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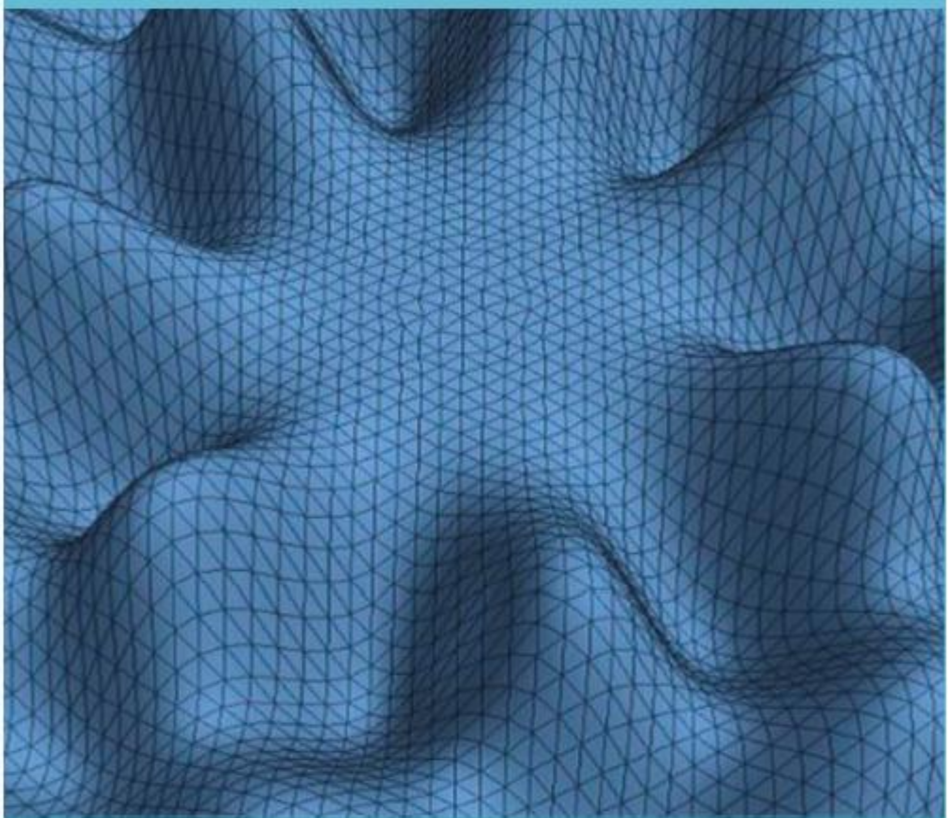

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CORRESPONDENCE ADDRESS	Institute of Mathematics and Mathematical Modeling, 125 Pushkin Str., 050010, Almaty, Kazakhstan Phone/Fax: +7 727 272-70-93
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On correct boundary value problems for nonclassical sixth order differential equations

Alexandr I. Kozhanov[1,a], Bakytbek D. Koshanov[2,3,b], Gulzhazira D. Smatova[4,c]

[1]Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences, Novosibirsk, Russian
[2]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[3]Abai Kazakh National Pedagogical University, Almaty, Kazakhstan
[4]Satbaev University, Almaty, Kazakhstan
[a] e-mail: kozhanov@math.nsc.ru, [b]e-mail: koshanov@math.kz, [c] e-mail: smatova1977@mail.ru

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Abstract. In this article we investigate the correctness of boundary value problems for the sixth order quasi-hyperbolic equation in Sobolev space

$$Lu = -D_t^6 u + \Delta u - \lambda u$$

($D_t = \frac{\partial}{\partial t}$, $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, λ is a real parameter). For the given operator L two spectral problems are introduced and the uniqueness of these problems is established. The eigenvalues and eigenfunctions of the first spectral problem are calculated for the sixth order quasi-hyperbolic equation. In this work we show that the equation $Lu = 0$ for $\lambda < 0$ under uniform conditions has a countable set of nontrivial solutions. Usually, this does not happen when the operator L is an ordinary hyperbolic operator.

Keywords. Sixth order quasi-hyperbolic equation, boundary value problems, eigenvalues, eigenfunctions, nontrivial solutions.

1 Introduction and Formulation of the problem

Let Ω be a limited area of space $\mathbb{R}^n$ of variables $x_1, x_2, ..., x_n$ with a smooth compact boundary $\Gamma = \partial\Omega$. Let us consider the following differential operator in the cylindrical area $Q = \Omega \times (0,T)$, $S = \Gamma \times (0,T)$, $0 < T < +\infty$,

$$Lu \equiv -\frac{\partial^6 u}{\partial t^6} + \Delta u - \lambda u = f(x,t), \quad x \in \Omega, \quad t \in (0,T), \tag{1}$$

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where $f(x,t)$ is a given function.

Boundary value problem $I_{3,\lambda}$. It is required to find a function $u(x,t)$ which is a solution to equation (1) in the cylinder Q that satisfies the following conditions:

$$u(x,t)|_S = 0, \tag{2}$$

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = \frac{\partial^2 u}{\partial t^2}(x,0) = \frac{\partial^3 u}{\partial t^3}(x,0) = 0,\ x \in \Omega, \tag{3}$$

$$\frac{\partial u}{\partial t}(x,T) = \frac{\partial^2 u}{\partial t^2}(x,T) = 0,\ x \in \Omega. \tag{4}$$

Boundary value problem $II_{3,\lambda}$. It is required to find a function $u(x,t)$ which is a solution to equation (1) in the cylinder Q that satisfies conditions (2), (3) and

$$D_t^4 u(x,t)|_{t=T} = D_t^5 u(x,t)|_{t=T} = 0,\ x \in \Omega. \tag{5}$$

The study of the solvability of boundary value problems for quasi-hyperbolic equations began, apparently, with the works of V.N. Vragov [1], [2]. Studies in [3]–[7] are related to further investigations of operators similar to L. One of the main conditions for correctness in these studies was the condition that parameter λ is non-negative. Investigations of nonlocal problems with integral conditions for linear parabolic equations, for differential equations of the odd order, and for some classes of non-stationary equations have been actively carried out recently in the works of A.I. Kozhanov [4], [6], [7]. In [5], the solvability of problem (2), (3), (5) for the fourth order quasi-hyperbolic equations with $p = 2$ is investigated. In the work [8] boundary value problems with normal derivatives were studied for elliptic equations of the $(2l)$-st order with constant real coefficients. For these problems, sufficient conditions for the Fredholm solvability of the problem are obtained and formulas for the index of this problem are given. An explicit form of the Green's function of the Dirichlet problem for the model-polyharmonic equation $\Delta^l u = f$ in a multidimensional sphere was constructed in [9]. [10], [11] are devoted to investigations of the solvability of various boundary value problems of the orders $0 \le k_1 < k_2 < ... < k_l \le 2l-1$ for the polyharmonic equation in a multidimensional ball.

In this paper, we describe calculation of eigenvalues $\lambda_m^{(1)}$ ($\lambda_m^{(2)}$) of spectral problems $I_{3,\lambda}$ ($II_{3,\lambda}$) for the sixth order quasi-hyperbolic equation and study the solvability of boundary value problems $I_{3,\lambda}$ ($II_{3,\lambda}$) for the cases when λ coincides or does not coincide with $\lambda_m^{(1)}$ ($\lambda_m^{(2)}$).

2 Supporting statement

We denote by V_3 the linear set of functions $v(x,t)$, belonging to the space $L_2(Q)$ and having generalized derivatives with respect to spatial variable up to the second order inclusively belonging to the same space and with respect to the variable t up to the order 6 inclusively, with the norm

$$\|v\|_{V_3} = \left(\int\limits_Q \Big[v^2 + \sum_{i,j=1}^n \Big(\frac{\partial^2 v}{\partial x_i \partial x_j}\Big)^2 + \Big(\frac{\partial^6 v}{\partial t^6}\Big)^2\Big] dxdt\right)^{\frac{1}{2}}.$$

Obviously, the space V_3 with this norm is Banach space.

Let $v(x)$ be function from the space $\overset{\circ}{W}{}_2^1(\Omega)$. The following inequality is true:

$$\int\limits_\Omega v^2(x)dx \le c_0 \int\limits_\Omega \sum_{i=1}^n v_{x_i}^2(x)dx, \tag{6}$$

where constant c_0 defined only by the area Ω (see, example in [12]).

For the function from the space V_3 satisfying condition (3), the following inequality holds:

$$\int\limits_\Omega v^2(x,t_0)dx \le T^3 \int\limits_0^T \int\limits_\Omega v_{ttt}^2(x,t)dxdt, \quad t_0 \in [0,T], \tag{7}$$

$$\int\limits_0^T \int\limits_\Omega v^2(x,t)dxdt \le \frac{T^6}{8} \int\limits_0^T \int\limits_\Omega v_{ttt}^2(x,t)dxdt. \tag{8}$$

Let $\omega_j(x)$ be the eigenfunction of the Dirichlet problem for the Laplace operator corresponding to the eigenvalue μ_j:

$$\Delta\omega_j(x) = \mu_j \omega_j(x), \quad \omega_j(x)|_\Gamma = 0.$$

3 Main results

Theorem 1. *Let* $\lambda > c_1, c_1 = \min\{-\frac{1}{c_0}, -\frac{40}{T^6}\}$, c_0 *from (6). Then the homogeneous boundary value problem* $I_{3,\lambda}$ *has only zero solution in the space* V_3. *On the interval* $(-\infty, c_1)$ *there exists a countable set of numbers* $\lambda_m^{(1)}$ *such that for* $\lambda = \lambda_m^{(1)}$ *the homogeneous boundary value problem* $I_{3,\lambda}$ *has a non-trivial solution.*

Proof. First, we prove the uniqueness of the solution to the problem $I_{3,\lambda}$. Let $A > T$. We consider the equality

$$\int\limits_0^T \int\limits_\Omega (A-t)Lu \cdot u_t dxdt = 0.$$

Integrating by parts and using conditions (2), (3), we get

$$\frac{A-T}{2}\int\limits_{\Omega}[u_{ttt}^2(x,T)+\sum_{i=1}^{n}u_{x_i}^2(x,T)]dx+\frac{5}{2}\int\limits_0^T\int\limits_{\Omega}u_{ttt}^2dxdt$$

$$+\frac{1}{2}\sum_{i=1}^{n}\int\limits_0^T\int\limits_{\Omega}u_{x_i}^2dxdt=-\frac{\lambda(A-T)}{2}\int\limits_{\Omega}u^2(x,T)dx-\frac{\lambda}{2}\int\limits_0^T\int\limits_{\Omega}u^2dxdt=I. \qquad (9)$$

When $\lambda\geq 0$ it follows from this equality that $u(x,t)\equiv 0$.

We now consider the case of negative values of λ. On the one hand due to expressions (6) and (7), there is an inequality

$$|I|=|-\frac{\lambda(A-T)}{2}\int\limits_{\Omega}u^2(x,T)dx-\frac{\lambda}{2}\int\limits_0^T\int\limits_{\Omega}u^2dxdt|$$

$$\leq\frac{|\lambda|(A-T)}{2}T^3\int\limits_0^T\int\limits_{\Omega}u_{ttt}^2dxdt+\frac{|\lambda|}{2}c_0\sum_{i=1}^{n}\int\limits_0^T\int\limits_{\Omega}u_{x_i}^2dxdt. \qquad (10)$$

On the other hand, due to inequalities (7) and (8) we get

$$|I|\leq\frac{|\lambda|(A-T)}{2}T^3\int\limits_0^T\int\limits_{\Omega}u_{ttt}^2dxdt+\frac{|\lambda|T^6}{2\cdot 2^3}\int\limits_0^T\int\limits_{\Omega}u_{ttt}^2dxdt.$$

If $c_1=-\frac{1}{c_0}$, then by evaluating the right side of (9) by (10), we get

$$\frac{A-T}{2}\int\limits_{\Omega}[u_{ttt}^2(x,T)+\sum_{i=1}^{n}u_{x_i}^2(x,T)]dx$$

$$+\frac{5-|\lambda|(A-T)T^3}{2}\int\limits_0^T\int\limits_{\Omega}u_{ttt}^2dxdt+\frac{1-|\lambda|c_0}{2}\sum_{i=1}^{n}\int\limits_0^T\int\limits_{\Omega}u_{x_i}^2dxdt\leq 0. \qquad (11)$$

Since inequality $|\lambda|c_0<1$ holds and we can choose number A close to number T, the inequality

$$5-|\lambda|(A-T)T^3>0$$

holds for fixed values of λ. Then, from (11) it follows that $u(x,t)\equiv 0$.

In the case of $c_1 = -\frac{40}{T^6}$, we have

$$\frac{A-T}{2}\int\limits_{\Omega}[u_{ttt}^2(x,T)+\sum_{i=1}^{n}u_{x_i}^2(x,T)]dx$$

$$+\frac{40-8|\lambda|(A-T)T^3-|\lambda|T^6}{2\cdot 2^3}\int\limits_0^T\int\limits_{\Omega}u_{ttt}^2dxdt+\frac{1}{2}\sum_{i=1}^{n}\int\limits_0^T\int\limits_{\Omega}u_{x_i}^2dxdt\leq 0. \quad (12)$$

Since $40-|\lambda|T^6>0$, then choosing again A close to the T, inequality

$$40-8|\lambda|(A-T)T^3-|\lambda|T^6>0$$

can be achieved. Then, from (12) we also get $u(x,t)\equiv 0$.

The solution to equation (1) is sought in the form $u(x,t)=\varphi(t)\omega_j(x)$. Then the function $\varphi(t)$ must be a solution to the equation

$$-D_t^6\varphi(t)+[\mu_j-\lambda]\varphi(t)=0, \quad (13)$$

satisfying condition

$$\varphi(0)=\varphi'(0)=\varphi''(0)=\varphi'''(0)=\varphi'(T)=\varphi''(T)=0. \quad (14)$$

a) If $\mu_j-\lambda>0$, then general solution (13) has the form

$$\varphi(t)=C_1e^{\gamma_jt}+C_2e^{\frac{\gamma_jt}{2}}cos\frac{\sqrt{3}}{2}\gamma_jt+C_3e^{\frac{\gamma_jt}{2}}sin\frac{\sqrt{3}}{2}\gamma_jt$$

$$+C_4e^{-\gamma_jt}+C_5e^{-\frac{\gamma_jt}{2}}cos\frac{\sqrt{3}}{2}\gamma_jt+C_6e^{-\frac{\gamma_jt}{2}}sin\frac{\sqrt{3}}{2}\gamma_jt, \quad (15)$$

where $\gamma_j=(\mu_j-\lambda)^{\frac{1}{6}}$. Taking into account (14), the numbers C_j, $j=\overline{1,6}$, should be a solution to the algebraic system

$$\begin{cases} C_1 + C_2 + C_4 + C_5 = 0, \\ C_1 + \frac{1}{2}C_2 + \frac{\sqrt{3}}{2}C_3 - C_4 - \frac{1}{2}C_5 + \frac{\sqrt{3}}{2}C_6 = 0, \\ C_1 - \frac{1}{2}C_2 + \frac{\sqrt{3}}{2}C_3 + C_4 - \frac{1}{2}C_5 - \frac{\sqrt{3}}{2}C_6 = 0, \\ C_1 - C_2 - C_4 + C_5 = 0, \\ E^2C_1 + E(\frac{1}{2}C - \frac{\sqrt{3}}{2}S)C_2 + E(\frac{\sqrt{3}}{2}C + \frac{1}{2}S)C_3 \\ -E^{-2}C_4 - E^{-1}(\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_5 + E^{-1}(\frac{\sqrt{3}}{2}C - \frac{1}{2}S)C_6 = 0, \\ E^2C_1 - E(\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_2 + E(\frac{\sqrt{3}}{2}C - \frac{1}{2}S)C_3 \\ +E^{-2}C_4 + E^{-1}(-\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_5 - E^{-1}(\frac{\sqrt{3}}{2}C + \frac{1}{2}S)C_6 = 0, \end{cases}$$

where

$$E = e^{\frac{\gamma_j T}{2}},\ C = cos\frac{\sqrt{3}}{2}\gamma_j T,\ S = sin\frac{\sqrt{3}}{2}\gamma_j T.$$

The determinant of this system will be equal to

$$D(\gamma_j) = \frac{3}{2}\left[2E^3C - 3E^2 - 6EC + 10 + 4C^2 - 6E^{-1}C - 3E^{-2} + 2E^{-3}C\right],$$

and it can not be zero, therefore, in this case, problem (13), (14) has not non-trivial solutions.

b) If $\mu_j - \lambda < 0$, then general solution (13) has a form

$$\varphi(t) = C_1e^{\frac{\sqrt{3}}{2}\gamma_j t}cos\frac{\gamma_j t}{2} + C_2e^{\frac{\sqrt{3}}{2}\gamma_j t}sin\frac{\gamma_j t}{2} + C_3e^{-\frac{\sqrt{3}}{2}\gamma_j t}cos\frac{\gamma_j t}{2}$$

$$+C_4e^{-\frac{\sqrt{3}}{2}\gamma_j t}sin\frac{\gamma_j t}{2} + C_5cos\gamma_j t + C_6sin\gamma_j t, \tag{16}$$

where $\gamma_j = (\lambda - \mu_j)^{\frac{1}{6}}$. Considering (14), the numbers $C_j, j = \overline{1,6}$, should be a solution to the algebraic system

$$\begin{cases} C_1+C_3+C_5=0, \\ \frac{\sqrt{3}}{2}C_1+\frac{1}{2}C_2-\frac{\sqrt{3}}{2}C_3+\frac{1}{2}C_4+C_6=0, \\ \frac{1}{2}C_1+\frac{\sqrt{3}}{2}C_2+\frac{1}{2}C_3-\frac{\sqrt{3}}{2}C_4-C_5=0, \\ C_2+C_4-C_6=0, \\ E(\frac{\sqrt{3}}{2}C-\frac{1}{2}S)C_1+E(\frac{1}{2}C+\frac{\sqrt{3}}{2}S)C_2-E^{-1}(\frac{\sqrt{3}}{2}C+\frac{1}{2}S)C_3 \\ +E^{-1}(\frac{1}{2}C-\frac{\sqrt{3}}{2}S)C_4-2CSC_5+(C^2-S^2)C_6=0, \\ E(\frac{1}{2}C-\frac{\sqrt{3}}{2}S)C_1+E(\frac{\sqrt{3}}{2}C+\frac{1}{2}S)C_2+E^{-1}(\frac{1}{2}C+\frac{\sqrt{3}}{2}S)C_3 \\ +E^{-1}(-\frac{\sqrt{3}}{2}C+\frac{1}{2}S)C_4+(-C^2+S^2)C_5-2CSC_6=0, \end{cases}$$

where $E=e^{\frac{\sqrt{3}}{2}\gamma_j T},\ C=cos\frac{\gamma_j T}{2},\ S=sin\frac{\gamma_j T}{2}$.

This system has a nontrivial solution if the determinant

$$D(\gamma_j)=-C^2S^2=-\frac{1}{4}sin^2\gamma_jT=0 \tag{17}$$

is equal to zero. From (17) we get desired set of eigenvalues

$$\lambda_{jk}^{(1)}=\mu_{jk}+\left(\frac{k\pi}{T}\right)^6,\quad k=1,2,\dots\ . \tag{18}$$

Theorem 1 is proved.

Corollary 1. *The problem $I_{3,\lambda}$ does not have real eigenvalues other than the numbers $\lambda_{jk}^{(1)}$ from (18) and the family $\{\lambda_{jk}^{(1)}\}_{j,k=1}^{\infty}$ does not have finite limit points. All eigenvalues of $\{\lambda_{jk}^{(1)}\}_{j,k=1}^{\infty}$ are finite multiplicity.*

Proof. The fact that the problem $I_{3,\lambda}$ does not have real eigenvalues other than the numbers $\lambda_{jk}^{(1)}$, follows from the basis of the system of functions

$$\{\omega_j(x)\}_{j=1}^{\infty}$$

in the space $W_2^2(\Omega)$.

Suppose that the family $\{\lambda_{jk}^{(1)}\}_{j,k=1}^{\infty}$ has a finite limit point. Then there is a family (j_i, k_i) of pairs of natural numbers such that $j_i + k_i \to \infty$ as $i \to \infty$ and the sequence $\lambda_{jk}^{(1)}$ will be fundamental. Note that the indices j_i, cannot be limited together, since in this case $\lambda_{jk} = \mu_{jk} + \left(\frac{k\pi}{T}\right)^6,\ k = 1, 2, ...$, which cannot be true for a fundamental sequence.

Further, the indices k_i also cannot be limited together, since in this case the sequence $\{\mu_{j_i} - \mu_{j_{i+m}}\}$ will be limited, which is not that case. Therefore, for the indices j_i and k_i, $j_i \to \infty$, $k_i \to \infty$ hold as $i \to \infty$. But then $\lambda_{j_k k_i} \to -\infty$, which again does not hold for a fundamental sequence. From the above, the validity of the second part of consequence follows. The finite multiplicity of each eigenvalue $\lambda_{jk}^{(1)}$ follows from the fact that for fixed numbers j and k the equality $\lambda_{jk}^{(1)} = \lambda_{j_1 k_1}^{(1)}$ is only possible for a finite set of indices j_1 and k_1. Consequence proved.

Note that for the case $n = 1$ the eigenvalues μ_j could be in exact form, and then it is easy to give constructive conditions for the simplicity of each eigenvalue $\lambda_{jk}^{(1)}$ or to provide examples in which the eigenvalues will have a multiplicity greater than one. In the general case, it is also easy to give simplicity conditions, but it seems that they will not be constructive.

Corollary 2. *The eigenvalues $\lambda_{jk}^{(1)}$ of the problem $I_{3,\lambda}$ correspond to the eigenfunctions*

$$u_{jk}^{(1)}(x,t) = \omega_j(x)\varphi_k^{(1)}(t),$$

where function $\varphi_k^{(1)}(t)$ represented as

$$\varphi_k^{(1)}(t) = \frac{C}{12S_k(E_k - E_k^{-1})}\Big[-(3C_k(E_k - E_k^{-1}) + 5\sqrt{3}S_k(E_k + E_k^{-1}) + 6)e^{\frac{\sqrt{3}}{2}\gamma_k t}cos\frac{\gamma_k t}{2}$$
$$-(3\sqrt{3}C_k(E_k + E_k^{-1}) - 15S_k(E_k - E_k^{-1}) + 3\sqrt{3})e^{\frac{\sqrt{3}}{2}\gamma_k t}sin\frac{\gamma_k t}{2}$$
$$+(-3C_k(E_k + E_k^{-1}) + (4 + 5\sqrt{3})S_k(E_k - E_k^{-1}) - 6)e^{-\frac{\sqrt{3}}{2}\gamma_k t}cos\frac{\gamma_k t}{2}$$
$$+(3\sqrt{3}C_k(E_k + E_k^{-1}) + 15S_k(E_k - E_k^{-1}) - 6\sqrt{3})e^{-\frac{\sqrt{3}}{2}\gamma_k t}sin\frac{\gamma_k t}{2}$$
$$+(6C_k(E_k + E_k^{-1}) - 6\sqrt{3}S_k(E_k - E_k^{-1}) + 12)cos\gamma_k t + 12S_k(E_k - E_k^{-1})sin\gamma_k t\Big],$$

$$E_k = e^{\frac{\sqrt{3}\pi k}{2}},\ C_k = cos\frac{\pi k}{2},\ S_k = sin\frac{\pi k}{2},\ C = Const,\ k = 1, 2, ...$$

Now consider the problem II_3. The study of the problem II_3 is similar to I_3. The following theorem holds.

Theorem 2. *For $\lambda > c_1, c_1 = \min\{-\frac{1}{c_0}, -\frac{40}{T^6}\}$, the homogeneous boundary problem $II_{3,\lambda}$ has only zero solution in the space V_3. On the interval $(-\infty, c_1)$ there does not exist a countable set of numbers $\lambda_m^{(2)}$ such that for $\lambda = \lambda_m^{(2)}$ the homogeneous boundary problem $II_{3,\lambda}$ has only trivial solution.*

The solution to equation (1) is sought in the form $u(x,t) = \varphi(t)\omega_j(x)$. Then, the function $\varphi(t)$ must be a solution to equation (13) that satisfies conditions

$$\varphi(0) = \varphi'(0) = \varphi''(0) = \varphi'''(0) = \varphi''''(T) = \varphi'''''(T) = 0. \tag{19}$$

a) If $\mu_j - \lambda > 0$, then the general solution $\varphi(t)$ has the form

$$\varphi(t) = C_1 e^{\gamma_j t} + C_2 e^{\frac{\gamma_j t}{2}} cos\frac{\sqrt{3}}{2}\gamma_j t + C_3 e^{\frac{\gamma_j t}{2}} sin\frac{\sqrt{3}}{2}\gamma_j t$$

$$+C_4 e^{-\gamma_j t} + C_5 e^{-\frac{\gamma_j t}{2}} cos\frac{\sqrt{3}}{2}\gamma_j t + C_6 e^{-\frac{\gamma_j t}{2}} sin\frac{\sqrt{3}}{2}\gamma_j t,$$

where $\gamma_j = (\mu_j - \lambda)^{\frac{1}{6}}$. Considering (15), $C_j, j = \overline{1,6}$, should be a solution to the algebraic system

$$\begin{cases} C_1 + C_2 + C_4 + C_5 = 0, \\ C_1 + \frac{1}{2}C_2 + \frac{\sqrt{3}}{2}C_3 - C_4 - \frac{1}{2}C_5 + \frac{\sqrt{3}}{2}C_6 = 0, \\ C_1 - \frac{1}{2}C_2 + \frac{\sqrt{3}}{2}C_3 + C_4 - \frac{1}{2}C_5 - \frac{\sqrt{3}}{2}C_6 = 0, \\ C_1 - C_2 - C_4 + C_5 = 0, \\ E^2 C_1 + E(-\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_2 - E(\frac{\sqrt{3}}{2}C + \frac{1}{2}S)C_3 \\ +E^{-2}C_4 - E^{-1}(\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_5 + E^{-1}(\frac{\sqrt{3}}{2}C - \frac{1}{2}S)C_6 = 0, \\ E^2 C_1 + E(\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_2 + E(-\frac{\sqrt{3}}{2}C + \frac{1}{2}S)C_3 \\ -E^{-2}C_4 + E^{-1}(-\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_5 - E^{-1}(\frac{\sqrt{3}}{2}C + \frac{1}{2}S)C_6 = 0, \end{cases}$$

where $E = e^{\frac{\gamma_j T}{2}}$, $C = cos\frac{\sqrt{3}}{2}\gamma_j T$, $S = sin\frac{\sqrt{3}}{2}\gamma_j T$. The determinant of this system will be equal to

$$D(\gamma_j) = -\frac{3}{2}\left[2E^3C + 3E^2 + 6EC + 10 + 4C^2 + 6E^{-1} + 3E^{-2} + 2E^{-3}C\right],$$

and it can not be zero, therefore, in this case, there are no non-trivial solutions.

b) If $\mu_j - \lambda < 0$, then the function $\varphi(t)$ has the form

$$\varphi(t) = C_1 e^{\frac{\sqrt{3}}{2}\gamma_j t} cos\frac{\gamma_j t}{2} + C_2 e^{\frac{\sqrt{3}}{2}\gamma_j t} sin\frac{\gamma_j t}{2} + C_3 e^{-\frac{\sqrt{3}}{2}\gamma_j t} cos\frac{\gamma_j t}{2}$$

$$+C_4 e^{-\frac{\sqrt{3}}{2}\gamma_j t} sin\frac{\gamma_j t}{2} + C_5 cos\gamma_j t + C_6 sin\gamma_j t,$$

where $\gamma_j = (\lambda - \mu_j)^{\frac{1}{6}}$. In this case, $C_j,\ j = \overline{1,6}$, should be a solution to the algebraic system

$$\begin{cases} C_1 + C_3 + C_5 = 0, \\ \frac{\sqrt{3}}{2}C_1 + \frac{1}{2}C_2 - \frac{\sqrt{3}}{2}C_3 + \frac{1}{2}C_4 + C_6 = 0, \\ \frac{1}{2}C_1 + \frac{\sqrt{3}}{2}C_2 + \frac{1}{2}C_3 - \frac{\sqrt{3}}{2}C_4 - C_5 = 0, \\ C_2 + C_4 - C_6 = 0, \\ -E(\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_1 + E(\frac{\sqrt{3}}{2}C - \frac{1}{2}S)C_2 + E^{-1}(-\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_3 \\ -E^{-1}(\frac{\sqrt{3}}{2}C + \frac{1}{2}S)C_4 + (C^2 - S^2)C_5 + 2CSC_6 = 0, \\ -E(\frac{\sqrt{3}}{2}C + \frac{1}{2}S)C_1 + E(\frac{1}{2}C - \frac{\sqrt{3}}{2}S)C_2 + E^{-1}(\frac{\sqrt{3}}{2}C - \frac{1}{2}S)C_3 \\ +E^{-1}(\frac{1}{2}C + \frac{\sqrt{3}}{2}S)C_4 - 2CSC_5 + (C^2 - S^2)C_6 = 0, \end{cases}$$

where $E = e^{\frac{\sqrt{3}}{2}\gamma_j T},\ C = cos\frac{\gamma_j T}{2},\ S = sin\frac{\gamma_j T}{2}$. The determinant of this system will be equal to

$$D(\gamma_j) = \frac{3}{4}\left[E^2 + 8EC^3 + 6 + 12C^2 + 8E^{-1}C^3 + E^{-2}\right],$$

also can not be zero.

In conclusion, the problem $II_{3,\lambda}$ does not have real eigenvalues $\lambda_{jk}^{(2)}$. Theorem 2 is proved.

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Кожанов А.И., Қошанов Б.Д., Сматова Г.Д. АЛТЫНШЫ РЕТТІ КЛАССИКАЛЫҚ ЕМЕС ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ҮШІН ҚИСЫНДЫ ШЕТТІК ЕСЕПТЕР ТУРАЛЫ

Бұл мақалада келесі алтыншы ретті квазигиперболалық теңдеу үшін

$$Lu = -D_t^6 u + \Delta u - \lambda u$$

шеттік есептердің Соболев кеңістігіндегі қисынды шешілімділігі зерттелген, мұнда $D_t = \frac{\partial}{\partial t}$, $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ – Лаплас операторы, λ – нақты параметр. Берілген L операторы үшін екі классикалық емес спектрлік есеп қойылған. Қойылған есептердің шешімінің жалғыздығы дәлелденген. Бірінші есептің меншікті мәндері мен меншікті функцияларының бар екендігі дәлелденген, яғни бұл есептің нөлдік емес шешімдері табылған. Бұл жұмыста $Lu = 0$ теңдеуі үшін $\lambda < 0$ болғанда және біртектілік шарттары орындалғанда спектрлік есептің нөлден өзгеше шешімдерінің, яғни меншікті функцияларының саналымды жүйесінің бар екендігі көрсетілген. L операторы кәдуілгі гиперболалық оператор болғанда мұндай жағдай әдетте орын алмайды.

Кілттік сөздер. Алтыншы ретті квазигиперболалық теңдеу, шеттік есептер, меншікті мәндер, меншікті функциялар, нөлдік емес шешімдер.

Кожанов А.И., Кошанов Б.Д., Сматова Г.Д. О КОРРЕКТНЫХ КРАЕВЫХ ЗАДАЧАХ ДЛЯ НЕКЛАССИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ШЕСТОГО ПОРЯДКА

В данной статье исследуется корректная разрешимость краевых задач для квазигиперболического уравнения шестого порядка в пространстве Соболева:

$$Lu = -D_t^6 u + \Delta u - \lambda u$$

($D_t = \frac{\partial}{\partial t}$, $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ – оператор Лапласа, λ – вещественный параметр). Ставятся две неклассические спектральные задачи для данного оператора L. Доказывается единственность поставленных задач. Доказывается существование собственных чисел и собственных функций поставленной первой задачи. В работе будет показано, что для уравнения $Lu = 0$ при $\lambda < 0$ и при выполнении однородных условий спектральная задача обладает счетной системой нетривиальных решений – собственных функций. Обычно такое не имеет место, когда оператор L есть обычный гиперболический оператор.

Ключевые слова. Квазигиперболические уравнения шестого порядка, краевые задачи, собственные значения, собственные функции, нетривиальные решения.

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Mild solution to integro-differential diffusion system with nonlocal source

Meiirkhan B. Borikhanov

Al–Farabi Kazakh National University, Almaty, Kazakhstan
Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan

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Abstract. In the present paper initial problem for the integro-differential diffusion system with nonlocal nonlinear source is considered. The results on the existence of local mild solutions to the nonlinear integro-differential diffusion system are presented.

Keywords. Local existence, mild solution, integro-differential diffusion system.

The main goal of the present paper is to obtain results on local existence of mild solution to the integro-differential diffusion system

$$\begin{cases} u_t(x,t) - \dfrac{\partial^2}{\partial x^2} D_{0|t}^{1-\alpha} u(x,t) = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} |v|^{p-1} v(s) ds, \\ \\ v_t(x,t) - \dfrac{\partial^2}{\partial x^2} D_{0|t}^{1-\beta} v(x,t) = \dfrac{1}{\Gamma(1-\delta)} \displaystyle\int_0^t (t-s)^{-\delta} |u|^{q-1} u(s) ds, \end{cases} \tag{1}$$

for $(x,t) \in \mathbb{R} \times (0,T) = \Omega_T$, subject to the initial conditions

$$u(x,0) = u_0(x) \geq 0, \; v(x,0) = v_0(x) \geq 0, \; x \in \mathbb{R}, \tag{2}$$

where $\alpha, \beta, \gamma, \delta \in (0,1)$, $p > 1$, $q > 1$, $D_{0|t}^{\mu}$ is the left-handed Riemann-Liouville fractional derivative of order $\mu \in (0,1)$ and Γ is the gamma function of Euler.

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Recently, Kirane et al. in [1] concerned the Cauchy problem for the fractional diffusion equation with a time nonlocal nonlinearity of exponential growth

$$\begin{cases} \mathcal{D}^{\alpha}_{0|t}u(x,t) + (-\Delta)^{\frac{\beta}{2}}u(x,t) = I^{1-\alpha}_{0|t}(e^u), \quad x \in \mathbb{R}^N, \ t > 0, \\ \\ u(x,0) = u_0(x), \ x \in \mathbb{R}^N, \end{cases} \tag{3}$$

where $N \geq 1, \ 0 < \alpha < 1, \ 0 < \beta \leq 2, \ \mathcal{D}^{\alpha}_{0|t}$ is the Caputo fractional derivative operator of order α, $I^{1-\alpha}_{0|t}(e^u)$ is the Riemann-Liouville fractional integral of order $1-\alpha$ for e^u.

They proved the existence and uniqueness of the local solution by the Banach contraction mapping principle. Then, the blowup result of the solution in finite time is established by the test function method with a judicious choice of the test function.

Later on, Ahmad et al. in [2] considered the following problem

$$\begin{cases} u_t(x,t) + (-\Delta)^{\frac{\beta}{2}}u(x,t) = I^{1-\alpha}_{0|t}(e^u), \quad x \in \mathbb{R}^N, \ t > 0, \\ \\ u(x,0) = u_0(x), \ x \in \mathbb{R}^N, \end{cases} \tag{4}$$

and when the problem (3) is also considered with a nonlinearity of the form $I^{1-\alpha}_{0|t}(|u|^{p-1}u)$, it reads

$$\begin{cases} u_t(x,t) + (-\Delta)^{\frac{\beta}{2}}u(x,t) = I^{1-\alpha}_{0|t}(|u|^{p-1}u), \quad x \in \mathbb{R}^N, \ t > 0, \\ \\ u(x,0) = u_0(x), \ x \in \mathbb{R}^N, \end{cases} \tag{5}$$

has been considered by Fino and Kirane in [3].

Also, Fino and Kirane in [4] studied the Cauchy problem for the semi-linear parabolic system with a nonlinear memory

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma}|v|^{p-1}v(s)ds, \quad x \in \mathbb{R}^N, \ t > 0, \\ \\ v_t(x,t) - \Delta v(x,t) = \dfrac{1}{\Gamma(1-\delta)} \displaystyle\int_0^t (t-s)^{-\delta}|u|^{q-1}u(s)ds, \quad x \in \mathbb{R}^N, \ t > 0, \end{cases} \tag{6}$$

supplemented with the initial conditions

$$u\,(x,0) = u_0\,(x) \geq \ 0, \ v\,(x,0) = v_0\,(x) \geq 0, \ x \in \mathbb{R}^N, \tag{7}$$

where $u_0(x), v_0(x) \in C_0(\mathbb{R}^N)$, $\gamma,\ \delta \in (0,1)$ and Γ is the Euler gamma function.

In these papers, they proved the existence of a unique local solution and under some suitable conditions on the initial data, they proved that the solution blows up in a finite time and studied its time blow-up profile.

In [5], Zhang and Sun investigated the blow-up and the global existence of solutions of the Cauchy problem for a time fractional nonlinear diffusion equation

$$\begin{cases} \mathcal{D}^{\alpha}_{0|t}u(x,t) - \Delta u(x,t) = |u|^{p-1}u, \quad x \in \mathbb{R}^N,\ t > 0, \\ \\ u(x,0) = u_0(x),\ x \in \mathbb{R}^N, \end{cases} \tag{8}$$

where $p > 1, 0 < \alpha < 1, u_0(x) \in C_0(\mathbb{R}^N)$ and $\mathcal{D}^{\alpha}_{0|t}$ is the Caputo fractional derivative operator of order α.

Definition 1. *The left and right Riemann-Liouville fractional integrals $I^{\alpha}_{0|t}f(t)$ and $I^{\alpha}_{t|T}f(t)$ of order $\alpha \in \mathbb{R}\,(\alpha > 0)$, for all $f(t) \in L^q(0,T))(1 \leq q \leq \infty)$, we defined as [see p. 69 in [6]]*

$$I^{\alpha}_{0|t}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$

and

$$I^{\alpha}_{t|T}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1} f(s)ds,$$

respectively.

Definition 2. *If $f(t) \in C([0,T])$, the left-handed and right-handed Riemann-Liouville fractional derivatives $D^{\alpha}_{0|t}f(t)$ and $D^{\alpha}_{t|T}f(t)$ of order $\alpha \in (0,1)$ are defined by [see p. 70 in [6]]*

$$D^{\alpha}_{0|t}f(t) = \frac{d}{dt}I^{1-\alpha}_{0|t}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha} f(s)\,ds,$$

and

$$D^{\alpha}_{t|T}f(t) = -\frac{d}{dt}I^{1-\alpha}_{t|T}f(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T (s-t)^{-\alpha} f(s)\,ds,$$

for all $f(t) \in [0,T]$.

Definition 3. *The Mittag-Leffler function is given by [see p. 40 in [6]]*

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \ \alpha > 0, \ z \in \mathbb{C}.$$

Lemma 1 [7]. *For every* $\alpha \in (0,1)$, *the uniform bilateral estimate*

$$\frac{1}{1+\Gamma(1-\alpha)x} \leq E_{\alpha,1}(-x) \leq \frac{1}{1+[\Gamma(1+\alpha)]^{-1}x}$$

holds over $\mathbb{R}^+$.

Lemma 2 [8]. *The Fourier transform of Dirac delta function* $\delta(x)$ *in* $\mathbb{R}$ *defined by*

$$F\{\delta(x);\xi\} = \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-ix\xi}\delta(x)dx = 1, \ \xi \in \mathbb{R},$$

and the inverse Fourier transform of $\delta(x)$ *can be written as*

$$\delta(x) = F^{-1}\{1\} = \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{ix\xi}d\xi, \ \xi \in \mathbb{R}.$$

The Dirac delta function $\delta(x)$, where $x \in \mathbb{R}$, in [9]:

$$\delta(x) = \begin{cases} +\infty \ \text{ for } \ x = 0, \\ 0 \ \text{ for } \ x \neq 0, \end{cases}$$

and

$$\int\limits_{\mathbb{R}} \delta(x)dx = 1.$$

Definition 4 (Mild solution). *Let* $u_0, v_0 \in C_0(\mathbb{R}), T > 0$ *and* $p, q > 1$.

We say that $(u,v) \in C_0(\mathbb{R}; C[0,T]) \times C_0(\mathbb{R}; C[0,T])$ *is a mild solution of the system* (1)–(2), *if* u *and* v *satisfy the following integral equations [see [10], Th. 2.5]:*

$$\begin{cases} u(x,t) = \int\limits_{\mathbb{R}} G(x-y,t)u_0(y)\,dy + \int\limits_0^t \int\limits_{\mathbb{R}} G(x-y,t-\tau) I_{0|s}^{1-\gamma}(|v|^{p-1}v)dyd\tau, \\ \\ v(x,t) = \int\limits_{\mathbb{R}} G(x-y,t)v_0(y)\,dy + \int\limits_0^t \int\limits_{\mathbb{R}} G(x-y,t-\tau) I_{0|s}^{1-\delta}(|u|^{q-1}u)dyd\tau, \end{cases} \tag{9}$$

for $t \in [0,T), x \in \mathbb{R}$, *where*

$$G\left(x,t\right)=\frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-ix\xi}E_{\alpha,1}\left(-\xi^{2}t^{\alpha}\right)d\xi$$

is a heat kernel of problem (1)–(2) *[10].*

Lemma 3. *$G(x,t)$ function in* (9) *has the following estimate:*

$$\int\limits_{\mathbb{R}} G(x,t)dx < 1,\ t > 0. \tag{10}$$

Proof. Accordingly to Lemma 1, we have that

$$G\left(x,t\right)=\frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-ix\xi}E_{\alpha,1}\left(-\xi^{2}t^{\alpha}\right)d\xi \le \frac{1}{2\pi}\left|\int\limits_{\mathbb{R}} e^{-ix\xi}E_{\alpha,1}\left(-\xi^{2}t^{\alpha}\right)d\xi\right|$$

$$\le \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-ix\xi}\left|E_{\alpha,1}\left(-\xi^{2}t^{\alpha}\right)\right|d\xi < \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-ix\xi}\cdot 1d\xi=\delta(x),$$

where $\delta(x)$ is the Dirac delta function.

From Lemma 2 we obtain

$$\int\limits_{\mathbb{R}} G(x,t)dx < \int\limits_{\mathbb{R}} \delta(x)dx = 1,\ \ t > 0.$$

Theorem 1 (Local existence). *Given $u_0, v_0 \in C_0(\mathbb{R})$ and $p, q > 1$. Then, there exists a maximal time $T > 0$ such that the system* (1)–(2) *has a unique mild solution $(u,v) \in C_0(\mathbb{R}; C[0,T)) \times C_0(\mathbb{R}; C[0,T))$. Furthermore, either $T = \infty$ or $T < \infty$ and $\|u(t)\|_{L^\infty(\mathbb{R}\times(0,T))} + \|v(t)\|_{L^\infty(\mathbb{R}\times(0,T))} \to \infty$, as $t \to T$.*

Proof. For arbitrary $T > 0$, we define the Banach space

$$B_T = \{(u,v) \in C_0(\mathbb{R}; C[0,T)) \times C_0(\mathbb{R}; C[0,T));$$
$$||(u,v)||_{B_T} \le 2(\| u_0 \|_{L^\infty(\mathbb{R})} + \| v_0 \|_{L^\infty(\mathbb{R})})\}, \tag{11}$$

where $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\mathbb{R})}$ and $||\cdot||_{B_T}$ is the norm of B_T defined by

$$||(u,v)||_{B_T} = \| u \|_1 + \| v \|_1 = \| u \|_{L^\infty(\mathbb{R}\times(0,T))} + \| v \|_{L^\infty(\mathbb{R}\times(0,T))},$$

and

$$d(u,v) = \max_{t\in[0,T)} ||u(t) - v(t)||_{L^\infty(\mathbb{R})} \text{ for } u, v \in B_T.$$

Since $C_0(\mathbb{R}; C[0,T))$ is the Banach space, $(B_T; d)$ is a complete metric space.

Next, for every $(u,v) \in B_T$, we introduce the map Ψ defined on B_T by

$$\Psi(u,v) := (\Psi_1(u,v), \Psi_2(u,v)),$$

where

$$\Psi_1(u,v) = \int\limits_{\mathbb{R}} G\,(x-y,t)u_0\,(y)\,dy + \int\limits_0^t \int\limits_{\mathbb{R}} G\,(x-y,t-\tau)I_{0|s}^{1-\gamma}(|v|^{p-1}v)dyd\tau,\ t \in [0,T),$$

and

$$\Psi_2(u,v) = \int\limits_{\mathbb{R}} G\,(x-y,t)v_0\,(y)\,dy$$

$$+\int\limits_0^t \int\limits_{\mathbb{R}} G\,(x-y,t-\tau)I_{0|s}^{1-\delta}(|u|^{q-1}u)dyd\tau,\ t \in [0,T).$$

We will prove the local existence by the Banach fixed point theorem.

• $\Psi : B_T \to B_T$.

If $(u,v) \in B_T$, using Lemma 3, we obtain

$$||\Psi(u,v)||_{B_T} \le \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)}\left\|\int\limits_0^t \int\limits_0^s (s-\tau)^{-\gamma}\|v(\tau)\|_\infty^p d\tau ds\right\|_{L^\infty(0,T)}$$

$$+\|v_0\|_\infty + \frac{1}{\Gamma(1-\delta)}\left\|\int\limits_0^t \int\limits_0^s (s-\tau)^{-\delta}\|u(\tau)\|_\infty^q d\tau ds\right\|_{L^\infty(0,T)}$$

$$\le \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)}\left\|\int\limits_0^t \int\limits_\tau^t (s-\tau)^{-\gamma}\|v(\tau)\|_\infty^p ds d\tau\right\|_{L^\infty(0,T)}$$

$$+\|v_0\|_\infty + \frac{1}{\Gamma(1-\delta)}\left\|\int\limits_0^t \int\limits_\tau^t (s-\tau)^{-\delta}\|u(\tau)\|_\infty^q ds d\tau\right\|_{L^\infty(0,T)}$$

$$\le \|u_0\|_\infty + C_1 T^{2-\gamma}\|v\|_1^p + \|v_0\|_\infty + C_2 T^{2-\delta}\|u\|_1^q,$$

where

$$C_1 := \frac{1}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} = \frac{1}{\Gamma(3-\gamma)},$$

$$C_2 := \frac{1}{(1-\delta)(2-\delta)\Gamma(1-\delta)} = \frac{1}{\Gamma(3-\delta)}.$$

As $(u,v) \in B_T$, we get

$$||\Psi(u,v)||_{B_T} \leq \|u_0\|_\infty + C_1T^{2-\gamma}\|v\|_1^p + \|v_0\|_\infty + C_2T^{2-\delta}\|u\|_1^q$$
$$\leq \|u_0\|_\infty + \|v_0\|_\infty + \max\{C_1T^{2-\gamma}\|v\|_1^{p-1}; C_2T^{2-\delta}\|u\|_1^{q-1}\}(\|v\|_1 + \|u\|_1)$$
$$\leq (\|u_0\|_\infty + \|v_0\|_\infty) + 2T(u_0,v_0)(\|u_0\|_\infty + \|v_0\|_\infty),$$

where

$$T(u_0,v_0) = \max\{C_1T^{2-\gamma}2^{p-1}(\|u_0\|_\infty + \|v_0\|_\infty)^{p-1}; C_2T^{2-\delta}2^{q-1}(\|u_0\|_\infty + \|v_0\|_\infty)^{q-1}\}.$$

If we choose T small enough such that

$$2T(u_0,v_0) \leq 1, \tag{12}$$

we conclude that $\|\Psi(u)\|_1 \leq 2(\|u_0\|_\infty + \|v_0\|_\infty)$ and hence $\Psi(u,v) \in B_T$.

- Let Ψ be a contraction map.

For $(u,v), (\tilde{u},\tilde{v}) \in B_T$, we have the estimate

$$||\Psi(u,v) - \Psi(\tilde{u},\tilde{v})||_{B_T}$$
$$\leq \frac{1}{\Gamma(1-\gamma)}\left\|\int_0^t\int_0^s (s-\tau)^{-\gamma}\||v|^{p-1}v(\tau) - |\tilde{v}|^{p-1}\tilde{v}(\tau)\|_\infty d\tau ds\right\|_{L^\infty(0,T)}$$
$$+\frac{1}{\Gamma(1-\delta)}\left\|\int_0^t\int_0^s (s-\tau)^{-\delta}\||u|^{q-1}u(\tau) - |\tilde{u}|^{q-1}\tilde{u}(\tau)\|_\infty d\tau ds\right\|_{L^\infty(0,T)}$$
$$= \frac{1}{\Gamma(1-\gamma)}\left\|\int_0^t\int_\tau^t (s-\tau)^{-\gamma}\||v|^{p-1}v(\tau) - |\tilde{v}|^{p-1}\tilde{v}(\tau)\|_\infty ds d\tau\right\|_{L^\infty(0,T)}$$
$$+\frac{1}{\Gamma(1-\delta)}\left\|\int_0^t\int_\tau^t (s-\tau)^{-\delta}\||u|^{q-1}u(\tau) - |\tilde{u}|^{q-1}\tilde{u}(\tau)\|_\infty ds d\tau\right\|_{L^\infty(0,T)}$$

$$= C_1 T^{2-\gamma} \| |v|^{p-1}v - |\tilde{v}|^{p-1}\tilde{v}\|_1 + C_2 T^{2-\delta} \| |u|^{q-1}u - |\tilde{u}|^{q-1}\tilde{u}\|_1.$$

Now, by the same computations as above, we have

$$\begin{aligned}||\Psi(u,v) - \Psi(\tilde{u},\tilde{v})||_{B_T} &\le C_1 T^{2-\gamma} \| |v|^{p-1}v - |\tilde{v}|^{p-1}\tilde{v}\|_1 \\ &+ C_2 T^{2-\delta} \| |u|^{q-1}u - |\tilde{u}|^{q-1}\tilde{u}\|_1 \\ &\le C(p) C_1 T^{2-\gamma} (\|v^{p-1}\|_1 + \|\tilde{v}|^{p-1}\|_1)\|v - \tilde{v}\|_1 \\ &+ C(q) C_2 T^{2-\delta} (\|u^{q-1}\|_1 + \|\tilde{u}|^{q-1}\|_1)\|u - \tilde{u}\|_1 \\ &\le 2C(p,q) T(u_0, v_0) \|| (u,v) - (\tilde{u},\tilde{v}) \|| \le \frac{1}{2} \|| (u,v) - (\tilde{u},\tilde{v}) \||,\end{aligned}$$

thanks to the following inequality

$$||u|^{p-1}u - |v|^{p-1}v| \le C(p)|u - v| \left(|u|^{p-1} + |v|^{p-1}\right), \tag{13}$$

T is chosen such that

$$\max\{2C(p,q), 1\} T(u_0, v_0) \le \frac{1}{2}. \tag{14}$$

According to the Banach fixed point theorem, system (1)–(2) admits a unique mild solution $(u,v) \in B_T$.

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Бөріханов М.Б.

БЕЙЛОКАЛ ДЕРЕККӨЗДІ ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ДИФФУЗИЯЛЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІНІҢ ТЕГІС ШЕШІМІ

Бұл жұмыста бейлокал бейсызықты дереккөзді интегралдық-дифференциалдық диффузиялық теңдеулер жүйесі үшін Коши есебінің локалды тегіс шешімі зерттелген. Берілген теңдеулер жүйесі Фурье түрлендіруі арқылы шешіліп, оның Грин функциясы құрылған және қасиеттері келтірілген. Жалғыз локалды шешімнің бар екендігі Банахтың жылжымайтын нүкте туралы теоремасы негізінде дәлелденеді.

Кілттік сөздер. Локалды шешімнің бар болуы, тегіс шешім, интегралдық-дифференциалдық диффузиялық теңдеулер жүйесі.

Бориханов М.Б.

ГЛАДКОЕ РЕШЕНИЕ СИСТЕМЫ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ ДИФФУЗИОННЫХ УРАВНЕНИЙ С НЕЛОКАЛЬНЫМ ИСТОЧНИКОМ

В этой работе изучено локальное гладкое решение задачи Коши для системы интегро-дифференциальных диффузионных уравнений с нелокальным нелинейным источником. С помощью преобразования Фурье решена заданная система уравнений, построена функция Грина и приведены ее свойства. Соответственно доказано существование единственного локального решения на основе теоремы Банаха о неподвижной точке.

Ключевые слова. Существование локального решения, гладкое решение, система интегро-дифференциальных диффузионных уравнений.

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Two-phase spherical Stefan problem with non-linear thermal conductivity

Stanislav N. Kharin[1,3,a], Targyn A. Nauryz[1,2,3,4,b]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[2]Al-Farabi Kazakh National University, Almaty, Kazakhstan
[3]Kazakh-British Technical University, Almaty, Kazakhstan
[4]Satbayev University, Almaty, Kazakhstan
[a] e-mail: staskharin@yahoo.com, [b]e-mail: targyn.nauryz@gmail.com

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Abstract. The existence of the solution of two-phase spherical Stefan problem with temperature dependence thermal coefficients is considered. Using the similarity principle this problem is reduced to a nonlinear ordinary differential equation, and then to a nonlinear integral equation of the Volterra type. It is proved that the obtained operator is an abstraction type, therefore the integral equation can be solved by the iteration method.

Keywords. Stefan problem, similarity solution, nonlinear ordinary differential equation, thermal coefficients, nonlinear integral equation.

1 Introduction

In the Stefan problem with nonlinear thermal coefficients, it is important to give attention to the temperature dependence of the specific heat and thermal conductivity to determine the heat process between the melting and boiling isotherms [1]. One-dimensional Stefan problem with a thermal coefficient at a fixed face is considered in papers [2]–[4].

The process of a closure of electrical contacts is accompanied by an explosion of a micro-asperity at the attaching point, ignition of an electrical arc and the formation of three zones, metallic vapor zone, liquid and solid zones, which start to move simultaneously. The temperature fields in all can be described by the heat equations. For the vapor zone we have

$$c_1(T_1)\gamma_1(T_1)\frac{\partial T_1}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[\lambda_1(T_1)r^2\frac{\partial T_1}{\partial r}\right], \quad 0 < r < \alpha(t), \quad t > 0, \tag{1}$$

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for the liquid zone

$$c_2(T_2)\gamma_2(T_2)\frac{\partial T_2}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left[\lambda_2(T_2)r^2\frac{\partial T_2}{\partial r}\right],\quad \alpha(t)<r<\beta(t),\quad t>0, \tag{2}$$

and for the solid zone

$$c_3(T_3)\gamma_3(T_3)\frac{\partial T_3}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left[\lambda_3(T_3)r^2\frac{\partial T_3}{\partial r}\right],\quad \beta(t)<r<\infty,\quad t>0. \tag{3}$$

At the initial time the vapor and liquid zones collapse into a point

$$\alpha(0)=\beta(0)=0$$

and initial conditions for the temperatures are

$$T_1(0,0)=T_2(r,0)=T_3(r,0)=T_0=const \tag{4}$$

and the arc heat source with the temperature of metallic vapor ionization T_i placed at the point $r=0$ is

$$T_1(0,t)=T_i. \tag{5}$$

Finally, the Stefan conditions should be written on the surfaces of the phase transformations:

$$T_1(\alpha(t),t)=T_2(\alpha(t),t)=T_b, \tag{6}$$

$$-\lambda_1(T_b)\frac{\partial T_1(\alpha(t),t)}{\partial r}=-\lambda_2(T_b)\frac{\partial T_2(\alpha(t),t)}{\partial r}+L_b\gamma_1(T_b)\frac{d\alpha}{dt}, \tag{7}$$

$$T_2(\beta(t),t)=T_3(\beta(t),t)=T_m, \tag{8}$$

$$-\lambda_2(T_m)\frac{\partial T_2(\beta(t),t)}{\partial r}=-\lambda_3(T_m)\frac{\partial T_3(\beta(t),t)}{\partial r}+L_m\gamma_2(T_m)\frac{d\beta}{dt}, \tag{9}$$

where $T_1(r,t)$ is temperature of vapor zone, $T_2(r,t)$ is temperature of liquid zone and $T_3(r,t)$ is temperature of solid zone. $c_i(Ti),\gamma_i(Ti)$ and $\lambda_i(Ti)$ are material's density, specific heat and thermal conductivity. T_b,T_m are boiling and melting temperature, $\alpha(t),\beta(t)$ are free boundaries.

If the value of the heat flux entering into the solid zone from the liquid zone is small in comparison with the value of the heat flux consumed for the phase transformation of the solid into the liquid, then the conditions (8)–(9) transform into the one-phase conditions

$$T_2(\beta(t),t)=T_m, \tag{10}$$

$$-\lambda_2(T_m)\frac{\partial T_2(\beta(t),t)}{\partial r}=L_m\gamma_2(T_m)\frac{d\beta}{dt}, \tag{11}$$

while the temperature of the solid zone remains the same value T_0 like at the initial time, and equation (3) should be omitted.

Thus, the final version of the problem includes equations (1)–(2), (4)–(7), (10)–(11). It should be noted that the problem is a classical Stefan problem without fitting conditions (4) and (5) which was introduced and considered by Stefan, Lame and Clapeyron.

2 Similarity solution of the problem

To solve problem (1)–(11) we use the substitution $\theta(r,t) = \dfrac{T(r,t) - T_m}{T_b - T_m}$ and get the following problem

$$c_1(\theta_1)\gamma_1(\theta_1)\frac{\partial \theta_1}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[\lambda_1(\theta_1) r^2 \frac{\partial \theta_1}{\partial r}\right], \quad 0 < r < \alpha(t), \quad t > 0, \tag{12}$$

$$c_2(\theta_2)\gamma_2(\theta_2)\frac{\partial \theta_2}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[\lambda_2(\theta_2) r^2 \frac{\partial \theta_2}{\partial r}\right], \quad \alpha(t) < r < \beta(t), \quad t > 0, \tag{13}$$

$$\theta_2(0,0) = \theta_2(r,0) = \theta_0 = const, \quad \alpha(0) = \beta(0) = 0, \tag{14}$$

$$\theta_1(0,t) = \theta_i \tag{15}$$

$$\theta_1(\alpha(t),t) = \theta_2(\alpha(t),t) = 1, \tag{16}$$

$$-\lambda_1 \frac{\theta_1(\alpha(t),t)}{\partial r} = -\lambda_2 \frac{\theta_2(\alpha(t),t)}{\partial r} + L_b\gamma_b \frac{d\alpha}{dt}, \tag{17}$$

$$\theta_2(\beta(t),t) = 0, \tag{18}$$

$$-\lambda_2 \frac{\theta_2(\beta(t),t)}{\partial r} = L_m \gamma_m \frac{d\beta}{dt}. \tag{19}$$

Now we focus on to obtain similarity solution to problem (12)–(19). If we take by similarity principle as following form

$$\theta_i(r,t) = u_i(\eta), \quad \eta = \frac{r}{2\alpha_0\sqrt{t}}, \quad i = 1,2, \tag{20}$$

and free boundaries are considered in the form $\alpha(t) = \alpha_0\sqrt{t}$ and $\beta(t) = \beta_0\sqrt{t}$, then we obtain the following free boundary problem with non-linear ordinary differential equations

$$[L(u_1)\eta^2 u_1']' + 2\alpha_0^2\eta^3 N(u_1)u_1' = 0, \quad 0 < \eta < \frac{1}{2}, \tag{21}$$

$$[L(u_2)\eta^2 u_2']' + 2\alpha_0^2\eta^3 N(u_2)u_2' = 0, \quad \frac{1}{2} < \eta < \frac{\beta_0}{2\alpha_0}, \tag{22}$$

$$u_1(0) = u_i, \tag{23}$$

$$u_1(1/2) = u_2(1/2) = 1, \tag{24}$$

$$-\lambda_1 \frac{du_1(1/2)}{d\eta} = -\lambda_2 \frac{du_2(1/2)}{d\eta} + L_m \gamma_m \alpha_0^2, \tag{25}$$

$$u_2(\beta_0/2\alpha_0) = 0, \tag{26}$$

$$-\lambda_2 \frac{du_2(\beta_0/2\alpha_0)}{d\eta} = L_m \gamma_m \alpha_0 \beta_0, \tag{27}$$

where $L(u_i) = \lambda_i((T_b - T_m)u_i + T_m), N(u_i) = c_i((T_b - T_m)u_i + T_m)\gamma_i((T_b - T_m)u_i + T_m), \quad i = 1, 2$. To solve the non-linear ordinary differential equation $[L[u_i]\eta^2 u_i']' + 2\alpha_0^2 \eta^3 N(u_i)u_i' = 0, i = 1, 2,$ we use substitution

$$L(u_i)\eta^2 u_i' = \nu_i(\eta) \tag{28}$$

and we have the following equation

$$\nu_i'(\eta) + P(\eta, u_i)\nu_i(\eta) = 0, \tag{29}$$

where $P(\eta, u_i) = \dfrac{2\alpha_0^2 \eta N(u_i)}{L(u_i)}$. By solving equation (29) for $i = 1, 2$, we have the solutions

$$\nu_1(\eta) = \nu_1(0) \exp\left(-2\alpha_0^2 \int_0^\eta \eta \frac{N(u_1(\eta))}{L(u_1(\eta))} d\eta \right), \tag{30}$$

$$\nu_2(\eta) = \nu_2(1/2) \exp\left(-2\alpha_0^2 \int_{1/2}^\eta \eta \frac{N(u_2(\eta))}{L(u_2(\eta))} d\eta \right). \tag{31}$$

By making substitution (30) and (31) to (28) and using the conditions (23)–(24) and (26), we have the following solutions

$$u_1(n) = 1 - \Phi_1[1/2, L(1), N(1)] + \Phi_1[\eta, L(u_1), N(u_1)], \tag{32}$$

where $\Phi_1[1/2, L(1), N(1)] = 1 - u_i$ and

$$u_2(n) = 1 - \frac{\Phi_2[\eta, L(u_2), N(u_2)]}{\Phi_2[\beta_0/2\alpha_0, L(0), N(0)]}, \tag{33}$$

where

$$\Phi_1[\eta, L(u_1), N(u_1)] = \nu_1(0) \int_0^\eta \frac{E_1[\eta, u_1]}{v^2 L(u_1(v))} dv,$$

$$\Phi_2[\eta, L(u_2), N(u_2)] = \nu_2(1/2) \int\limits_{1/2}^{\eta} \frac{E_2[\eta, u_2]}{v^2 L(u_2(v))} dv,$$

$$E_1[\eta, u_1] = \exp\bigg(- 2\alpha_0^2 \int\limits_0^{\eta} \eta \frac{N(u_1)}{L(u_1)} d\eta \bigg),$$

$$E_2[\eta, u_2] = \exp\bigg(- 2\alpha_0^2 \int\limits_{1/2}^{\eta} \eta \frac{N(u_2)}{L(u_2)} d\eta \bigg).$$

Equations (32) and (33) satisfy problem (21)–(27). From Stefan's condition (25) and (27) we obtain

$$-4\nu_1(0)E_1[1/2, 1] = \frac{4\nu_2(1/2)E_2[1/2, 1]}{\Phi_2[\beta/2\alpha_0, L(0), N(0)]} + L_b\gamma_b\alpha^2, \tag{34}$$

$$\frac{4\alpha_0\nu_2(1/2)E_2[\beta_0/2\alpha_0, 0]}{\Phi_2[\beta_0/2\alpha_0, L(0), N(0)]} = L_m\gamma_m\beta_0^3. \tag{35}$$

The coefficients of free boundaries $\alpha(t)$ and $\beta(t)$ can be found from the expressions (34)–(35). In the next section, we will prove the existence of similarity solutions (32) and (33).

3 Existence of similarity solutions of the problem

To prove the existence of solutions to of the non-linear integral equations (32) and (33) we use the fixed point theorem. We suppose that there exist constants L_m, L_M, N_m and N_M which satisfy the inequalities

$$L_m \le L(T) \le L_M \text{ and } N_m \le N(T) \le N_M. \tag{36}$$

We consider that thermal conductivity and specific heat are Lipchitz functions and satisfy the following inequality

$$|h(f) - h(g)| \le \bar{h}||f - g|| \tag{37}$$

by contraction mapping to ordinary differential equation. Let denote $\Phi[\eta, u_i] \equiv \Phi[\eta, L(u_i), N(u_i)]$, $i = 1, 2$, for convenient proving. Before proving the existence of a unique solution of similarity solutions (32)–(33) we must consider the following lemmas.

Lemma 1. *If for any positive η (36) and (37) hold, then the following inequalities*

1. $\exp\left(- \dfrac{\alpha_0^2 N_M}{L_m}\eta^2\right) \le E_1[\eta, u_1] \le \exp\left(- \dfrac{\alpha_0^2 N_m}{L_M}\eta^2\right),$

2. $\exp\left(- \dfrac{\alpha_0^2 N_M}{L_m}\left(\eta^2 - \dfrac{1}{4}\right)\right) \le E_2[\eta, u_2] \le \exp\left(- \dfrac{\alpha_0^2 N_m}{L_M}\left(\eta^2 - \dfrac{1}{4}\right)\right)$

hold for $\eta > 0$.

Proof. For the second inequality we have the following prove

$$E_2[\eta, u_2] \le \exp\left(- 2\alpha_0^2 \frac{N_m}{L_M} \int\limits_{1/2}^{\eta} s ds \right) = \exp\left(- \frac{\alpha_0^2 N_m}{L_M}\left(\eta^2 - \frac{1}{4}\right)\right).$$

The first inequality can be proved similarly.

Lemma 2. *If* (36)–(37) *hold, then*

1. *for* $0 < \eta < \dfrac{1}{2}$ *we have*

$$\frac{\nu_1(0)\sqrt{\pi L_m}}{2\alpha_0 L_M\sqrt{N_M}} \operatorname{erf}\left(\eta\sqrt{\frac{N_M}{L_m}}\alpha_0\right) \le \Phi_1[\eta, u_1] \le \frac{\nu_1(0)\sqrt{\pi L_M}}{2\alpha_0 L_m\sqrt{N_m}} \operatorname{erf}\left(\eta\sqrt{\frac{N_m}{L_M}}\alpha_0\right),$$

2. *for* $\dfrac{1}{2} < \eta < \dfrac{\beta_0}{2\alpha_0}$ *we have*

$$\frac{\nu_2(1/2)\alpha_0\sqrt{N_M}}{L_M\sqrt{L_m}} \exp\left(\frac{\alpha_0^2 N_M}{4L_m}\right)\left\{\sqrt{\pi}\operatorname{erf}\left(\frac{\alpha_0\sqrt{N_M}}{2\sqrt{L_m}}\right) - \sqrt{\pi}\operatorname{erf}\left(\frac{\alpha_0\eta\sqrt{N_M}}{L_m}\right)\right.$$
$$\left. - \frac{\sqrt{L_m}}{\alpha_0\eta\sqrt{N_M}}\exp\left(-\frac{\alpha_0^2\eta^2 N_M}{L_m}\right) + \frac{2\sqrt{L_m}}{\alpha_0\sqrt{N_M}}\exp\left(-\frac{\alpha_0^2 N_M}{4L_m}\right)\right\} \le \Phi_2[\eta, u_2]$$
$$\le \frac{\nu_2(1/2)\alpha_0\sqrt{N_m}}{L_m\sqrt{L_M}} \exp\left(\frac{\alpha_0^2 N_m}{4L_M}\right)\left\{\sqrt{\pi}\operatorname{erf}\left(\frac{\alpha_0\sqrt{N_m}}{2\sqrt{L_M}}\right) - \sqrt{\pi}\operatorname{erf}\left(\frac{\alpha_0\eta\sqrt{N_m}}{L_M}\right)\right.$$
$$\left. - \frac{\sqrt{L_M}}{\alpha_0\eta\sqrt{N_m}}\exp\left(-\frac{\alpha_0^2\eta^2 N_m}{L_M}\right) + \frac{2\sqrt{L_M}}{\alpha_0\sqrt{N_m}}\exp\left(-\frac{\alpha_0^2 N_m}{4L_M}\right)\right\}.$$

Proof. By using Lemma 1 let us try to prove the second inequality

$$\Phi_2[\eta, u_2] \le \frac{\nu_2\left(\dfrac{1}{2}\right)}{L_m} \int\limits_{1/2}^{\eta} \frac{1}{v^2}\exp\left(-\frac{\alpha_0^2 N_m}{L_M}\left(v^2 - \frac{1}{4}\right)\right) dv$$

$$= \frac{\nu_2(1/2)}{L_m}\exp\left(\frac{\alpha_0^2 N_m}{4L_M}\right)\int\limits_{1/2}^{\eta} \frac{\exp\left(-\dfrac{\alpha_0^2 N_m v^2}{L_M}\right)}{v^2} dv.$$

After making substitution $t = \alpha_0 v\sqrt{\dfrac{N_m}{L_M}}$ and solving this integral, we finished proving the second inequality. The one is proved analogously.

Lemma 3. *If inequalities* (36)–(37) *hold, then*

1. *for all* $u_1, u_1^* \in C^0\left[0, \frac{1}{2}\right]$ *we have*
$$|E_1[\eta, u_1] - E_1[\eta, u_1^*]| \leq \frac{\alpha_0^2}{L_m}\eta^2\left(\bar{N} + \frac{N_M\bar{L}}{L_m}\right)||u_1^* - u_1||,$$
2. *for all* $u_2, u_2^* \in C^0\left[\frac{1}{2}, \frac{\beta_0}{2\alpha_0}\right]$ *we have*
$$|E_2[\eta, u_2] - E_2[\eta, u_2^*]| \leq \frac{\alpha_0^2}{L_m}\left(\eta^2 - \frac{1}{4}\right)\left(\bar{N} + \frac{N_M\bar{L}}{L_m}\right)||u_2^* - u_2||.$$

Proof. For the second inequality we have

$$|E_2[\eta, u_2] - E_2[\eta, u_2^*]| \leq \left|\exp\left(-2\alpha_0^2\int_{\frac{1}{2}}^{\eta} s\frac{N(u_2)}{L(u_2)}ds\right) - \exp\left(-2\alpha_0^2\int_{\frac{1}{2}}^{\eta} s\frac{N(u_2^*)}{L(u_2^*)}ds\right)\right|,$$

by using $|\exp(-x) - \exp(-y)| \leq |x - y|$ we get

$$|E_2[\eta, u_2] - E_2[\eta, u_2^*]| \leq 2\alpha_0^2\left|\int_{\frac{1}{2}}^{\eta} s\frac{N(u_2)}{L(u_2)}ds - \int_{\frac{1}{2}}^{\eta} s\frac{N(u_2^*)}{L(u_2^*)}ds\right| \leq 2\alpha_0^2\int_{\frac{1}{2}}^{\eta}\left|\frac{N(u_2)}{L(u_2)} - \frac{N(u_2^*)}{L(u_2^*)}\right|sds$$

$$\leq \frac{\alpha_0^2}{L_m}\left(\bar{N} + \frac{N_M\bar{L}}{L_m}\right)||u_2^* - u_2||\int_{\frac{1}{2}}^{\eta} sds = \frac{\alpha_0^2}{L_m}\left(\eta^2 - \frac{1}{4}\right)\left(\bar{N} + \frac{N_M\bar{L}}{L_m}\right)||u_2^* - u_2||.$$

The first inequality is proved analogously as the second.

Lemma 4. *If* (36)–(37) *hold, then*

1. *for all* $u_1, u_1^* \in C^0\left[0, \frac{1}{2}\right]$ *and* $0 < \eta < \frac{1}{2}$ *we get* $|\Phi_1[\eta, u_1] - \Phi_2[\eta, u_1^*]| \leq \infty$ *as integral defined for* $\Phi_1[\eta, u_1]$ *is divergent at* $\eta = 0$,
2. *for all* $u_2, u_2^* \in C^0\left[\frac{1}{2}, \frac{\beta_0}{2\alpha_0}\right]$ *and* $\frac{1}{2} < \eta < \frac{\beta_0}{2\alpha_0}$ *we get*
$$|\Phi_2[\eta, u_2] - \Phi_2[\eta, u_2^*]| \leq \frac{\left|\nu_2\left(\frac{1}{2}\right)\right|}{L_m^2}||u_2^* - u_2||\left[\alpha_0^2\left(\bar{N} + \frac{N_M\bar{L}}{L_m}\right)\left(\eta + \frac{1}{4\eta} - 1\right) + \bar{L}\left(2 - \frac{1}{\eta}\right)\right].$$

Proof. By using Lemma 2 and Lemma 3 for the second inequality, we obtain

$$|\Phi_2[\eta, u_2] - \Phi_2[\eta, u_2^*]| \le T_1(\eta) + T_2(\eta),$$

where

$$T_1(\eta) \le \frac{\left|\nu_2\left(\frac{1}{2}\right)\right|}{L_m} \int\limits_{\frac{1}{2}}^{\eta} \frac{|E_2[\eta, u_2] - E_2[\eta, u_2^*]|}{s^2} ds$$

$$= \frac{\left|\nu_2\left(\frac{1}{2}\right)\right|\alpha_0^2}{L_m}\left(\bar{N} + \frac{N_M \bar{L}}{L_m}\right)||u_2^* - u_2|| \int\limits_{\frac{1}{2}}^{\eta} \frac{s^2 - \frac{1}{4}}{s^2} ds$$

$$= \frac{\left|\nu_2\left(\frac{1}{2}\right)\right|\alpha_0^2}{L_m}\left(\bar{N} + \frac{N_M \bar{L}}{L_m}\right)||u_2^* - u_2||\left(\eta + \frac{1}{4\eta} - 1\right)$$

and

$$T_2(\eta) \le \left|\nu_2\left(\frac{1}{2}\right)\right| \int\limits_{\frac{1}{2}}^{\eta} \frac{\left|\frac{1}{L(u_2)} - \frac{1}{L(u_2^*)}\right|}{s^2} ds \le \left|\nu_2\left(\frac{1}{2}\right)\right| \int\limits_{\frac{1}{2}}^{\eta} \frac{|L(u_2^*) - L(u_2)|}{s^2 |L(u_2) L(u_2^*)|} ds$$

$$= \frac{\left|\nu_2\left(\frac{1}{2}\right)\right|\bar{L}}{L_m^2}||u_2^* - u_2|| \int\limits_{\frac{1}{2}}^{\eta} \frac{ds}{s^2} = \frac{\left|\nu_2\left(\frac{1}{2}\right)\right|\bar{L}}{L_m^2}||u_2^* - u_2||\left(2 - \frac{1}{\eta}\right).$$

By making summation, we can prove the second inequality. The one has an analogous proof. Now we try to prove the theorem on the existence of a unique solution to the integral equation (26).

Theorem 1. *Let* η_0 *be a given positive real number and suppose that* (36)–(37) *hold. If* η_0 *satisfies the following inequality*

$$\sigma(\eta_0) := \frac{2L_M^{3/2}\sqrt{N_m}\exp\left(\frac{\alpha_0^2 N_m}{4L_M}\right)\mu_1(\eta_0)}{L_m \alpha_0 N_M \exp\left(\frac{\alpha_0^2 N_M}{2L_m}\right)[\mu_2(\eta_0)]^2}$$

$$\times||u_2^* - u_2||\left[\alpha_0^2\left(\bar{N} + \frac{N_M \bar{L}}{L_m}\right)\left(\eta_0 + \frac{1}{4\eta_0} - 1\right) + \bar{L}\left(2 - \frac{1}{\eta_0}\right)\right] < 1, \tag{38}$$

where

$$\mu_1(\eta_0)=\sqrt{\pi}\,\mathrm{erf}\left(\frac{\alpha_0}{2}\sqrt{\frac{N_m}{L_M}}\right)-\sqrt{\pi}\,\mathrm{erf}\left(\frac{\alpha_0\eta_0\sqrt{N_m}}{L_M}\right)-\frac{\sqrt{L_M}}{\alpha_0\eta_0\sqrt{N_m}}\exp\left(-\frac{\alpha_0^2\eta_0^2N_M}{L_M}\right)$$
$$+\frac{2\sqrt{L_M}}{\alpha_0\sqrt{N_m}}\exp\left(-\frac{\alpha_0^2N_m}{4L_M}\right),$$
$$\mu_2(\eta_0)=\sqrt{\pi}\,\mathrm{erf}\left(\frac{\alpha_0}{2}\sqrt{\frac{N_M}{L_m}}\right)-\sqrt{\pi}\,\mathrm{erf}\left(\frac{\alpha_0\eta_0\sqrt{N_M}}{L_m}\right)-\frac{\sqrt{L_m}}{\alpha_0\eta_0\sqrt{N_M}}\exp\left(-\frac{\alpha_0^2\eta_0^2N_m}{L_m}\right)$$
$$+\frac{2\sqrt{L_m}}{\alpha_0\sqrt{N_M}}\exp\left(-\frac{\alpha_0^2N_M}{4L_m}\right),$$

then there exists a unique solution $u_2\in C^0\left[\frac{1}{2},\eta_0\right]$ *to the integral equation* (33).

Proof. We have the operator $W:C^0\left[\frac{1}{2},\eta_0\right]\to C^0\left[\frac{1}{2},\eta_0\right]$ which can be defined as

$$W(u_2(\eta))=1-\frac{\Phi_2[\eta,L(u_2)]}{\Phi_2[\eta_0,L(u_2)]}.$$

The solution to equation (33) is a fixed point of the operator W, that is

$$W(u_2(\eta))=u_2(\eta),\ \ \frac{1}{2}<\eta<\eta_0.$$

We suppose that $u_2,u_2^*\in C^0\left[\frac{1}{2},\eta_0\right]$, then by using Lemmas 2-4, we get

$$||W(u_2)-W(u_2^*)||=\max_{\eta\in[0,\eta_0]}|W(u_2(\eta))-W(u_2^*(\eta))|$$
$$\le\max_{\eta\in[0,\eta_0]}|(\Phi_2[\eta,u_2^*]\Phi_2[\eta_0,u_2]-\Phi_2[\eta_0,u_2^*]\Phi_2[\eta,u_2])/(\Phi_2[\eta_0,u_2]\Phi_2[\eta_0,u_2^*])|$$
$$\le A\max_{\eta\in[0,\eta_0]}|\Phi_2[\eta,u_2^*]\Phi_2[\eta_0,u_2]-\Phi_2[\eta_0,u_2^*]\Phi_2[\eta,u_2]|$$
$$\le A\max_{\eta\in[0,\eta_0]}(|\Phi_2[\eta,u_2^*]||\Phi_2[\eta_0,u_2]-\Phi_2[\eta_0,u_2^*]|$$
$$+|\Phi_2[\eta_0,u_2^*]||\Phi_2[\eta,u_2^*]-\Phi_2[\eta,u_2]|),$$

where

$$A=\frac{L_M^2L_m}{(\nu_2(1/2))^2\alpha_0^2N_M\exp\left(\frac{\alpha_0^2N_M}{2L_m}\right)[\mu(\eta_0)]^2}>0.$$

Finally, from Lemmas 3, 4 we have that

$$|W(u_2) - W(u_2^*)| \leq \sigma(\eta_0)||u_2^* - u_2||.$$

We can see that W is a contraction operator and if the inequality (38) holds, then there exists a unique solution for integral equation (33). The existence of a unique solution to the integral equation (32) can also be proved similarly to Theorem 1.

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Харин С.Н., Наурыз Т.А. СЫЗЫҚТЫҚ ЕМЕС ЖЫЛУ ӨТКІЗГІШТІГІ БАР ЕКІ ФАЗАЛЫҚ СФЕРАЛЫҚ СТЕФАН ЕСЕБІ

Бұл мақалада температураға тәуелді жылу коэффициенттері бар екі фазалық сфералық Стефан есебінің ұқсастық шешімі бар екендігі дәлелденген. Есеп сызықтық емес жәй дифференциалдық теңдеу үшін еркін шекаралық есепке келтірілген, содан кейін Вольтерра тектес сызықтық емес интегралдық теңдеуі алынады. Ұқсастық принципі балқу мен қайнау изотермалары арасындағы шекаралары еркін болатын сұйық және қатты аймақтың температурасын моделдеу үшін қолданылған.

Кілттік сөздер. Стефан есебі, ұқсастық шешімі, сызықтық емес жәй дифференциалдық теңдеу, жылу коэффициенттері, сызықтық емес интегралдық теңдеу.

Харин С.Н., Наурыз Т.А. ДВУХФАЗНАЯ СФЕРИЧЕСКАЯ ЗАДАЧА СТЕФАНА С НЕЛИНЕЙНОЙ ТЕПЛОПРОВОДНОСТЬЮ

В данной работе доказано существование решения подобия двухфазной сферической задачи Стефана с температурными зависимостями тепловых коэффициентов. Задача сводится к задаче со свободной границей нелинейного обыкновенного дифференциального уравнения, затем получается нелинейное интегральное уравнение типа Вольтерра. Принцип подобия используется для моделирования температуры жидкой и твердой зон со свободными границами между изотермами плавления и кипения.

Ключевые слова. Задача Стефана, решение подобия, нелинейное обыкновенное дифференциальное уравнение, тепловые коэффициенты, нелинейное интегральное уравнение.

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Numerical simulation of the supersonic airflow with hydrogen jet injection at various Mach number

Gulzana Ashirova[1,2,a], Asel Beketaeva[1,2,b], Altynshash Naimanova[1,c]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[2]Al-Farabi Kazakh National University, Almaty, Kazakhstan
[a] e-mail: gulzana.ashyrova@yandex.ru,[b]e-mail: azimaras10@gmail.com,[c] e-mail: alt_naimanova@yahoo.com

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Abstract. The multispecies supersonic airflow in a planar channel with transverse hydrogen jet injection is simulated. The Favre averaged Navier-Stokes equations coupled with $k-\omega$ turbulence model are solved using a third order ENO scheme. The main attention is paid to the influence of the flow Mach number to the interaction of the shock wave structure with boundary layers on the upper and the lower channel walls under the conditions of an internal turbulent flow. In particular, a detailed study of the shock wave structure, separation zones, jet penetration are investigated at the various Mach number. It is established that the shock wave structures appearing on the upper and the lower walls and the vortex zones resulting from the interaction of the shock wave structures with the boundary layers (SWBLI) decrease due to an increase of the Mach number. For small values of the flow Mach number, an additional interaction of the shock waves structures on the bottom wall behind the jet is revealed. Also the decrease of the jet penetration with increasing Mach number is revealed and the dependencies are obtained. The comparison with an experimental data is implemented.

Keywords. Navier-Stokes equations, supersonic flow shock wave, boundary layer, flow separation, Mach number.

1 Introduction

The fuel-air mixing and combustion in the scramjet combustor are implemented with supersonic speed. The jet injection in a cross-flow (JICF) leads to the formation of system shock wave structures, where a shock wave boundary layer interaction (SWBLI) near walls of the combustion chamber is the most complex. Such flow with injected jet has been extensively studied as experimentally [1]–[6] and theoretically [7]–[13].

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There are a few investigations studied the influence of the effect SWBLI to the mixing of the fuel and the airflow and the combustion as a result [14], [15]. The formed bow shock wave (resulting from the jet and flow interaction) reaches the upper boundary layer and causes separation of the boundary layer. Thus, formed SWBLI phenomenon on the top wall can significantly influence on the structure of the flowfield and, as a consequence, to the processes of mixing the jet and the flow. It should be noted that in the most of experimental [16]–[18] and theoretical [19]–[25] works, SWBLI process is studied on the basis of interaction of the boundary layer of a flat plate with the incident shock wave, generated by the wedge (shock generator), i.e. the case of SWBLI during the JICF is almost not considered.

During the numerical solving some research [26]–[27] observed the flow unsteadiness caused the intrinsic flow instabilities in flowfield which is Richtmyer-Meshkov instability in shock-wave/shear-layer interactions. While the mixing of the airflow with the fuel and the combustion of the mixture occurs at supersonic speeds. This is the stringent condition for the time of the oxidant-fuel mixing and the combustion in the channel. Thus, the Mach number is one of the important flow parameters, since in the chambers the combustion process is very dependent on the flow speed. The analysis of the researches performed the numerical simulation of supersonic multispecies gas flows shows that the detailed study of the dependence

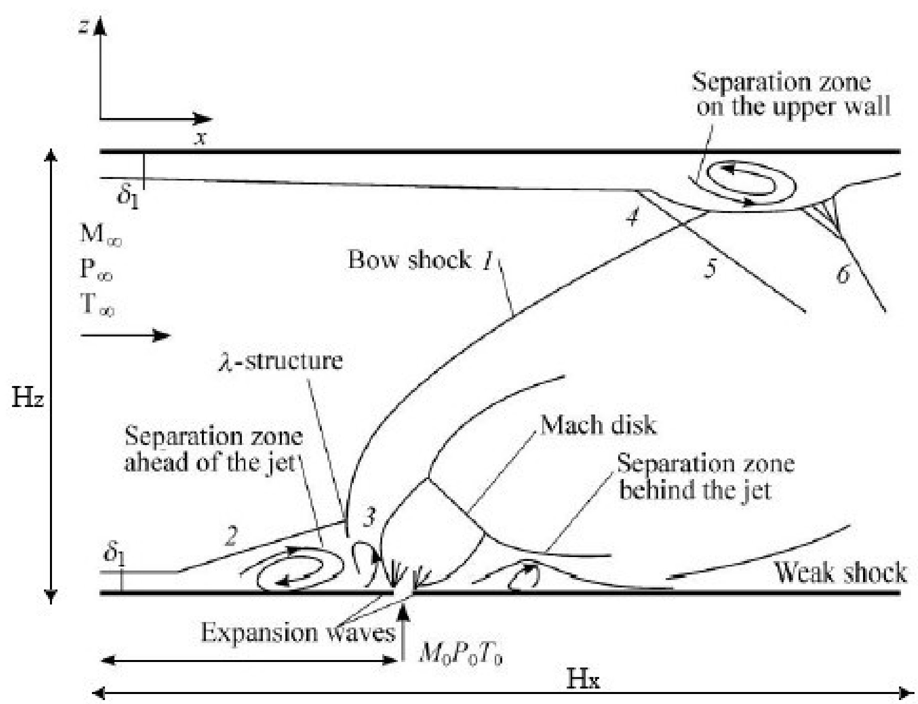


Figure 1 – Scheme of the flow

of the flow structures on the parameters is needed. The purpose of the present work is the numerical simulation of a planar supersonic turbulent airflow in a channel with a transverse injection of the hydrogen jet. The study of the interaction of a shock – wave with boundary layers (SWBLI) on the bottom and upper walls as well as the conditions of the boundary layer separations and their influence to the mixture of airflow and hydrogen for a broad range of the Mach number is performed. The scheme of the flow is shown in Figure 1.

2 Problem statement

Basic equations for the stated problem are the system of two-dimensional Favre averaged Navier-Stokes equations for multispecies gaseous mixture in Cartesian coordinate system in conservative form as:

$$\frac{\partial \vec{U}}{\partial t} + \frac{\partial \left(\vec{E} - \vec{E}_v\right)}{\partial x} + \frac{\left(\vec{F} - \vec{F}_v\right)}{\partial z} = S, \tag{1}$$

where the vectors of dependent variables and vector fluxes are defined in the form

$$\vec{U} = (\rho, \rho u, \rho w, E_t, \rho Y_k, \rho k, \rho\omega)^T,$$

$$\vec{E} = \left(\rho u, \rho u^2 + p, \rho uw, (E_t + p)\, u, \rho u Y_k, \rho uk, \rho u\omega\right)^T,$$

$$\vec{F} = \left(\rho w, \rho uw, \rho w^2 + p, (E_t + p)\, w, \rho w Y_k, \rho wk, \rho w\omega\right)^T,$$

$$\vec{E}_v = \left(0, \tau_{xx}, \tau_{xz}, u\tau_{xx} + w\tau_{xz} - q_x, J_{kx}, \frac{1}{Re}(\mu + \sigma_k\mu_t)\frac{\partial k}{\partial x}, \frac{1}{Re}(\mu + \sigma_\omega\mu_t)\frac{\partial \omega}{\partial x}\right)^T,$$

$$\vec{F}_v = \left(0, \tau_{xz}, \tau_{zz}, u\tau_{xz} + w\tau_{zz} - q_z, J_{kz}, \frac{1}{Re}(\mu + \sigma_k\mu_t)\frac{\partial k}{\partial z}, \frac{1}{Re}(\mu + \sigma_\omega\mu_t)\frac{\partial \omega}{\partial z}\right)^T.$$

Viscous stress tensor components are given as

$$\tau_{xx} = \frac{2\mu}{3Re}(3u_x - w_z),\ \tau_{zz} = \frac{2\mu}{3Re}(3w_z - u_x),\ \tau_{xz} = \tau_{zx} = \frac{\mu}{Re}(u_z + w_x).$$

The heat flux is defined in a form

$$q_x = \left(\frac{\mu}{PrRe}\right)\frac{\partial T}{\partial x} + \frac{1}{\gamma_\infty M_\infty^2}\sum_{k=1}^{N} h_k J_{kx},\ q_z = \left(\frac{\mu}{PrRe}\right)\frac{\partial T}{\partial z} + \frac{1}{\gamma_\infty M_\infty^2}\sum_{k=1}^{N} h_k J_{kz}.$$

The diffusion flux is determined as

$$J_{kx} = -\frac{\mu}{ScRe}\frac{\partial Y_k}{\partial x},\ J_{kz} = -\frac{\mu}{ScRe}\frac{\partial Y_k}{\partial z}.$$

The pressure and the total energy are given as

$$P = \frac{\rho T}{\gamma_\infty M_\infty^2 W},\ W = \left(\sum_{k=1}^{N_p} \frac{Y_k}{W_k}\right)^{-1},\ \sum_{k=1}^{N_p} Y_k = 1,$$

$$E_t = \frac{\rho}{\gamma_\infty M_\infty^2} \sum_{k=1}^{N} Y_k h_k - P + \frac{1}{2}\rho(u^2 + w^2).$$

The specific enthalpy and the specific heat at a constant pressure of the k^{th} species are

$$h_k = h_k^0 + \int_{T_0}^{T} c_{pk} dT,\ c_{pk} = C_{pk}/W_k,$$

where the molar specific heat is written in the polynomial form as

$$C_{pk} = \sum_{i=1}^{5} \bar{a}_{ki} T^{(i-1)},$$

where the coefficients $\overline{a}_{jk}$ are taken from the table JANAF [28] at a normal pressure ($p = 1$ atm) and temperature $T^0 = 293$ K.

The vector of additional terms is as follows:

$$\vec{S} = \left(0, 0, 0, 0, (P_k - \beta^* \rho \omega k),\ \left(\gamma^* \rho P_k/\mu_t - \beta \rho \omega^2\right)\right)^T,$$

$$P_k = \tau_{\text{ij}} \frac{\partial u_i}{x_j},\ i, j = 1, 2,$$

$$\sigma_k = 0.5,\ \sigma_\omega = 0.5,\ \beta^* = 0.09,\ \beta = 0.075,\ \gamma^* = 5/9,$$

k, ω are the turbulent kinetic energy and its dissipation rate, P_k is the term defining the turbulence generation, the turbulent viscosity is determined by $\mu_t = \frac{\rho k}{\omega}$ [29] and μ_l is determined by the Sutherlend formula.

The system of equations (1) is written in non – dimensional form. The input parameters of airflow $u_\infty, \rho_\infty, T_\infty, W_\infty$ are taken as reference parameters, the pressure and the total energy are normalized by $\rho_\infty u_\infty^2$, for the specific enthalpy h_k are $R^0 T_\infty / W_\infty$, for the molar specific heats C_{pk} are R^0, and the slot width is chosen as the reference length scale. In the mass fraction Y_k $k = 1$ corresponds to O_2, $k = 2$ – H_2, $k = 3$ – N_2. W_k is the molecular weight of a component; Re, Pr, Sc, M are Reynolds, Prandtl, Schmidt and Mach numbers respectively.

3 The initial and boundary conditions

At the entrance, the parameters of flow are taken as

$$P = P_\infty,\ T = T_\infty,\ u = M_\infty\sqrt{\frac{\gamma_\infty R_0 T_\infty}{W_\infty}},\ w = 0,\ Y = Y_{k\infty},\ W = W_{k\infty},\ x = 0,\ 0 \le z \le H,$$

where the boundary layer is specified near the walls in which longitudinal velocity component is determined as

$$u = \begin{cases} 0.1\left(\frac{z}{\delta_2}\right) + 0.9\left(\frac{z}{\delta_2}\right)^2, & x = 0,\ 0 \le z \le \delta_2, \\ \left(\frac{z}{\delta_1}\right)^{1/7}, & x = 0,\ \delta_2 \le z \le \delta_1, \end{cases}$$

here $\delta_1 = 0.37x(Re_x x)^{-0.2}$ is the boundary layer thickness [30] and $\delta_2 = 0.2\delta_1$ is the viscous sublayer thickness [31].

The profile of temperature and density are taken as [32]

$$T = T_W + u\left(1 - T_W\right),\ \rho = \frac{1}{T},$$

where $T_W = \left(1 + r\frac{(\gamma-1)}{2}M_\infty^2\right)$ is the temperature on the wall and $r = 0.88$.

On the bottom and top walls:

$$u = w = 0,\ \frac{\partial T}{\partial z} = 0,\ \frac{\partial P}{\partial z} = 0,\ \frac{\partial Y_k}{\partial z} = 0,\ 0 \le x \le L,\ z = 0 \text{ and } z = H.$$

In the slot:

$$W = W_{k0},\ P = nP,\ T = T_0,\ w = M_0\sqrt{\frac{\gamma_0 R_0 T_0}{W_0}},\ u = 0,\ Y = Y_{k0},\ z = 0,\ L_b \le x \le L_b + d,$$

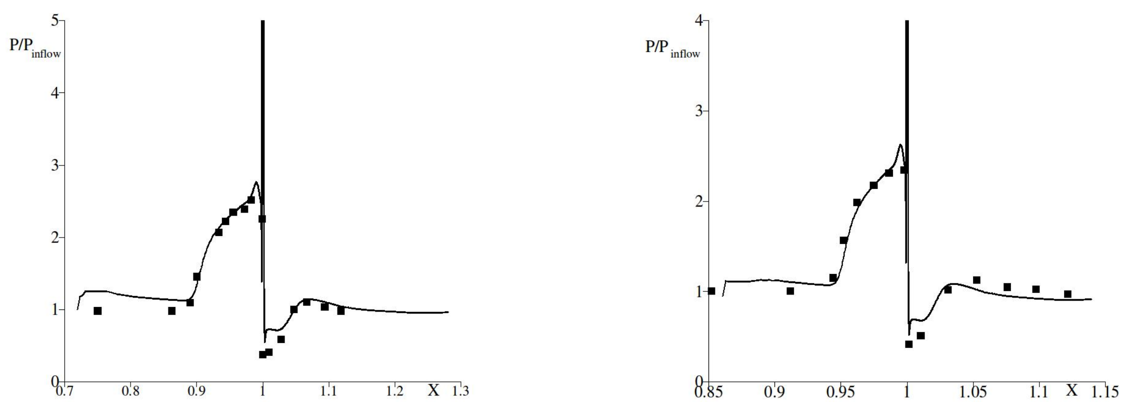
where L_b is the distance from the entrance to the slot, d is the width of slot, $n = P_0/P_\infty$ is the pressure ratio, M_0 and M_∞ are the Mach numbers of the jet and the flow respectively, $0, \infty$ refers to the jet and flow parameters; H_x, H_z is the length and the height of domain. The initial conditions are taken the same as the boundary conditions at the entrance. The non-reflection boundary conditions are specified at the outlet boundary [33].

4 Solution method

The methodology of the numerical solving the system (1) is described in [7], [8]. Numerical solution of the system (1) is performed in two stages. A coordinate transformation is preliminarily done, where a grid thickening is made in the region of high gradients. At first stage the thermodynamic parameters ρ, u, w, E_t are defined.The third order Essentially Nonoscillatory Scheme are applied for approximation inviscid terms [34]–[36]. The central differences of the second order of accuracy have been used for the approximation of the second derivatives. The obtaining system of equations is solved using the matrix sweep method for the vector of the thermodynamic parameters. The equations of the mass fractions Y_k are similarly solved at the second stage. The temperature field is calculated from the known values of the variables $\vec{U}$ using of the Newton-Raphson iterative method with the quadratic rate of convergence [37].

5 Analysis of results

The validation of numerical model is performed by comparison between the experimental data [2] and the numerical solution of a supersonic airflow with transverse jet injection of nitrogen. The next parameters of the supersonic airflow are given: $M_\infty = 3.5$, $P_\infty = 3145\ Pa$, $T_\infty = 309K$, $Y_{\infty\ O_2} = 0.2$, $Y_{\infty\ N_2} = 0.8$. The nitrogen sonic jet is injected with parameters: $M_\infty = 1, T_0 = 292K$, $Y_{0\ N_2} = 1$, $L_b = 228.6\ mm$ through a slot of width $d = 0.2667\ mm$ on the bottom wall. The pressure distribution on the wall in the jet region is defined with the pressure ratios $n = 8.74$ and $n = 17.12$. Figure 2 shows the result of comparison with experiment for the pressure distribution on the wall near the jet. Here the "curve" is a numerical result and "■■■" are an experimental data [2]. As it is seen from Figure 2 the good agreement is obtained for the pressure distribution parameter.



a) b)

Figure 2 – The pressure distribution on the wall in the region of jet for pressure ratio $n = 8.74$ (a) and $n = 17.12$ (b)

The stated problem of a planar supersonic flow in channel with transverse sound jet injection of hydrogen from the bottom wall is numerically simulated for studying the influence of the flow Mach number on the interaction of the shock wave system and the boundary layers near walls. The dimensionless parameters in this case are: $H_x = 90$ is the channel length, $H_z = 30$ is the height and the center of the jet is located at the distance of 32.5 from the entrance. Airflow and jet parameters are: $P_\infty = 1000\ Pa,\ T_\infty = 800\ K,\ Re = 10^6,\ Pr = 0.9,\ Y_{\infty O_2} = 0.2,\ Y_{\infty N_2} = 0.8, M_0 = 1,\ T_0 = 627\ K,\ Y_{0H_2} = 1, n = 15$. The boundary layer thickness $\delta_1 = 1.28$ is computed for $x = 145$ and specified at the inlet section. The near-wall layer height corresponds to the laminar-turbulent sublayer $z^+ = 70$, where $z^+ = \delta_2 u_\tau Re$, and the boundary layer height is $z^+ = 3700$, where $z^+ = \delta_1 u_\tau Re$. Here $u_\tau = \sqrt{\frac{C_f}{2}}$ is the dynamic viscosity, C_f is the flow friction coefficient on the wall. The numerical grid is 401×351. The grid refinement near the wall gives the first node near the wall equal to $z^+ = 1.5$. At the entrance nodes 5-8 lie in the near-wall layer along the z-axis and entire boundary layer is calculated with the use of 35-40 nodes of the numerical grid. The flow Mach number of flowfield is varied in the range $2.5 \leq M_\infty \leq 4.5$.

The isobar distribution is presented in Figure 3 (a) $M_\infty = 2.5$, b) $M_\infty = 3.0$, c) $M_\infty = 3.5$, d) $M_\infty = 4.0$, e) $M_\infty = 4.5$). The well-known and widely represented in various papers [7]–[9], [38] ahead of the jet shock – wave structure is visible for all values Mach number. From Figures 3a–3e it is seen that the inclination angle of the bow shock wave 1 and size of the λ – shape shock (which formed because intersection of the bow shock 1, oblique shock 2 and reflected shock 3) are decreased with growth of M_∞. Such behavior is apparently due to growth of incoming flow velocity. After reaching the upper wall, the bow shock 1 creates positive pressure gradient (Figures 3a–3e), leading to the separation of the boundary layer near upper wall, moreover, the larger the angle of inclination bow shock wave 1, the larger the pressure gradient. From Figure 3 one can see that the supersonic part of the upper boundary layer deviates and generates the system of converging compression wave 4, which propagates as the reflected shock wave 5. And the secondary system of compression waves is appeared as a result of reattachment of the separated flow to the streamlined wall, which is the reflected shock wave 6. It is visible (Figures 3a–3e) the bow shock 1, the compression wave 4 and the reflected shock 5 intersect at a single point and form λ – shaped system. The size of this λ – shaped structure reduces with increasing the Mach number, and this can be observed through comparing Figures 3a–3e. In Figure a for an additional λ – shaped structure is appeared near bottom wall behind the jet. Shock wave 6 reaches the bottom boundary layer behind the jet, where creates compression wave 7, which propagates in the form of shock 8. The weak reflected shock 9 is can also be seen here.

The behavior of a flowfield for different M_∞ is demonstrated the iso – Mach line contours in the jet injection region in Figures 4a–4e (a) $M_\infty = 2.5$, b) $M_\infty = 3.0$, c) $M_\infty = 3.5$, d) $M_\infty = 4.0$, e) $M_\infty = 4.5$). For all cases, the sonic velocity of the jet becomes supersonic because of the acceleration after injection and as can be observed from the Figure 4, a barrel structure is formed. It is visible from Figures 4a–4e that the barrel-shock structure in the jet decreases with increasing Mach number. Hence, jet penetration decreases too. It is due to the reduction of the hydrogen momentum with respect to the incoming airflow momentum. Consequently,

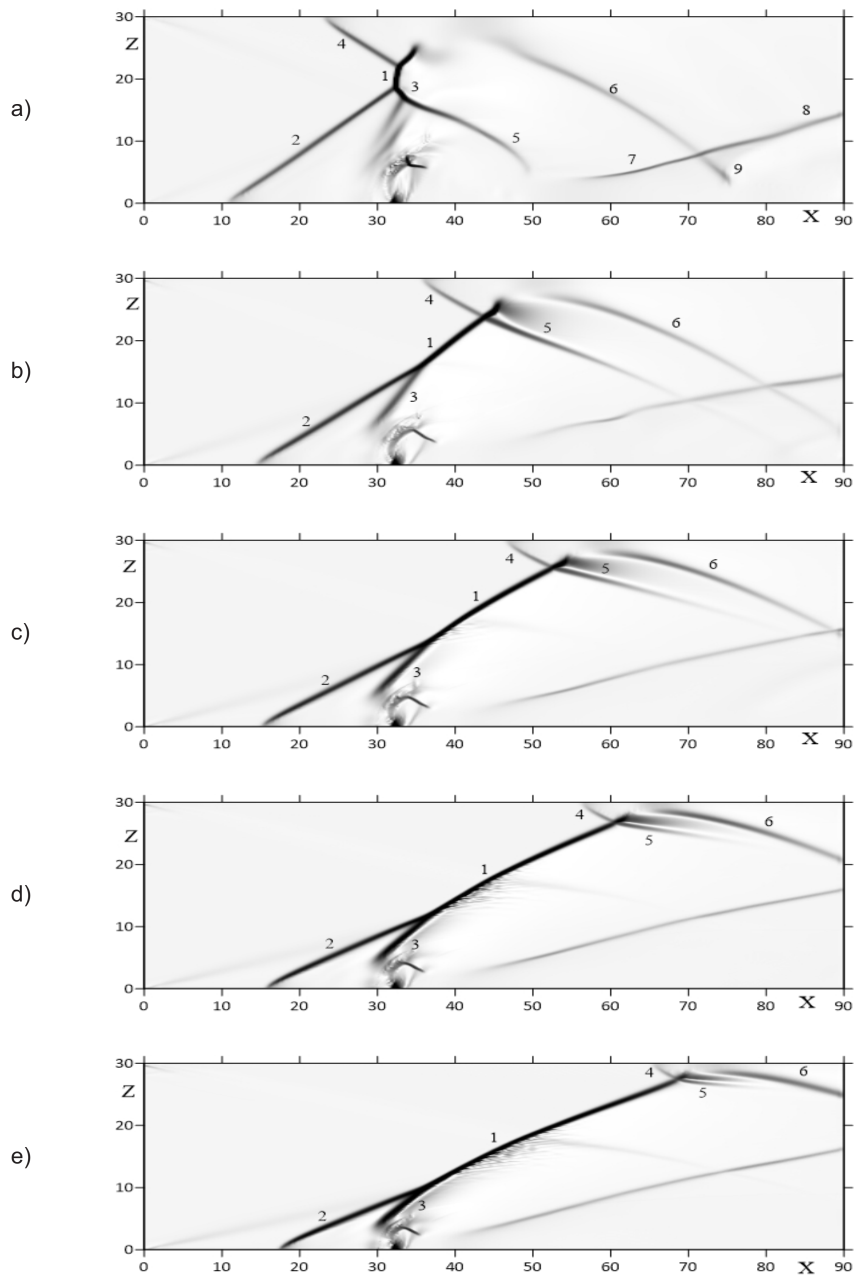


Figure 3 – Distribution of isobars at various Mach number: $a)$ $M_\infty = 2.5,$ $b)$ $M_\infty = 3.0,$ $c)$ $M_\infty = 3.5,$ $d)$ $M_\infty = 4.0,$ $e)$ $M_\infty = 4.5$

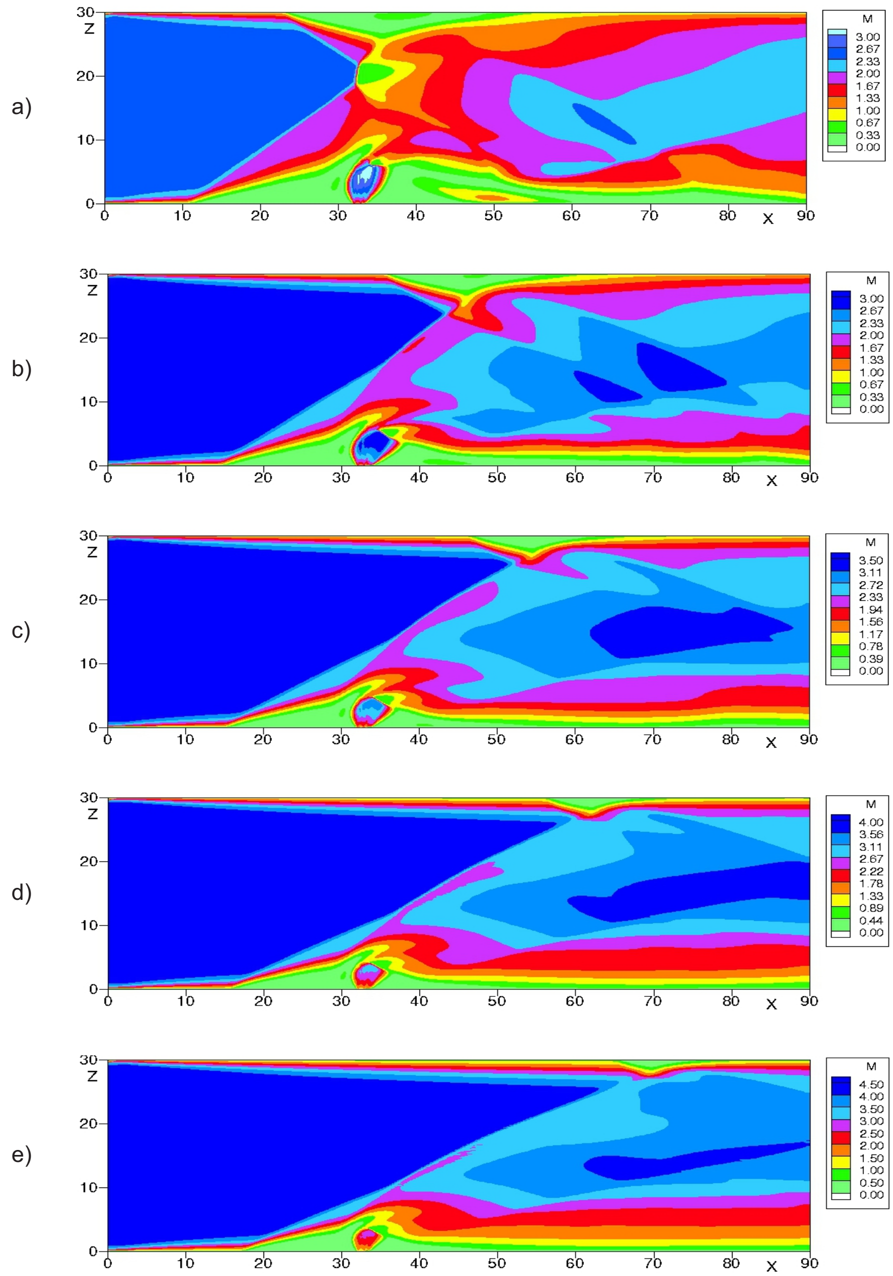


Figure 4 – The local Mach number contour at various Mach number: *a)* $M_\infty = 2.5$, *b)* $M_\infty = 3.0$, *c)* $M_\infty = 3.5$, *d)* $M_\infty = 4.0$, *e)* $M_\infty = 4.5$

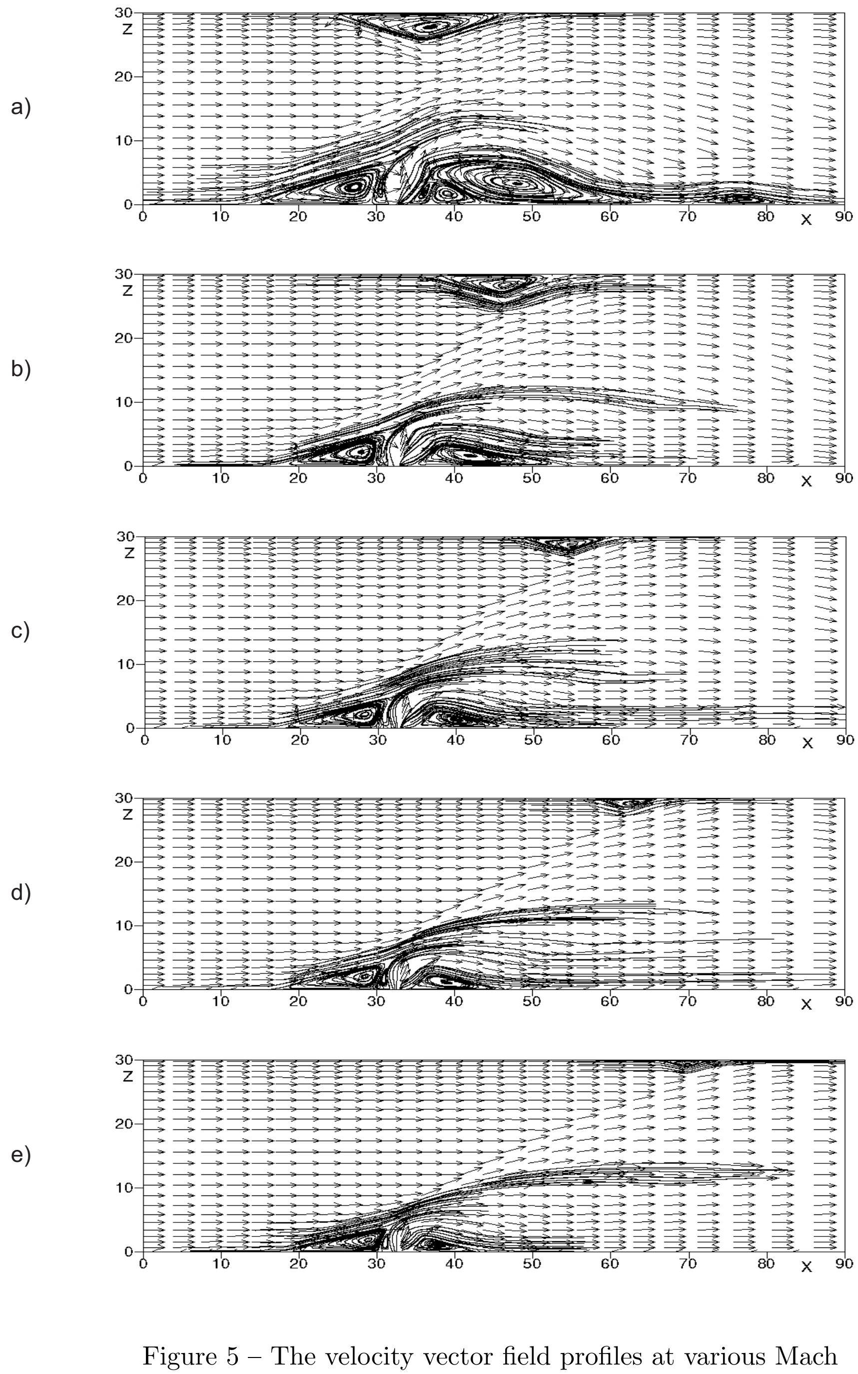


Figure 5 – The velocity vector field profiles at various Mach number: $a)$ $M_\infty = 2.5,$ $b)$ $M_\infty = 3.0,$ $c)$ $M_\infty = 3.5,$ $d)$ $M_\infty = 4.0,$ $e)$ $M_\infty = 4.5$

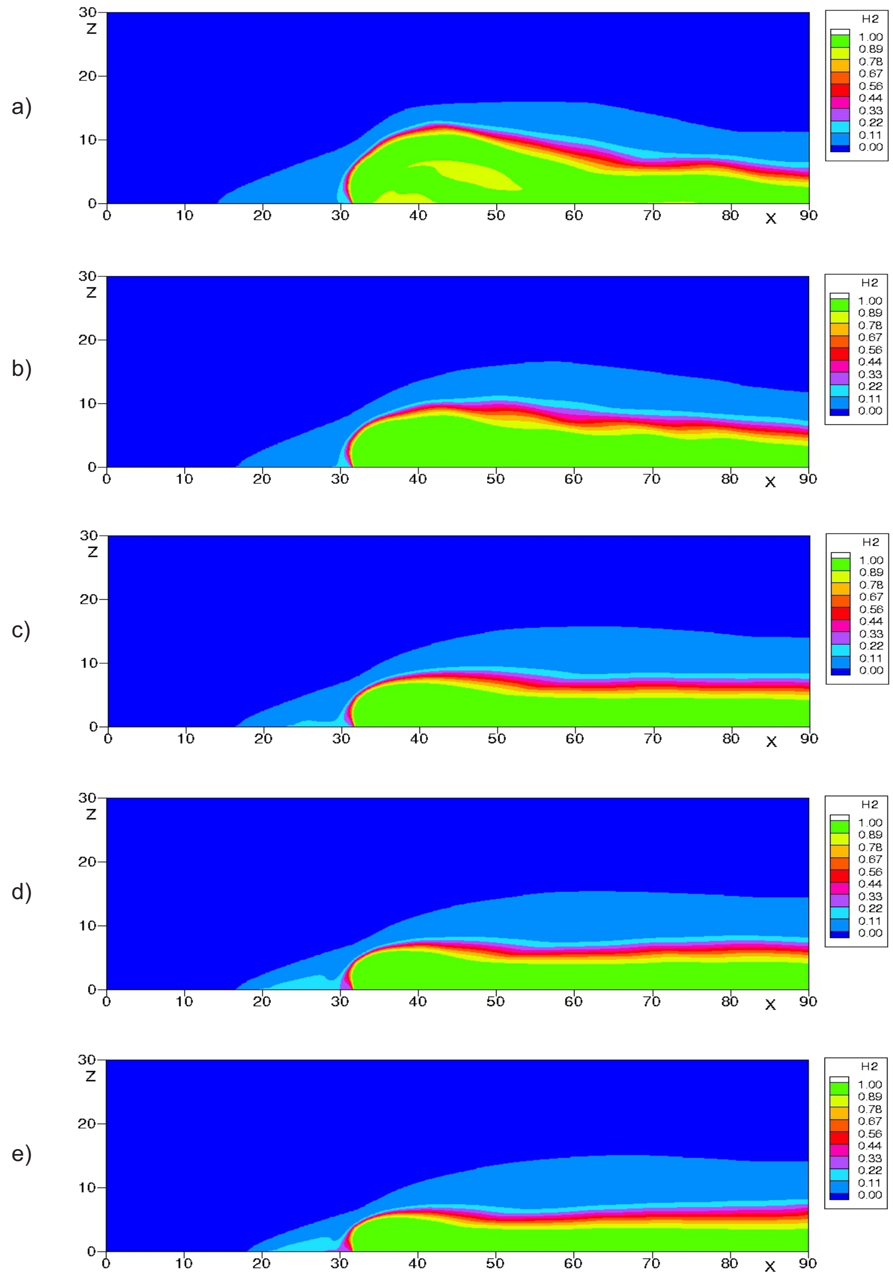


Figure 6 – The distribution of hydrogen mass fraction at various Mach number: *a*) $M_\infty = 2.5$, *b*) $M_\infty = 3.0$, *c*) $M_\infty = 3.5$, *d*) $M_\infty = 4.0$, *e*) $M_\infty = 4.5$

the barrel size is diminished.

The graph of velocity vector field which is represented in Figures 5a–5e (*a*) $M_\infty = 2.5$, *b*) $M_\infty = 3.0$, *c*) $M_\infty = 3.5$, *d*) $M_\infty = 4.0$, *e*) $M_\infty = 4.5$) demonstrates that the recirculation zones ahead and behind the jet are become smaller with the growth of Mach number. Figure 5a shows for $M_\infty = 2.5$, besides the well-known behind the jet vorticity zone, additional separation zone is formed on the bottom wall behind jet at the distance $45 < x < 60$. This separation is due to the interaction of the shock wave 6 with the boundary layer (SWBLI) on the bottom wall at distance $x = 75$. The size of separation bubble at the upper wall is reducing and moving upstream growing Mach number. It can be noticed comparing Figures 5a–5e that the jet penetration increases with growth of M_∞. This is also confirmed by the mass fraction of species contours shown in Figures 6a–6e. As can be seen, this is verified by Figure 7, which presents the influence of various Mach number on the jet penetration. The hydrogen jet penetration decreases sharply from $M_\infty = 2.5$ to $M_\infty = 3.0$, then declines moderately between $M_\infty = 3.0$ and $M_\infty = 4.5$ (Figure 7).

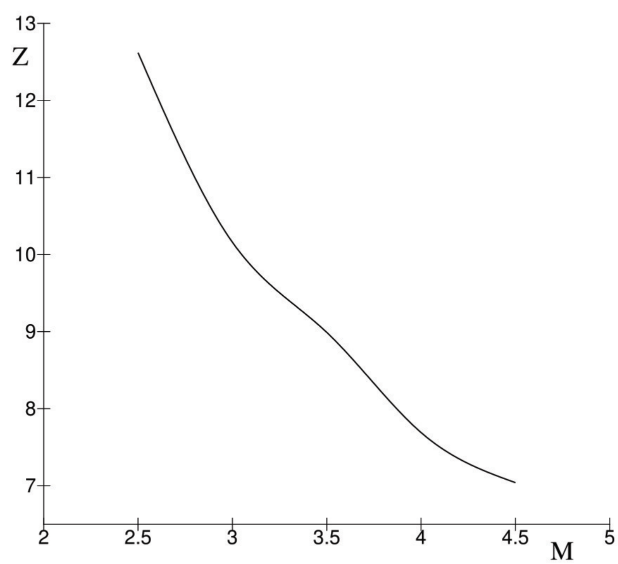


Figure 7 – Effect of various Mach number on the jet penetration

6 Conclusion

The influence of the Mach number on the supersonic flow dynamics with transverse hydrogen jet injection is numerically studied in detail. It is revealed that inclination angle of the bow shock wave 1 and size of the λ – shape shock (which is formed because of the intersection of the bow shock 1, the oblique shock 2 and the reflected shock 3) decrease with growth of M_∞. On the upper wall it is formed one more additional λ – shaped system (the bow shock 1, the compression wave 4 and the reflected shock 5 are intersected at a single point). The size of this λ – shaped structure reduces simultaneously with the increase of the Mach number. For $M_\infty = 2.5$ an additional λ – shaped structure appears near the bottom wall behind the jet due to the shock wave 6 reaching the bottom boundary layer behind the jet, where it creates the compression wave 7, which propagates in a form of the shock 8. Consequently all vortex structures at the upper and the bottom walls resulting from the interaction of the shock-wave structures with the boundary layers (SWBLI) increase with declining of the Mach number. The additional λ – shaped structure near the bottom wall behind the jet for the Mach number 2.5 generates the additional separation zone on the lower wall at a distance $x = 75$. It is received that the barrel – shock structure in the jet decreases with increasing of the Mach number. Hence, jet penetration decreases and this is also confirmed by results of the mass fraction of species. The influence of the Mach number on the hydrogen jet penetration is determined. The result shows a sharply decrease in penetration from $M_\infty = 2.5$ to $M_\infty = 3.0$, then with the Mach number greater than three it is declined moderately. A comparison of computations with experimental data shows a satisfactory agreement of results.

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Аширова Г.А., Бекетаева А.О., Найманова А.Ж. СУТЕГІ АҒЫНЫМЕН ҮРЛЕНЕТІН ЖОҒАРЫ ДЫБЫСТЫ АУА АҒЫСЫНЫҢ МАХ САНДАРЫ ӘРТҮРЛІ БОЛҒАНДАҒЫ САНДЫҚ МОДЕЛДЕУІ

Көп компонентті жоғары дыбысты газ ағысы сутегі ағыны көлденең үрленетін тегіс арнада моделденеді. $k-\omega$ турбуленттік моделімен тұйықталған Фавр бойынша орташаланған Навье-Стокс теңдеулері үшінші ретті ENO сызбасын қолдану арқылы шешіледі. Ағыстың Мах санының соққы толқыны құрылымының каналдың жоғарғы және төменгі қабырғаларындағы шекаралық қабаттармен ішкі турбулентті ағын жағдайындағы өзара әрекеттесуіне әсер етуіне басты назар аударылады. Атап айтқанда, Мах сандары әртүрлі болғандағы соққы толқынының құрылымы, ажырау аймақтары, ағыстың кіріп кетуі егжей-тегжейлі зерттеледі. Мах санын өсіргенде жоғарғы және төменгі қабырғаларда және құйынды аймақтарда пайда болатын, соққы толқындарының құрылымдарының шекаралық қабаттармен (SWBLI) өзара әрекеттесуі нәтижесінде пайда болатын соққы толқындарының құрылымдарының азаятындығы анықталды. Ағыстың Мах санының кішігірім мәндері үшін соққы толқындарының құрылымдарының ағынның сыртындағы төменгі қабырғадағы қосымша өзара әрекеттесуі анықталды. Сондай-ақ, Мах санының өсуі кезінде ағыстың кіріп кетуінің кемуі байқалды. Тәжірибелік мәліметтермен салыстыру жасалды.

Кілттік сөздер. Навье-Стокс теңдеулері, дыбыстан жоғары ағын, соққы толқыны, ажырау аймағы, шекаралық қабат, Мах саны.

Аширова Г.А., Бекетаева А.О., Найманова А.Ж. ЧИСЛЕННОЕ МОДЕЛИРОВАНИЕ СВЕРХЗВУКОВОГО ПОТОКА ВОЗДУХА С ВДУВОМ СТРУИ ВОДОРОДА ПРИ РАЗЛИЧНЫХ ЧИСЛАХ МАХА

Моделируется течение многокомпонентного сверхзвукового газа в плоском канале с поперечным вдувом струи водорода. Решение осредненных по Фавру уравнений Навье–Стокса, замкнутых $k-\omega$ моделью турбулентности, осуществляются с использованием схемы ENO третьего порядка. Основное внимание уделено влиянию числа Маха потока на взаимодействие структуры ударной волны с пограничными слоями на верхней и нижней стенках канала в условиях внутреннего турбулентного потока. В частности, детально исследуются структура ударной волны, зоны отрыва, проникновение струи при различных числах Маха. Установлено, что структуры ударных волн, возникающие на верхней и нижней стенках и вихревых зонах, возникающие в результате взаимодействия структур ударных волн с пограничными слоями (SWBLI), уменьшаются при увеличении числа Маха. При малых значениях числа Маха потока обнаружено дополнительное взаимодействие структур ударных волн на нижней стенке за струей. Также обнаружено уменьшение проникновения струи с увеличением числа Маха. Проведено сравнение с экспериментальными данными.

Ключевые слова. Уравнения Навье–Стокса, сверзвуковое течение, ударная волна, отрывная зона, пограничный слой, число Маха.

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Mixed value problem for nonstationary nonlinear one-dimensional Boltzmann moment system of equations in the first and third approximations with macroscopic boundary conditions

Y. Auzhani[a], A. Sakabekov[b]

Satbayev Kazakh National Technical University, Almaty, Kazakhstan
[a] e-mail: erkawww@gmail.com, [b]e-mail: auzhani@gmail.com

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Abstract. We approximate the microscopic Maxwell boundary condition for one-dimensional Boltzmann equation when some of molecules are reflected from the surface specularly and some diffusely with Maxwell distribution. We formulate the mixed value problem for the first and third moments of Boltzmann system of equations with macroscopic boundary conditions. We prove the existence and uniqueness of the solution of mixed value problem for one-dimensional nonlinear nonstationary Boltzmann moment system of equations in first and third approximations with macroscopic boundary conditions at in space of functions continuous in time and summable in square by spatial variable.

Keywords. Boltzmann moment system of equations, microscopic Maxwell boundary condition, macroscopic boundary conditions.

1 Introduction

Many problems of rarefied gas dynamics require solving problems for Boltzmann equation. Prediction of the aerodynamic characteristics of aircraft at very high speeds and at high altitudes is an important problem in aerospace engineering. In case of a gas flow near a solid body or inside a region bounded by a solid surface, the boundary conditions describe the interaction of gas molecules with solid walls. Unfortunately, it is almost impossible to conduct experiments to study the interaction of gas with a surface at very high speeds and at high altitudes. The aerodynamic characteristics of aircraft at very high speeds and at high altitudes can be determined by the methods of the theory of rarefied gas [1]. For analyzing aerodynamic characteristics of aircraft in transient regime the complete integro-differential Boltzmann equation

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is used with appropriate boundary conditions. Determination of the boundary conditions on surfaces that are streamlined with rarefied gas is one of the most important questions of the in kinetic theory of gases. In high-altitude aerodynamics the interaction of gas with surface of a streamlined body plays an important role [2]. The aerothermodynamic characteristics of bodies in a gas flow are determined by transfer of momentum and energy to the surface of the body, that is, the relationship between velocities and energies of molecules incident on the surface and molecules reflected from it, which is the essence of the kinetic boundary conditions on the surface. Maxwell boundary condition for solving specific problems more accurately describes the interaction of gas molecules with the surface. One of the approximate methods for solving the initial-boundary value problem for Boltzmann equation is the moment method. Using this method, it becomes possible to determine the aerodynamic characteristics of aircraft such as atmospheric parameters, flight speed, geometric parameters, and like that. In the work [3], two new models of boundary conditions were proposed: diffusive-moment and mirror-moment, generalizing the known boundary conditions of Cherchinyani; in work [4], the aerodynamic characteristics of space vehicles were studied by the method of direct static modeling (Monte Carlo method) and various models of the interaction of gas molecules with a surface and their effect on aerodynamic characteristic. Moment methods are the different from each other as sets of various systems of basis functions. For example, Grad in works [5] and [6] obtained a moment system through decomposition of particles distribution function by Hermitte polynomials near the local Maxwell distributions. Grad used Cartesian coordinates of velocities and his moment system contained unknown hydrodynamic characteristics such as density, temperature, average speed, etc. In work [7] we obtained moment system which differs from Grad's system of equations. We used spherical coordinates of velocity and distribution function was decomposed into series by eigenfunctions of linearized collision operator [1], [8], which is the product of Sonin polynomials and spherical functions. The expansion coefficients, the moments of distribution function are defined differently from Grad. The resulting system of equations corresponding to a partial sum of series, which we call Boltzmann moment system of equations, is nonlinear hyperbolic system relative to the moments of particles distribution function. Differential part of the resulting system is linear and quadratic nonlinearity has the form of moments of a distribution function. Quadratic forms, that is the moments of nonlinear collision ingerals, are calculated in [9] and are expressed in terms of coefficients of Talmi [10] and Klebsh-Gordon [11].

In the works [12]–[13] there were obtained moment systems for spatially homogeneous Boltzmann equation and conditions for representability of the solution of spatially homogeneous Boltzmann equation in the form of Poincaré series. The method proposed in [12] (application of Fourier transform with respect to velocity variable in isotropic case) greatly simplified the collision integral and, hence, calculation of moments from of collision integral. In work [13] the results of [12] were generalized for in case of anisotropic scattering.

Levermore C.D. in the work [14] presented systematic nonperturbative derivation of hierarchy of closed systems of moment equations corresponding to any classical theory. This paper is a fundamental work where in which closed systems of moment equations describe the transition regime.

The Boltzmann equation is equivalent to an infinite system of differential equations for the moments of the particle distribution function in the complete system of eigenfunctions of linearized operator. As a rule, we limit study to the finite moment system of equations as solving the infinite system of equations is not possible.

The finite system of moment equations for a specific task with a certain degree of accuracy replaces the Boltzmann equation. It is necessary, also roughly, to replace boundary conditions for the particle distribution function by a number of macroscopic conditions for moments, i.e. there arises a problem of boundary conditions for a finite system of equations that approximate microscopic boundary conditions for the Boltzmann equation. The problem of boundary conditions for a finite system of moment equations can be divided into two parts: how many conditions must be imposed and how they should be prepared. From microscopic boundary conditions for the Boltzmann equation there can be obtained an infinite set of boundary conditions for each type of decomposition. However, the number of boundary conditions is not determined by the number of moment equations, i.e. it is impossible to take as many boundary conditions as equations, although the number of moment equations affect the number of boundary conditions. In addition, the boundary conditions must be consistent with moment equations and the resulting problem must be correct.

Grad in [5] described the construction of an infinite sequence of boundary conditions without consent of the order of approximation for decomposition of boundary conditions and expansion of the Boltzmann equation. Construction of boundary conditions (even one-dimensional Grad's moment system of equations) is a difficult task as Grad's moment system of equations is a hyperbolic system and this system contains unknown parameters for coefficients, such as density, temperature, average speed, etc. In this case, the characteristic equation also depends on unknown parameters and it appears to be difficult to formulate the boundary conditions for the moment system. In the work [15] there were discussed the boundary conditions for the 13-moment Grad system.

In the work [7] we showed approximation of homogeneous boundary condition for particle distribution function and proved the correctness of the initial-boundary value problem for nonstationary nonlinear Boltzmann moment system of equations in three-dimensional space. More precisely, we proved the existence of a unique generalized solution for the initial-boundary value problem for Boltzmann moment system of equations in the space of functions continuous in time and summable by square in the space of variables. In addition, an approximation of microscopic boundary condition for three-dimensional Boltzmann equation was given. The boundary condition is given in a form of integral relation between particles incident on the boundary of particles and reflected from the boundary of particles.

The boundary condition can be formulated as follows: determine the mirrored half of the distribution function from the known half, corresponding to the incident particles. The boundary condition is specified as an integral relation between particles incident on the boundary and particles reflected from the boundary (assuming that we know the probability of an event that a particle incident on the boundary with velocity v_i is reflected with velocity v_r).

However, in practice, the fluxes of particles incident on boundary and reflected from it are determined by numerically solving the corresponding mixed problem for various approximations of Boltzmann moment system of equations. Therefore, the study of mixed problems for moment equations is an urgent and important problem of the in dynamics of a rarefied gas.

In this work, we give an approximation of the microscopic boundary condition when part of molecules is reflected from the surface specularly and part is diffused by the Maxwell distribution. For this case, macroscopic boundary conditions for two-moment and six-moment system of equations were obtained from microscopic Maxwell boundary conditions. Let us prove the existence of a unique solution of the mixed value problem for one-dimensional Boltzmann moment system of equations in the first and third approximations (two-moment and six-moment system of equations) in the space of functions continuous in time and summable in square by spatial variable.

2 Investigation of the existence and uniqueness of solution of mixed value problem for non-stationary nonlinear one-dimensional system of Boltzmann moment system of equations in the first and third approximations under macroscopic Maxwell boundary conditions

In case of gas flow inside a region bounded by a closed or open surface, or near a solid body, the boundary conditions are specified in the form of ratio between particles incident on the boundary and reflected from it. If the initial distribution of gas molecules is known, then the further evolution of the gas is described by the Boltzmann integro-differential equation. So, the problem reduces to solving initial-boundary value problem for the Boltzmann equation. Here we show the formulation of the initial-boundary value problem for the one-dimensional Boltzmann equation under Maxwell boundary conditions without going into details of interaction of gas with wall. We will approximate initial-boundary value problem for the Boltzmann equation by the corresponding problem for the system of Boltzmann moment equations in first and third approximations and show the correctness of the obtained problems.

We note that Mischler S. in work [16] proved a theorem on the existence of a global solution to the initial-boundary value problem for the 3-dimensional nonlinear Boltzmann equation under the Maxwell boundary conditions.

Statement of the problem. Find a solution to the initial-boundary value problem for a homogeneous one-dimensional Boltzmann equation

$$\frac{\partial f}{\partial t} + |v|\cos\theta\frac{\partial f}{\partial x} = J(f,f),\ t\in(0,T],\ x\in(-a,a),\ v\in R_3^v, \tag{1}$$

$$f|_{t=0} = f^0(x,v),\ (x,v) \in [-a,a] \times R_3^v, \tag{2}$$

$$f^+(t,x,v_1,v_2,v_3) = \beta f^-(t,x,v_1,v_2,-v_3) + (1-\beta)\eta \exp\Big(-\frac{|v|^2}{2RT_0}\Big),$$

$$v_3 = |v|\cos\theta,\ (n,v) = (n,|v|\cos\theta) > 0, x = -a \quad \text{or} \quad x = a, \tag{3}$$

where $f \equiv f(t,x,v)$ is a particle distribution function in the space of velocity and time; $f^0(x,v)$ is a distribution of particles at the initial time (fixed function); $J(f,f) \equiv \int [f(v')f(w') - f(v)f(w)]\sigma(\cos x)dwd\varepsilon$ is a nonlinear collision operator recorded for Maxwell molecules, n is unit external normal vector of boundary, $v, w(v',w')$ are velocities of particles before (after) a collision; θ is the angle between v and x axis.

The condition (3) is a natural boundary condition for the Boltzmann equation, which makes it possible to determine the reflected half of distribution function f, if we know the half corresponding to the incident particles. According to (3) some of the incident particles are reflected specularly and others are absorbed by the wall and emitted with a Maxwell distribution with the corresponding wall temperature T_0.

Formula (3) refers to the case of wall at rest; otherwise v must be replaced by $v - u_0, u_0$ being the velocity of wall. β, T_0, u_0 may vary from point to point and with time [8].

For one-dimensional problem eigenfunctions of linearized operator are [1], [8]:

$$g_{nl}(\alpha v) = \left(\frac{\sqrt{\pi}n!(2l+1)}{2\Gamma(n+l+3/2)}\right)^{1/2}\left(\frac{\alpha|v|}{\sqrt{2}}\right)^l S_n^{l+1/2}\left(\frac{\alpha^2|v|^2}{2}\right)P_l(\cos\theta),\ 2n+l = 0,1,2,\ldots,$$

where $S_n^{l+1/2}(\frac{\alpha^2|v|^2}{2})$ are Sonin polynomials, $P_l(\cos\theta)$ are Legendre polynomials, Γ is Gamma function.

To find an approximate solution of problem (1)–(3) we apply the Galerkin method. We define an approximate solution to problem (1)–(3) as follows:

$$f_{2N+1}(t,x,v) = \sum_{2n+l=0}^{2N+1} f_{nl}(t,x)g_{nl}(\alpha v), \tag{4}$$

$$\int_{R_3^v}\left(\frac{\partial f_{2N+1}}{\partial t} + |v|\cos\theta\frac{\partial f_{2N+1}}{\partial x} - J(f_{2N+1},f_{2N+1})\right)f_0(\alpha|v|)g_{nl}(\alpha v)dv = 0, \tag{5}$$

$$2n+l = 0,1,...,2N+1,\ (t,x) \in (0,T] \times (-a,a),$$

$$\int_{R_3^v}[f_{2N+1}(0,x,v) - f_{2N+1}^0(x,v)]f_0(\alpha|v|)g_{nl}(\alpha v)dv = 0,\ 2n+l = 0,1,...,2N+1,\ x \in (-a,a), \tag{6}$$

$$\int_{(n,v)>0}(n,v)f_0(\alpha|v|)f_{2N+1}^+(t,x,v)g_{n,2l}(\alpha v)dv - \beta\int_{(n,v)<0}(n,-v)f_0(\alpha|v|)f_{2N+1}^-(t,x,v)g_{n,2l}(\alpha v)dv$$

$$-(1-\beta)\int\limits_{(n,v)<0}(n,-v)f_0(\alpha|v|)\exp\left(-\frac{|v|^2}{2RT_0}\right)g_{n,2l}(\alpha v)dv=0, \tag{7}$$

$$2(n+l)=0,2,...,2N,\ x=-a \quad \text{or} \quad x=a,$$

where $n=(0,0,1)$ with $x=a$ and $n=(0,0,-1)$ with $x=-a$;

$$f_0(\alpha|v|)=\left(\frac{\alpha^2}{2\pi}\right)^{3/2}\exp\left(-\frac{\alpha^2v^2}{2}\right)$$

is a global Maxwell distribution, $\alpha^2=\frac{1}{RT_0}$;

$$f_{nl}(t,x)=\int\limits_{R_3^v}f_{2N+1}(t,x,v)f_0(\alpha|v|)g_{nl}(\alpha v)dv,$$

$$f^0_{2N+1}(x,v)=\sum_{2n+l=0}^{2N+1}f^0_{nl}(x)g_{nl}(\alpha v)dv,$$

$$f^0_{nl}(x)=\int\limits_{R_3^v}f^0_{2N+1}(x,v)f_0(\alpha|v|)g_{nl}(\alpha v)dv. \tag{8}$$

In general, the approximation of the boundary condition (3) depends on the parity or oddness of approximation of the Boltzmann moment system of equations [17]. In approximating the microscopic boundary condition we took into account the approximation of the Boltzmann equation by the moment equations corresponding to the odd approximation of the Boltzmann moment system of equations. Thus, the approximation orders for the expansion of the boundary condition and the expansion of the Boltzmann equation are consistent. The macroscopic conditions (7) we called the Maxwell boundary conditions [17].

The Boltzmann system of moment equations (5) corresponding to decomposition (4) can be written in extended form:

$$\frac{\partial f_{nl}}{\partial t}+\frac{1}{\alpha}\frac{\partial}{\partial x}\left[l\left(\sqrt{\frac{2(n+l+1/2)}{(2l-1)(2l+1)}}f_{n,l-1}-\sqrt{\frac{2(n+1)}{(2l-1)(2l+1)}}f_{n+1,l-1}\right)\right.$$

$$\left.+(l+1)\left(\sqrt{\frac{2(n+l+3/2)}{(2l+1)(2l+3)}}f_{n,l+1}-\sqrt{\frac{2n}{(2l+1)(2l+3)}}f_{n-1,l+1}\right)\right]=I_{nl}, \tag{9}$$

$$2n+l=0,1,\dots,2N+1,$$

where the moments of collision integral can be expressed in terms of coefficients of Talmi and Klebsh-Gordon as follows [6]:

$$I_{nl} = \sum \langle N_3L_3n_3l_3 : l|nl00 : l\rangle \langle N_3L_3n_3l_3 : l|n_1l_1n_2l_2 : l\rangle (l_10l_20/l0)(\sigma_{l_3} - \sigma_0) f_{n_1l_1} f_{n_2l_2},$$

$\langle N_3L_3n_3l_3 : l|n_1l_1n_2l_2 : l\rangle$ are generalized Talmi coefficients, $(l_10l_20/l0)$ are Klebsh-Gordon coefficients. In this formula summation is carried out over all repeating indices $N_3L_3n_3l_3, n_1l_1$ n_2l_2, and they take a number of values which determined from the following restrictions:

1. energy conservation law $2n_1 + l_1 + 2n_2 + l_2 = 2N_3 + L_3 + 2n_3 + l_3$;
2. parity conservation law $(-1)^{l_1+l_2} = (-1)^{L_3+l_3}$.

A program was also compiled for calculating the values of Talmi coefficients. If in (9) $2n+l$ takes values from 0 to 1, then we obtain the following system of equations corresponding to the first approximation of the Boltzmann moment system of equations or the two-moment system of the Boltzmann equations

$$\frac{\partial f_{00}}{\partial t} + \frac{1}{\alpha}\frac{\partial f_{01}}{\partial x} = 0,$$

$$\frac{\partial f_{01}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}(f_{00}) = 0. \tag{10}$$

We introduce the following designations: $u = f_{00},\ w = f_{01},\ A = \frac{1}{\alpha}(1),\ B = \frac{1}{\alpha\sqrt{\pi}}(\sqrt{2})$.

Here, a mixed value problem for two-moment system of Boltzmann equations under the Maxwell boundary conditions is formulated. Find a solution to the system of equations

$$\frac{\partial u}{\partial t} + A\frac{\partial \omega}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + A'\frac{\partial u}{\partial x} = 0,\ t \in (0, T],\ x \in (-a, a), \tag{11}$$

satisfying the following initial condition

$$u|_{t=0} = u_0(x),\ \omega|_{t=0} = \omega_0(x),\ x \in (-a, a), \tag{12}$$

and boundary conditions

$$(Aw^+ - Bu^+)|_{x=-a} = \beta(Aw^- + Bu^-)|_{x=-a} + \frac{1}{\alpha\sqrt{\pi}}(1-\beta)F,\ t \in [0, T], \tag{13}$$

$$(Aw^+ + Bu^+)|_{x=a} = \beta(Aw^- - Bu^-)|_{x=a} + \frac{1}{\alpha\sqrt{\pi}}(1-\beta)F,\ t \in [0, T], \tag{14}$$

where $u_0(x), w_0(x)$ are given functions, $F = \frac{1}{4\sqrt{2}}$.

Problem (11)–(14) represents a linear hyperbolic system of equations regarding u, w.

Similarly, if in (9) $2n + l$ takes values from 0 to 3, then we obtain the following system of equations corresponding to the third approximation of the Boltzmann moment system of equations or the six-moment system of Boltzmann equations

$$\frac{\partial f_{00}}{\partial t} + \frac{1}{\alpha}\frac{\partial f_{01}}{\partial x} = 0,$$

$$\frac{\partial f_{02}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(\frac{2}{\sqrt{3}}f_{01} + \frac{3}{\sqrt{5}}f_{03} - \frac{2\sqrt{2}}{\sqrt{15}}f_{11}\right) = J_{02},$$

$$\frac{\partial f_{10}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(-\sqrt{\frac{2}{3}}f_{01} + \sqrt{\frac{5}{3}}f_{11}\right) = 0,$$

$$\frac{\partial f_{01}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(f_{00} + \frac{2}{\sqrt{3}}f_{02} - \sqrt{\frac{2}{3}}f_{10}\right) = 0,$$

$$\frac{\partial f_{03}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\frac{3}{\sqrt{5}}f_{02} = J_{03},$$

$$\frac{\partial f_{11}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left(-\frac{2\sqrt{2}}{\sqrt{15}}f_{02} + \sqrt{\frac{5}{3}}f_{10}\right) = J_{11},\ x \in (-a, a),\ t > 0, \tag{15}$$

where $f_{00} = f_{00}(t, x),\ f_{01} = f_{01}(t, x), \ldots, f_{11} = f_{11}(t, x)$ are moments of particle distribution function;

$$J_{02} = (\sigma_2 - \sigma_0)(f_{00}f_{02} - f_{01}^2/\sqrt{3})/2,$$

$$J_{03} = \frac{1}{4}(\sigma_3 + 3\sigma_1 - 4\sigma_0)f_{00}f_{03} + \frac{1}{4\sqrt{5}}(2\sigma_1 + \sigma_0 - 3\sigma_3)f_{01}f_{02},$$

$$J_{11} = (\sigma_1 - \sigma_0)(f_{00}f_{01} + \frac{1}{2}\sqrt{\frac{5}{3}}f_{10}f_{01} - \frac{\sqrt{2}}{\sqrt{15}}f_{01}f_{02}$$

are the moments of collision integral, where $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are constants.

The mixed value problem for the Boltzmann six-moment system of equations under the Maxwell boundary conditions is as follows: find solution to the system of equations

$$\frac{\partial u}{\partial t} + A\frac{\partial \omega}{\partial x} = J_1(u, \omega),$$

$$\frac{\partial u}{\partial t} + A'\frac{\partial u}{\partial x} = J_2(u, \omega),\ x \in (-a, a), \tag{16}$$

satisfying the following initial condition

$$u|_{t=0} = u_0(x),\ \omega|_{t=0} = \omega_0 x),\ x \in (-a, a), \tag{17}$$

and boundary conditions

$$(Aw^+ - Bu^+)|_{x=-a} = \beta(Aw^- + Bu^-)|_{x=-a} + \frac{1}{\alpha\sqrt{\pi}}(1-\beta)F,\ t \in [0,T], \tag{18}$$

$$(Aw^+ + Bu^+)|_{x=a} = \beta(Aw^- - Bu^-)|_{x=a} + \frac{1}{\alpha\sqrt{\pi}}(1-\beta)F,\ t \in [0,T], \tag{19}$$

where

$$A = \frac{1}{\alpha}\begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{\sqrt{3}} & \frac{3}{\sqrt{5}} & -\frac{2\sqrt{2}}{\sqrt{15}} \\ -\sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{5}{3}} \end{pmatrix},\ B = \frac{1}{\alpha\sqrt{\pi}}\begin{pmatrix} \sqrt{2} & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & 2\sqrt{2} & -1 \\ -\frac{1}{\sqrt{3}} & -1 & 3\sqrt{2} \end{pmatrix},$$

$$J_1(u,w) = (0, J_{02}, 0)',\ J_2(u,w) = (0, J_{03}, J_{11})',$$

$u = (f_{00}, f_{02}, f_{10})'$, $w = (f_{01}, f_{03}, f_{11})'$, $F = \left(\frac{1}{4\sqrt{2}}; \frac{1}{8\sqrt{6}}; \frac{1}{8\sqrt{3}}\right)'$, A' is a transposed matrix, B is a positive defined matrix; $u_0(x) = (f^0_{00}(x), f^0_{02}(x), f^0_{10}(x))'$, $w_0(x) = (f^0_{01}(x), f^0_{03}(x), f^0_{11}(x))'$ are given initial vector functions; w^+, u^+ are the moments of incident on the boundary particle distribution function, w^-, u^- are moments of distribution function of particles reflected from the boundary. (16) is a vector matrix form recording of the system of equations (15).

Due to the cumbersome computations, we omit the derivation of the boundary conditions (13)–(14) and (18)–(19) and rationale for the number of boundary conditions is given in conclusion.

We prove the following theorem.

Theorem 1. *If $U_0 = (u_0(x),\ w_0(x)) \in L^2[-a,a]$, then problem (11)–(14) has a unique solution in domain $[-a,a]\times[0,T]$, belonging to the space $C([0,T]; L^2[-a,a])$, moreover*

$$\|U\|_{C([0;T];L^2[-a,a])} \le C_1(\|U_0\|_{L^2[-a,a]} + \|f\|_{C([0;T];L^2[-a,a])}, \tag{20}$$

where C_1 is a constant independent of U, function f will be defined below.

Proof. Let $U_0 \in L^2[-a,a]$. Let us prove estimation (20). We multiply the first equation of the system (11) by u and the second equation by w, and integrate from $-a$ to a :

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a}[(u,u)+(w,w)]dx + \int_{-a}^{a}\Big[(A\frac{\partial w}{\partial x}, u) + (A'\frac{\partial u}{\partial x}, w)\Big]dx = 0.$$

After integration by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a}[(u,u)+(w,w)]dx + (u^-, Aw^-)|_{x=a} - (u^-, Aw^-)|_{x=-a} = 0. \tag{21}$$

Taking into account the boundary conditions (13)–(14) we rewrite equality (21) in the following form

$$\frac{1}{2}\frac{d}{dt}\int\limits_{-a}^{a}[(u,u)+(w,w)]dx+(Bu^-,u^-)|_{x=a}+(Bu^-,u^-)|_{x=-a}-\frac{1}{\beta}((Aw^+-Bu^+),u^-)|_{x=-a}$$

$$+\frac{1}{\beta}((Aw^++Bu^+),u^-)|_{x=a}+(F_1,u^-)|_{x=a}+(F_1,u^-)|_{x=-a}=0, \qquad (22)$$

where $F_1=\frac{(1-\beta)\eta}{\alpha\beta\sqrt{\pi}}F$.

Let us use spherical representation [18] of the vector $U(t,x)=r(t)\omega(t,x)$, where

$$\omega(t,x)=(\omega_1(t,x),\omega_2(t,x))', r(t)=\|U(t,.)\|_{L^2[-a,a]}, \|\omega\|_{L^2[-a,a]}=1.$$

Substituting values $u=r(t)\omega_1(t,x),\ w=r(t)\omega_2(t,x)$ into (22), we have that

$$\frac{dr}{dt}+rP(t)=-f(t), \qquad (23)$$

where

$$P(t)=(B\omega_1^-,\omega_1^-)|_{x=a}+(B\omega_1^-,\omega_1^-)|_{x=-a}$$

$$+\frac{1}{\beta}[(A\omega_2^+,\omega_1^-)|_{x=a}+(B\omega_1^+,\omega_1^-)|_{x=a}+(B\omega_1^+,\omega_1^-)|_{x=-a}-(A\omega_2^+,\omega_1^-)|_{x=-a}],$$

$$f(t)=(F_1,\omega_1^-)|_{x=a}+(F_1,\omega_1^-)|_{x=-a}.$$

Let us study equation (23) with the initial condition

$$r(0)=\|U_0\|=\|U_0\|_{L^2[-a,a]}. \qquad (24)$$

The solution of problem (23)–(24) has following form

$$r(t)=exp\Big(-\int\limits_0^t P(\tau)d\tau\Big)\Big[\|U_0\|-\int\limits_0^t f(\tau)exp\Big(-\int\limits_0^\tau P(\xi)d\xi\Big)d\tau\Big]. \qquad (25)$$

In equality (25) integrand $f(\tau)exp(-\int\limits_0^\tau P(\xi)d\xi)$ is bounded. Therefore, $\forall t\in[0,T]$ apriori estimation (20) is valid, where T is any bounded real number. We can prove existence of the solution to of the problem (11)–(14) by Galerkin method. The uniqueness of the solution to the of problem (11)–(14) followed from apriori estimation (20).

Theorem is proved.

For problem (16)–(19) the following theorem takes place [19].

Theorem 2. *If $U_0 = (u_0(x), w_0(x)) \in L^2[-a,a]$, then problem (15)–(19) has a unique solution in the domain $[-a,a]\times[0,T]$, belonging to the space $C([0,T];L^2[-a,a])$, moreover*

$$\|\|U\|_{L^2[-a,a]} - r_1\|_{C([0;T];L^2[-a,a])} \leq C_2(\|U_0\|_{L^2[-a,a]} - r_1(0)), \tag{26}$$

where C_2 is a constant independent of U and $T \sim \mathrm{O}(\|U_0\|_{L^2[-a,a]} - r_1(0))^{-1})$, $r_1(t)$ is a partial solution of the Riccati equation $\frac{dr}{dt} + rP(t) = r^2Q(t) - f(t)$, $P(t)$, $Q(t)$, $f(t)$ are given functions.

For proving this theorem the methods of apriori estimation, Galerkin method and Tartar's compactness compensated method were used [20]. This theorem describes the existence and uniqueness of a local on time solution to the of initial-boundary value problem (16)–(19).

3 Conclusion

1. The system of equations (11) contains two equations corresponding to the laws of conservation of mass and momentum, and represents a linear hyperbolic system of equations regarding u, w. Matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has two eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$. Therefore, for correcting the problem two boundary conditions must be specified – one boundary condition with outgoing characteristic and the other one for incoming characteristic. For initial-boundary value problem two boundary conditions (13) and (14) are specified, which correspond to the number of eigenvalues of matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In Theorem 1 the existence of global in time solution for the initial-boundary value problem (11)–(14).

2. System (16) is a symmetric hyperbolic nonlinear system of partial differential equations. Indeed, direct calculations show that

$$\det A_1 = \det \begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix} \neq 0$$

and matrix A_1 has three positive and the same number of negative nonzero eigenvalues, namely $-\sqrt{(3+\sqrt{6})}, -1, -\sqrt{(3-\sqrt{6})}, \sqrt{(3-\sqrt{6})}, 1, \sqrt{(3+\sqrt{6})}$. It follows from (18)–(19), that the number of boundary conditions on the at left and right ends of the interval $(-a,a)$ are equal to the number of positive and negative eigenvalues of matrix A_1. Theorem 2 claims the existence of a unique local on time solution for problem (16)–(19), since the length of time during on which there is the solution to the of problem (16)–(19) the depends on the difference in norm of the initial vector function and value of a particular solution of the Riccati equation at the initial time in degree -1.

3. The moments f_{00}, f_{01}, f_{10} are expressed by macroscopic characteristics of gas such as density, average speed and temperature. More exactly, we have following equalities

$$f_{00} = \rho, f_{01} = \alpha\rho V, f_{10} = \sqrt{\frac{3}{2}}\rho - \sqrt{\frac{2}{3}}\alpha^2\rho\Big(\frac{3}{2}k\theta + \frac{1}{2}V^2\Big),$$

where ρ is a density of gas, V is an average speed of gas, θ is a temperature of gas and α is a constant (in special case $\alpha = 1$). Moreover, we have the following equality

$$f_{00} + \frac{2}{\sqrt{3}} f_{02} - \sqrt{\frac{2}{3}} f_{10} = \alpha(P_{33} + \rho V^2),$$

where P_{33} is a component of a stress tensor.

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Аужани Е., Сакабеков А.С. БОЛЬЦМАН МОМЕНТТІК ТЕҢДЕУЛЕРІНІҢ СТАЦИОНАР ЕМЕС БІР ӨЛШЕМДІ БЕЙСЫЗЫҚ ЖҮЙЕСІ ҮШІН МАКРОСКОПИЯЛЫҚ ШЕКАРАЛЫҚ ШАРТТАРЫ БАР БІРІНШІ ЖӘНЕ ҮШІНШІ ЖУЫҚТАУЛАРДАҒЫ АРАЛАС ЕСЕБІ

Жұмыста бір өлшемді Больцман теңдеуі үшін микроскопиялық Максвелл шекаралық шарты аппроксимацияланады, мұнда молекулалардың бір бөлігі беттен айналы шағылысса, ал қалған бөлігі Максвелл үлестирімділігі бойынша диффузиялы шағылысады. Бір өлшемді бейсызық стационар емес Больцман теңдеулерінің жүйесінің бірінші және үшінші жуықтаулары үшін макроскопиялық шекаралық шарты бар аралас есеп тұжырымдалған. Бір өлшемді бейсызық стационар емес Больцман теңдеулерінің жүйесінің бірінші және үшінші жуықтаулары үшін макроскопиялық шекаралық шарты бар аралас есептің уақыт айнымалысы бойынша үзіліссіз, ал кеңістіктік айнымалысы бойынша квадрат қосындылынатын функциялар кеңістігінде шешімінің бар екендігі және жалғыздығы дәлелденген.

Кілттік сөздер. Больцман моменттік теңдеулерінің жүйесі, Максвелл микроскопиялық шекаралық шарты, макроскопиялық шекаралық шарт.

Аужани Е., Сакабеков А.С. СМЕШАННАЯ ЗАДАЧА ДЛЯ НЕСТАЦИОНАРНОЙ НЕЛИНЕЙНОЙ ОДНОМЕРНОЙ СИСТЕМЫ МОМЕНТНЫХ УРАВНЕНИЙ БОЛЬЦМАНА В ПЕРВОМ И ТРЕТЬЕМ ПРИБЛИЖЕНИЯХ С МАКРОСКОПИЧЕСКИМИ ГРАНИЧНЫМИ УСЛОВИЯМИ

В работе мы аппроксимируем микроскопическое граничное условие Максвелла для одномерного уравнения Больцмана, когда часть молекул отражается от поверхности зеркально, а часть диффузионно по Максвелловскому распределению. Сформулирована смешанная задача для первого и третьего приближений систем уравнения Больцмана с макроскопическими граничными условиями. Доказаны существование и единственность решения смешанной задачи для одномерной нелинейно нестационарной системы уравнений Больцмана в первом и третьем приближениях при макроскопических граничных условиях в пространстве функций, непрерывных по времени и суммируемых в квадрате по пространственной переменной.

Ключевые слова. Система моментных уравнений Больцмана, микроскопическое граничное условие Максвелла, макроскопическое граничное условие.

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On solvability of one nonlinear boundary value problem of heat conductivity in degenerating domains

M.T. Jenaliyev[1,a], K.B. Imanberdiyev[1,2,b], A.S. Kasymbekova[1,2,c], M.G. Yergaliyev[1,2,d]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[2]Al-Farabi Kazakh National University, Almaty, Kazakhstan
[a] e-mail: muvasharkhan@gmail.com, [b]e-mail: kanzharbek75ikb@gmail.com,
[c]e-mail: kasar1337@gmail.com, [d]e-mail: ergaliev.madi.g@gmail.com

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Abstract. The paper is devoted to problems of solvability of nonlinear heat conduction problem in a degenerating non-rectangular domain in Sobolev classes, the degeneration point of which located at the origin. By using methods of a priori estimates and Faedo-Galerkin method, we prove theorems on the existence and uniqueness of the solution for the boundary value problem under consideration, and also for the one-dimensional boundary problem we prove its regularity with increasing smoothness of given functions. We also obtained further development of these results for the multidimensional version (in a multidimensional cone with a degeneration point at the vertex of the cone) of the boundary value problems under consideration. Here it have also been shown the existence and uniqueness, but of a weaker solution than in one-dimensional case.

Keywords. Second-order parabolic equations, nonlinear parabolic equations.

1 Introduction

The range of application of boundary value problems for parabolic equations in a domain with a boundary that changes over time is quite wide. Such problems arise in the study of thermal processes in electrical contacts [1], the processes of ecology and medicine [2], in solving some problems of hydromechanics [3], thermomechanics in thermal shock [4] and so on.

Extensive literature is devoted to the study of the solvability of linear and nonlinear equations in cylindrical domains. However, as for nonlinear boundary value problems in degenerating non-cylindrical domains, they have been studied relatively little.

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In the works [5] and [6], the solvability of boundary value problems for Burgers equation in the non-rectangular domain was investigated. In the first work [5], it is required that the domain (non-degenerated domain) can be transformed into a rectangular domain by regular replacement of (independent) variables; in the second work [6], this requirement is excluded (the domain of independent variables degenerates at the initial moment of time). On the basis of the results of the work [7] in Sobolev spaces, the existence and uniqueness of the regular solution of the considered boundary problems are established by the methods of a Faedo-Galerkin and a priori estimates.

In [8] and [9] we show that homogeneous boundary value problems for one nonlinear equation and Burgers equation in the (degenerating) angular domain along with the zero solution have non-zero solutions. In [10] we have studied various cases of inhomogeneity at the boundary. In these cases, it is shown that for the corresponding boundary value problems there are both unique solvability and non-unique solvability.

In this paper, in Sobolev classes we study the solvability of a nonlinear equation with homogeneous Dirichlet boundary conditions in a degenerating non-rectangular domain represented by a triangle, one of the corners of which is located at the origin and is a point of degeneracy. In Section 1, we give a statement of the boundary value problem under study, which in Section 2 is transformed by one-to-one nonlinear substitution for an unknown function to a linear boundary value problem in a degenerating triangular domain. In Section 3, for the linear boundary value problem we collate a family of boundary value problems in non-degenerated quadrangular domains represented by the corresponding trapezoids. Here, this family of boundary-value problems is transformed by the replacement of independent variables into the corresponding family of boundary-value problems in rectangular domains, and also here a number of theorems are formulated on their unique solvability. In Section 4, a priori estimates for the solution of boundary value problems in trapezoids are established. In the same Section, the main results of the work are formulated in the form of two theorems for linear and initial nonlinear boundary value problems in a degenerating triangular domain. The proofs of these theorems are given in Sections 5 and 6.

These results in Sections 7–11 are further developed for a multidimensional version of the boundary value of problems under consideration. Here it have also been shown the existence and uniqueness, but of a weaker solution than in the previous sections. It is not yet possible to show the regularity of the weak solution. The work concludes with a brief conclusion.

1 Statement of the boundary value problem

Let $Q_{xt_1} = \{x, t_1 \mid 0 < x < t_1,\ 0 < t_1 < T_1 < \infty\}$ be a triangular domain, one of the vertices of which is located at the origin, and also let Ω_{t_1} be a section of the domain Q_{xt_1} for a fixed time variable $t_1 \in (0, T_1)$. In the domain Q_{xt_1} we consider the following boundary value problem:

$$\partial_{t_1} u - \nu \partial_x^2 u + (\partial_x u)^2 = f, \ \ (\nu > 0), \tag{1}$$

$$u(x,t_1)|_{x=0} = 0,\ u(x,t_1)|_{x=t_1} = 0, \tag{2}$$

where

$$f \in L_\infty(Q_{xt_1}),\ f \geq 0. \tag{3}$$

In this paper, we study the question of the existence and uniqueness of a solution for boundary value problem (1)–(3) in Sobolev space (throughout the work, the space designations correspond to those accepted in the book [11]):

$$u \in H_0^{2,1}(Q_{xt_1}) \equiv L_2(0,T_1; H^2(0,t_1) \cap H_0^1(0,t_1)) \cap H^1(0,T_1; L_2(0,t_1)). \tag{4}$$

2 Converting (1)–(3) to a linear boundary value problem

We transform (1)–(3) to a linear boundary value problem for an unknown function $w(x,t_1)$. Using the following one-to-one transformation:

$$w(x,t_1) = \exp\{-u/\nu\} - 1,\ \ u = -\nu \ln(w+1), \tag{5}$$

we obtain

$$\partial_{t_1} w - \nu \partial_x^2 w + f_\nu w = -f_\nu, \tag{6}$$

$$w(x,t_1)|_{x=0} = 0,\ w(x,t_1)|_{x=t_1} = 0, \tag{7}$$

$$f_\nu \equiv f/\nu \in L_\infty(Q_{xt_1}),\ f_\nu \geq 0. \tag{8}$$

3 On a family of auxiliary boundary value problems in quadrangular domains (in the form of trapezoids)

For problem (6)–(8), we set a family of the boundary value problems, each of which is considered in the domain representing by the corresponding trapezoid.

So, let $n \in \mathbb{N}^* \equiv \{n \in \mathbb{N} : n \geq n_1, 1/n_1 < T_1\}$, $Q_{xt_1}^n = \{x, t_1 :\ 0 < x < t_1,\ 1/n < t_1 < T_1 < \infty\}$ be a trapezoid, and let Ω_{xt_1} be a section at fixed $t_1 \in (1/n, T_1)$. Note that at the point $t_1 = 1/n$ the domain $Q_{xt_1}^n$ no longer degenerates into a point, moreover, between the original domain Q_{xt_1} and domains $Q_{xt_1}^n$ the strict inclusions $Q_{xt_1}^{n_1} \subset Q_{xt_1}^{n_1+1} \subset ... \subset Q_{xt_1}$ take place and, obviously, $\lim\limits_{n\to\infty} Q_{xt_1}^n = Q_{xt_1}$.

In the non-degenerating domain $Q_{xt_1}^n$ (for each finite $n \in \mathbb{N}^*$) we consider the following boundary value problem:

$$\partial_{t_1} w_n - \nu \partial_x^2 w_n + f_{\nu,n} w_n = -f_{\nu,n}, \tag{9}$$

$$w_n(x,t_1)|_{x=0} = 0,\ w_n(x,t_1)|_{x=t_1} = 0,\ w_n(x,t_1)|_{t_1=1/n} = 0, \tag{10}$$

$$f_{\nu,n} \equiv f_n/\nu \in L_\infty(Q_{xt_1}^n),\ f_{\nu,n} \geq 0. \tag{11}$$

We want to transform boundary value problem (9)–(11) so that it would be set in a rectangular domain. For this purpose we will make the transformation of independent variables: we pass from the variables $\{x, t_1\}$ to variables $\{y, t\}$. We have

$$x = \frac{y}{n-t},\ t_1 = \frac{1}{n-t};\ y = \frac{x}{t_1},\ t = n - \frac{1}{t_1};$$

$Q_{yt}^n = \{y, t :\ 0 < y < 1,\ 0 < t < T\}$ is a rectangular domain, and Ω is a section of the rectangle Q_{yt}^n for any fixed $t \in [0, T]$,

$$t_1 = 1/n \Leftrightarrow t = 0,\ t_1 = T_1 \Leftrightarrow t = T = n - \frac{1}{T_1}.$$

Since

$$\tilde{w}_n(y,t) \triangleq w_n\left(\frac{y}{n-t}, \frac{1}{n-t}\right),\ \tilde{f}_{\nu,n}(y,t) = f_{\nu,n}\left(\frac{y}{n-t}, \frac{1}{n-t}\right), \tag{12}$$

then for the derivative with respect to t_1 of function $w_n(x, t_1)$ (12) we obtain

$$\frac{\partial w_n}{\partial t_1} = \frac{\partial \tilde{w}_n(y,t)}{\partial t}(n-t)^2 - \frac{\partial \tilde{w}_n(y,t)}{\partial y}(n-t)y.$$

Now we find the derivative of function $w_n(x, t_1)$ (12) with respect to the variable x :

$$\frac{\partial w_n}{\partial x} = \frac{\partial \tilde{w}_n}{\partial y}(n-t), \quad \frac{\partial^2 w_n}{\partial x^2} = \frac{\partial^2 \tilde{w}_n}{\partial y^2}(n-t)^2.$$

We write down boundary value problem (9)–(11) in the domain Q_{yt}^n:

$$\partial_t \tilde{w}_n - \nu \partial_y^2 \tilde{w}_n - \frac{y}{n-t}\partial_y \tilde{w}_n + \frac{1}{(n-t)^2}\tilde{f}_{\nu,n}\tilde{w} = -\frac{1}{(n-t)^2}\tilde{f}_{\nu,n}, \tag{13}$$

$$\tilde{w}_n(y,t) = 0,\ \{y,t\} \in \Sigma_{yt}^n = \{y, t :\ y \in \{0\} \cup \{1\},\ 0 < t < T\}, \tag{14}$$

$$\tilde{w}_n(y,0) = 0,\ y \in \Omega = \{y :\ 0 < y < 1,\ t = 0\}. \tag{15}$$

Instead of (13)–(15) in the domain Q_{yt}^n, following [5] and [6], we will consider a more general boundary value problem:

$$\partial_t \tilde{w}_n - \nu \partial_y^2 \tilde{w}_n - \gamma_n(y,t)\partial_y \tilde{w}_n + \alpha_n(t)\tilde{f}_{\nu,n}\tilde{w}_n = -\beta_n(t)\tilde{f}_{\nu,n},\ (\nu > 0), \tag{16}$$

$$\tilde{w}_n(y,t)|_{y=0} = 0,\ \tilde{w}_n(y,t)|_{y=1} = 0,\ \tilde{w}_n(y,t)|_{t=0} = 0, \tag{17}$$

where the given functions $\alpha_n(t)$, $\beta_n(t)$, $\gamma_n(y,t)$ for any fixed number $n \in \mathbb{N}^*$ satisfy the following conditions

$$\begin{aligned} &\alpha_{1n} \le \alpha_n(t) \le \alpha_{2n},\ \beta_{1n} \le \beta_n(t) \le \beta_{2n}, \quad \forall\, t \in [0,T], \\ &|\gamma_n(y,t)| \le \gamma_{1n},\ |\partial_y \gamma_n(y,t)| \le \gamma_{1n}, \qquad \forall\,\{y,t\} \in Q^n_{yt}, \end{aligned} \tag{18}$$

with given positive constants $\alpha_{1n},\ \alpha_{2n},\ \beta_{1n},\ \beta_{2n},\ \gamma_{1n}$.

The following theorem is valid.

Theorem 1. *Suppose we have a fixed number $n \in \mathbb{N}^*$. Then, if $\tilde f_{\nu,n} \in L_\infty(Q^n_{yt})$ and $\alpha_n(t)$, $\beta_n(t)$, $\gamma_n(y,t)$ satisfy conditions* (18)*, then boundary value problem* (16)–(17) *has a unique solution*

$$\tilde w_n \in H^{2,1}_0(Q^n_{yt}) \equiv L_2(0,T; H^2(0,1) \cap H^1_0(0,1)) \cap H^1(0,T; L_2(0,1)), \tag{19}$$

which satisfies the following estimate:

$$\|\tilde w_n\|_{H^{2,1}_0(Q^n_{yt})} \le K\left(\|\tilde f_{\nu,n}\|_{L_\infty(Q^n_{yt})}, \nu\right), \quad \textit{moreover,} \quad K(0,\nu) = 0. \tag{20}$$

The proof of Theorem 1 can be obtained by Faedo-Galerkin method (for example, as in [11]).

Since coefficients of equations (13)–(15) meet conditions (18), then for boundary value problem (13)–(15) from Theorem 1 we obtain, as a corollary, the following theorem.

Theorem 2. *Suppose we have a fixed number $n \in \mathbb{N}^*$. Then, if $\tilde f_{\nu,n} \in L_\infty(Q^n_{yt})$, then boundary value problem* (13)–(15) *has a unique solution $\tilde w_n \in H^{2,1}_0(Q^n_{yt})$* (19)*, which satisfies the following estimate:*

$$\|\tilde w_n\|_{H^{2,1}_0(Q^n_{yt})} \le K\left(\|\tilde f_{\nu,n}\|_{L_\infty(Q^n_{yt})}, \nu\right), \quad \textit{moreover,} \quad K(0,\nu) = 0. \tag{21}$$

We give the correspondence of functional spaces in terms of independent variables $\{y,t\} \in Q^n_{yt}$ and $\{x,t_1\} \in Q^n_{xt_1}$:

$$\tilde f_{\nu,n} \in L_\infty(Q^n_{yt}) \equiv L_\infty(0,T; L_\infty(0,1)) \Leftrightarrow f_{\nu,n} \in L_\infty(Q^n_{xt_1}) \equiv L_\infty(1/n, T_1; L_\infty(0,t_1)), \tag{22}$$

$$\begin{aligned} &\tilde w(y,t) \in H^{2,1}_0(Q^n_{yt}) \equiv L_2(0,T; H^2(0,1) \cap H^1_0(0,1)) \cap H^1(0,T; L_2(0,1)) \Leftrightarrow \\ &\Leftrightarrow w(x,t_1) \in H^{2,1}_0(Q^n_{xt_1}) \equiv L_2(1/n, T_1; H^2(0,t_1) \cap H^1_0(0,t_1)) \cap H^1(1/n, T_1; L_2(0,t_1)). \end{aligned} \tag{23}$$

Further, taking into account the correspondence of spaces (22)–(23), in accordance with Theorem 2 we can formulate the following statement:

Theorem 3. *Suppose we have a fixed number $n \in \mathbb{N}^*$. Then, if $f_{\nu,n} \in L_\infty(Q^n_{xt_1})$ (22), then boundary value problem* (9)–(11) *has a unique solution $w_n \in H_0^{2,1}(Q^n_{xt_1})$ (23) that satisfies the following estimate*

$$\|w_n\|_{H_0^{2,1}(Q^n_{xt_1})} \le K\left(\|f_{\nu,n}\|_{L_\infty(Q^n_{xt_1})}, \nu\right)$$

$$\le K_0\left(\|f_\nu\|_{L_\infty(Q_{xt_1})}, \nu\right), \quad \textit{moreover,} \;\; K(0,\nu) = K_0(0,\nu) = 0. \tag{24}$$

The proof of this theorem will be given in the next section.

4 A priori estimates for a solution of problem (9)–(11). Formulation of the main result for one-dimensional problem

Lemma 1. *There exists a positive constant K_1 independent of n, such that for all $t_1 \in (1/n, T_1]$ the following inequality takes place:*

$$\|w_n(x,t_1)\|^2_{L_2(0,t_1)} + \int_{1/n}^{t_1} \|\partial_x w_n(x,\tau_1)\|^2_{L_2(0,\tau_1)} d\tau_1 \le K_1\left(\|f_\nu(x,t_1)\|_{L_\infty(Q_{xt_1})}, \nu\right). \tag{25}$$

Proof. Multiplying equation (9) by $w_n(x,t_1)$ in the space $L_2(0,t_1)$, we obtain

$$\frac{1}{2}\frac{d}{d\,t_1}\|w_n(x,t_1)\|^2_{L_2(0,t_1)} + \nu\|\partial_x w_n(x,t_1)\|^2_{L_2(0,t_1)}$$

$$\le \|f_{\nu,n}(x,t_1)\|_{L_\infty(0,t_1)}\|w_n(x,t_1)\|^2_{L_2(0,t_1)} + \|f_{\nu,n}(x,t_1)\|_{L_\infty(0,t_1)}\|w_n(x,t_1)\|_{L_1(0,t_1)}.$$

Now by using Gronwall's inequality and the following obvious inequality

$$\|f_{\nu,n}\|_{L_\infty(Q^n_{xt_1})} \le \|f_\nu\|_{L_\infty(Q_{xt_1})}, \tag{26}$$

we get required statement of Lemma 1. Note that the equality $K_1(0,\nu) = 0$ holds.

Lemma 2. *For a positive constant K_2 independent of n, for all $t_1 \in (1/n, T_1]$ the following inequality takes place:*

$$\|\partial_x w_n(x,t_1)\|^2_{L^2(0,t_1)} + \int_{1/n}^{t_1} \|\partial_x^2 w_n(x,\tau_1)\|^2_{L^2(0,\tau_1)} d\tau_1 \le K_2\left(\|f_\nu(x,t_1)\|_{L_\infty(Q_{xt_1})}, \nu\right). \tag{27}$$

Proof. Multiplying equation (9) by $-\partial_x^2 w_n(x,t_1)$ in the space $L_2(0,t_1)$, we obtain

$$\frac{1}{2}\frac{d}{d\,t_1}\|\partial_x w_n(x,t_1)\|^2_{L_2(0,t_1)} + \nu\|\partial_x^2 w_n(x,t_1)\|^2_{L_2(0,t_1)}$$

$$\leq \|f_{\nu,n}(x,t_1)\|_{L_\infty(0,t_1)}\|w_n(x,t_1)\|_{L_2(0,t_1)}\|\partial_x^2 w_n(x,t_1)\|_{L_2(0,t_1)}$$

$$+\|f_{\nu,n}(x,t_1)\|_{L_\infty(0,t_1)}\|\partial_x^2 w_n(x,t_1)\|_{L_1(0,t_1)}.$$

Hence, by using Gronwall's inequality, Cauchy ε-inequality and (26), we get required statement of Lemma 2. Note that the equality $K_2(0,\nu)=0$ holds.

Lemma 3. *For a positive constant K_3 independent of n, for all $t_1 \in (1/n, T_1]$ the following inequality takes place:*

$$\|\partial_{t_1} w_n(x,t_1)\|^2_{L_2(Q^n_{xt_1})} \leq K_3\Big(\|f_\nu(x,t_1)\|_{L_\infty(Q_{xt_1})},\nu\Big). \tag{28}$$

Proof. The statement of Lemma 3 follows from Lemmas 1–2 and equation (9), moreover, the equality $K_3(0,\nu)=0$ holds.

Thus, from Lemmas 1–3 we directly obtain the validity of the statement of Theorem 3 and a priori estimate (24).

Now we can formulate the following two theorems:

Theorem 4. *Let $f_\nu(x,t_1) \in L_\infty(0,T_1;L_\infty(0,t_1))$. Then problem (6)–(8) has a unique solution $w(x,t_1) \in H_0^{2,1}(Q_{xt_1})$.*

Theorem 5 (Main result)**.** *Let $f(x,t_1) \in L_\infty(0,T_1;L_\infty(0,t_1))$. Then problem (1)–(3) has a unique solution $u(x,t_1) \in H_0^{2,1}(Q_{xt_1})$.*

Proofs of Theorems 4–5 will be given in the following two sections.

5 Proof of Theorem 4

Let $w_n(x,t_1)$ be a solution to boundary value problem (9)–(11), which exists and is unique according to Theorem 3 on the corresponding trapezoid $Q^n_{xt_1}$ ($n \in \mathbb{N}^*$) and belongs to the space $H_0^{2,1}(Q^n_{xt_1})$. Denote by $\{\widetilde{w_n}(x,t_1),\widetilde{f_n}(x,t_1)\}$ the extensions of the mentioned solution $w_n(x,t_1)$ and the given function $f_n(x,t_1)$ by zeros to the entire triangular domain Q_{xt_1}. It is obvious that a priori estimate (24) will remain true for extensions $\{\widetilde{w_n}(x,t_1),\widetilde{f_n}(x,t_1)\}$. Thus, we obtain a bounded sequence of functions $\{\widetilde{w_n}(x,t_1)\}_{n\in\mathbb{N}^*}$, from which we can extract weakly convergent subsequence (we preserve the notation of the index n for the subsequence):

$$\widetilde{w_n}(x,t_1) \to z(x,t_1) \quad \text{weakly in} \quad H_0^{2,1}(Q_{xt_1}).$$

Hence, in the integral identity (for any $\theta(x,t_1) \in L_2(Q_{xt_1})$)

$$\int_0^{T_1}\int_0^{t_1}\Big[\partial_{\tau_1}\widetilde{w_n}(x,\tau_1)-\nu\partial_x^2\widetilde{w_n}(x,\tau_1)+\widetilde{f_{\nu,n}}(x,\tau_1)\widetilde{w_n}(x,\tau_1)+\widetilde{f_{\nu,n}}(x,\tau_1)\Big]\theta(x,\tau_1)d\,x\,d\tau_1=0,$$

we can pass to the limit as $n \to \infty$. For any $\theta(x,t_1) \in L_2(Q_{xt_1})$ we have

$$\int_0^{T_1}\int_0^{t_1} \left[\partial_{\tau_1} z(x,\tau_1) - \nu\partial_x^2 z(x,\tau_1) + f_\nu(x,\tau_1)z(x,\tau_1) + f_\nu(x,\tau_1)\right]\theta(x,\tau_1)d\,x\,d\tau_1 = 0.$$

This means that the limit function $z(x,t_1)$ satisfies equation (6) in the space $L_2(Q_{xt_1})$ and boundary condition (24).

Thus, Theorem 4 is completely proved.

6 Proof of Theorem 5

First of all, we note that by virtue of condition (8) the weak maximum principle holds for the solution of boundary value problem (6)–(7) ([12], chapter III, p. 2: Corollary), i.e. we will have

$$w(x,t_1) \le 0, \ \{x,t_1\} \in Q_{xt_1} \cup \Omega_{t_1}. \tag{29}$$

From (29) according to transformation (5) we will also have

$$-1 < w(x,t_1), \ u(x,t_1) \ge 0, \ \{x,t_1\} \in Q_{xt_1} \cup \Omega_{t_1}. \tag{30}$$

Let us prove the following lemma.

Lemma 4. *The following estimate holds*

$$\|u\|_{H^{2,1}(Q_{xt_1})} \le C_1\Big(\|w\|_{H^{2,1}(Q_{xt_1})}, \nu\Big), \quad moreover\ , \ \ C_1(0,\nu) = 0. \tag{31}$$

Proof. From relation (5) we directly have

$$\|u\|_{L_2(Q_{xt_1})} \le \sqrt{T_1}\|\partial_x u\|_{L_2(Q_{xt_1})} \le \nu\sqrt{T_1}\|\partial_x w\|_{L_2(Q_{xt_1})}, \tag{32}$$

$$\|\partial_x u\|_{L_2(Q_{xt_1})} \le \nu\|\partial_x w\|_{L_2(Q_{xt_1})}, \tag{33}$$

$$\|\partial_{t_1} u\|_{L_2(Q_{xt_1})} \le \nu\|\partial_{t_1} w\|_{L_2(Q_{xt_1})}, \tag{34}$$

and since, according to the statement of Theorem 4: $w(x,t_1) \in H_0^{2,1}(Q_{xt_1})$, from this we additionally obtain the estimate

$$\|\partial_x u\|_{L_4(0,t_1)} \le \nu\|\partial_x w\|_{L_4(0,t_1)}, \ \forall\, t_1 \in (0,T_1). \tag{35}$$

It remains for us to estimate the second derivative with respect to the variable x from $u(x,t_1)$. To do this, we multiply equation (1) by $-\partial_x^2 u(x,t_1)$ in the space $L_2(0,t_1)$. We will have

$$\frac{1}{2}\frac{d}{dt_1}\|\partial_x u(x,t_1)\|^2_{L_2(0,t_1)} + \nu\left\|\partial_x^2 u(x,t_1)\right\|^2_{L_2(0,t_1)}$$

$$\leq \left|\left([\partial_x u(x,t_1)]^2, \partial_x^2 u(x,t_1)\right)\right| + \left|\left(f(x,t_1), \partial_x^2 u(x,t_1)\right)\right|$$

$$\leq \frac{2}{\nu}\|f(x,t_1)\|^2_{L_2(0,t_1)} + \frac{2}{\nu}\|[\partial_x u(x,t_1)]^2\|^2_{L_2(0,t_1)} + \frac{\nu}{2}\|\partial_x^2 u(x,t_1)\|^2_{L_2(0,t_1)},$$

or

$$\frac{d}{dt_1}\|\partial_x u(x,t_1)\|^2_{L_2(0,t_1)} + \nu\left\|\partial_x^2 u(x,t_1)\right\|^2_{L_2(0,t_1)}$$
$$\leq \frac{4}{\nu}\left\{\|f(x,t_1)\|^2_{L_2(0,t_1)} + \|[\partial_x u(x,t_1)]^2\|^2_{L_2(0,t_1)}\right\}. \quad (36)$$

Taking into account (35) and the embedding $H_0^{2,1}(Q_{xt_1}) \subset L_2(0,T_1; H^2(0,t_1) \cap H_0^1(0,t_1))$, we derive the following inequality

$$\|[\partial_x u(x,t_1)]^2\|^2_{L_2(0,t_1)} \equiv \|\partial_x u\|^4_{L_4(0,t_1)} \leq \nu^4\|\partial_x w(x,t_1\|^4_{L_4(0,t_1)} \leq K_4\|w(x,t_1)\|^4_{H_0^{2,1}(Q_{xt_1})}. \quad (37)$$

Thus, from (32)–(37) we obtain the required estimate (31). Lemma 4 is completely proved.

Finally, Lemma 4 gives us for boundary value problem (1)–(3) the uniqueness and the fact that its solution $u(x,t_1)$ belongs to the space $H_0^{2,1}(Q_{xt_1})$ under the conditions of Theorem 5. This lemma also gives us the completion of the proof of Theorem 5.

7 Statement of multidimensional boundary value problem

Let $x = \{x_1, ..., x_m\}$, $Q_{xt_1} = \{x, t_1 \mid |x| < t_1,\ 0 < t_1 < T_1 < \infty\}$ be a cone with the vertex at the origin and let Ω_{t_1} be a section of the cone Q_{xt_1} for the fixed time variable $t_1 \in (0, T_1)$. In the cone Q_{xt_1} we consider the following boundary value problem:

$$\partial_{t_1} u - \nu\Delta u + |\nabla u|^2 = f,\ \ (\nu > 0), \quad (38)$$

$$u(x,t_1)|_{|x|=t_1} = 0, \quad (39)$$

where

$$f \in L_\infty(Q_{xt_1}),\ \ f \geq 0. \quad (40)$$

In this work, we study the question of the existence and uniqueness of a solution of boundary value problem (38)–(40) in Sobolev space:

$$u \in H_0^{1,0}(Q_{xt_1}) \equiv L_2(0,T_1; H_0^1(\Omega_{t_1})) \cap H^1(0,T_1; H^{-1}(\Omega_{t_1})). \quad (41)$$

8 Converting (38)–(40) to linear boundary value problem

We transform (38)–(40) to a linear boundary value problem for an unknown function $w(x,t_1)$. Using the following one-to-one transformation:

$$w(x,t_1) = \exp\{-u/\nu\} - 1,\ \ u = -\nu\ln(w+1), \quad (42)$$

we obtain

$$\partial_{t_1} w - \nu\Delta w + f_\nu w = -f_\nu, \tag{43}$$

$$w(x,t_1)|_{|x|=t_1} = 0, \tag{44}$$

$$f_\nu \equiv f/\nu \in L_\infty(Q_{xt_1}),\ f_\nu \geq 0. \tag{45}$$

9 On a family of auxiliary boundary value problems in domains represented by truncated cones

To problem (43)–(45), we will set a family of boundary value problems, each of which is considered in the domain representing the corresponding truncated cone.

So, let $n \in \mathbb{N}^* \equiv \{n \in \mathbb{N} : n \geq n_1, 1/n_1 < T_1\}$, $Q^n_{xt_1} = \{x, t_1 :\ |x| < t_1,\ 1/n < t_1 < T_1 < \infty\}$ be a cone, and let Ω_{t_1} be a section at fixed $t_1 \in (1/n, T_1)$. Note that at the point $t_1 = 1/n$ the domain $Q^n_{xt_1}$ no longer degenerates into a point, moreover, between the original domain Q_{xt_1} and domains $Q^n_{xt_1}$ the strict inclusions $Q^{n_1}_{xt_1} \subset Q^{n_1+1}_{xt_1} \subset ... \subset Q_{xt_1}$ take place and, obviously, $\lim\limits_{n\to\infty} Q^n_{xt_1} = Q_{xt_1}$.

In the non-degenerating domain $Q^n_{xt_1}$ (for each finite $n \in \mathbb{N}^*$) we consider the following boundary value problem:

$$\partial_{t_1} w_n - \nu\Delta w_n + f_{\nu,n} w_n = -f_{\nu,n}, \tag{46}$$

$$w_n(x,t_1)|_{|x|=t_1} = 0,\ w_n(x,t_1)|_{t_1=1/n} = 0, \tag{47}$$

$$f_{\nu,n} \equiv f_n/\nu \in L_\infty(Q^n_{xt_1}),\ f_{\nu,n} \geq 0. \tag{48}$$

We want to transform boundary value problem (46)–(48), so that it would be set in a cylindrical domain. For this purpose we will make the transformation of independent variables: we pass from the variables $\{x, t_1\}$ to variables $\{y = y_1, ..., y_m, t\}$. We have

$$x_i = \frac{y_i}{n-t},\ t_1 = \frac{1}{n-t},\ y_i = \frac{x_i}{t_1},\ t = n - \frac{1}{t_1};$$

$Q^n_{yt} = \{y, t :\ |y| < 1,\ 0 < t < T\}$ is a cylindrical domain, and Ω is a section of the cylinder Q^n_{yt} for any fixed $t \in [0,T]$,

$$t_1 = 1/n \Leftrightarrow t = 0,\ t_1 = T_1 \Leftrightarrow t = T = n - \frac{1}{T_1}.$$

Since

$$\tilde{w}_n(y,t) \triangleq w_n\left(\frac{y}{n-t}, \frac{1}{n-t}\right),\ \tilde{f}_{\nu,n}(y,t) = f_{\nu,n}\left(\frac{y}{n-t}, \frac{1}{n-t}\right), \tag{49}$$

then for the derivative with respect to t_1 of function $w_n(x,t_1)$ (49) we obtain

$$\frac{\partial w_n}{\partial t_1} = \frac{\partial \tilde{w}_n(y,t)}{\partial t}(n-t)^2 - \sum_{i=1}^m \frac{\partial \tilde{w}_n(y,t)}{\partial y_i}(n-t)y_i.$$

Now we find the derivative of function $w_n(x, t_1)$ (49) with respect to the variable x_i :

$$\frac{\partial w_n}{\partial x_i} = \frac{\partial \tilde{w}_n}{\partial y_i}(n-t), \quad \frac{\partial^2 w_n}{\partial x_i^2} = \frac{\partial^2 \tilde{w}_n}{\partial y_i^2}(n-t)^2.$$

We write down boundary value problem (46)–(48) in the domain Q^n_{yt}:

$$\partial_t \tilde{w}_n - \nu\Delta\tilde{w}_n - \sum_{i=1}^{m} \frac{y_i}{n-t}\partial_{y_i}\tilde{w}_n + \frac{1}{(n-t)^2}\tilde{f}_{\nu,n}\tilde{w} = -\frac{1}{(n-t)^2}\tilde{f}_{\nu,n}, \tag{50}$$

$$\tilde{w}_n(y,t) = 0, \ \{y,t\} \in \Sigma^n_{yt} = \{y, t : |y| = 1, \ 0 < t < T\}, \tag{51}$$

$$\tilde{w}_n(y,0) = 0, \ y \in \Omega = \{y : |y| < 1\}. \tag{52}$$

Instead of (50)–(52) in the domain Q^n_{yt}, following [5] and [6], we will consider a more general boundary value problem:

$$\partial_t \tilde{w}_n - \nu\Delta\tilde{w}_n - \sum_{i=1}^{m} \gamma_{in}(y_i,t)\partial_{y_i}\tilde{w}_n + \alpha_n(t)\tilde{f}_{\nu,n}\tilde{w}_n = -\beta_n(t)\tilde{f}_{\nu,n}, \ (\nu > 0), \tag{53}$$

$$\tilde{w}_n(y,t)|_{|y|=1} = 0, , \ \tilde{w}_n(y,t)|_{t=0} = 0, \tag{54}$$

where the given continuous functions $\alpha_n(t)$, $\beta_n(t)$, $\gamma_{in}(y,t)$ satisfy the following conditions for any fixed number $n \in \mathbb{N}^*$

$$\begin{aligned} &\alpha_{1n} \le \alpha_n(t) \le \alpha_{2n}, \ \beta_{1n} \le \beta_n(t) \le \beta_{2n}, \quad \forall\, t \in [0,T], \\ &|\gamma_{in}(y,t)| \le \gamma_{1n}, \ |\partial_y \gamma_{in}(y,t)| \le \gamma_{1n}, \qquad \forall\, \{y,t\} \in Q^n_{yt}, \end{aligned} \tag{55}$$

with given positive constants $\alpha_{1n}, \ \alpha_{2n}, \ \beta_{1n}, \ \beta_{2n}, \ \gamma_{1n}$.

The following theorem is valid.

Theorem 6. *Suppose we have a fixed number $n \in \mathbb{N}^*$. Then, if $\tilde{f}_{\nu,n} \in L_\infty(Q^n_{yt})$ and $\alpha_n(t)$, $\beta_n(t)$, $\gamma_{in}(y,t)$ satisfy conditions (55), then boundary value problem (53)–(54) has a unique solution*

$$\tilde{w}_n \in H^{1,0}_0(Q^n_{yt}) \equiv L_2(0,T; H^1_0(\Omega)) \cap H^1(0,T; H^{-1}(\Omega)), \tag{56}$$

which satisfies the following estimate:

$$\|\tilde{w}_n\|_{H^{1,0}_0(Q_{yt})} \le K\left(\|\tilde{f}_{\nu,n}\|_{L_\infty(Q_{yt})}, \nu\right), \quad \textit{moreover,} \quad K(0,\nu) = 0. \tag{57}$$

The proof of Theorem 6 can be obtained by Faedo-Galerkin method (for example, as in [11]).

Since coefficients of equations (50)–(52) meet conditions (55), then for boundary value problem (50)–(52) from Theorem 6 we obtain, as a corollary, the following theorem.

Theorem 7. *Suppose we have a fixed number* $n \in \mathbb{N}^*$*. Then, if* $\tilde{f}_{\nu,n} \in L_\infty(Q^n_{yt})$*, then boundary value problem* (50)–(52) *has a unique solution* $\tilde{w}_n \in H^{1,0}_0(Q^n_{yt})$ (56)*, which satisfies the following estimate:*

$$\|\tilde{w}_n\|_{H^{1,0}_0(Q_{yt})} \leq K\left(\|\tilde{f}_{\nu,n}\|_{L_\infty(Q_{yt})}, \nu\right), \quad \textit{moreover,} \quad K(0,\nu) = 0. \tag{58}$$

We give the correspondence of functional spaces in terms of the independent variables $\{y,t\} \in Q^n_{yt}$ and $\{x,t_1\} \in Q^n_{xt_1}$:

$$\tilde{f}_{\nu,n} \in L_\infty(Q^n_{yt}) \equiv L_\infty(0,T;L_\infty(\Omega)) \Leftrightarrow f_{\nu,n} \in L_\infty(Q^n_{xt_1}) \equiv L_\infty(1/n,T_1;L_\infty(\Omega_{t_1})), \tag{59}$$

$$\tilde{w}(y,t) \in H^{1,0}_0(Q^n_{yt}) \equiv L_2(0,T;H^1_0(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)) \Leftrightarrow$$
$$\Leftrightarrow w(x,t_1) \in H^{1,0}_0(Q^n_{xt_1}) \equiv L_2(1/n,T_1;H^1_0(\Omega_{t_1})) \cap H^1(1/n,T_1;H^{-1}(\Omega_{t_1})). \tag{60}$$

Further, taking into account the correspondence of spaces (59)–(60), in accordance with Theorem 7 we can formulate the following statement:

Theorem 8. *Suppose we have a fixed number* $n \in \mathbb{N}^*$*. Then, if* $f_{\nu,n} \in L_\infty(Q^n_{xt_1})$ (59)*, then boundary value problem* (46)–(48) *has a unique solution* $w_n \in H^{1,0}_0(Q^n_{xt_1})$ (60) *that satisfies the following estimate:*

$$\|w_n\|_{H^{1,0}_0(Q^n_{xt_1})} \leq K\left(\|f_{\nu,n}\|_{L_\infty(Q^n_{xt_1})}, \nu\right)$$
$$\leq K_0\left(\|f_\nu\|_{L_\infty(Q_{xt_1})}, \nu\right), \quad \textit{moreover,} \quad K(0,\nu) = K_0(0,\nu) = 0. \tag{61}$$

The proof of this theorem will be given in the next section.

10 A priori estimates for the solution of problem (46)–(48). Formulation of the main result for the multidimensional problem

Lemma 5. *There exists a positive constant* K_1 *independent of* n*, such that for all* $t_1 \in (1/n, T_1]$ *the following inequality takes place:*

$$\|w_n(x,t_1)\|^2_{L_2(\Omega_{t_1})} + \int_{1/n}^{t_1} \|\nabla w_n(x,\tau_1)\|^2_{L_2(\Omega_{\tau_1})} d\tau_1 \leq K_1\left(\|f_\nu(x,t_1)\|_{L_\infty(Q_{xt_1})}, \nu\right). \tag{62}$$

Proof. Multiplying equation (46) by $w_n(x,t_1)$ in the space $L_2(\Omega_{t_1})$, we obtain

$$\frac{1}{2}\frac{d}{d\,t_1}\|w_n(x,t_1)\|^2_{L_2(\Omega_{t_1})}+\nu\|\nabla w_n(x,t_1)\|^2_{L_2(\Omega_{t_1})}$$

$$\leq \|f_{\nu,n}(x,t_1)\|_{L_\infty(\Omega_{t_1})}\|w_n(x,t_1)\|^2_{L_2(\Omega_{t_1})}+\|f_{\nu,n}(x,t_1)\|_{L_\infty(\Omega_{t_1})}\|w_n(x,t_1)\|_{L_1(\Omega_{t_1})}.$$

Now by using Gronwall's inequality and the following obvious inequality

$$\|f_{\nu,n}\|_{L_\infty(Q^n_{xt_1})}\leq\|f_\nu\|_{L_\infty(Q_{xt_1})}, \tag{63}$$

we get the required statement of Lemma 5. Note that the equality $K_1(0,\nu)=0$ holds.

From the linear continuity of the Laplace operator $\Delta:\ H^1_0(\Omega_{t_1})\to H^{-1}(\Omega_{t_1})$ it follows the validity of the following lemma.

Lemma 6. *For a positive constant K_2 independent of n, for all $t_1\in(1/n,T_1]$ the following inequality takes place:*

$$\int\limits_{1/n}^{t_1}\|\Delta w_n(x,\tau_1)\|^2_{H^{-1}(\Omega_{\tau_1})}d\tau_1\leq K_2\Big(\|f_\nu(x,t_1)\|_{L_\infty(Q_{xt_1})},\nu\Big),\quad moreover,\quad K_2(0,\nu)=0. \tag{64}$$

Lemma 7. *For a positive constant K_3 independent of n, for all $t_1\in(1/n,T_1]$ the following inequality takes place:*

$$\int\limits_{1/n}^{t_1}\|\partial_{\tau_1}w_n(x,\tau_1)\|^2_{H^{-1}(\Omega_{\tau_1})}d\tau_1\leq K_3\Big(\|f_\nu(x,t_1)\|_{L_\infty(Q_{xt_1})},\nu\Big). \tag{65}$$

Proof. The statement of Lemma 7 follows from Lemmas 5–6 and equation (46), moreover, the equality $K_3(0,\nu)=0$ holds.

Thus, from Lemmas 5–7 we directly obtain the validity of the statement of Theorem 8 and a priori estimate (61).

Now we can formulate the following two theorems:

Theorem 9. *Let $f_\nu(x,t_1)\in L_\infty(0,T_1;L_\infty(\Omega_{t_1}))$. Then problem* (43)–(45) *has a unique solution $w(x,t_1)\in H^{1,0}_0(Q_{xt_1})$.*

Theorem 10 (Main result)**.** *Let $f(x,t_1)\in L_\infty(0,T_1;L_\infty(\Omega_{t_1}))$. Then problem* (38)–(40) *has a unique solution $u(x,t_1)\in H^{1,0}_0(Q_{xt_1})$.*

Proofs of Theorems 9–10 will be given in the following two sections.

11 Proof of Theorem 9

Let $w_n(x,t_1)$ be a solution to boundary value problem (46)–(48), which exists and is unique according to Theorem 8 at the corresponding truncated cone $Q^n_{xt_1}$ $(n \in \mathbb{N}^*)$ and belongs to the space $H^{1,0}_0(Q^n_{xt_1})$. Denote by $\{\widetilde{w_n}(x,t_1), \widetilde{f_n}(x,t_1)\}$ the extensions of the mentioned solution $w_n(x,t_1)$ and the given function $f_n(x,t_1)$ by zeros to the entire cone Q_{xt_1}. It is obvious that a priori estimate (61) will remain true for extensions $\{\widetilde{w_n}(x,t_1), \widetilde{f_n}(x,t_1)\}$. Thus, we obtain a bounded sequence of functions $\{\widetilde{w_n}(x,t_1)\}_{n\in\mathbb{N}^*}$, from which we can extract weakly convergent subsequence (we preserve the notation of the index n for the subsequence):

$$\widetilde{w_n}(x,t_1) \to z(x,t_1) \quad \text{weakly in} \quad H^{1,0}_0(Q_{xt_1}).$$

Hence, in the integral identity (for any $\theta(x,t_1) \in L_2(0,T_1;H^1_0(\Omega_{t_1}))$)

$$\int\limits_0^{T_1}\int\limits_0^{t_1} \Big[\partial_{\tau_1}\widetilde{w_n}(x,\tau_1) - \nu\Delta\widetilde{w_n}(x,\tau_1) + \widetilde{f_{\nu,n}}(x,\tau_1)\widetilde{w_n}(x,\tau_1) + \widetilde{f_{\nu,n}}(x,\tau_1)\Big]\,\theta(x,\tau_1)d\,x\,d\tau_1 = 0,$$

we can pass to the limit as $n \to \infty$. For any $\theta(x,t_1) \in L_2(0,T_1;H^1_0(\Omega_{t_1}))$ we have

$$\int\limits_0^{T_1}\int\limits_0^{t_1} \left[\partial_{\tau_1}z(x,\tau_1) - \nu\Delta z(x,\tau_1) + f_\nu(x,\tau_1)z(x,\tau_1) + f_\nu(x,\tau_1)\right]\theta(x,\tau_1)d\,x\,d\tau_1 = 0.$$

This means that the limit function $z(x,t_1)$ satisfies equation (43) in the space $L_2(0,T_1;H^{-1}(\Omega_{t_1}))$ and boundary condition (44). Thus, Theorem 9 is completely proved.

12 Proof of Theorem 10

First of all, we note that by virtue of condition (45) the weak maximum principle holds for a solution of boundary value problem (43)–(44) ([12], chapter III, p. 2: Corollary), i.e. we will have

$$w(x,t_1) \le 0,\ \{x,t_1\} \in Q_{xt_1} \cup \Omega_{t_1}. \tag{66}$$

From (66) according to transformation (42) we will also have

$$-1 < w(x,t_1),\ \ u(x,t_1) \ge 0,\ \{x,t_1\} \in Q_{xt_1} \cup \Omega_{t_1}. \tag{67}$$

Let us prove the following lemma.

Lemma 8. *The following estimate holds*

$$\|u\|_{H^{1,0}_0(Q_{xt_1})} \le C_1\Big(\|w\|_{H^{1,0}_0(Q_{xt_1})}, \nu\Big), \quad \textit{moreover}\ ,\ \ C_1(0,\nu) = 0. \tag{68}$$

Proof. From relation (42) we directly have

$$\|u\|_{L_2(Q_{xt_1})} \leq \sqrt{T_1}\|\nabla u\|_{L_2(Q_{xt_1})} \leq \nu\sqrt{T_1}\|\nabla w\|_{L_2(Q_{xt_1})}, \tag{69}$$

$$\|\nabla u\|_{L_2(Q_{xt_1})} \leq \nu\|\nabla w\|_{L_2(Q_{xt_1})}, \tag{70}$$

$$\|\partial_{t_1} u\|_{L_2(0,T_1;H^{-1}(\Omega_{t_1}))} \leq \nu\|\partial_{t_1} w\|_{L_2(0,T_1;H^{-1}(\Omega_{t_1}))}, \tag{71}$$

and according to inequality (64) from Lemma 6 we additionally obtain estimate

$$\|\Delta u\|_{H^{-1}(\Omega_{t_1})} \leq \nu\|\Delta w\|_{H^{-1}(\Omega_{t_1})}, \ \forall\, t_1 \in (0, T_1). \tag{72}$$

Finally, from equation (38) we will directly have that

$$(\nabla u(x,t_1))^2 \quad \text{bounded in} \quad L_2(0,T_1;H^{-1}(\Omega_{t_1})). \tag{73}$$

Thus, from (69)–(73) we obtain required estimate (68). Lemma 8 is completely proved.

Finally, Lemma 8 gives us for boundary value problem (38)–(40) the uniqueness and the fact that its solution $u(x,t_1)$ belongs to the space $H_0^{1,0}(Q_{xt_1})$ under the conditions of Theorem 10. This lemma also gives us the completion of the proof of Theorem 10.

Conclusion

In this paper, we have established theorems on solvability of nonlinear heat conduction problem in a degenerating domain in Sobolev classes, the degeneracy point of which located at the origin.

The results of the work for the one-dimensional version can be generalized to the case when we have the domain of independent variables $Q_{xt_1} = \{x, t_1 : 0 < x < \varphi(t_1),\ 0 < t_1 < T_1 < \infty\}$ represented by curvilinear triangle moving boundary of which can change according to the rule $x = \varphi(t_1),\ t_1 \in [0, T_1]$, and the condition $\varphi(0) = 0$ holds. Moreover, for the function $\varphi(t_1)$ it is required to meet certain natural conditions. For example, the function $\varphi(t_1)$ must satisfy the following two conditions: 1^0 in a sufficiently short period of time $(0, t_1^*)$ the function $\varphi(t_1)$ could have the representation $\varphi(t_1) = \mu\, t_1$, where μ would be a given positive constant (in our work it was equal to one); 2^0 on the interval $[t_1^*, T_1]$ the function $\varphi(t_1)$ would be continuously differentiable and possess the property of monotonicity, providing one-to-one transformation from the independent variables $\{x, t_1\}$ to variables $\{y, t\}$.

Similar considerations take place for boundary value problems in the multidimensional case. Indeed, in the multidimensional case, when we have the domain of independent variables $Q_{xt_1} = \{x = x_1, ..., x_m, t_1 : |x| < \varphi(t_1),\ 0 < t_1 < T_1 < \infty\}$ represented by "curvilinear cone". Moreover, the "moving" lateral surface of this domain for each fixed t_1 can be changed according to the rule $|x| = \sqrt{x_1^2 + ... + x_m^2} = \varphi(t_1),\ t_1 \in [0, T_1]$, and the condition $\varphi(0) = 0$ holds. Moreover, for the function $\varphi(t_1)$ it is required to meet certain conditions. For example, the function $\varphi(t_1)$ must satisfy the following two conditions: 1^0 in a sufficiently short period of time $(0, t_1^*)$ the function $\varphi(t_1)$ could have the representation $\varphi(t_1) = \mu\, t_1$, where μ would be a given positive constant (in our work it was equal to one); 2^0 on the interval $[t_1^*, T_1]$ the function $\varphi(t_1)$ would be continuously differentiable and possess the property of monotonicity, providing one-to-one transformation of each circular section of the "curvilinear cone" in the independent variables $\{x, t_1\}$ to the corresponding circular section of the cylinder in variables $\{y, t\}$.

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Жиенәлиев М.Т., Иманбердиев Қ.Б., Қасымбекова А.С., Ерғалиев М.Ғ. ӨЗГЕШЕЛЕТІН ОБЛЫСТАРДАҒЫ ЖЫЛУӨТКІЗГІШТІК ТЕҢДЕУІ ҮШІН БІР СЫЗЫҚТЫҚ ЕМЕС ШЕКАРАЛЫҚ ЕСЕПТІҢ ШЕШІМДІЛІГІ ТУРАЛЫ

Жұмыс өзгешелену нүктесі координаталар басында орналасқан өзгешеленетін облыстардағы жылуөткізгіштік теңдеуіне қойылған бір сызықтық емес шекаралық есептің Соболев кластарындағы шешілімділік мәселелеріне арналған. Фаедо-Галеркин мен априорлы бағалаулар әдістерін пайдалану арқылы қарастырылып отырған шекаралық есептің шешімінің бар болуы мен жалғыздығы туралы теоремалар, әрі, оған қоса, бірөлшемді шекаралық есеп үшін берілген функциялардың тегістігінің өсуі кезіндегі регулярлығы дәлелденеді. Сонымен қатар бұл нәтижелердің қарастырылып отырған шекаралық есептердің көпөлшемді (өзгешелену нүктесі конус төбесінде орналасқан көпөлшемді конустағы) жағдайы үшін әрі қарай дамытылуы алынған. Бұл жерде тек бірөлшемді жағдаймен салыстырғанда әлсізірек шешімнің ғана бар болуы мен жалғыздығы көрсетілді.

Кілттік сөздер. Екінші ретті параболалық теңдеулер, сызықтық емес параболалық теңдеулер.

Дженалиев М.Т., Иманбердиев К.Б., Касымбекова А.С., Ергалиев М.Г. О РАЗРЕШИМОСТИ ОДНОЙ НЕЛИНЕЙНОЙ ГРАНИЧНОЙ ЗАДАЧИ ДЛЯ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ В ВЫРОЖДАЮЩИХСЯ ОБЛАСТЯХ

Работа посвящена вопросам разрешимости в соболевских классах одной нелинейной задачи теплопроводности в вырождающихся областях, точка вырождения которой находится в начале координат. С использованием методов Фаэдо-Галеркина и априорных оценок доказываются теоремы о существовании и единственности решения рассматриваемой граничной задачи, а также его регулярность при повышении гладкости заданных функций для одномерной граничной задачи. Также получено дальнейшее развитие этих результатов для многомерного варианта (в многомерном конусе с точкой вырождения на вершине конуса) рассматриваемых граничных задач. Здесь показаны существование и единственность, но только более слабого решения, чем в одномерном случае.

Ключевые слова. Параболические уравнения второго порядка, нелинейные параболические уравнения.

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On boundary value problem of the Samarskii-Ionkin type for the Laplace operator in a ball

Aishabibi A. Dukenbayeva[1,2,a], Makhmud A. Sadybekov[2,b]

[1]Al-Farabi Kazakh National University, Almaty, Kazakhstan
[2]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[a] e-mail: dukenbayeva@math.kz, [b]e-mail: sadybekov@math.kz

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Abstract. In this paper we consider a nonlocal boundary value problem for the Laplace operator in a ball, which is a multidimensional generalisation of the Samarskii-Ionkin problem. The well-posedness of the problem is investigated, and an integral representation of the solution is obtained.

Keywords. Laplace operator, Poisson's equation, Boundary value problem, Nonlocal boundary value problem, Samarskii-Ionkin problem

1 Introduction

It is well known that Dirichlet and Neumann boundary value problems play important roles in the theory of harmonic functions [1]. In one-dimensional case, or when considering the problem in a multidimensional parallelepiped, the main problems include also periodic boundary value problems. In the works [2], [3], for the first time, a new class of boundary value problems for the Poisson's equation in a multidimensional ball $\Omega \subset \mathbb{R}^n$ was introduced $(k = 1, 2)$:

The problem P_k. *Find a solution of the Poisson's equation*

$$-\Delta u(x) = f(x), \quad x \in \Omega,$$

satisfying the following periodic boundary conditions

$$u(x) - (-1)^k u(x^*) = \tau(x), \quad x \in \partial\Omega_+,$$

$$\frac{\partial u}{\partial r}(x) + (-1)^k \frac{\partial u}{\partial r}(x^*) = \mu(x), \quad x \in \partial\Omega_+.$$

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Here, $\partial\Omega_+$ is a part of the sphere $\partial\Omega$, for which $x_1 \geq 0$; each point $x = (x_1, x_2, ..., x_n) \in \Omega$ is matched by its *"opposite"* point $x^* = (-x_1, \alpha_2 x_2, ..., \alpha_n x_n) \in \Omega$, where the indices $\alpha_j \in \{-1, 1\}, j = 2, ..., n$. Clearly, if $x \in \partial\Omega_+$, then $x^* \in \partial\Omega_-$.

These problems are analogous to the classical periodic boundary value problems. In [2], [3], the well-posedness of these problems were investigated. Moreover, there, the authors showed the existence and uniqueness of the solution to the problem P_1, while the solution of the problem P_2 is unique up to a constant term and exists if the necessary condition of the well-posedness holds. The uniqueness and existence were shown by using the extremum principle and Green's function, respectively. In [3], the authors considered the problem P_k in the two-dimensional case and showed the possibility of using the method of separation of variables. Moreover, in this case, the self-adjointness of these problems and its spectral properties were studied.

If we turn to the non-classical problems, then one of the most popular problems is the Samarskii-Ionkin problem, arisen in connection with the study of the processes occurring in the plasma in the 70s of the last century by physicists (see e.g. [5], [6]). In [7], [8], an analog of the Samarskii-Ionkin type boundary value problem for the Poisson's equation in a disk was considered. We also refer to [9]–[12] for the problems generalising the periodic problem P_k. We also note that nonlocal boundary value problems of periodic type were developed for the case of problems with integro-differential boundary operators: for Poisson's equation [13], [14] and biharmonic equation [15], [16]. In [17], a nonlocal problem for the Laplace equation generalising the periodic P_k and Robin problems were considered.

In this paper we study a nonlocal boundary value problem for the Laplace operator in a ball, which is a multidimensional generalisation of the Samarskii-Ionkin problem.

2 Statement of the problem

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be an arbitrary point of the unit ball $\Omega = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : |x| < 1\} \subset \mathbb{R}^n$. Let $\alpha_k \in \{-1, 1\}$. Then $(\alpha_k)^2 = 1$. Denote $x^* = (-x_1, \alpha_2 x_2, \ldots, \alpha_n x_n)$, and $\partial\Omega_+ \, (\partial\Omega_-)$ is a part of the sphere $\partial\Omega$, for which $x_1 > 0 \, (x_1 < 0)$. We also denote a part of the sphere $\partial\Omega$, for which $x_1 = 0$, by $\partial\Omega_0$.

Let us consider the following nonlocal boundary value problem for the Laplace operator in the ball, which is a multidimensional generalisation of the Samarskii-Ionkin problem.

The problem $S_{\alpha 1}$. *Find a function $u(x) \in C^2(\Omega) \cap C^1(\bar{\Omega} \backslash \partial\Omega_0)$ satisfying the Poisson's equation*

$$-\Delta u(x) = f(x), \quad x \in \Omega, \tag{1}$$

and the following boundary conditions

$$u(x) - \alpha u\left(x^*\right) = \tau(x), \quad x \in \partial\Omega_+, \tag{2}$$

$$\frac{\partial u}{\partial n}(x) - \frac{\partial u}{\partial n}\left(x^*\right) = \mu(x), \quad x \in \partial\Omega_+, \tag{3}$$

where $f(x) \in C^{\varepsilon}(\bar{\Omega})$, $\tau(x) \in C^{1+\varepsilon}[\partial\Omega_+], \mu(x) \in C^{\varepsilon}[\partial\Omega_+]$, $0 < \varepsilon < 1$, *and* α *is a fixed real number. Here,* $\frac{\partial}{\partial n}$ *is a derivative with respect to the direction of the outer normal to* $\partial\Omega$.

In the case when $\alpha = -1$, we obtain antiperiodic boundary problem, which was studied earlier in the works [1]–[2]. We refer to [7]–[8] for the case $\alpha = 0$. The two-dimensional case of the problem $S_{\alpha 1}$ was studied in [10]–[12].

3 Fredholm property of the problem $S_{\alpha 1}$

In this section we show that the problem $S_{\alpha 1}$ is not even Noetherian when $\alpha = 1$, that is, the homogeneous problem $S_{\alpha 1}$

$$\begin{cases} \Delta u(x) = 0, \quad x \in \Omega, \\ u(x) - u(x^*) = 0, \quad x \in \partial\Omega_+, \\ \frac{\partial u}{\partial n}(x) - \frac{\partial u}{\partial n}(x^*) = 0, \quad x \in \partial\Omega_+, \end{cases} \tag{4}$$

has an infinite number of linearly independent solutions.

For this, let us introduce the auxiliary functions $c(x)$ and $s(x)$ as follows

$$c(x) = u(x) + u(x^*), \quad s(x) = u(x) - u(x^*).$$

Substituting the function $s(x)$ in the homogeneous problem (4), we have

$$\Delta s(x) = 0, \quad x \in \Omega, \quad s(x) = 0, \quad x \in \partial\Omega,$$

which means $s(x) \equiv 0$ for all $x \in \Omega$. This implies $u(x) = u(x^*)$ for all $x \in \Omega$. Hence, we obtain $c(x) = 2u(x)$.

By the construction of the function $c(x)$, it must have the symmetric property

$$c(x) = c(x^*). \tag{5}$$

So, this function automatically satisfies boundary conditions of (4).

Thus, the function $c(x)$ is harmonic ($\Delta c(x) = 0$) satisfying the symmetric condition (5). Since there are infinite number of such linearly independent harmonic functions, the problem $S_{\alpha 1}$ is not even Noetherian when $\alpha = 1$. Therefore, in this case the problem $S_{\alpha 1}$ is not Fredholm.

Throughout this paper, we consider the Fredholm case of the problem $S_{\alpha 1}$, that is, the case $\alpha \neq 1$.

4 Uniqueness of the solution to the problem $S_{\alpha 1}$

Theorem 1. *Let $\alpha \neq 1$. Then the problem $S_{\alpha 1}$ has no more than one solution.*

Proof. Suppose that there are two functions $u_1(x)$ and $u_2(x)$ satisfying the conditions of the problem $S_{\alpha 1}$. We show that the function $u(x) = u_1(x) - u_2(x)$ is equal to zero. It is obvious that the function $u(x)$ is harmonic and satisfies the following homogeneous conditions

$$u(x) - \alpha u(x^*) = 0, \quad x \in \partial\Omega_+, \tag{6}$$

$$\frac{\partial u}{\partial n}(x) - \frac{\partial u}{\partial n}(x^*) = 0, \quad x \in \partial\Omega_+. \tag{7}$$

Denote

$$v(x) = u(x) - u(x^*). \tag{8}$$

It is clear that $v(x)$ is a harmonic function with the symmetric property

$$v(x) = -v(x^*), \quad x \in \Omega. \tag{9}$$

Hence, in view of the boundary condition (7) we get the following classical Neumann problem

$$\Delta v(x) = 0, \quad x \in \Omega; \quad \frac{\partial v}{\partial n}(x) = 0, \quad x \in \partial\Omega.$$

Consequently, $v = const$.

Therefore, from (8) we obtain $v \equiv 0$, $x \in \Omega$. It implies that $u(x) = u(x^*)$, $x \in \Omega$. Moreover, we get $u(x) - u(x^*) = 0$, $x \in \partial\Omega_+$, which, together with the boundary condition (6), show that

$$u(x) = 0, \quad x \in \partial\Omega, \tag{10}$$

since $\alpha \neq 1$. By the uniqueness of the solution to the Dirichlet problem for the Laplace equation, we have $u(x) \equiv 0$, $x \in \bar{\Omega}$, that is, $u_1(x) = u_2(x)$.

Thus, we have completed the proof of Theorem 1. □

5 Construction of the adjoint problem to the problem $S_{\alpha 1}$

Let us denote by $W_{\alpha 1}$ the linear manifold of functions $u(x) \in C^2(\Omega) \cap C^1(\bar{\Omega} \backslash \partial\Omega_0)$ satisfying the boundary conditions (6) and (7).

Let $L_{\alpha 1}$ be a closure of the linear operator in $L_2(\Omega)$ given by the differential expression

$$Lu = -\Delta u(x), \quad x \in \Omega, \tag{11}$$

on the linear manifold $W_{\alpha 1}$.

It is easy to see that the domain of the definition of the given operator consists of strong solutions to the problem $S_{\alpha 1}$. Clearly, this domain of the definition is dense in $L_2(\Omega)$. Hence, the adjoint operator to the operator $L_{\alpha 1}$ exists. Since the initial operator is given by the boundary conditions, then its adjoint operator should also be given by the boundary conditions. Moreover, the adjoint operator is given by the differential expression (11).

In order to construct the adjoint operator, let us consider the following difference

$$(L_{\alpha 1}u, v) - (u, Lv) = 0 \tag{12}$$

for all $u \in W_{\alpha 1}$ and $v \in C^2(\Omega) \cap C^1\left(\bar{\Omega}\backslash\partial\Omega_0\right)$.

We apply the Green's theorem in a plane to (12) to get

$$\oint\limits_{\partial\Omega} \left\{ u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right\} ds = 0, \tag{13}$$

where $\dfrac{\partial}{\partial n}$ is a derivative with respect to the direction of the outer normal to $\partial\Omega$.

Hence, taking into account the boundary conditions (6) and (7), to which functions $u \in W_{\alpha 1}$ satisfy, we get from (13) that

$$\int\limits_{\partial\Omega_+} \left\{ u\left(x^*\right)\left[\frac{\partial v}{\partial n}\left(x^*\right) + \alpha\frac{\partial v}{\partial n}(x)\right] - \frac{\partial u}{\partial n}\left(x^*\right)\left[v(x) + v\left(x^*\right)\right] \right\} ds = 0.$$

Since $u(x)$ and $\dfrac{\partial u}{\partial n}(x)$ are independent of each other, we obtain the boundary conditions for the functions $v \in C^2(\Omega) \cap C^1\left(\bar{\Omega}\backslash\partial\Omega_0\right)$, which belong to the domain of the definition of the adjoint operator

$$v(x) + v\left(x^*\right) = 0, \quad x \in \partial\Omega_+, \tag{14}$$

$$\alpha\frac{\partial v}{\partial n}(x) + \frac{\partial v}{\partial n}\left(x^*\right) = 0, \quad x \in \partial\Omega_+. \tag{15}$$

Taking the limit of the sequences corresponding to the strong solutions, it is immediately to see that equality (12) holds for all $u \in D\left(L_{\alpha 1}\right)$ and $v \in D\left(L^*_{\alpha 1}\right)$.

As in Section 3, it is easy to show that the problem with the boundary conditions (14)–(15) is Fredholm. Consequently, this problem is formal adjoint to $S_{\alpha 1}$. In the next section, the well-posedness of $S_{\alpha 1}$ with $\alpha \neq 1$ will be justified in the sense of both classical and strong solutions. Hence, the inverse operator $L^{-1}_{\alpha 1}$ exists and is defined everywhere in $L_2(\Omega)$.

Here, by standard arguments related to the coincidence of the adjoint operator to the inverse one and the inverse operator to the adjoint one for the linear closed operators, we obtain that the adjoint problem to $S_{\alpha 1}$ is a problem for the Poisson's equation

$$-\Delta v = g(x), \quad x \in \Omega, \tag{16}$$

with the boundary conditions (14)–(15).

Thus, the adjoint problem (in the sense of classical solutions) is the following problem:

The problem $S^*_{\alpha 1}$. *Find a function $v(x) \in C^2(\Omega) \cap C^1\left(\bar{\Omega} \backslash \partial\Omega_0\right)$ satisfying the Poisson's equation* (16) *in the ball $\Omega = \{x : |x| < 1\} \subset \mathbb{R}^n$ and the boundary conditions*

$$v(x) + v\left(x^*\right) = \tau(x), \quad x \in \partial\Omega_+, \tag{17}$$

$$\alpha \frac{\partial v}{\partial n}(x) + \frac{\partial v}{\partial n}\left(x^*\right) = \mu(x), \quad x \in \partial\Omega_+, \tag{18}$$

where $g(x) \in C^{\varepsilon}(\bar{\Omega})$, $\tau(x) \in C^{1+\varepsilon}\left[\partial\Omega_+\right]$, $\mu(x) \in C^{\varepsilon}\left[\partial\Omega_+\right]$, $0 < \varepsilon < 1$, α is a fixed real number from (2) *of the problem $S_{\alpha 1}$.*

Thus, we have obtained the following result:

Theorem 2. *The boundary value problems $S_{\alpha 1}$ and $S^*_{\alpha 1}$ form a Fredholm pair.*

5 The well-posedness of the problem $S_{\alpha 1}$

By Theorem 1 we know that the well-posedness case is the case when $\alpha \neq 1$.

For convenience, let us formulate this problem again.

The problem $S_{\alpha 1}$. *Find a function $u(x) \in C^2(\Omega) \cap C^1(\bar{\Omega} \backslash \partial\Omega_0)$ satisfying the Poisson's equation*

$$-\Delta u(x) = f(x), \quad x \in \Omega, \tag{19}$$

and the boundary conditions

$$u(x) - \alpha u\left(x^*\right) = \tau(x), \quad x \in \partial\Omega_+, \tag{20}$$

$$\frac{\partial u}{\partial n}(x) - \frac{\partial u}{\partial n}\left(x^*\right) = \mu(x), \quad x \in \partial\Omega_+, \tag{21}$$

where $f(x) \in C^{\varepsilon}(\bar{\Omega})$, $\tau(x) \in C^{1+\varepsilon}[\partial\Omega_+], \mu(x) \in C^{\varepsilon}[\partial\Omega_+]$, $0 < \varepsilon < 1$ and α is a fixed real number. Here, $\frac{\partial}{\partial n}$ is a derivative with respect to the direction of the outer normal to $\partial\Omega$.

It is clear that a necessary condition for the existence of the solution in the class $C^1(\bar{\Omega})$ is the fulfillment of the following conditions

$$\begin{gathered} \mu\left(0, x_2, \ldots, x_n\right) = \mu\left(0, \alpha_2 x_2, \ldots, \alpha_n x_n\right) = 0, \quad x \in \partial\Omega_+, \\ \tau\left(0, x_2, \ldots, x_n\right) = \tau\left(0, \alpha_2 x_2, \ldots, \alpha_n x_n\right) = 0, \quad x \in \partial\Omega_+, \quad \text{when} \quad \alpha = 1. \end{gathered} \tag{22}$$

Let us briefly demonstrate that problem (19)–(21) can be reduced to two boundary value problems for Poisson's equation with self-adjoint boundary conditions.

Note that when we change to a new variable $x^* = (-x_1, \alpha_2 x_2, \ldots, \alpha_n x_n)$, the "radial derivative" in spherical coordinates does not change its sign:

$$\frac{\partial}{\partial r^*} = \sum_{j=1}^{n} \frac{x_j^*}{|x^*|} \frac{\partial}{\partial x_j^*} = \sum_{j=1}^{n} \frac{\alpha_j x_j}{|x|} \frac{\partial x_j}{\partial x_j^*} \frac{\partial}{\partial x_j} = \sum_{j=1}^{n} \frac{x_j}{|x|} \frac{\partial}{\partial x_j} = \frac{\partial}{\partial r}.$$

So, we have

$$\left(\frac{\partial u}{\partial n}\right)(x^*) = \frac{\partial}{\partial n}\left(u\left(x^*\right)\right), \quad x \in \partial\Omega. \tag{23}$$

Let us now introduce the auxiliary functions $U(x)$ and $V(x)$:

$$u(x) - u\left(x^*\right) = 2U(x), \quad u(x) + u\left(x^*\right) = 2V(x).$$

Clearly,

$$u(x) = U(x) + V(x), \tag{24}$$

and

$$U(x) = -U\left(x^*\right), \quad V(x) = V\left(x^*\right), \quad x \in \Omega. \tag{25}$$

By the direct calculation, one can verify that the function $U(x)$ is a solution of the Neumann problem:

$$-\Delta U = f_-(x), \quad x \in \Omega, \tag{26}$$

$$\frac{\partial U}{\partial n}(x) = \mu_-(x), \quad x \in \partial\Omega; \tag{27}$$

while $V(x)$ is a solution to the Dirichlet problem:

$$-\Delta V = f_+(x), \quad x \in \Omega, \tag{28}$$

$$V(x) = \tau_+(x), \quad x \in \partial\Omega. \tag{29}$$

Here,

$$f_\pm(x) = \frac{1}{2}\left\{f(x) \pm f\left(x^*\right)\right\}, \tag{30}$$

$$\mu_-(x) = \frac{1}{2}\begin{cases} \mu(x), & x \in \partial\Omega_+, \\ -\mu\left(x^*\right), & x \in \partial\Omega_-, \end{cases} \tag{31}$$

$$\tau_+(x) = \frac{1}{1-\alpha}\begin{cases} \tau(x) - (1+\alpha)U(x), & x \in \partial\Omega_+, \\ \tau\left(x^*\right) - (1+\alpha)U\left(x^*\right), & x \in \partial\Omega_-. \end{cases} \tag{32}$$

We note that the function $\tau_+(x)$ depends not only on $\tau(x)$, but also on $U(x)$ on the part of the boundary $\partial\Omega_+$. Therefore, these two problems should be solved sequentially: first, the Neumann problem for $U(x)$, then, using the obtained solution, we solve the Dirichlet problem for $V(x)$.

The Neumann (26), (27) and Dirichlet (28), (29) problems are classical boundary value problems. So, nowadays, the well-posedness of these problems and smoothness of solutions are well-known. By the assumption of fulfillment of the matching conditions (22), it is easy to verify availability of the required smoothness of the boundary functions $\tau_+(x)$ and $\mu_-(x)$.

For the Neumann problem (26), (27), by (30) and (31) we see that the necessary and sufficient conditions for the existence of the solution hold:

$$\int\limits_{\Omega} f_-(x)dx + \int\limits_{\partial\Omega} \mu_-(x)dS_x = 0.$$

Therefore, the solution $U(x)$ to the Neumann problem (26), (27) exists for all $f(x) \in C^{\varepsilon}(\bar{\Omega})$ and $\mu \in C^{\varepsilon}\left[\partial\Omega_+\right]$, and belongs to $U(x) \in C^{2+\varepsilon}(\Omega) \cap C^{1+\varepsilon}(\bar{\Omega})$.

Consequently, the boundary function $\tau_+(x)$ from (32) belongs to $C^{1+\varepsilon}[\partial\Omega_+]$ and $C^{1+\varepsilon}[\partial\Omega_-]$. Therefore, the solution to the Dirichlet problem (28), (29) exists and is unique. This solution belongs to $C^{2+\varepsilon}(\Omega) \cap C^{1+\varepsilon}(\bar{\Omega}\backslash\partial\Omega_0)$.

The solution to the Neumann problem (26)–(27) has the form

$$U(x) = \int\limits_{\Omega} G_N(x,y)f_-(y)dy + \int\limits_{\partial\Omega} G_N(x,y)\mu_-(y)dS_y + C_1, \tag{33}$$

while the solution to the Dirichlet problem (28)–(29) is

$$V(x) = \int\limits_{\Omega} G_D(x,y)f_+(y)dy - \int\limits_{\partial\Omega} \frac{\partial G_D(x,y)}{\partial n_y}\tau_+(y)dS_y, \tag{34}$$

where $G_D(x,y)$ and $G_N(x,y)$ are Green's functions of the Dirichlet and Neumann problems for Poisson's equation in Ω, respectively. By the construction of the function $U(x)$, it must have the symmetric property $U(x) = -U\left(x^*\right)$, which means that $C_1 = 0$. Therefore, we further assume that this condition is fulfilled.

By substituting the functions $f_-(y)$ and $\mu_-(y)$ in the representation of $U(x)$, we get

$$U(x) = \int\limits_{\Omega} G_N(x,y)f_-(y)dy + \int\limits_{\partial\Omega} G_N(x,y)\mu_-(y)dS_y$$

$$= \frac{1}{2}\int\limits_{\Omega} G_N(x,y)\left(f(y) - f\left(y^*\right)\right)dy + \frac{1}{2}\int\limits_{\partial\Omega_+} G_N(x,y)\mu(y)dS_y$$

$$-\frac{1}{2}\int\limits_{\partial\Omega_-} G_N(x,y)\mu\left(y^*\right)dS_y = \frac{1}{2}\int\limits_{\Omega}\left(G_N(x,y) - G_N\left(x,y^*\right)\right)f(y)dy + \frac{1}{2}\int\limits_{\partial\Omega_+} G_N(x,y)\mu(y)dS_y$$

$$-\frac{1}{2}\int\limits_{\partial\Omega_+} G_N\left(x,y^*\right)\mu(y)dS_y = \frac{1}{2}\int\limits_{\Omega}\left(G_N(x,y) - G_N\left(x,y^*\right)\right)f(y)dy$$

$$+\frac{1}{2}\int\limits_{\partial\Omega_+}\left(G_N(x,y) - G_N\left(x,y^*\right)\right)\mu(y)dS_y.$$

Next, plugging the functions $f_+(y)$ and $\tau_+(y)$ into the representation of $V(x)$, we obtain

$$V(x) = \frac{1}{2}\int\limits_{\Omega} G_D(x,y)\left(f(y) + f\left(y^*\right)\right)dy - \frac{1}{1-\alpha}\left(\int\limits_{\partial\Omega_+}\frac{\partial G_D(x,y)}{\partial n_y}(\tau(y) - (1+\alpha)U(y))dS_y\right.$$

$$\left. + \int\limits_{\partial\Omega_-}\frac{\partial G_D(x,y)}{\partial n_y}\left(\tau\left(y^*\right) - (1+\alpha)U\left(y^*\right)\right)dS_y\right) = \frac{1}{2}\int\limits_{\Omega}\left(G_D(x,y) + G_D\left(x,y^*\right)\right)f(y)dy$$

$$-\frac{1}{1-\alpha}\int\limits_{\partial\Omega_+}\left(\frac{\partial G_D(x,y)}{\partial n_y} + \frac{\partial G_D\left(x,y^*\right)}{\partial n_y}\right)\tau(y)dS_y$$

$$+\frac{1+\alpha}{1-\alpha}\int\limits_{\partial\Omega_+}\left(\frac{\partial G_D(x,y)}{\partial n_y} + \frac{\partial G_D(x,y^*)}{\partial n_y}\right)U(y)dS_y.$$

Now, we combine them to get

$$u(x) = U(x) + V(x) = \frac{1}{2}\int\limits_{\Omega}\left(G_N(x,y) - G_N\left(x,y^*\right) + G_D(x,y) + G_D\left(x,y^*\right)\right)f(y)dy$$

$$+\frac{1+\alpha}{1-\alpha}\int\limits_{\partial\Omega_+}\left(\frac{\partial G_D(x,y)}{\partial n_y} + \frac{\partial G_D(x,y^*)}{\partial n_y}\right)$$

$$\times\left(\frac{1}{2}\int\limits_{\Omega}\left(G_N(y,z) - G_N\left(y,z^*\right)\right)f(z)dz\right)dS_y + \frac{1}{2}\int\limits_{\partial\Omega_+}\left(G_N(x,y) - G_N\left(x,y^*\right)\right)\mu(y)dS_y$$

$$+\frac{1+\alpha}{1-\alpha}\int\limits_{\partial\Omega_+}\left(\frac{\partial G_D(x,y)}{\partial n_y} + \frac{\partial G_D(x,y^*)}{\partial n_y}\right)$$

$$\times \left(\frac{1}{2} \int\limits_{\partial\Omega_+} \left(G_N(y,z) - G_N\left(y,z^*\right)\right) \mu(z) dS_z \right) dS_y$$

$$-\frac{1}{1-\alpha} \int\limits_{\partial\Omega_+} \left(\frac{\partial G_D(x,y)}{\partial n_y} + \frac{\partial G_D\left(x,y^*\right)}{\partial n_y} \right) \tau(y) dS_y. \qquad (35)$$

Thus, we have proved the following theorem.

Theorem 3. *Let* $\alpha \neq 1$ *and let the natural matching conditions* (22) *be satisfied. Then for all* $f(x) \in C^{\varepsilon}(\bar{\Omega})$, $\tau(x) \in C^{1+\varepsilon}[\partial\Omega_+]$, $\mu \in C^{\varepsilon}[\partial\Omega_+]$, $0 < \varepsilon < 1$, *the solution to the problem* $S_{\alpha 1}$ (19)-(21) *exists, is unique and can be represented in the form* (35). *This solution belongs to* $C^{2+\varepsilon}(\Omega) \cap C^{1+\varepsilon}(\bar{\Omega} \backslash \partial\Omega_0)$.

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Дукенбаева А.А., Садыбеков М.А. ШАРДАҒЫ ЛАПЛАС ОПЕРАТОРЫ ҮШІН САМАРСКИЙ-ИОНКИН ТЕКТЕС ШЕТТІК ЕСЕБІ ЖАЙЛЫ

Бұл жұмыста шардағы Лаплас операторы үшін Самарский-Ионкин есебінің көп өлшемді жалпыламасы болып табылатын бейлокал шеттік есебі қарастырылды. Есептің қисындылығы зерттелді және шешімнің интегралдық кейіптемесі алынды.

Кілттік сөздер. Лаплас операторы, Пуассон теңдеуі, шекаралық есеп, бейлокал шеттік есеп, Самарский-Ионкин есебі.

Дукенбаева А.А., Садыбеков М.А. ОБ ОДНОЙ КРАЕВОЙ ЗАДАЧЕ ТИПА САМАРСКОГО-ИОНКИНА ДЛЯ ОПЕРАТОРА ЛАПЛАСА В ШАРЕ

В данной работе рассматривается нелокальная краевая задача для оператора Лапласа в шаре, являющаяся многомерным обобщением задачи Самарского-Ионкина. Исследована корректность задачи и получено интегральное представление решения.

Ключевые слова. Оператор Лапласа, уравнение Пуассона, краевая задача, нелокальная краевая задача, задача Самарского-Ионкина.

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An algorithm for solving a nonlinear boundary value problem with parameter for the Mathieu equation

Dulat S. Dzhumabaev[1,a], Yekaterina S. La[2,b]
Akmaral A. Pussurmanova[2,c], Zhanerke Zh. Kisash[2,d]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[2]International Information Technology University, Almaty, Kazakhstan
[a] e-mail: ddzhumabaev54@gmail.com, [b]e-mail: layekaterina99@gmail.com,
[c]e-mail: akmaral.pussurmanova@gmail.com, [d]e-mail: zhanerkekisash@gmail.com

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Abstract. In this paper, we study a boundary value problem with a parameter for the Mathieu equation with cubic nonlinearity. The boundary condition of this problem is periodic. An additional condition is given to determine the unknown parameter. We present a numerical algorithm to solve the problem under consideration.

Keywords. Mathieu's equation with parameter, numerical algorithm, Newton's method.

The Mathieu equation appears in applied mathematics and many engineering fields; see [1] and references therein. This equation includes numerical parameters which characterize effect of various factors on the behavior of processes studied. For finding their values we need to impose some additional conditions. Various problems for differential problems with parameters have been investigated in [2]–[8].

We consider the Mathieu equation with a parameter that is expressed via two quasilinear ordinary differential equations of the first order

$$\frac{dx_1}{dt} = x_2, \quad t \in (0,T), \tag{1}$$

$$\frac{dx_2}{dt} = -x_1 + \varepsilon[\alpha\cos(2t)x_1 + \beta x_1^3] + \mu\sin(2t) + g(t), \quad t \in (0,T), \tag{2}$$

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subject to the periodic boundary conditions

$$x_1(0) = x_1(T), \quad x_2(0) = x_2(T), \tag{3}$$

and additional condition

$$x_1(0) = x_1^0, \tag{4}$$

where $\varepsilon > 0$, α, β, and x_1^0 are some given numbers, and g(t) is a continuous on [0, T] function.

By a solution to problem (1)–(4) we mean a triple $(\mu^*, x_1^*(t), x_2^*(t))$, where $\mu^* \in R$ and $x_1^*(t)$, $x_2^*(t)$ are continuous on [0, T] and continuously differentiable on (0,T) functions, satisfying conditions (3), (4) and system (1), (2) with $\mu = \mu^*$.

The aim of this paper is to develop a numerical algorithm for solving problem (1)–(4) based on the method proposed in [9], [10].

We reduce the problem under consideration to a problem with an additional parameter λ that is chosen as the value of the function $x_2(t)$ at the point $t = 0$: $\lambda \doteq x_2(0)$. Then, by substituting $u_1(t) = x_1(t) - x_1^0$, $u_2(t) = x_2(t) - \lambda$, problem (1)–(4) is transformed into the following problem:

$$\frac{du_1}{dt} = u_2 + \lambda, \quad t \in (0, T) \tag{5}$$

$$\frac{du_2}{dt} = -(u_1 + x_1^0) + \varepsilon[\alpha \cos(2t)(u_1 + x_1^0) + \beta(u_1 + x_1^0)^3] + \mu \sin(2t) + g(t), \quad t \in (0, T), \tag{6}$$

$$u_1(0) = 0, \quad u_2(0) = 0, \tag{7}$$

$$u_1(T) = 0, \quad u_2(T) = 0. \tag{8}$$

A solution to problem (5)–(8) is a quadruple $(\mu^*, \lambda^*, u_1^*(t), u_2^*(t))$, where $\mu^*, \lambda^* \in R$ and functions $u_1^*(t)$, $u_2^*(t)$ satisfy the system of nonlinear differential equations (5), (6) and conditions (7), (8) with $\mu = \mu^*$ and $\lambda = \lambda^*$. Obviously, if this quadruple is a solution to problem (5)–(8), then the triple $(\mu^*, x_1^*(t), x_2^*(t))$ with $x_1^*(t) = u_1^*(t) - x_1^0$ and $x_2^*(t) = u_2^*(t) - \lambda^*$ is a solution to problem (1)–(4).

Let us choose some numbers $\lambda^{(0)}$, $\mu^{(0)}$, $\rho_\lambda > 0$, and $\rho_\mu > 0$. Suppose that the Cauchy problem (5)–(7) has a unique solution $u(t, \lambda, \mu) = (u_1(t, \lambda, \mu), u_2(t, \lambda, \mu))$ for all $\lambda \in (\lambda_0 - \rho_\lambda, \lambda_0 + \rho_\lambda)$ and $\mu \in (\mu_0 - \rho_\mu, \mu_0 + \rho_\mu)$. By substituting the value $u(T, \lambda, \mu)$ into the boundary condition (8), we get the following system of nonlinear algebraic equations in parameters λ and μ:

$$u_1(T, \lambda, \mu) = 0, \tag{9}$$

$$u_2(T, \lambda, \mu) = 0. \tag{10}$$

Problem (5)–(8) is solvable if the system of algebraic equations (9), (10) has a solution $(\lambda^*, \mu^*) \in (\lambda^{(0)} - \rho_\lambda, \lambda^{(0)} + \rho_\lambda) \times (\mu^{(0)} - \rho_\mu, \mu^{(0)} + \rho_\mu)$.

We represent system (9)–(10) in the vector form

$$Q^*(\lambda, \mu) = 0, \tag{11}$$

and find its solution applying Newton's method taking as an initial assumption the pair $(\lambda^{(0)}, \mu^{(0)})$ that we have arbitrarily chosen as the centers of the above-mentioned intervals. The question now arises: how to choose a good initial assumption that is close enough to the exact solution? If we take into account that $\varepsilon > 0$ is a small number, it is reasonable to determine the values of $\lambda^{(0)}$ and $\mu^{(0)}$ by solving problem (5)–(8) for $\varepsilon = 0$. In this case, we get the linear boundary value problem with parameters

$$\frac{du}{dt} = A(t)(u + \tilde{\lambda}) + B(t)\mu + f(t), \quad t \in (0, T), \tag{12}$$

$$u(0) = 0, \tag{13}$$

$$u(T) = 0, \tag{14}$$

where $\tilde{\lambda} = \begin{pmatrix} x_1^0 \\ \lambda \end{pmatrix}$, $A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B(t) = \begin{pmatrix} 0 \\ sin(2t) \end{pmatrix}$, $f(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$.

As is known, for fixed $\tilde{\lambda}$ and μ, the Cauchy problem for the linear differential equation (12) subject to the initial condition (13) has a unique solution $u(t, \tilde{\lambda}, \mu)$. Let $\Phi(t)$ be a fundamental matrix of the differential equation $\frac{dx}{dt} = A(t)x$, $t \in [0, T]$. We can then represent the solution to (12), (13) in the form

$$u(t, \tilde{\lambda}, \mu) = \Phi(t)\int_0^t \Phi^{-1}(\tau)A(\tau)d\tau\tilde{\lambda} + \Phi(t)\int_0^t \Phi^{-1}(\tau)B(\tau)d\tau\mu$$

$$+\Phi(t)\int_0^t \Phi^{-1}(\tau)f(\tau)d\tau, \quad t \in [0, T]. \tag{15}$$

Let us consider an auxiliary Cauchy problem

$$\frac{dz}{dt} = A(t)z + P(t), \quad t \in [0, T], \quad z(0) = 0, \tag{16}$$

where $P(t)$ is a (2×2) matrix or a vector of the dimension 2 that is continuous on [0,T]. By $a(P, t)$ we denote the unique solution to problem (16), which can be written as

$$a(P, t) = \Phi(t)\int_0^t \Phi^{-1}(\tau)P(\tau)d\tau, \ t \in [0, T].$$

Then, solution (15) to the Cauchy problem (12), (13) can be represented through $a(P,t)$ as

$$u(t,\tilde{\lambda},\mu) = a(A,t)\tilde{\lambda} + a(B,t)\mu + a(f,t). \quad (17)$$

Substituting the right-hand side of (17) into the boundary condition (14), we get the system of linear algebraic equation in parameters λ and μ:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}\begin{pmatrix} x_1^0 \\ \lambda \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}\mu + \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = 0,$$

where $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = a(A,T)$, $\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = a(B,T)$ and $\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = a(f,T)$.

Rewrite this system in an equivalent form

$$\alpha_{12}\lambda + \beta_1\mu = -\gamma_1 - \alpha_{11}x_1^0, \quad \alpha_{22}\lambda + \beta_2\mu = -\gamma_2 - \alpha_{21}x_1^0. \quad (18)$$

We will use the solution to (18), the pair $(\lambda^{(0)},\mu^{(0)})$, as an initial assumption for the solution of (11) when applying Newton's method.

Now that we have got the initial approximation, we are in position to continue the iterative process for finding the solution to the system of nonlinear algebraic equations (11). The successive approximations are determined by the formula

$$\begin{pmatrix} \lambda^{(n+1)} \\ \mu^{(n+1)} \end{pmatrix} = \begin{pmatrix} \lambda^{(n)} \\ \mu^{(n)} \end{pmatrix} - \left[\frac{\partial Q_*(\lambda^{(n)},\mu^{(n)})}{\partial y}\right]^{-1} Q_*(\lambda^{(n)},\mu^{(n)}), \quad n = 0,1,\ldots, \quad (19)$$

where

$$\frac{\partial Q_*(\lambda^{(n)},\mu^{(n)})}{\partial y} = \begin{pmatrix} \frac{\partial u_1(T,\lambda^{(n)},\mu^{(n)})}{\partial \lambda} & \frac{\partial u_1(T,\lambda^{(n)},\mu^{(n)})}{\partial \mu} \\ \frac{\partial u_2(T,\lambda^{(n)},\mu^{(n)})}{\partial \lambda} & \frac{\partial u_2(T,\lambda^{(n)},\mu^{(n)})}{\partial \mu} \end{pmatrix}.$$

The vector $Q_*(\lambda^{(n)},\mu^{(n)})$ in (19) is determined as

$$Q_*(\lambda^{(n)},\mu^{(n)}) = \begin{pmatrix} u_1(T,\lambda^{(n)},\mu^{(n)}) \\ u_2(T,\lambda^{(n)},\mu^{(n)}) \end{pmatrix},$$

where $u_1(t,\lambda^{(n)},\mu^{(n)})$ and $u_2(t,\lambda^{(n)},\mu^{(n)})$ are the solutions to the Cauchy problems (5), (7) and (6), (7) with given $\lambda = \lambda^{(n)}$ and $\mu = \mu^{(n)}$.

In order to determine the elements of the Jacobi matrix $\frac{\partial Q_*(\lambda^{(n)},\mu^{(n)})}{\partial y}$ in (19), we will use the following equalities

$$\frac{du_1(t,\lambda^{(n)},\mu^{(n)})}{dt} = u_2(t,\lambda^{(n)},\mu^{(n)}) + \lambda^{(n)}, \quad t \in (0,T), \quad (20)$$

$$\frac{du_2(t,\lambda^{(n)},\mu^{(n)})}{dt} = -(u_1(t,\lambda^{(n)},\mu^{(n)}) + x_1^0) + \varepsilon[\alpha\cos(2t)(u_1(t,\lambda^{(n)},\mu^{(n)}) + x_1^0)$$

$$+\beta(u_1(t,\lambda^{(n)},\mu^{(n)}) + x_1^0)^3] + \mu\sin(2t) + g(t), \quad t \in (0,T), \tag{21}$$

$$u_1(0,\lambda^{(n)},\mu^{(n)}) = 0, \quad u_2(0,\lambda^{(n)},\mu^{(n)}) = 0, \tag{22}$$

which hold true for any pair $(\lambda^{(n)},\mu^{(n)}) \in (\lambda^{(0)} - \rho_\lambda, \lambda^{(0)} + \rho_\lambda) \times (\mu^{(0)} - \rho_\mu, \mu^{(0)} + \rho_\mu)$.

To find the elements of the first column of the Jacobi matrix, we differentiate both sides of each of the equations (20)–(22) with respect to λ. We get

$$\frac{d}{dt}\left(\frac{\partial u_1(t,\lambda^{(n)},\mu^{(n)})}{\partial\lambda}\right) = \frac{\partial u_2(t,\lambda^{(n)},\mu^{(n)})}{\partial\lambda} + 1, \quad t \in [0,T],$$

$$\frac{d}{dt}\left(\frac{\partial u_2(t,\lambda^{(n)},\mu^{(n)})}{\partial\lambda}\right) = (-1 + \varepsilon\alpha\cos(2t) + 3\varepsilon\beta[u_1(t,\lambda^{(n)},\mu^{(n)}) + x_1^0]^2)\frac{\partial u_1(t,\lambda^{(n)},\mu^{(n)})}{\partial\lambda},$$

$$\frac{\partial u_1(0,\lambda^{(n)},\mu^{(n)})}{\partial\lambda} = 0, \quad \frac{\partial u_2(0,\lambda^{(n)},\mu^{(n)})}{\partial\lambda} = 0.$$

It can be seen from these equations that the functions $v_1^{(n)}(t) = \frac{\partial u_1(t,\lambda^{(n)},\mu^{(n)})}{\partial\lambda}$, $v_2^{(n)}(t) = \frac{\partial u_2(t,\lambda^{(n)},\mu^{(n)})}{\partial\lambda}$ satisfy the following Cauchy problems for ordinary differential equations

$$\frac{dv_1}{dt} = v_2 + 1, \quad t \in [0,T], \tag{23}$$

$$\frac{dv_2}{dt} = (-1 + \varepsilon\alpha\cos(2t) + 3\varepsilon\beta[x_1^{(n)}(t)]^2)v_1, \quad t \in [0,T], \tag{24}$$

$$v_1(0) = 0, \quad v_2(0) = 0, \tag{25}$$

where $x_1^{(n)}(t) = u_1(t,\lambda^{(n)},\mu^{(n)}) + x_1^0$.

In the same way we determine the elements of the second column of the Jacobi matrix. Differentiating both sides of (20)–(22) with respect to μ, we conclude that the functions $w_1^{(n)}(t) = \frac{\partial u_1(t,\lambda^{(n)},\mu^{(n)})}{\partial\mu}$, $w_2^{(n)}(t) = \frac{\partial u_2(t,\lambda^{(n)},\mu^{(n)})}{\partial\mu}$ are the solutions to the Cauchy problems for ordinary differential equations

$$\frac{dw_1}{dt} = w_2, \quad t \in [0,T], \tag{26}$$

$$\frac{dw_2}{dt} = (-1 + \varepsilon\alpha\cos(2t) + 3\varepsilon\beta[x_1^{(n)}(t)]^2)w_1 + sin(2t), \quad t \in [0,T], \tag{27}$$

$$w_1(0) = 0, \quad w_2(0) = 0. \tag{28}$$

Thus, the Jacobi matrix is determined as

$$\frac{\partial Q_*(\lambda^{(n)},\mu^{(n)})}{\partial y} = \begin{pmatrix} v_1^{(n)}(T) & v_2^{(n)}(T) \\ w_1^{(n)}(T) & w_2^{(n)}(T) \end{pmatrix},$$

where $(v_1^{(n)}(t), v_2^{(n)}(t))$ and $(w_1^{(n)}(t), w_2^{(n)}(t))$ are the solutions to the Cauchy problems (23)–(25) and (26)–(28), respectively.

Summarizing the above, we propose the following numerical algorithm for solving the quasi-linear boundary value problem with parameter (1)–(4).

Step 0. Set the initial assumption $(\lambda^{(0)}, \mu^{(0)})$ for parameters (λ, μ) as a solution to the system of linear algebraic equations (18) with coefficients determined by solving the auxiliary Cauchy problems (16).

Step 1.

(a) Solve the Cauchy problem (5)–(7) with $\lambda = \lambda^{(0)}$ and $\mu = \mu^{(0)}$ by the fourth-order Runge-Kutta method with a step size $h > 0 : 2Nh = T$. Use the numerical solution $u(\hat{t}, \lambda^{(0)}, \mu^{(0)})$, where $\hat{t} = \{0, h, \ldots, (2N-1)h, 2Nh\}$, to construct the vector $Q_*(\lambda^{(0)}, \mu^{(0)}) = \begin{pmatrix} u_1(T, \lambda^{(0)}, \mu^{(0)}) \\ u_2(T, \lambda^{(0)}, \mu^{(0)}) \end{pmatrix}$ and the function $x_1^{(0)}(t) = x_1^0 + u_1(\hat{t}, \lambda^{(0)}, \mu^{(0)})$.

(b) Solve the Cauchy problems (23)–(25) and (26)–(28) by the fourth-order Runge-Kutta method with the step size $h_1 = 2h$ (we have to double the step size since we know the values of $x_1^{(0)}(t)$ only on the grid $\{0, h, \ldots, (2N-1)h, 2Nh\}$). Use the numerical solutions $v_1^{(0)}(\hat{t})$, $v_2^{(0)}(\hat{t})$ and $w_1^{(0)}(\hat{t})$, $w_2^{(0)}(\hat{t})$ to construct the Jacobi matrix

$$\frac{\partial Q_*(\lambda^{(0)},\mu^{(0)})}{\partial y} = \begin{pmatrix} v_1^{(0)}(T) & v_2^{(0)}(T) \\ w_1^{(0)}(T) & w_2^{(0)}(T) \end{pmatrix}.$$

(c) Assuming that the matrix $\frac{\partial Q_*(\lambda^{(0)},\mu^{(0)})}{\partial y}$ is invertible, determine the next approximation to the solution of (11) by the formula

$$\begin{pmatrix} \lambda^{(1)} \\ \mu^{(1)} \end{pmatrix} = \begin{pmatrix} \lambda^{(0)} \\ \mu^{(0)} \end{pmatrix} - \left[\frac{\partial Q_*(\lambda^{(0)},\mu^{(0)})}{\partial y}\right]^{-1} Q_*(\lambda^{(0)}, \mu^{(0)}).$$

As we continue this process, at the **n-th step** we find $(\lambda^{(n)}, \mu^{(n)})$, $u_1(t, \lambda^{(n)}, \mu^{(n)})$ and $u_2(t, \lambda^{(n)}, \mu^{(n)})$, $n = 1, 2, \ldots$. The convergence conditions for the iterative process in terms of $Q_*(\lambda, \mu)$ and its Jacobi matrix are given in Theorem 4.1 [10, p.1019].

Example. Let us consider problem (1)–(4) with $T = 2$, $\alpha = 0$, $\beta = 1$, $\varepsilon = 0.1$, $g(t) = (1 - \pi^2)\sin(\pi t) + 1 - 0.1(\sin \pi t + 1)^3 - 2\sin(2t)$, $x_1^0 = 1$. The exact solution to the problem is the pair $(\mu^*, x^*(t))$, where $\mu^* = 2$, $x^*(t) = \begin{pmatrix} \sin(\pi t) + 1 \\ \pi\cos(\pi t) \end{pmatrix}$.

At each step of the algorithm proposed, we solve Cauchy problems for ordinary differential equations by the fourth-order Runge-Kutta method. Let us take the step size $h = 0.1$.

By solving the linear boundary value problem (12)–(14) we obtain $(\lambda^{(0)}, \mu^{(0)}) = (3.16785, 2.52888)$. Starting with this initial approximation, we perform the iterative process.

Iteration 1. $Q_*(\lambda^{(0)}, \mu^{(0)}) = \begin{pmatrix} 0.529103 \\ 0.100919 \end{pmatrix}$, $\frac{\partial Q_*(\lambda^{(0)},\mu^{(0)})}{\partial y} = \begin{pmatrix} 1.50188 & 0.958459 \\ -1.05907 & 0.322619 \end{pmatrix}$,

$$(\lambda^{(1)}, \mu^{(1)}) = (3.11852, 2.05414).$$

Iteration 2. $Q_*(\lambda^{(1)}, \mu^{(1)}) = \begin{pmatrix} 0.0188694 \\ 0.0408092 \end{pmatrix}$, $\frac{\partial Q_*(\lambda^{(1)},\mu^{(1)})}{\partial y} = \begin{pmatrix} 1.3985 & 0.924797 \\ -1.24978 & 0.221606 \end{pmatrix}$,

$$(\lambda^{(2)}, \mu^{(2)}) = (3.14141, 1.99911).$$

Iteration 3. $Q_*(\lambda^{(2)}, \mu^{(2)}) = \begin{pmatrix} -0.00107237 \\ 0.00000000 \end{pmatrix}$, $\frac{\partial Q_*(\lambda^{(2)},\mu^{(2)})}{\partial y} = \begin{pmatrix} 1.4036 & 0.925345 \\ -1.25054 & 0.220128 \end{pmatrix}$,

$$(\lambda^{(3)}, \mu^{(3)}) = (3.1416, 1.99999).$$

Iteration 8. $Q_*(\lambda^{(7)}, \mu^{(7)}) = \begin{pmatrix} -0.00000000 \\ -0.00000000 \end{pmatrix}$, $\frac{\partial Q_*(\lambda^{(7)},\mu^{(7)})}{\partial y} = \begin{pmatrix} 1.4038 & 0.925399 \\ -1.25027 & 0.22027 \end{pmatrix}$,

$$(\lambda^{(8)}, \mu^{(8)}) = (3.1416, 2.00002).$$

The comparison with the exact solution $(\lambda^*, \mu^*) = (\pi, 2)$ gives

$$\|\lambda^{(8)} - \lambda^*\| < 0.0011, \quad \|\mu^{(8)} - \mu^*\| < 0.00003.$$

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Жұмабаев Д.С., Ла Е.С., Пұсұрманова А.А., Кисаш Ж.Ж. МАТЬЕ ТЕҢДЕУІ ҮШІН ПАРАМЕТРІ БАР СЫЗЫҚТЫҚ ЕМЕС ШЕТТІК ЕСЕПТІ ШЕШУ АЛГОРИТМІ

Бұл мақалада кубтық сызықсыздығы бар Матье теңдеуі үшін параметрі бар шеттік есепті қарастырамыз. Бұл есептің шеттік шарты периодты болып табылады. Белгісіз параметрді анықтау үшін қосымша шарт беріледі. Қарастырылып отырған есепті шешуге арналған сандық алгоритмді ұсынамыз.

Кілттік сөздер. Параметрі бар Матье теңдеуі, сандық алгоритм, Ньютон әдісі.

Джумабаев Д.С., Ла Е.С., Пусурманова А.А., Кисаш Ж.Ж. АЛГОРИТМ РЕШЕНИЯ НЕЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ С ПАРАМЕТРОМ ДЛЯ УРАВНЕНИЯ МАТЬЕ

В этой статье мы рассматриваем краевую задачу с параметром для уравнения Матье с кубической нелинейностью. Краевое условие данной задачи является периодическим. Задано дополнительное условие для определения неизвестного параметра. Мы предлагаем численный алгоритм решения рассматриваемой задачи.

Ключевые слова. Уравнение Матье с параметром, численный алгоритм, метод Ньютона.

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Novel approach for solving multipoint boundary value problem for integro-differential equation

Anar T. Assanova[1,a], Elmira A. Bakirova[1,b], Roza E. Uteshova[1,2,c]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[2]International Information Technology University, Almaty, Kazakhstan
[a] e-mail: anartasan@gmail.com, [b]e-mail: bakiroba1974@mail.ru, [c] e-mail: ruteshova1@gmail.com

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Abstract. In the present paper, we study a multipoint boundary value problem for a system of Fredholm integro-differenial equations by the method of parameterization.The case of a degenerate kernel is studied separately, for which we obtain well-posedness conditions and propose some algorithms to find approximate and numerical solutions to the problem. Then we establish necessary and sufficient conditions for the well-posedness of the multipoint problem for the system of Fredholm integro-differential equations and develop some algorithms for finding its approximate solutions. These algorithms are based on the solutions of an approximating problem for the system of integro-differential equations with degenerate kernel.

Keywords. Fredholm integro-differential equation, multipoint problem, parameterization method, algorithm, solvability criteria.

1 Introduction

Various types of multipoint problems for differential and integro-differential equations have been studied by many researchers, see [1]–[15]. A number of methods have been applied to solve these problems, e.g., methods of qualitative theory of differential equations, the method of Green's functions, the method of upper and lower solutions, numerical-analytical methods. However, the problem of establishing effective criteria for the unique solvability of multipoint problems for integro-differential equations, as well as developing algorithms for finding their approximate and numerical solutions, still remains open.

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One of the constructive methods for investigation and solving boundary value problems for ordinary differential equations and integro-differential equations is the method of parameterization proposed by Dzhumabaev [16]. This method was originally developed for studying and solving boundary value problems for the systems of ordinary differential equations. In [16], coefficient criteria were established for the unique solvability of linear boundary value problems. An algorithm for finding their approximate solutions was developed. The method of parameterization was later extended to linear multipoint boundary value problems [10], [11], for which necessary and sufficient conditions were obtained for the unique solvability in terms of the initial data and the algorithm for finding their approximate solutions was proposed. In [17]–[20], the method of parameterization was applied to the two-point boundary value problems for Fredholm integro-differential equations to establish criteria for their solvability and the unique solvability. For these problems, based on the method of parameterization and a new concept of a general solution, novel algorithms for approximate and numerical solutions were developed, see [21]–[23]. The results obtained in the above-mentioned papers were used to investigate a multipoint boundary value problem for loaded differential equations [4] and the boundary value problem with a parameter for Fredholm integro-differential equations [3].

Consider the multipoint boundary value problem for the system of integro-differential equations

$$\frac{dx}{dt} = A(t)x + \int_0^T K(t,\tau)x(\tau)d\tau + f(t), \quad x \in R^n, \quad t \in (0,T), \tag{1.1}$$

$$\sum_{i=0}^{m} B_i x(t_i) = d, \quad d \in R^n. \tag{1.2}$$

Here $x(t) = col(x_1(t), x_2(t), ..., x_n(t))$ is an unknown function, $(n \times n)$ matrix $A(t)$ and n-vector $f(t)$ are continuous on $[0,T]$, $(n \times n)$ matrix $K(t,\tau)$ is continuous on $[0,T] \times [0,T]$, B_i are constant $(n \times n)$ matrices, $0 = t_0 < t_1 < t_2 < ... < t_{m-1} < t_m = T$, $\|x\| = \max_{i=\overline{1,n}} |x_i|$.

The solution to multipoint problem (1.1), (1.2) is a function $x^*(t) : [0,T] \to \mathbb{R}^n$ that is continuous on $[0,T]$, continuously differentiable on $(0,T)$ and satisfies integro-differential equations (1.1) and multipoint condition (1.2).

The aim of the present paper is to obtain criteria for the unique solvability of problem (1.1), (1.2) and develop algorithms for finding its approximate solutions. To this end, the parameterization method is used. The interval $[0,T]$ is partitioned and additional parameters are introduced as the values of the desired solution at the left endpoints of the partition subintervals. When applying the method of parameterization to problem (1.1), (1.2), some intermediate problems occur, so called special Cauchy problems for integro-differential equations with parameters. The questions of solvability and unique solvability of such problems were thoroughly investigated in [17]–[23].

Section 2 is devoted to the study of Fredholm integro-differential equations with degenerate kernel. We divide $[0, T]$ into m parts and introduce additional parameters as the values of the desired solution at the left endpoints $t = t_i$, $i = \overline{0, m-1}$, of the subintervals. The unique solvability of a special Cauchy problem for the Δ_m partition is equivalent to the invertibility of a matrix $I - G(\Delta_m)$ constructed through a fundamental matrix of the differential part and the matrices of the integral kernel. The Δ_m partition is called regular if the matrix $I - G(\Delta_m)$ is invertible (see [21]). For the regular Δ_m partition, a system of linear algebraic equations in the parameters introduced is constructed using $[I - G(\Delta_m)]^{-1}$, the multipoint condition (1.2), and the continuity conditions at the interior partition points $t = t_i$, $i = \overline{1, m-1}$. It is shown that the invertibility of the matrix of the system constructed is equivalent to the unique solvability of the multipoint boundary value problem under consideration.

In Section 3, we develop the algorithms for finding a solution to a multipoint boundary value problem for the integro-differential equation with degenerate kernel. For the chosen Δ_m partition, the matrix $G(\Delta_m)$ is calculated. If there is an inverse of $I - G(\Delta_m)$, then we construct a system of linear algebraic equations. The elements of $G(\Delta_m)$, the coefficients and right-hand side of the system are determined by the solutions of the Cauchy problems for ordinary differential equations and the values of the definite integrals of some functions over the partition subintervals. By solving the system of algebraic equations, we determine the values of the solution at the left endpoints of the subintervals. Next, using the values obtained and the data of the integro-differential equation we compose a function $\mathcal{F}^*(t)$ that is continuous on $[0, T]$. Solving the Cauchy problems for ordinary differential equations with the right-hand side $\mathcal{F}^*(t)$, we get the values of the desired solution at the remaining points of the interval $[0, T]$. If a fundamental matrix of the differential part is found explicitly and the integrals are evaluated exactly, then the algorithm allows us to find a closed-form solution as well. As is known, it is usually impossible to explicitly find the fundamental matrix for a system of ordinary differential equations with variable coefficients, and, in general, only approximate values of definite integrals can be obtained. For this reason, in this section we propose a numerical implementation of the algorithm. The Cauchy problems for ordinary differential equations on the subintervals are solved by the fourth-order Runge-Kutta method; the integrals are calculated by the Simpson formula. It should be noted that the elements of the matrix $G(\Delta_m)$, the coefficients and the right-hand side of the system of algebraic equations in parameters can be evaluated by parallel computing on the partition subintervals.

In Section 4, a multipoint boundary value problem is considered for the Fredholm integro-differential equation when the integral kernel is not degenerate. We approximate the kernel by the degenerate one and then use the results obtained in Section 2. At each step of the process, the multipoint boundary value problem for the integro-differential equation with degenerate kernel is solved. We establish sufficient conditions for the convergence of the iterative process to a solution of the multipoint boundary value problem for the Fredholm integro-differential equation with non-degenerate kernel. The accuracy of the approximate solution depends on the choice of the approximating kernel and the number of iteration steps. The necessary and sufficient conditions for the well-posedness of the multipoint problem (1.1), (1.2) are obtained in terms of the properties of solutions to approximating problems for integro-differential equations with degenerate kernels.

2 The well-posedness of multipoint problems for Fredholm integro-differential equations with degenerate kernel

Consider the integro-differential equation

$$\frac{dx}{dt} = A(t)x + \sum_{j=1}^{k} \int_{0}^{T} \varphi_j(t)\psi_j(\tau)x(\tau)d\tau + f(t), \quad t \in (0,T), \quad x \in R^n, \tag{2.1}$$

subject to the multipoint condition

$$\sum_{i=0}^{m} B_i x(t_i) = d, \quad d \in R^n, \tag{2.2}$$

where the matrices $A(t)$, $\varphi_j(t)$, $\psi_j(\tau)$, $j = \overline{1,k}$, and the vector $f(t)$ are continuous on $[0,T]$, $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

The interval $[0,T)$ is divided into m parts by the points $t_0 = 0 < t_1 < \ldots < t_m = T$, and the partition $[0,T) = \bigcup\limits_{r=1}^{m} [t_{r-1}, t_r)$ is denoted by Δ_m. The case of no partitioning the interval $[0,T]$ is denoted by Δ_1.

We introduce the following spaces: $C([0,T], R^n)$ is the space of continuous functions $x : [0,T] \to R^n$ with the norm $\|x\|_1 = \max\limits_{t\in[0,T]} \|x(t)\|$;

$C([0,T], \Delta_N, R^{nm})$ is the space of function systems $x[t] = (x_1(t), x_2(t), \ldots, x_m(t))$, where functions $x_r : [t_{r-1}, t_r) \to R^n$ are continuous and have finite left-handed limits $\lim\limits_{t\to t_r-0} x_r(t)$ for all $r = \overline{1,m}$, with the norm $\|x[\cdot]\|_2 = \max\limits_{r=\overline{1,m}} \sup\limits_{t\in[t_{r-1},t_r)} \|x_r(t)\|$.

Let $x_r(t)$ be the restriction of the function $x(t)$ to the rth subinterval $[t_{r-1}, t_r)$, i.e. $x_r(t) = x(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,m}$.

We introduce additional parameters $\lambda_r = x_r(t_{r-1})$ and make the substitution $x_r(t) = u_r(t) + \lambda_r$ on each rth subinterval. Multipoint problem (2.1), (2.2) is then reduced to the following problem with parameters:

$$\frac{du_r}{dt} = A(t)(u_r + \lambda_r) + \sum_{s=1}^{m}\sum_{j=1}^{k} \int_{t_{s-1}}^{t_s} \varphi_j(t)\psi_j(\tau)[u_r(\tau) + \lambda_r]d\tau + f(t), \; t \in (t_{r-1}, t_r), \; r = \overline{1,m}, \tag{2.3}$$

$$u_r(t_{r-1}) = 0, \quad r = \overline{1,m}, \tag{2.4}$$

$$\sum_{i=0}^{m-1} B_i\lambda_{i+1} + B_m\lambda_m + B_m \lim_{t\to T-0} u_m(t) = d, \tag{2.5}$$

$$\lambda_p + \lim_{t\to t_p-0} u_p(t) - \lambda_{p+1} = 0, \quad p = \overline{1,m-1}, \tag{2.6}$$

where (2.6) are the continuity conditions for the solution at the interior points of the partition Δ_m. Note that conditions (2.6) and integro-differential equations (2.3) ensure the continuity of the derivative of the solution at those points.

If $x^*(t)$ is a solution to multipoint problem (2.1),(2.2), then the pair $(\lambda^*, u^*[t])$ with elements $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*) \in R^{nm}$, $u^*[t] = \big(u_1^*(t), u_2^*(t), \ldots, u_m^*(t)\big) \in C([0,T], \Delta_m, R^{nm})$, where $\lambda_r^* = x^*(t_{r-1})$, $u_r^*(t) = x^*(t) - x^*(t_{r-1})$, $[t_{r-1}, t_r)$, $r = \overline{1,m}$, is a solution to the problem with parameters (2.3)–(2.6). Vice versa, if a pair $(\widetilde{\lambda}, \widetilde{u}[t])$ with elements $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_m) \in R^{nm}$, $\widetilde{u}[t] = \big(\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_m(t)\big) \in C([0,T], \Delta_m, R^{nm})$, is a solution to problem with parameters (2.3)–(2.6), then the function $\widetilde{x}(t)$ defined as $\widetilde{x}(t) = \widetilde{\lambda}_r + \widetilde{u}_r(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,m}$, $\widetilde{x}(T) = \widetilde{\lambda}_m + \lim\limits_{t\to T-0} \widetilde{u}_m(t)$, is a solution to the original problem (2.1), (2.2).

If $X_r(t)$ is a fundamental matrix of the differential equation $\dfrac{dx_r}{dt} = A(t)x_r$ on $[t_{r-1}, t_r]$, then the special Cauchy problem for the system of integro-differential equations with parameters (2.3), (2.4) is reduced to the equivalent system of integral equations

$$u_r(t) =$$

$$= X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)A(\tau)d\tau\lambda_r + X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)\sum_{s=1}^{m}\sum_{j=1}^{k}\int_{t_{s-1}}^{t_s} \varphi_j(\tau)\psi_j(\tau_1)u_s(\tau_1)d\tau_1 d\tau$$

$$+X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)\sum_{s=1}^{m}\sum_{j=1}^{k}\int_{t_{s-1}}^{t_s} \varphi_j(\tau)\psi_j(\tau_1)d\tau_1 d\tau\lambda_s$$

$$+X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)f(\tau)d\tau, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,m}. \tag{2.7}$$

Setting $\mu_j = \sum\limits_{s=1}^{m}\int\limits_{t_{s-1}}^{t_s} \psi_j(\tau)u_s(\tau)d\tau$, rewrite (2.7) in the following way:

$$u_r(t) = \sum_{j=1}^{k} X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)\varphi_j(\tau)d\tau\mu_j + X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)\Big[A(\tau)\lambda_r$$

$$+\sum_{j=1}^{k}\varphi_j(\tau)\sum_{s=1}^{m}\int_{t_{s-1}}^{t_s}\psi_j(\tau_1)d\tau_1\lambda_s+f(\tau)\Big]d\tau,\quad t\in[t_{r-1},t_r),\quad r=\overline{1,m}. \tag{2.8}$$

Multiplying both sides of (2.8) by $\psi_p(t)$, integrating them over $[t_{r-1},t_r]$, and summing up with respect to r, we get the following system of linear algebraic equations in $\mu=(\mu_1,\ldots,\mu_k)\in R^{nk}$:

$$\mu_p=\sum_{l=1}^{k}G_{p,l}(\Delta_m)\mu_l+\sum_{r=1}^{m}V_{p,r}(\Delta_m)\lambda_r+g_p(f,\Delta_m),\ p=\overline{1,k}, \tag{2.9}$$

with $(n\times n)$ matrices

$$G_{p,l}(\Delta_m)=\sum_{r=1}^{m}\int_{t_{r-1}}^{t_r}\psi_p(\tau)X_r(\tau)\int_{t_{r-1}}^{\tau}X_r^{-1}(\tau_1)\varphi_l(\tau_1)d\tau_1d\tau, \tag{2.10}$$

$$V_{p,r}(\Delta_m)=\int_{t_{r-1}}^{t_r}\psi_p(\tau)X_r(\tau)\int_{t_{r-1}}^{\tau}X_r^{-1}(\tau_1)A(\tau_1)d\tau_1d\tau$$

$$+\sum_{s=1}^{m}\sum_{j=1}^{k}\int_{t_{s-1}}^{t_s}\psi_p(\tau)X_s(\tau)\int_{t_{s-1}}^{\tau}X_s^{-1}(\tau_1)\varphi_j(\tau_1)d\tau_1d\tau\int_{t_{r-1}}^{t_r}\psi_j(\tau_2)d\tau_2, \tag{2.11}$$

and vectors of the dimension n

$$g_p(f,\Delta_m)=\sum_{r=1}^{m}\int_{t_{r-1}}^{t_r}\psi_p(\tau)X_r(\tau)\int_{t_{r-1}}^{\tau}X_r^{-1}(\tau_1)f(\tau_1)d\tau_1d\tau,\quad p=\overline{1,k},\quad j=\overline{1,k}. \tag{2.12}$$

Using matrices $G_{p,l}(\Delta_m)$ and $V_{p,r}(\Delta_m)$, we construct matrices $G(\Delta_m)=(G_{p,l}(\Delta_m))$, $p,l=\overline{1,k}$, and $V(\Delta_m)=(V_{p,r}(\Delta_m))$, $p=\overline{1,k}$, $r=\overline{1,m}$. Then, system (2.9) can be rewritten in the form

$$[I-G(\Delta_m)]\mu=V(\Delta_m)\lambda+g(f,\Delta_m), \tag{2.13}$$

where I is the identity matrix of the dimension nk, $g(f,\Delta_m)=(g_1(f,\Delta_m),\ldots,g_k(f,\Delta_m))\in R^{nk}$.

Definition 1. *The partition Δ_m is said to be regular if the matrix $I-G(\Delta_m)$ is invertible.*

Any fundamental matrix of the differential equation $\dfrac{dx_r}{dt}=A(t)x_r$ on $[t_{r-1},t_r]$ can be represented as $X_r(t)=X_r^0(t)\cdot C_r$, where $X_r^0(t)$ is the normalized fundamental matrix

$(X_r^0(t_{r-1}) = I)$ and C_r is an arbitrary invertible matrix. Thus, the following equalities hold true

$$G_{p,l}(\Delta_m) = \sum_{r=1}^{m} \int\limits_{t_{r-1}}^{t_r} \psi_p(\tau) X_r^0(\tau) C_r \int\limits_{t_{r-1}}^{\tau} [X_r^0(\tau_1) C_r]^{-1} \varphi_l(\tau_1) d\tau_1 d\tau$$

$$= \sum_{r=1}^{m} \int\limits_{t_{r-1}}^{t_r} \psi_p(\tau) X_r^0(\tau) \int\limits_{t_{r-1}}^{\tau} [X_r^0(\tau_1)]^{-1} \varphi_l(\tau_1) d\tau_1 d\tau,$$

and the regularity of the Δ_m partition does not depend on the choice of the fundamental matrix for the differential part of the equation.

Let us denote by $\sigma(k, [0,T])$ the set of regular partitions Δ_m of the interval $[0,T]$ for equation (2.1).

Definition 2. *The special Cauchy problem (2.3),(2.4) is called uniquely solvable if it has a unique solution for any* $\lambda \in R^{nm}$ *and* $f(t) \in C([0,T], R^n)$.

The special Cauchy problem (2.3), (2.4) is equivalent to the system of integral equations (2.7). Since the kernel of (2.7) is degenerate, this system, in turn, is equivalent to the system of algebraic equations (2.9) in $\mu = (\mu_1, \dots, \mu_k) \in R^{nk}$. Therefore, the special Cauchy problem is uniquely solvable if and only if the Δ_m partition, generating this problem, is regular. Since the special Cauchy problem is uniquely solvable for a partition with a sufficiently small step size $h > 0$ (see [17, p.1152]), the set $\sigma(k, [0,T])$ is not empty.

Let us take a partition $\Delta_m \in \sigma(k, [0,T])$ and represent the matrix $[I - G(\Delta_m)]^{-1}$ in the form $[I - G(\Delta_m)]^{-1} = \Big(M_{j,p}(\Delta_m)\Big)$, $j, p = \overline{1,k}$, where $M_{j,p}(\Delta_m)$ are square matrices of the dimension n.

Then, taking into account (2.13), we can determine elements of the vector $\mu \in R^{nk}$ from the equalities

$$\mu_j = \sum_{i=1}^{m} \Big(\sum_{p=1}^{k} M_{j,p}(\Delta_m) V_{p,i}(\Delta_m) \Big) \lambda_i + \sum_{p=1}^{k} M_{j,p}(\Delta_m) g_p(f, \Delta_m), \quad j = \overline{1,k}. \tag{2.14}$$

In (2.8), by replacing μ_j with the right-hand side of (2.14), we get the representation of the functions $u_r(t)$ through λ_i, $i = \overline{1,m}$:

$$u_r(t) = \sum_{i=1}^{m} \Big\{ \sum_{j=1}^{k} X_r(t) \int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau) \varphi_j(\tau) d\tau \Big[\sum_{p=1}^{k} M_{j,p}(\Delta_m) V_{p,i}(\Delta_m) + \int\limits_{t_{i-1}}^{t_i} \psi_j(\tau_1) d\tau_1 \Big] \Big\} \lambda_i$$

$$+ X_r(t) \int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau) A(\tau) d\tau \lambda_r$$

$$+X_r(t)\int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau)\Big[\sum_{j=1}^{k}\varphi_j(\tau)\sum_{p=1}^{k}M_{j,p}(\Delta_m)g_p(f,\Delta_m)+f(\tau)\Big]d\tau,\ t\in[t_{r-1},t_r),\ r=\overline{1,m}. \quad (2.15)$$

We will use the following notation:

$$D_{r,i}(\Delta_m)=\sum_{j=1}^{k}X_r(t_r)\int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau)\varphi_j(\tau)d\tau\Big[\sum_{p=1}^{k}M_{j,p}(\Delta_m)V_{p,i}(\Delta_m)$$

$$+\int\limits_{t_{i-1}}^{t_i}\psi_j(\tau_1)d\tau_1\Big],\quad i\neq r,\quad r,j=\overline{1,m}, \quad (2.16)$$

$$D_{r,r}(\Delta_m)=\sum_{j=1}^{k}X_r(t_r)\int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau)\varphi_j(\tau)d\tau\Big[\sum_{p=1}^{k}M_{j,p}(\Delta_m)V_{p,r}(\Delta_m)+\int\limits_{t_{r-1}}^{t_r}\psi_j(\tau_1)d\tau_1\Big]$$

$$+X_r(t_r)\int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau)A(\tau)d\tau, \quad (2.17)$$

$$F_r(\Delta_m)=X_r(t)\int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau)\Big[\sum_{j=1}^{k}\varphi_j(\tau)\sum_{p=1}^{k}M_{j,p}(\Delta_m)g_p(f,\Delta_m)+f(\tau)\Big]d\tau,\ r=\overline{1,m}. \quad (2.18)$$

From (2.15), we find the limits

$$\lim_{t\to t_r-0}u_r(t)=\sum_{i=1}^{m}D_{r,i}(\Delta_m)\lambda_i+F_r(\Delta_m). \quad (2.19)$$

Substituting the right-hand side of (2.19) into condition (2.5) and continuity conditions (2.6), we get the following system of linear algebraic equations in parameters λ_r, $r=\overline{1,m}$:

$$\sum_{i=0}^{m-2}[B_i+B_mD_{r,i+1}(\Delta_m)]\lambda_{i+1}+[B_{m-1}+B_m+B_mD_{r,m}(\Delta_m)]\lambda_m=d-B_mF_m(\Delta_m), \quad (2.20)$$

$$[I+D_{p,p}(\Delta_m)]\lambda_p-[I-D_{p,p+1}(\Delta_m)]\lambda_{p+1}+\sum_{\substack{i=1\\ i\neq p,\, i\neq p+1}}^{m}D_{p,i}(\Delta_m)\lambda_i=-F_p(\Delta_m),\ p=\overline{1,m-1}. \quad (2.21)$$

Let $Q_*(\Delta_m)$ be the matrix corresponding to the left-hand side of system (2.20), (2.21). Then system (2.20), (2.21) can be written in the form

$$Q_*(\Delta_m)\lambda=-F_*(\Delta_m),\quad \lambda\in R^{nm}, \quad (2.22)$$

where $F_*(\Delta_m) = \Big(-d + B_m F_m(\Delta_m), F_1(\Delta_m), \dots, F_{m-1}(\Delta_m)\Big) \in R^{nm}$.

Lemma 1. *The following statements hold true for* $\Delta_m \in \sigma(k,[0,T])$*:*

(a) the vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_m^*) \in R^{nm}$*, composed of the values of a solution* $x^*(t)$ *to problem (2.1),(2.2) at the partition points,* $\lambda_r^* = x^*(t_{r-1})$, $r = \overline{1,m}$*, satisfies system (2.22);*

(b) if $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \dots, \widetilde{\lambda}_m) \in R^{nm}$ *is a solution to system (2.22) and a function system* $\widetilde{u}[t] = (\widetilde{u}_1(t), \widetilde{u}_2(t), ..., \widetilde{u}_m(t))$ *is a solution to the special Cauchy problem (2.3), (2.4) with* $\lambda_r = \widetilde{\lambda}_r$, $r = \overline{1,m}$*, then the function* $\widetilde{x}(t)$*, defined as* $\widetilde{x}(t) = \widetilde{\lambda}_r + \widetilde{u}_r(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,m}$, $\widetilde{x}(T) = \widetilde{\lambda}_m + \lim\limits_{t\to T-0} \widetilde{u}_m(t)$*, is a solution to problem (2.1),(2.2).*

The proof of Lemma 1 repeats the proof of Lemma 1 in [24, p. 1155] with minor changes.

We will use the following notation: $\alpha = \max\limits_{t\in[0,T]} \|A(t)\|$,

$$\overline{\omega} = \max_{r=\overline{1,m}} (t_r - t_{r-1}),\ \overline{\varphi}(k) = \max_{r=\overline{1,m}} \int\limits_{t_{r-1}}^{t_r} \sum_{j=1}^{k} \|\varphi_j(t)\| dt,\ \overline{\psi}(T) = \max_{p=\overline{1,k}} \int\limits_0^T \|\psi_p(t)\| dt.$$

Theorem 1. *Let* $\Delta_m \in \sigma(k,[0,T])$ *and let the matrix* $Q_*(\Delta_m) : R^{nm} \to R^{nm}$ *be invertible. Then problem (2.1), (2.2) has a unique solution* $x^*(t)$ *for any* $f(t) \in C([0,T], R^n)$, $d \in R^n$*, and the following estimate holds:*

$$\|x^*\|_1 \le \mathcal{N}(k, \Delta_m) \max(\|d\|, \|f\|_1), \tag{2.23}$$

where

$$\mathcal{N}(k,\Delta_m) = e^{\alpha\overline{\omega}} \Big\{ \overline{\varphi}(k) \Big[\|[I - G(\Delta_m)]^{-1}\| \cdot \overline{\psi}(T) \Big(e^{\alpha\overline{\omega}} - 1 + e^{\alpha\overline{\omega}} \cdot \overline{\varphi}(k) \cdot \overline{\psi}(T) \Big) + \overline{\psi}(T) \Big] + 1 \Big\}$$
$$\times \gamma_*(\Delta_m)(1 + \|C\|) \max\Big\{ 1, \overline{\omega} e^{\alpha\overline{\omega}} \Big[1 + e^{\alpha\overline{\omega}} \cdot \overline{\varphi}(k) \cdot \|[I - G(\Delta_m)]^{-1}\| \cdot \overline{\psi}(T) \Big\}$$
$$+ e^{\alpha\overline{\omega}} \overline{\omega} \Big[\overline{\varphi}(k) \cdot \|[I - G(\Delta_m)]^{-1}\| \cdot \overline{\psi}(T) \cdot e^{\alpha\overline{\omega}} + 1 \Big]. \tag{2.24}$$

Proof. Take a partition $\Delta_m \in \sigma(k,[0,T])$. Let $f(t) \in C([0,T], R^n)$ and $d \in R^n$. Since the matrix $Q_*(\Delta_m)$ is invertible, we can find the unique solution to the system of linear algebraic equations (2.22): $\lambda^* = -[Q_*(\Delta_m)]^{-1} F_*(\Delta_m)$.

By solving the special Cauchy problem (2.3), (2.4) with $\lambda = \lambda^*$, we get the function system $u^*[t] = (u_1^*(t), u_2^*(t), \dots, u_m^*(t))$. It follows from the regularity of the Δ_m partition that there exists a unique function system $u^*[t]$ with the elements $u_r^*(t)$ that are determined from the right-hand side of representation (2.15) with $\lambda = \lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_m^*) \in R^{nm}$. Then, by Lemma 1, the function $x^*(t)$ defined as $x^*(t) = \lambda_r^* + u_r^*(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,m}$, $x^*(T) = \lambda_m^* + \lim\limits_{t\to T-0} u_m^*(t)$, is a solution to problem (2.1), (2.2). The uniqueness of the solution can be proved by contradiction.

Let us verify the validity of the estimate (2.23). Using the equalities

$$X_r(t)\int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau)P(\tau)d\tau = \int\limits_{t_{r-1}}^{t} P(\tau_1)d\tau_1 + \int\limits_{t_{r-1}}^{t} A(\tau_1)\int\limits_{t_{r-1}}^{\tau_1} P(\tau_2)d\tau_2 d\tau_1$$

$$+\int\limits_{t_{r-1}}^{t} A(\tau_1)\int\limits_{t_{r-1}}^{\tau_1} A(\tau_2)\int\limits_{t_{r-1}}^{\tau_2} P(\tau_3)d\tau_3 d\tau_2 d\tau_1 + \ldots, \quad t\in[t_{r-1},t_r],$$

we get the estimates

$$\|X_r(t_r)\int\limits_{t_{r-1}}^{t_r} X_r^{-1}(\tau)\varphi_j(\tau)d\tau\| \le e^{\alpha(t_r-t_{r-1})}\int\limits_{t_{r-1}}^{t_r}\|\varphi_j(t)\|dt,\ r=\overline{1,m}. \tag{2.25}$$

It follows from (2.12), (2.18), (2.25) that

$$\|g_p(f,\Delta_m)\| \le \sum_{r=1}^{m}\int\limits_{t_{r-1}}^{t_r}\|\psi_p(\tau)\|\cdot\|X_r(\tau)\int\limits_{t_{r-1}}^{\tau} X_r^{-1}(\tau_1)f(\tau_1)d\tau_1\|d\tau$$

$$\le \sum_{r=1}^{m}\int\limits_{t_{r-1}}^{t_r}\|\psi_p(\tau)\|d\tau\cdot e^{\alpha\overline{\omega}}\cdot\overline{\omega}\cdot\|f\|_1 = \int\limits_{0}^{T}\|\psi_p(t)\|dt\cdot e^{\alpha\overline{\omega}}\cdot\overline{\omega}\cdot\|f\|_1, \quad p=\overline{1,k}, \tag{2.26}$$

$$\|F_r(\Delta_m)\| \le e^{\alpha(t_r-t_{r-1})}\sum_{j=1}^{k}\int\limits_{t_{r-1}}^{t_r}\|\psi_j(t)\|dt\|[I-G(\Delta_m)]^{-1}\|\max_{p=\overline{1,k}}\|g_p(f,\Delta_m)\|+e^{\alpha(t_r-t_{r-1})}\omega_r\|f\|_1.$$

By using $\|F_*(\Delta_m)\| \le (1+\|B_m\|)\max\Big(\|d\|,\max\limits_{r=\overline{1,m}}\|F_r(\Delta_m)\|\Big)$, and taking into account (2.16), (2.25), and (2.26), we get

$$\|F_*(\Delta_m)\| \le (1+\|B_m\|)\max\Big\{1,\overline{\omega}e^{\alpha\overline{\omega}}\Big[1$$

$$+e^{\alpha\overline{\omega}}\max_{r=\overline{1,m}}\int\limits_{t_{r-1}}^{t_r}\sum_{j=1}^{k}\|\varphi_j(t)\|dt\|[I-G(\Delta_m)]^{-1}\|\max_{p=\overline{1,k}}\int\limits_{0}^{T}\|\psi_p(t)\|dt\Big]\Big\}\max(\|d\|,\|f\|_1). \tag{2.27}$$

Inequalities (2.22), (2.27) and the invertibility of $Q_*(\Delta_m)$ yield the following estimate:

$$\|\lambda^*\| \le \|[Q_*(\Delta_m)]^{-1}\|\|F_*(\Delta_m)\| \le \gamma_*(\Delta_m)(1+\|B_m\|)\max\Big\{1,\overline{\omega}e^{\alpha\overline{\omega}}\Big[1$$

$$+e^{\alpha\overline{\omega}}\max_{r=\overline{1,m}}\int\limits_{t_{r-1}}^{t_r}\sum_{j=1}^{k}\|\varphi_j(t)\|dt\|[I-G(\Delta_m)]^{-1}\|\max_{p=\overline{1,k}}\int\limits_0^T\|\psi_p(t)\|dt\Big\}\max(\|d\|,\|f\|_1). \quad (2.28)$$

By (2.15) and (2.11), we get

$$\|u^*[\cdot]\|_2 \le \Big\{e^{\alpha\overline{\omega}}\max_{r=\overline{1,m}}\int\limits_{t_{r-1}}^{t_r}\sum_{j=1}^{k}\|\varphi_j(t)\|dt\|[I-G(\Delta_m)]^{-1}\|\max_{p=\overline{1,k}}\int\limits_0^T\|\psi_p(t)\|dt\Big(e^{\alpha\overline{\omega}}-1$$

$$+e^{\alpha\overline{\omega}}\max_{r=\overline{1,m}}\int\limits_{t_{r-1}}^{t_r}\max_{j=\overline{1,k}}\|\varphi_j(t)\|dt\max_{p=\overline{1,k}}\int\limits_0^T\|\psi_p(t)\|dt\Big)+\max_{p=\overline{1,k}}\int\limits_0^T\|\psi_p(t)\|dt\Big]+(e^{\alpha\overline{\omega}}-1)\Big\}\|\lambda\|$$

$$+e^{\alpha\overline{\omega}}\overline{\omega}\Big[\max_{r=\overline{1,m}}\int_{t_{r-1}}^{t_r}\sum_{j=1}^{k}\|\varphi_j(t)\|dt\|[I-G(\Delta_m)]^{-1}\|\max_{p=\overline{1,k}}\int_0^T\|\psi_p(t)\|dt\cdot e^{\alpha\overline{\omega}}+1\Big]\cdot\|f\|_1. \quad (2.29)$$

Finally, by using (2.28), (2.29) and $\|x^*\|_1 \le \|\lambda^*\| + \|u^*[\cdot]\|_2$, we arrive at the estimate (2.23). Theorem 1 is proved.

Definition 3. *Problem (2.1), (2.2) is said to be well-posed if for any pair $(f(t), d)$, with $f(t) \in C([0,T], R^n)$ and $d \in R^n$, it has a unique solution $x(t)$, and the estimate*

$$\|x\|_1 \le K\max(\|f\|_1, \|d\|)$$

holds, where K is a constant independent of $f(t)$ and of d.

Theorem 2. *Problem (2.1), (2.2) is well-posed if and only if the matrix $Q_*(\Delta_m) : R^{nm} \to R^{nm}$ is invertible for any partition $\Delta_m \in \sigma(k, [0,T])$.*

Proof. For a fixed k and $\Delta_m \in \sigma(k, [0,T])$ the number $\mathcal{N}(k, \Delta_m)$, defined by (2.24), does not depend on $f(t)$ and d. Thus the sufficiency of the conditions of Theorem 2.2 for the well-posedness of problem (2.1), (2.2) follows from Theorem 1.

Necessity. Let problem (2.1), (2.2) be well-posed and $\Delta_m \in \sigma(k, [0,T])$. Suppose to the contrary that the matrix $Q_*(\Delta_m) : R^{nm} \to R^{nm}$ is not invertible. This is possible only if the homogeneous system of equtions

$$Q_*(\Delta_N)\lambda = 0, \quad \lambda \in R^{nN}, \quad (2.30)$$

has a nonzero solution. Assuming that $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_m)$ is a nonzero solution (i.e. $\|\widetilde{\lambda}\| \neq 0$) to system (2.30), consider the homogeneous problem (2.1), (2.2) with $f(t) = 0$ and $d = 0$. For this problem, system (2.22) coincides with (2.30). Then, by Lemma 2.1, the function $\widetilde{x}(t)$

defined as $\widetilde{x}(t) = \widetilde{\lambda}_r + \widetilde{u}_r(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,m}$, $\widetilde{x}(T) = \widetilde{\lambda}_m + \lim\limits_{t \to T-0} \widetilde{u}_m(t)$, is a nonzero solution to the homogeneous problem. Here the function system $\widetilde{u}[t] = (\widetilde{u}(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_m(t))$ is a solution to the special Cauchy problem (2.3), (2.4) with $\lambda = \widetilde{\lambda}$ and $f(t) = 0$. This contradicts the well-posedness of problem (2.1), (2.2). Theorem 2 is proved.

3 An algorithm for solving multipoint problems for Fredholm integro-differential equations with degenerate kernel and its numerical implementation

The Cauchy problems for ordinary differential equations on the subintervals

$$\frac{dx}{dt} = A(t)x + P(t), \quad x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1,m}, \tag{3.1}$$

are an essential part of the algorithm proposed. Here $P(t)$ is $(n \times n)$ matrix or n vector that is continuous on $[t_{r-1}, t_r]$, $r = \overline{1,m}$. Hence, a solution to problem (3.1) is a matrix or a vector of the dimension n.

Let $E_{*,r}(A(\cdot), P(\cdot), t)$ denote a solution to the Cauchy problem (3.1). Clearly,

$$E_{*,r}(A(\cdot), P(\cdot), t) = X_r(t) \int\limits_{t_{r-1}}^{t} X^{-1}(\tau) P(\tau) d\tau, \quad t \in [t_{r-1}, t_r], \tag{3.2}$$

where $X_r(t)$ is a fundamental matrix of differential equation (3.1) on the r-th subinterval.

The choice of a regular partition is another important part of the algorithm. We can start with Δ_1, when the interval $[0, T]$ is not partitioned.

I. We divide $[0, T]$ into m parts by the points $t_0 = 0 < t_1 < \ldots < t_{m-1} < t_m = T$, involved in the multipoint condition. The resulting partition we denote by $\Delta_m, m = 1, 2, \ldots$.

II. By solving mk Cauchy problems for ordinary differential matrix equations

$$\frac{dx}{dt} = A(t)x + \varphi_k(t), \quad x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \tag{3.3}$$

we obtain the matrix functions

$$E_{*,r}(A(\cdot), \varphi_j(\cdot), t), \quad t \in [t_{r-1}, t_r], \quad r = \overline{1,m}, \quad j = \overline{1,k}. \tag{3.4}$$

III. We multiply each $(n \times n)$ matrix (3.4) by $(n \times n)$ matrix $\psi_p(t)$, $p = \overline{1,k}$, and integrate the products over $[t_{r-1}, t_r]$:

$$\widehat{\psi}_{p,r}(\varphi_j) = \int\limits_{t_{r-1}}^{t_r} \psi_p(t) E_{*,r}(A(\cdot), \varphi_j(\cdot), t) dt. \tag{3.5}$$

Summing up (3.5) with respect to r and taking into account (2.10), (3.2), we get $(n \times n)$ matrices $G_{p,j}(\Delta_m) = \sum_{r=1}^{m} \widehat{\psi}_{p,r}(\varphi_j)$, $p, j = \overline{1,k}$.

We then construct $(nk \times nk)$ matrix $G(\Delta_m) = (G_{p,j}(\Delta_m))$, $p, j = \overline{1,k}$, and determine whether the matrix $[I - G(\Delta_m)] : R^{nk} \to R^{nk}$ is invertible. If so, we find its inverse and represent it as $[I - G(\Delta_m)]^{-1} = (M_{p,j}(\Delta_m))$, where $M_{p,j}(\Delta_m))$ are $(n \times n)$ matrices, $p, j = \overline{1,k}$. We then move on to the next step of the algorithm.

If $[I - G(\Delta_m)]$ is not invertible, i.e. the Δ_m partition is not regular, then we take a new partition of $[0, T]$ and start the algorithm again. A simple way to choose a new partition is to take Δ_{2m}, dividing each subinterval Δ_m in half. We add to the points $t = t_i$ of the multipoint condition the points $(t_i - t_{i-1})/2$, $i = \overline{1,m}$. Then, redesignating all points as $\theta_0 = t_0 = 0$, $\theta_1 = (t_1 - t_0)/2$, $\theta_2 = t_1$, $\theta_3 = (t_2 - t_1)/2$, $\theta_4 = t_2$, ..., $\theta_{2m-1} = (t_m - t_{m-1})/2$, $\theta_{2m} = t_m = T$, we again get problem (2.1), (2.2) with multipoint conditions at the points $t = \theta_i$, $i = \overline{0, 2m}$.

IV. Solving again the Cauchy problems for ordinary differential equations

$$\frac{dx}{dt} = A(t)x + A(t), \quad x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r],$$

$$\frac{dx}{dt} = A(t)x + f(t), \quad x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1,m},$$

we obtain $E_{*,r}(A(\cdot), A(\cdot), t)$ and $E_{*,r}(A(\cdot), f(\cdot), t)$, $r = \overline{1,m}$.

V. We evaluate the integrals $\widehat{\psi}_{p,r} = \int\limits_{t_{r-1}}^{t_r} \psi_p(t)dt$,

$$\widehat{\psi}_{p,r}(A) = \int\limits_{t_{r-1}}^{t_r} \psi_p(t) E_{*,r}(A(\cdot), A(\cdot), t)dt, \ \widehat{\psi}_{p,r}(f) = \int\limits_{t_{r-1}}^{t_r} \psi_p(t) E_{*,r}(A(\cdot), f(\cdot), t)dt.$$

From (2.11), (2.12) and (3.2) we determine $(n \times n)$ matrices

$$V_{p,r}(\Delta_m) = \widehat{\psi}_{p,r}(A) + \sum_{i=1}^{m} \sum_{k=1}^{m} \widehat{\psi}_{p,i}(\varphi_j) \cdot \widehat{\psi}_{j,r}$$

and n vectors $g_p(f, \Delta_m) = \sum_{r=1}^{m} \widehat{\psi}_{p,r}(\Delta_m)$, $p = \overline{1,k}$, $r = \overline{1,m}$.

VI. We construct the system of linear algebraic equations in parameters

$$Q_*(\Delta_m)\lambda = -F_*(\Delta_m), \quad \lambda \in R^{nm}. \tag{3.6}$$

The elements of the matrix $Q_*(\Delta_m)$ and the vector $F_*(\Delta_m) = (-d + B_m F_m(\Delta_m), F_1(\Delta_m), \ldots, F_{m-1}(\Delta_m)) \in R^{nm}$ are determined by the equalities (2.16), (2.17), (2.18), where, by (3.2), we replace $X_r(t_r)\int\limits_{t_{r-1}}^{t_r} X_r^{-1}(\tau)\varphi_j(\tau)d\tau$ and $X_r(t_r)\int\limits_{t_{r-1}}^{t_r} X_r^{-1}(\tau)f(\tau)d\tau$ with $E_{*,r}(A(\cdot),\varphi_j(\cdot),t_r)$ and $E_{*,r}(A(\cdot),f(\cdot),t_r)$, respectively. It follows from Theorem 2 that the invertibility of the matrix $Q_*(\Delta_m)$ is equivalent to the well-posedness of problem (2.1), (2.2). By solving system (3.6), we get $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_m^*) \in R^{nm}$.

VII. From the equalities

$$\mu_s^* = \sum_{j=1}^{m}\Big(\sum_{p=1}^{k} M_{s,p}(\Delta_m)V_{p,j}(\Delta_m)\Big)\lambda_j^* + \sum_{p=1}^{k} M_{s,p}(\Delta_m)g_p(f,\Delta_m) \tag{3.7}$$

we determine the components $\mu^* = (\mu_1^*, \mu_2^*, \dots, \mu_k^*) \in R^{nk}$ and construct the function

$$\mathcal{F}^*(t) = \sum_{s=1}^{k}\varphi_s(t)\Big[\mu_s^* + \sum_{r=1}^{m}\int\limits_{t_{r-1}}^{t_r}\psi_s(\tau)d\tau\lambda_r^*\Big] + f(t). \tag{3.8}$$

Recall that $\lambda_r^* = x^*(t_{r-1})$, where $x^*(t)$ is a solution to problem (2.1), (2.2). Hence, by solving system (3.6), we get the values of the desired solution at the left endpoints of the partition subintervals. In order to determine the values of the function $x^*(t)$ at the remaining points of the subintervals $[t_{r-1}, t_r)$, we solve the following Cauchy problems for the ordinary differential equation:

$$\frac{dx}{dt} = A(t)x + \mathcal{F}^*(t), \quad x(t_{r-1}) = \lambda_r^*, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,m}.$$

Thus, the proposed algorithm contains seven interrelated parts.

If the fundamental matrices $X_r(t)$, $r = \overline{1,m}$, are known, then the equalities (2.16), (2.17), and (2.18) enable us to construct the system (3.6). Let $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_m^*) \in R^{nm}$ be a solution to (3.6). Then, using (3.7) and (3.8), we construct the function $\mathcal{F}^*(t)$ and determine a solution to problem (2.1), (2.2) by the equalities

$$x^*(t) = X_r(t)X_r^{-1}(t_{r-1})\lambda_r^* + X_r(t)\int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau)\mathcal{F}^*(\tau)d\tau, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,m}, \tag{3.9}$$

$$x^*(T) = X_m(T)X_m^{-1}(t_{m-1})\lambda_m^* + X_m(T)\int\limits_{t_{m-1}}^{T} X_m^{-1}(\tau)\mathcal{F}^*(\tau)d\tau. \tag{3.10}$$

So, in this case the proposed algorithm provides the solution to the linear multipoint boundary value problem for integro-differential equations (2.1), (2.2) in the form (3.9), (3.10). As is known, it is not always possible to construct a fundamental matrix for the system of ordinary differential equations with variable coefficients. For this reason, we propose the following numerical implementation of the algorithm that is based on the fourth-order Runge-Kutta method and Simpson's rule.

I. Let us take a partition $\Delta_m : t_0 = 0 < t_1 < \ldots < t_{m-1} < t_m = T$. We divide each rth subinterval $[t_{r-1}, t_r]$, $r = \overline{1,m}$, into m_r parts with the step size $h_r = (t_r - t_{r-1})/m_r$.

Suppose that on each subinterval $[t_{r-1}, t_r]$ a variable $\widehat{t}$ takes on discrete values: $\widehat{t} = t_{r-1}$, $\widehat{t} = t_{r-1} + h_r$, ..., $\widehat{t} = t_{r-1} + (m_r - 1)h_r$, $\widehat{t} = t_r$. Let $\{t_{r-1}, t_r\}$ denote the set of such points.

II. Using the fourth-order Runge-Kutta method, we obtain numerical solutions to the Cauchy problems (3.1) and determine the values of $(n \times n)$ matrix $E^{h_r}_{*,r}(A(\cdot), \varphi_j(\cdot), \widehat{t})$ on the set $\{t_{r-1}, t_r\}$, $r = \overline{1,m}$, $j = \overline{1,k}$.

III. Using the values of $(n \times n)$ matrices $\psi_j(s)$ and $E^{h_r}_{*,r}\Big(A(\cdot), \varphi_j(\cdot), \widehat{t}\Big)$ on $\{t_{r-1}, t_r\}$ and applying Simpson's rule, we determine $(n \times n)$ matrices

$$\widehat{\psi}^{h_r}_{p,r}(\varphi_j) = \int\limits_{t_{r-1}}^{t_r} \psi_p(\tau) E^{h_r}_{*,r}(A(\cdot), \varphi_j(\cdot), \tau) d\tau, \quad p, j = \overline{1,k}, \quad r = \overline{1,m}.$$

Summing up the matrices $\widehat{\varphi}^{h_r}_{p,r}(\psi_j)$ with respect to r, we get $(n \times n)$ matrices $G^{\widetilde{h}}_{p,j}(\Delta_m) = \sum\limits_{r=1}^{m} \widehat{\varphi}^{h_r}_{p,r}(\psi_j)$, where $\widetilde{h} = (h_1, h_2, \ldots, h_m) \in R^n$. We then construct $(nk \times nk)$ matrix $G^{\widetilde{h}}(\Delta_m) = (G^{\widetilde{h}}_{p,j}(\Delta_m))$, $p, j = \overline{1,k}$.

Determine whether the matrix $[I - G^{\widetilde{h}}(\Delta_m)] : R^{nk} \to R^{nk}$ is invertible. If so, we find $[I - G^{\widetilde{h}}(\Delta_m)]^{-1} = (M^{\widetilde{h}}_{p,j}(\Delta_m))$, $p, j = \overline{1,k}$.

In the case $[I - G^{\widetilde{h}}(\Delta_m)]$ is not invertible, we choose a new partition. In particular, as shown above, each subinterval can be divided in half.

IV. Solving the Cauchy problem (3.5), (3.6) by the fourth-order Runge-Kutta method, we get the values of $(n \times n)$ matrix $E_{*,r}(A(\cdot), A(\cdot), \widehat{t})$ and n vector $E_{*,r}(A(\cdot), f(\cdot), \widehat{t})$ on $\{t_{r-1}, t_r\}$, $r = \overline{1,m}$.

V. By using Simpson's rule on the grid $\{t_{r-1}, t_r\}$, we evaluate the definite integrals

$$\widehat{\psi}^{h_r}_{p,r} = \int\limits_{t_{r-1}}^{t_r} \psi_p(\tau) d\tau, \quad \widehat{\psi}^{h_r}_{p,r}(A) = \int\limits_{t_{r-1}}^{t_r} \psi_p(\tau) E^{h_r}_{*,r}(A(\cdot), A(\cdot), \tau) d\tau,$$

$$\widehat{\psi}^{h_r}_{p,r}(f) = \int\limits_{t_{r-1}}^{t_r} \psi_p(\tau) E^{h_r}_{*,r}(A(\cdot), f(\cdot), \tau) d\tau, \quad r = \overline{1,N}, \quad p = \overline{1,m}.$$

We determine $(n\times n)$ matrices $V_{p,r}^{\widetilde{h}}(\Delta_m)$ and n vectors $g_p^{\widetilde{h}}(f,\Delta_m)$, $r=\overline{1,m}$, $p=\overline{1,k}$, by the equalities $\quad V_{p,r}^{\widetilde{h}}(\Delta_m)=\widehat{\psi}_{p,r}^{h_r}(A)+\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{k}\widehat{\psi}_{p,j}^{h_j}(\varphi_i)\cdot\widehat{\psi}_{i,r}^{h_r}, \quad g_p^{\widetilde{h}}(f,\Delta_m)=\sum\limits_{r=1}^{m}\widehat{\psi}_{p,r}^{h_r}(f).$

VI. We construct the system of linear algebraic equations in parameters

$$Q_*^{\widetilde{h}}(\Delta_m)\lambda=-F_*^{\widetilde{h}}(\Delta_m),\quad \lambda\in R^{nm}. \tag{3.11}$$

The elements of the matrix $Q_*^{\widetilde{h}}(\Delta_m)$ and the vector $F_*^{\widetilde{h}}(\Delta_m)=(-d+B_mF_m^{\widetilde{h}}(\Delta_m),F_1^{\widetilde{h}}(\Delta_m),\ldots,F_{m-1}^{\widetilde{h}}(\Delta_m))$ are determined by the equalities

$$D_{r,i}^{\widetilde{h}}(\Delta_m)=\sum_{j=1}^{k}E_{*,r}^{h_r}(A(\cdot),\varphi_j(\cdot),t_r)\Big[\sum_{p=1}^{k}M_{j,p}^{\widetilde{h}}(\Delta_m)V_{p,i}^{\widetilde{h}}(\Delta_m)+\widehat{\psi}_{j,i}^{h_i}\Big],\quad i\neq r,\quad r,i=\overline{1,m},$$

$$D_{r,r}^{\widetilde{h}}(\Delta_m)=\sum_{j=1}^{k}E_{*,r}^{h_r}(A(\cdot),\varphi_j(\cdot),t_r)\Big[\sum_{p=1}^{k}M_{j,p}^{\widetilde{h}}(\Delta_m)V_{p,r}^{\widetilde{h}}(\Delta_m)+\widehat{\psi}_{j,r}^{h_r}\Big]+E_{*,r}^{h_r}(A(\cdot),A(\cdot),t_r),$$

$$F_r^{\widetilde{h}}(\Delta_m)=\sum_{j=1}^{k}E_{*,r}^{h_r}(A(\cdot),\varphi_j(\cdot),t_r)\sum_{p=1}^{k}M_{j,p}^{\widetilde{h}}(\Delta_m)g_p^{\widetilde{h}}(\Delta_m)+E_{*,r}^{h_r}(A(\cdot),f(\cdot),t_r),\quad r=\overline{1,m}.$$

Using the constructed matrix $(Q_*^{\widetilde{h}}(\Delta_m)$, we can establish the well-posedness of problem (2.1), (2.2). Suppose the matrix $Q_*^{\widetilde{h}}(\Delta_m)$ is invertible and the estimate $||Q_*(\Delta_m)-Q_*^{\widetilde{h}}(\Delta_m)||\leq\varepsilon(\widetilde{h})$ holds. If the inequality $||[Q_*^{\widetilde{h}}(\Delta_m)]^{-1}||\cdot\varepsilon(\widetilde{h})<1$ is true, then, by Theorem 4 [9, p.212], the matrix $Q_*(\Delta_m)$ is invertible. It follows then from Theorem 2.2 that problem (2.1), (2.2) is well-posed.

By solving (3.11) we determine $\lambda^{\widetilde{h}}\in R^{nm}$. As noted above, the elements $\lambda^{\widetilde{h}}=(\lambda_1^{\widetilde{h}},\lambda_2^{\widetilde{h}},\ldots,\lambda_m^{\widetilde{h}})$ are the values of the approximate solution to problem (2.1), (2.2) at the left endpoints of the subintervals: $x^{\widetilde{h}_r}(t_{r-1})=\lambda_r^{\widetilde{h}}$, $r=\overline{1,m}$.

VII. In order to calculate the values of the approximate solution at the remaining points of the set $\{t_{r-1},t_r\}$, we first find

$$\mu_i^{\widetilde{h}}=\sum_{j=1}^{m}\Big(\sum_{p=1}^{k}M_{i,p}^{\widetilde{h}}(\Delta_m)V_{p,j}^{\widetilde{h}}(\Delta_m)\Big)\lambda_i^{h}+\sum_{p=1}^{k}M_{i,p}^{\widetilde{h}}(\Delta_m)g_p^{\widetilde{h}}(f,\Delta_m),\quad i=\overline{1,k},$$

and then, using the fourth-order Runge-Kutta method, solve the Cauchy problems

$$\frac{dx}{dt}=A(t)x+\mathcal{F}^{\widetilde{h}}(t),\quad x(t_{r-1})=\lambda_r^{\widetilde{h}},\quad t\in[t_{r-1},t_r],\quad r=\overline{1,m},$$

where $\mathcal{F}^{\widetilde{h}}(t) = \sum\limits_{i=1}^{k} \varphi_i(t)\Big(\mu_i^{\widetilde{h}} + \sum\limits_{j=1}^{m} \widehat{\psi}_{i,j}^{h_j}\lambda_j^h\Big) + f(t).$

Thus the algorithm allows us to find a numerical solution to problem (2.1), (2.2).

4 Multipoint problem for integro-differential equation with non-degenerate kernel

Let us now turn to the original multipoint problem (1.1), (1.2). To solve the problem, we will approximate the kernel of the integral summand by a degenerate kernel [6], [8], [24], [25].

By the Weierstrass polynomial approximation theorem, for any $\varepsilon > 0$ there exist a number $k = k(\varepsilon)$ and continuous on $[0, T]$ matrices $\varphi_j(t)$, $\psi_j(\tau)$, $j = \overline{1, k}$, such that the following inequality holds

$$\max_{t\in[0,T]} \int\limits_0^T \|K(t,\tau) - \sum_{j=1}^{k} \varphi_j(t)\psi_j(\tau)\| d\tau < \varepsilon. \tag{4.1}$$

The set of matrices $\{\varphi_j(t), \psi_j(\tau), j = \overline{1, m}\}$, satisfying (4.1), we will call the ε-approximating set for $K(t, \tau)$. The multipoint problem with degenerate kernel (2.1), (2.2), corresponding to (1.1), (1.2), we will call the ε-approximating problem for problem (1.1), (1.2).

Assuming the ε-approximating multipoint problem (2.1), (2.2) to be well-posed with constant C_k, we find the solution to problem (1.1), (1.2) according to the following algorithm.

Step 0. By solving problem (2.1), (2.2), we get a function $x^{(0)}(t)$, which we take as an initial approximation of the solution to problem (1.1), (1.2).

Step 1. Using $x^{(0)}(t)$ and solving the ε-approximating problem

$$\frac{dx}{dt} = A(t)x + \sum_{j=1}^{k} \varphi_j(t) \int\limits_0^T \psi_j(\tau)x(\tau)d\tau$$

$$+ \int\limits_0^T \Big[K(t,\tau) - \sum_{j=1}^{k} \varphi_j(t)\psi_j(\tau)\Big] x^{(0)}(\tau)d\tau + f(t), \quad t \in (0, T), \tag{4.2}$$

$$\sum_{i=0}^{m} B_i x(t_i) = d, \quad d \in R^n, \tag{4.3}$$

we get the function $x^{(1)}(t)$.

Continue the algorithm, in the ith step ($i = 1, \dots$) we solve the problem

$$\frac{dx}{dt} = A(t)x + \sum_{j=1}^{k} \varphi_j(t) \int\limits_0^T \psi_j(\tau)x(\tau)d\tau$$

$$+\int_0^T \Big[K(t,\tau) - \sum_{j=1}^{k} \varphi_j(t)\psi_j(\tau)\Big] x^{(i-1)}(\tau)d\tau + f(t), \quad t \in (0,T), \tag{4.4}$$

$$\sum_{i=0}^{m} B_i x(t_i) = d, \tag{4.5}$$

and get the function $x^{(i)}(t)$.

The well-posedness of the approximating problem ensures the feasibility of the algorithm and allows us to construct the sequence $(x^{(i)}(t))$, $i = 0, 1, \dots$.

The following assertion provides conditions for the convergence of the algorithm to the unique solution of multipoint problem (1.1), (1.2) and the estimates for the difference between the exact and approximate solutions to the problem.

Theorem 3. *Let the ε-approximating problem (2.1), (2.2) be well-posed with constant C_k. Suppose that the following inequality holds:*

$$q_m^{\varepsilon} = K_m \cdot \varepsilon < 1. \tag{4.6}$$

Then the algorithm converges to $x^*(t)$ and the estimate

$$\|x^* - x^{(i)}\|_1 \le \frac{1}{1 - q_k^{\varepsilon}} (q_k^{\varepsilon})^i \cdot C_k \max(\|f\|_1, \|d\|) \tag{4.7}$$

is valid, where $x^*(t)$ and $x^{(i)}(t)$ are the unique solutions to problems (1.1), (1.2) and (4.4), (4.5), respectively.

Proof. By assumption, there exists a unique solution to problem (2.1), (2.2) and it satisfies the inequality

$$\|x^{(0)}\|_1 \le C_k \max(\|f\|_1, \|d\|).$$

By solving problem (4.2), (4.3), we get $x^{(1)}(t)$. The difference $\Delta x^{(1)}(t) = x^{(1)}(t) - x^{(0)}(t)$ satisfies the inequality

$$\|\Delta x^{(1)}\|_1 \le C_k \cdot \max_{t\in[0,T]} \int_0^T \|K(t,s) - \sum_{j=1}^{k} \varphi_j(t)\psi_j(\tau)\| d\tau \|x^{(0)}\|_1$$

$$\le C_k \cdot \varepsilon \cdot C_k \cdot \max(\|f\|_1, \|d\|). \tag{4.8}$$

In the same way, by solving problem (4.4), (4.5), we get $x^{(i)}(t)$, and for the difference $\Delta x^{(i)}(t) = x^{(i)}(t) - x^{(i-1)}(t)$ we have

$$\|\Delta x^{(i)}\|_1 \le C_k \cdot \varepsilon \cdot \|\Delta x^{(i-1)}\|_1 = q_k^{\varepsilon} \|\Delta x^{(i-1)}\|_1, \quad i = 2, 3, \dots . \tag{4.9}$$

The convergence of the sequence $(x^{(i)}(t))$, $i = 0, 1, \dots$, to the solution $x^*(t)$ of problem (1.1), (1.2), as well as the uniqueness of this solution, follow from the inequalities (4.6) and (4.9). The estimate is derived from (4.8) and (4.9). Theorem 3 is proved.

The conditions of Theorem 1 ensure the existence of a unique solution to problem (2.1), (2.2.) and the validity of the estimate (2.23). The number $\mathcal{N}(k, \Delta_m)$ in (2.23), as mentioned above, does not depend on $f(t)$ and d. We therefore can treat this number as a constant of well-posedness of problem (2.1), (2.2). Hence, by Theorems 1 and 3, the following statement holds true.

Theorem 4. *Suppose that*

(a) the set of the matrices $\{\varphi_j(t), \psi_j(\tau), j = \overline{1,k}\}$ *is an* ε*-approximating set for* $K(t, \tau)$*;*

(b) $\Delta_m \in \sigma(k, [0, T])$;

(c) the matrix $Q_*(\Delta_m) : R^{nm} \to R^{nm}$ *in (2.22) is invertible;*

(d) the inequality $\delta_m^\varepsilon = \mathcal{N}(k, \Delta_m) \cdot \varepsilon < 1$ *holds.*

Then problem (1.1), (1.2) is well-posed with constant $C = \dfrac{1}{1 - \delta_k^\varepsilon} \cdot \mathcal{N}(k, \Delta_m)$.

The conditions of Theorem 3 are not only necessary but also sufficient for the well-posedness of problem (1.1), (1.2).

Theorem 5. *Problem (1.1), (1.2) is well-posed if and only if there exists the* ε*-approximating multipoint problem (2.1), (2.2), that is well-posed with constant* C_k*, and the inequality (4.6) holds true.*

Proof. The sufficiency of the conditions of the theorem for the well-posedness of problem (1.1), (1.2) follows from Theorem 3.

Let us prove the necessity. Assume that problem (1.1), (1.2) is well-posed with a constant C. Take $\varepsilon > 0$ satisfying the inequality $\varepsilon \cdot C < 1/2$. For chosen ε take $k \in \mathbb{N}$ and continuous on $[0, T]$ matrices $\varphi_j(t)$, $\psi_j(\tau)$, $j = \overline{1,k}$, satisfying inequality (4.1). Let us show that multipoint problem (2.1), (2.2) with these matrices is well-posed and the constant C_k of well-posedness satisfies inequality (4.4). To this end, we use the following algorithm.

Step 0. By solving problem (1.1), (1.2), we get the function $x^{(0)}(t)$.

Step i. Assuming $x^{(i-1)}(t)$, $i = 1, 2, \dots$, to be known, we solve the problem

$$\frac{dx}{dt} = A(t)x + \int_0^T K(t, \tau)x(\tau)d\tau + f(t) + \int_0^T \Big[\sum_{j=1}^k \varphi_j(t)\psi_j(\tau) - K(t, \tau)\Big]x^{(i-1)}(\tau)d\tau, \quad t \in [0, T],$$

$$\sum_{i=0}^m B_i x(t_i) = d,$$

and get the function $x^{(i)}(t)$.

It is easy to check that the algorithm converges to $x^*(t)$ and the estimate

$$\|x^*\|_1 \leq \frac{C}{1 - C \cdot \varepsilon} \cdot \max(\|f\|_1, \|d\|), \tag{4.10}$$

holds, where $x^*(t)$ is the unique solution to problem (2.1), (2.2).

Since, by assumption, $C \cdot \varepsilon < 1/2$, the well-posedness of the ε-approximating problem (2.1), (2.2) with constant $C_k = 2C$ follows from (4.10). Taking into account the choice of $\varepsilon > 0$, we get $q_k^\varepsilon = C_k \cdot \varepsilon < 1$. Theorem 5 is proved.

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Асанова А.Т., Бакирова Э.А., Утешова Р.Е. ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУ ҮШІН КӨП НҮКТЕЛІ ЕСЕПТІ ШЕШУГЕ ЖАҢА ТӘСІЛ

Фредгольм интегралдық-дифференциалдық теңдеулер жүйесі үшін көпнүктелі есеп қарастырылады. Өзегі айныған интегралдық-дифференциалдық теңдеулер жүйесі үшін көпнүктелі есеп жеке зерттеледі. Параметрлеу әдісі арқылы өзегі айныған интегралдық-дифференциалдық теңдеулер жүйесі үшін көпнүктелі есептің қисынды шешілімділігінің шарттары алынды. Қарастырылып отырған есептің жуық және сандық шешімдерін табу алгоритмдері ұсынылды. Фредгольм интегралдық-дифференциалдық теңдеулер жүйесі үшін көпнүктелі есептің қисынды шешілімділігінің қажетті және жеткілікті шарттары тағайындалды. Зерттеліп отырған есептің жуық шешімдерін табу алгоритмдері аппроксимациялаушы өзегі айныған интегралдық-дифференциалдық теңдеулер жүйесі үшін есептің шешімдері негізінде тұрғызылды.

Кілттік сөздер. Фредгольм интегралдық-дифференциалдық теңдеуі, көп нүктелі есеп, параметрлеу әдісі, алгоритм, шешілімділік критерийі.

Асанова А.Т., Бакирова Э.А., Утешова Р.Е. НОВЫЙ ПОХОД К РЕШЕНИЮ МНОГОТОЧЕЧНОЙ ЗАДАЧИ ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ

Рассматривается многоточечная задача для системы интегро-дифференциальных уравнений Фредгольма. Отдельно изучается многоточечная задача для системы интегро-дифференциальных уравнений с вырожденным ядром. Получены условия корректной разрешимости многоточечной задачи для системы интегро-дифференциальных уравнений с вырожденным ядром методом параметризации. Предложены алгоритмы нахождения приближенных и численных решений рассматриваемой задачи. Установлены необходимые и достаточные условия корректной разрешимости многоточечной задачи для системы интегро-дифференциальных уравнений Фредгольма. Построены алгоритмы нахождения приближенных решений исследуемой задачи на основе решений аппроксимирующей задачи для системы интегро-дифференциальных уравнений с вырожденным ядром.

Ключевые слова. Интегро-дифференциальное уравнение Фредгольма, многоточечная задача, метод параметризации, алгоритм, критерий разрешимости.

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Адрес типографии:
Институт математики и математического моделирования
г. Алматы, ул. Пушкина, 125
Тел./факс: 8 (727) 2 72 70 93
e-mail: math_journal@math.kz
web-site: http://kmj.math.kz

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EDITOR IN CHIEF: Makhmud Sadybekov,
Institute of Mathematics and Mathematical Modeling

HEAD OFFICE: 125 Pushkin Str., 050010, Almaty, Kazakhstan

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