_q \| \,|\, L_p(\mathbb{I}^m)\|$);
$\ell_q(L_p) := \ell_q(L_p(\mathbb{R}^m))$, $\ell_q(\widetilde{L}_p) := \ell_q(L_p(\mathbb{T}^m))$, $L_p(\ell_q) = L_p(\mathbb{R}^m;\ell_q)$, $\widetilde{L}_p(\ell_q) = L_p(\mathbb{T}^m;\ell_q)$.

Let $n \le m$. We fix multi-index $\mathtt{m} = (m_1, ..., m_n) \in \mathbb{N}^n$ such that $|\mathtt{m}| = m$ along with representation $\mathbb{R}^m = \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$ and corresponding representation $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ of the form $x = (x^1, \ldots, x^n)$, where $x^\nu \in \mathbb{R}^{m_\nu}$ ($\mathtt{m} = m$ if $n = 1$ and $\mathtt{m} = \mathbf{1}$ if $n = m$).

We choose systems $\mathrm{H}^{(m_\nu)}(x^\nu)$ and $\widetilde{\mathrm{H}}^{(m_\nu)}(x^\nu)$ as in (4) ($\nu \in \mathrm{z}_n$) and define ($\mathtt{m}$-fold) systems $\mathrm{H}^{(\mathtt{m})}(x)$ and $\widetilde{\mathrm{H}}^{(\mathtt{m})}(x)$ as follows:

$$\mathrm{H}^{(\mathtt{m})}(x) := \otimes_{\nu\in \mathrm{z}_n}\mathrm{H}^{(m_\nu)}(x^\nu) \equiv \{\eta_k^{(\mathtt{m})}(x) := \prod_{\nu\in \mathrm{z}_n} \eta_{k_\nu}^{(m_\nu)}(x^\nu) \,|\, k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n\}(x \in \mathbb{R}^m),$$

$$\widetilde{\mathrm{H}}^{(\mathtt{m})}(x) := \otimes_{\nu\in \mathrm{z}_n}\widetilde{\mathrm{H}}^{(m_\nu)}(x^\nu) \equiv \{\widetilde{\eta}_k^{(\mathtt{m})}(x) := \prod_{\nu\in \mathrm{z}_n} \widetilde{\eta}_{k_\nu}^{(m_\nu)}(x^\nu) \,|\, k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n\}(x \in \mathbb{T}^m).$$

Next we define operators $\Delta_k^\eta = \Delta_k^{\eta,\mathtt{r}}$ on $\mathcal{S}'$ and $\widetilde{\Delta}_k^\eta = \Delta_k^{\eta,\mathtt{t}}$ on $\widetilde{\mathcal{S}}'$ ($k \in \mathbb{N}_0^n$) as follows: for $f \in \mathcal{S}'$ and $g \in \widetilde{\mathcal{S}}'$

$$\Delta_k^\eta(f,x) = \Delta_k^{\eta,\mathtt{r}}(f,x) = f * \eta_k^{(\mathtt{m})}(x) = \langle f, \eta_k^{(\mathtt{m})}(x-\cdot)\rangle, \tag{5}$$

$$\widetilde{\Delta}_k^\eta(g,x) = \Delta_k^{\eta,\mathtt{t}}(g,x) = g * \widetilde{\eta}_k^{(\mathtt{m})}(x) = \langle g, \widetilde{\eta}_k^{(\mathtt{m})}(x-\cdot)\rangle = \sum_{\xi\in\mathbb{Z}^m} \widehat{\eta}_k(\xi)\widehat{g}(\xi)e^{2\pi i\,\xi x}. \tag{6}$$

Definition 1. *Let $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$, $0 < p, q \le \infty$; $(\mathtt{i}, \mathbb{I}) \in \{(\mathtt{t}, \mathbb{T}), (\mathtt{r}, \mathbb{R})\}$.*

I. The Nikol'skii – Besov type space $B_{pq}^{s\mathtt{m}}(\mathbb{I}^m)$ consists of all distributions $f \in \mathcal{S}'(\mathbb{I}^m)$, for which the quasi-norm

$$\| f \,|\, B_{pq}^{s\,m}(\mathbb{I}^m) \| \;=\; \|(2^{sk}\Delta_k^{\eta,\mathtt{i}}(f,x)) \,|\, \ell_q(L_p(\mathbb{I}^m))\|$$

is finite.

II. The Lizorkin – Triebel type space $L_{pq}^{s\mathtt{m}}(\mathbb{I}^m)$ ($p < \infty$) consists of all distributions $f \in \mathcal{S}'(\mathbb{I}^m)$, for which the quasi-norm

$$\| f \,|\, L_{pq}^{s\mathtt{m}}(\mathbb{I}^m) \| \;=\; \|(2^{sk}\Delta_k^{\eta,\mathtt{i}}(f,x)) \,|\, L_p(\mathbb{I}^m;\ell_q)\|$$

is finite.

We will call the unit balls $\mathrm{B}_{pq}^{s\mathtt{m}}(\mathbb{I}^m)$ and $\mathrm{L}_{pq}^{s\mathtt{m}}(\mathbb{I}^m)$ of those spaces the Nikol'skii-Besov and Lizorkin-Triebel classes, respectively.

In what follows, for brevity, we will often use the notation $F_{pq}^{s\mathtt{m}} = F_{pq}^{s\mathtt{m}}(\mathbb{R}^m)$, $\widetilde{F}_{pq}^{s\mathtt{m}} = F_{pq}^{s\mathtt{m}}(\mathbb{T}^m)$, here $F \in \{B, L, \mathrm{B}, \mathrm{L}\}$.

Remark 1 Comments and bibliography on spaces $B_{pq}^{s\mathtt{m}}(\mathbb{I}^m)$ and $L_{pq}^{s\mathtt{m}}(\mathbb{I}^m)$ can be found in [17]. Here we note only the following. When $n = m$ ($\Rightarrow \mathtt{m} = \mathbf{1}$), $\widetilde{B}_{pq}^{s\,\mathbf{1}}$ and $\widetilde{L}_{pq}^{s\,\mathbf{1}}$ are spaces of ("pure") mixed smoothness; in particular, for $s \in \mathbb{R}^m_+$, $M\widetilde{W}^s_p = \widetilde{L}^{s\mathbf{1}}_{p2}$ is the space of functions with dominating mixed derivative bounded in $\widetilde{L}_p$ (if $1 < p < \infty$) and $M\widetilde{H}^s_p \equiv \widetilde{B}^{s\mathbf{1}}_{p\infty}$ is the space of functions with dominating mixed difference bounded in $\widetilde{L}_p$ (if $1 \le p \le \infty$). When $n = 1 (\Rightarrow \ \mathtt{m} = m)$, $\widetilde{B}^s_{p\,q} \equiv \widetilde{B}^{s\,m}_{p\,q}$ and $\widetilde{L}^s_{p\,q} \equiv \widetilde{L}^{s\,m}_{p\,q}$ are the isotropic Nikol'skii-Besov and Lizorkin-Triebel spaces, respectively; in particular, when $1 < p < \infty$ $\widetilde{W}^s_p \equiv \widetilde{L}^s_{p\,2}$ is the isotropic Sobolev space and $\widetilde{H}^s_p \equiv \widetilde{B}^s_{p\infty} (1 \le p \le \infty)$ is the isotropic Nikol'skii space.

2 Optimal error of numerical integration on classes $\widetilde{\mathrm{B}}^{s\,\mathtt{m}}_{p\,q}$ and $\widetilde{\mathrm{L}}^{s\,\mathtt{m}}_{p\,q}$

In this section, we formulate and discuss the main result on estimates exact in order for optimal errors of numerical integration on the Nikol'skii – Besov and Lizorkin – Triebel classes $\mathrm{B}^{s\,\mathtt{m}}_{p\,q}(\mathbb{T}^m)$ and $\mathrm{L}^{s\,\mathtt{m}}_{p\,q}(\mathbb{T}^m)$ for a number of relations between parameters $s, p, q, \mathtt{m}$ ($s \in \mathbb{R}^n_+$, $1 \le p, q \le \infty$, $\mathtt{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, $m = m_1 + \cdots + m_n$).

For given $s = (s_1, ..., s_n) \in \mathbb{R}^n_+$, $\mathtt{m} = (m_1, .., m_n) \in \mathbb{N}^n$, we put $\varsigma_\nu = \frac{s_\nu}{m_\nu}$ ($\nu \in \mathrm{z}_n$); without loss of generality, we will assume that

$$\varsigma \equiv \min \{ \varsigma_\nu \mid \nu \in \mathrm{z}_n \} = \varsigma_1 = \ldots = \varsigma_\iota < \varsigma_\nu, \quad \nu \in \mathrm{z}_n \setminus \mathrm{z}_\iota$$

(with some $\iota \in \mathrm{z}_n$).

In what follows, we will use the signs $\ll$ and $\asymp$ of the ordinal inequality and equality: for functions $F : \mathbb{R}_+ \to \mathbb{R}_+$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$, we write $F(u) \ll H(u)$ as $u \to \infty$, if there exists a constant $C = C(F, H) > 0$ such that the inequality $F(u) \le CH(u)$ holds for $u \ge u_0 > 0$; $F(u) \asymp H(u)$, if $F(u) \ll H(u)$ $H(u) \ll F(u)$ simultaneously.

In what follows, $\log \equiv \log_2$. When $\Omega = \mathbb{T}^m$ or $[\mathbf{0}, \mathbf{1}]$ we will often write simply $\mathscr{R}_N(\mathrm{F})$ instead of $\mathscr{R}_N(\mathrm{F}, \Omega)$.

Theorem 1. *Let* $1 \le p, q \le \infty$, $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_+$. *Then*

I. for $\varsigma > 1/p$, *the relation*

$$\mathscr{R}_N(\widetilde{\mathrm{B}}^{s\,\mathtt{m}}_{p\,q}, \mathbb{T}^m) \asymp N^{-\varsigma} (\log N)^{(\iota-1)(1-1/q)}$$

holds;

II. for $p < \infty$ *and* $\varsigma > \max(1/p, 1/q)$, *the relation*

$$\mathscr{R}_N(\widetilde{\mathrm{L}}^{s\,\mathtt{m}}_{p\,q}, \mathbb{T}^m) \asymp N^{-\varsigma} (\log N)^{(\iota-1)(1-1/q)}$$

holds.

Remark 2. By theorem C from [18] the condition $\varsigma > \frac{1}{p}$ provides the embedding $\widetilde{F}^{s\,\mathtt{m}}_{p\,q} \hookrightarrow C(\mathbb{T}^m)$, which is necessary in problems of numerical integration ($F \in \{B, L\}$).

Remark 3. As noted above, there is an extensive literature devoted to optimal cubature formulas for classes of functions of several variables. Here we discuss results directly related to Theorem 1, namely, results on function classes on the torus included in the Nikol'skii – Besov and Lizorkin – Triebel scales from definition 1. But here we do not touch on the case of low smoothness ($p > q, 1/p < \varsigma \le 1/q$) of the Lizorkin – Triebel classes; for this, see Remark 4 below.

The estimates of $\mathscr{R}_N(\mathrm{F})$ exact in order of the isotropic Sobolev and Nikol'skii classes are given in [3, ch.3] (in fact, the anisotropic case is considered there).

For $\mathscr{R}_N(\widetilde{\mathrm{MW}}^s_p)$, the right (in order) upper bounds were proved by N.S. Bakhvalov [2] (the case $p = m = 2, s_1 = s_2 \in \mathbb{N}$; there Fibonacci's cubature formulas were used for the first time), V.N. Temlyakov (1989) (the general case $p = m = 2, 1/2 < s_1 = s_2 \in \mathbb{R}_+$; see [3]), K.K. Frolov [12] (the case $p = 2, s = s_1\mathbf{1}, s_1 \in \mathbb{N}, m \ge 2$ is arbitrary; a new construction of cubature formulas was invented), further, V.A. Bykovskii (1985) ($p = 2, m \ge 2, s = s_1\mathbf{1}, s_1 \in \mathbb{R}, s_1 \ge 1$), then V.N. Temlyakov [13] ($p \ge 2, m \ge 2, s \in \mathbb{R}^m_+ : 1/2 < s_1 = s_\iota, s_\nu > s_1(1 + 1/\lfloor s_1 \rfloor)$), M.M. Skriganov [15] ($1 < p \le \infty, m \ge 2, s = s_1\mathbf{1}, s_1 \in \mathbb{N}$), and the right (in order) lower bounds were established by V.A. Bykovskii (1985) (the case $p = 2, m \ge 2, s = s_1\mathbf{1}, s_1 > 1/2$) and V.N. Temlyakov [13] (the general case $1 \le p < \infty, m \ge 2, s = s_1\mathbf{1}, s_1 > 1/p$).

For $\mathscr{R}_N(\widetilde{\mathrm{MH}}^s_p)$, the right (in order) lower bounds were established by N.S. Bakhvalov [11] in the general case, the right (in order) upper bounds were proved by N.S. Bakhvalov [2, 6] (the case $m = 2, s_1 = s_2$; Fibonacci's cubature formulas) and V.V. Dubinin [14] (the general case, Frolov's cubature formulas). Exact order of the quantity $\mathscr{R}_N(\widetilde{\mathrm{B}}^{s\mathbf{1}}_{p\,q})$ was found by V.V. Dubinin [16]. Finally, exact order of the quantity $\mathscr{R}_N(\widetilde{\mathrm{L}}^{s\mathbf{1}}_{p\,q})$ in the case $s_1 = \ldots = s_m$ was obtained by V.K. Nguyen, M. Ullrich, T. Ullrich [9].

3 Estimates from below

There are two main methods for obtaining lower bounds of $\mathscr{R}_N(\mathrm{F}, \Omega)$. The first one was proposed by N.S. Bakhvalov [11]. His idea is for a given N and any cubature formula (1) to construct a "bad" function g_{Λ_N}, $\|g_{\Lambda_N} \,|\, F\| = 1$, vanishing at its nodes, in the form of a sum with equal positive coefficients of special shifts of contractions of a suitable fixed smooth bump function for which

$$\mathscr{R}(g_{\Lambda_N}, \Omega, C_N, \Lambda_N) = \int\limits_{\Omega} g_{\Lambda_N}(x)dx = \|g_{\Lambda_N} \,|\, \widetilde{L}_1\|$$

has the required order. In the second one, proposed by V.N. Temlyakov [13] for $\Omega = \mathbb{T}^m$, the function g_{Λ_N} with those properties is sought among trigonometric polynomials with spectrum in the "hyperbolic layer" depending on N. The existence of such a polynomial is established using deep estimates for the volumes of the sets of Fourier coefficients of such polynomials. Lower bounds in Theorem 1 is proved by Bakhvalov's method.

Denote by $\#\Gamma$ the number of elements of a finite set Γ ($\Gamma = \varnothing \Leftrightarrow \#\Gamma = 0$) and by $|P|$ the volume of a parallelepiped P.

Let $\mathfrak{R}^{\mathfrak{m}} \equiv \mathfrak{R}^{\mathfrak{m}\mathfrak{r}}$ be the collection of all half-open dyadic parallelepipeds from $\mathbb{R}^m$ of the form $P = P^{\mathtt{m}}_{k\xi} = \{\, x \in \mathbb{R}^m \,:\, 2^k \cdot x - \xi \in [\mathbf{0},\mathbf{1})\,\}$ $(k \in \mathbb{N}_0^n, \xi \in \mathbb{Z}^m)$,

$$\widetilde{\mathfrak{R}}^{\mathtt{m}} \equiv \mathfrak{R}^{\mathtt{mt}} = \{P \in \mathfrak{R}^{\mathtt{m}} \,|\, P \subset [\mathbf{0},\mathbf{1})\} = \{P^{\mathtt{m}}_{k\xi} \,|\, k \in \mathbb{N}_0^n,\, \xi \in \mathbb{Z}^m : \mathbf{0} \le \xi < 2^k \cdot \mathbf{1}\},$$

here $a^z = (a^{z_1}, \ldots, a^{z_n})$, $z \cdot x = (z_1 x^1, \ldots, z_n x^n)$ for $a \in \mathbb{R}$, $z \in \mathbb{R}^n$, $x \in \mathbb{R}^m$. Below $x_P := 2^{-k} \cdot \xi$, $k(P) := k$, if $P = P^{\mathtt{m}}_{k\xi}$. It is clear that

$$\{P \in \widetilde{\mathfrak{R}}^{\mathtt{m}} \,|\, k(P) = k\} = \{P^{\mathtt{m}}_{k\xi} \,|\, \xi \in \mathbb{Z}^m : \mathbf{0} \le \xi < 2^k \cdot \mathbf{1}\},$$

is the partition of the torus $[\mathbf{0},\mathbf{1})$, $\#\{P \in \widetilde{\mathfrak{R}}^{\mathtt{m}} \,|\, k(P) = k\} = 2^{k\mathtt{m}}$, $|P| = 2^{-k(P)\mathtt{m}}$.

Key ingredient in the estimating from below is atomic characterization of the Nikol'skii – Besov and Lizorkin – Triebel spaces from proposition 1.

Under hypotheses of Theorem 1, we call a collection of functions $(\mathscr{A}_P^{(\mathtt{r})}) \equiv (\mathscr{A}_P : P \in \mathfrak{R}^{\mathtt{m}}) \subset \mathcal{S}(\mathbb{R}^m)$ *a family of atoms* for $F^{s\,\mathtt{m}}_{p\,q}$, if for each $P \in \mathfrak{R}^{\mathtt{m}}$ the conditions

$$\operatorname{supp} \mathscr{A}_P \subset 3P, \qquad |\partial^\alpha \mathscr{A}_P(x)| \le |P|^{-1/2} 2^{|k(P)\cdot\alpha|}, \quad x \in \mathbb{R}^m, \ \alpha \le \mathtt{K} \cdot \mathbf{1} \tag{7}$$

are fulfilled, and a collection $(\mathscr{A}_P^{(\mathtt{t})}) \equiv (\mathscr{B}_P : P \in \widetilde{\mathfrak{R}}^{\mathtt{m}}) \subset \mathcal{S}(\mathbb{T}^m)$ *a family of atoms* for $\widetilde{F}^{s\,\mathtt{m}}_{p\,q}$, if for each $P \in \widetilde{\mathfrak{R}}^{\mathtt{m}}$ $\mathscr{B}_P$ is periodization of a function $\mathscr{A}_P \in \mathcal{S}(\mathbb{R}^m)$ (i.e. $\mathscr{B}_P = \widetilde{\mathscr{A}_P}$) satisfying the conditions (7) (here $3P$ is the dilation of P with the same center).

For a sequence $(c_P^{(\mathtt{i})}) \equiv (c_P^{(\mathtt{i})} : P \in \mathfrak{R}^{\mathtt{mi}}) \subset \mathbb{C}$, we put

$$\|(c_P^{(\mathtt{i})}) \,|\, \mathtt{B}^{s\,\mathtt{mi}}_{p\,q}\| := \|(2^{sk} \sum_{P \in \mathfrak{R}^{\mathtt{mi}} : k(P) = k} c_P |P|^{-1/2} \chi_P(\cdot)) \,|\, \ell_q(L_p(\mathbb{I})^m)\|,$$

$$\|(c_P^{(\mathtt{i})}) \,|\, \mathtt{L}^{s\,\mathtt{mi}}_{p\,q}\| := \|(2^{sk} \sum_{P \in \mathfrak{R}^{\mathtt{mi}} : k(P) = k} c_P |P|^{-1/2} \chi_P(\cdot)) \,|\, L_p((\mathbb{I})^m; \ell_q)\|$$

(χ_P is the indicator of P).

Proposition 1. *Let* $(\mathtt{i}, \mathbb{I}) \in \{(\mathtt{r}, \mathbb{R}), (\mathtt{t}, \mathbb{T})\}$, $(F, \mathtt{F}) \in \{(B, \mathtt{B})(L, \mathtt{L})\}$. *Then, under hypotheses of Theorem 1,* $f \in F^{s\,\mathtt{m}}_{p\,q}(\mathbb{I}^m)$, *if and only if there exist a family of atoms* $(\mathscr{A}_P^{(\mathtt{i})})$ *for* $F^{s\,\mathtt{m}}_{p\,q}(\mathbb{I}^m)$ *and a sequence* $(c_P^{(\mathtt{i})}) \in \mathtt{L}^{s\,\mathtt{mi}}_{p\,q}$ *such that*

$$f = \sum_{P \in \mathfrak{R}^{\mathtt{mi}}} c_P^{(\mathtt{i})} \mathscr{A}_P^{(\mathtt{i})} \qquad (\textit{convergence in } \ L_p(\mathbb{I})^m), \tag{8}$$

moreover,

$$\|\, f \,|\, F^{s\,\mathtt{m}}_{p\,q}(\mathbb{I}^m)\| \asymp \inf \|(c_P^{(\mathtt{i})}) \,|\, \mathtt{F}^{s\,\mathtt{mi}}_{p\,q}\|, \tag{9}$$

where inf *is taken over all representations* (8).

4 Estimates from above

Under hypotheses of definition 1($F \in \{B, L\}$, $u, v \in \mathbb{R}, u < v$), we consider the space

$$\dot{F}^{s\,\mathtt{m}}_{p\,q}([\mathbf{u},\mathbf{v}]) := \{g \in F^{s\,\mathtt{m}}_{p\,q}(\mathbb{R}^m)\ :\ \operatorname{supp} g \subset [\mathbf{u},\mathbf{v}]\}.$$

and its unit ball (class) $\dot{\mathrm{F}}^{s\,\mathtt{m}}_{p\,q}([\mathbf{u},\mathbf{v}])$.

Scheme for proving upper bounds of $\mathscr{R}_N(\mathrm{F},\Omega)$ when $\Omega = [\mathbf{0},\mathbf{1}]$ or $\mathbb{T}^m$, $m \geq 2$, for classes of functions with mixed smoothness was proposed by K.K. Frolov [12]. The scheme is as follows: i) by a suitable smooth change of variables, the class F is mapped to the class $\mathrm{G} = \mathrm{G}([0,1]^m)$ of functions vanishing on the cube boundary, ii) the inequality $\mathscr{R}_N(\mathrm{F},\Omega) \ll \mathscr{R}_N(\mathrm{G},[\mathbf{0},\mathbf{1}])$ is established, iii) a special lattice Λ is chosen such that the number of its nodes falling into an arbitrary parallelepiped with sides parallel to the coordinate axes is proportional to its volume, iv) a cubature formula with equal weights equal to $\frac{1}{N}$ and a grid of nodes $\Lambda^{\circ}_N := (N\det(\Lambda))^{-1/m}\Lambda \cap (\mathbf{0},\mathbf{1})$ (Frolov's cubature formula) has the number of nodes of order N and the required order of error for G, v) and the cubature formula induced by it gives the same order of error for F on Ω. This approach has been applied and developed in [13], [14], [15], [16]. In [8], [9], the minimal smoothness conditions for the change of variables are substantiated; the scheme was simplified (using the characterizations of the spaces $B^{s\,\mathtt{m}}_{p\,q}(\mathbb{R}^m)$ and $L^{s\,\mathtt{m}}_{p\,q}(\mathbb{R}^m)$ by so-called local means); for $\Omega = \mathbb{T}^m$ a simple way of passing from F to G is proposed using a smooth periodic partition of unity instead of changing variables.

When proving upper bounds, we adhere to Frolov's scheme with modifications and simplifications from [8], [9].

5 Estimates from above for Lizorkin – Triebel classes in the case of small smoothness

Here we consider upper estimates for error of optimal numerical integration for classes $\dot{\mathrm{L}}^{s\,\mathtt{m}}_{p\,q}([\mathbf{0},\mathbf{1}])$ in the case of small smoothness: $p > q$ and $1/p < \varsigma \leq 1/q$.

Theorem 2. *Let $1 \leq q < p < \infty$, $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_+$. Then*

I. for $1/p < \varsigma < 1/q$, the relation

$$\mathscr{R}_N(\dot{\mathrm{L}}^{s\,\mathtt{m}}_{p\,q}([\mathbf{0},\mathbf{1}]),[\mathbf{0},\mathbf{1}]) \ll N^{-\varsigma}(\log N)^{(\iota-1)(1-\varsigma)}$$

holds;

II. for $\varsigma = 1/q$, the relation

$$\mathscr{R}_N(\dot{\mathrm{L}}^{s\,\mathtt{m}}_{p\,q}([\mathbf{0},\mathbf{1}]),[\mathbf{0},\mathbf{1}]) \ll N^{-\varsigma}(\log N)^{(\iota-1)(1-\varsigma)}(\log\log N)^{1-\varsigma}$$

holds.

Remark 4. V.N. Temlyakov [19, 20] was the first to discover and fully investigate the phenomenon of "small smoothness" in problems of optimal numerical integration for the classes $\mathrm{M}\widetilde{\mathrm{W}}_p^s$ of functions of two variables with a bounded mixed derivative for the case of $2 < p \leq \infty$ and $1/p < s_1 \leq 1/2$. Note that the Fibonacci cubature formulas again turned out to be optimal in this case. Theorem 2 is a generalization to the case of the Lizokin−Triebel classes $\dot{\mathrm{L}}_{p\,q}^{s\,\mathbf{m}}([\mathbf{0},\mathbf{1}])$ of the recent result of M. Ullrich and T. Ullrich [8] for classes $\dot{\mathrm{L}}_{p\,q}^{s\mathbf{1}}([\mathbf{0},\mathbf{1}])$.

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Базарханов Д.Б. КӨП АЙНЫМАЛЫЛЫ ТЕГІС ФУНКЦИЯЛАР КЛАСТАРЫНДА ЕҢ ТИІМДІ ИНТЕГРАЛДЫ ЖУЫҚТАУ

Бұл жұмыста s, p, q, $\mathtt{m}$ ($s = (s_1, \ldots, s_n) \in \mathbb{R}^n_+$, $1 \leq p, q \leq \infty$, $\mathtt{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n, m = m_1 + \cdots + m_n$) параметрлері арасындағы бірқатар қатынастары үшін Никольский–Бесов $B^{s\,\mathtt{m}}_{p\,q}(\mathbb{T}^m)$ және Лизоркин–Трибель $L^{s\,\mathtt{m}}_{p\,q}(\mathbb{T}^m)$ кеңістіктері үшін ең тиімді кубатуралық формуланың қателігінің реті бойынша нақты бағалауы алынған.

Кілттік сөздер. Интегралды жуықтау, ең тиімді кубатуралық формула, тор, Фроловтың кубатуралық формуласы, Никольский–Бесов/Лизоркин–Трибель кеңістігі/класы, аралас тегістігі.

Базарханов Д.Б. ОПТИМАЛЬНОЕ ЧИСЛЕННОЕ ИНТЕГРИРОВАНИЕ НА КЛАССАХ ГЛАДКИХ ФУНКЦИЙ НЕСКОЛЬКИХ ПЕРЕМЕННЫХ

В предлагаемой работе установлены точные в смысле порядка оценки погрешности оптимальных кубатурных формул для пространств типа Никольского–Бесова $B^{s\,\mathtt{m}}_{p\,q}(\mathbb{T}^m)$ и Лизоркина – Трибеля $L^{s\,\mathtt{m}}_{p\,q}(\mathbb{T}^m)$ для ряда соотношений между параметрами s, p, q, $\mathtt{m}$ ($s = (s_1, \ldots, s_n) \in \mathbb{R}^n_+$, $1 \leq p, q \leq \infty$, $\mathtt{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n, m = m_1 + \cdots + m_n$).

Ключевые слова. Численное интегрирование, оптимальная кубатурная формула, решётка, кубатурная формула Фролова, пространство/класс Никольского–Бесова/Лизоркина–Трибеля, смешанная гладкость, многомерный тор.

KAZAKH MATHEMATICAL JOURNAL

20:3 (2020)

Собственник "Kazakh Mathematical Journal":
Институт математики и математического моделирования

Журнал подписан в печать
и выставлен на сайте http://kmj.math.kz / Института математики и
математического моделирования
30.09.2020 г.

Тираж 300 экз. Объем 111 стр.
Формат 70×100 1/16. Бумага офсетная № 1

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