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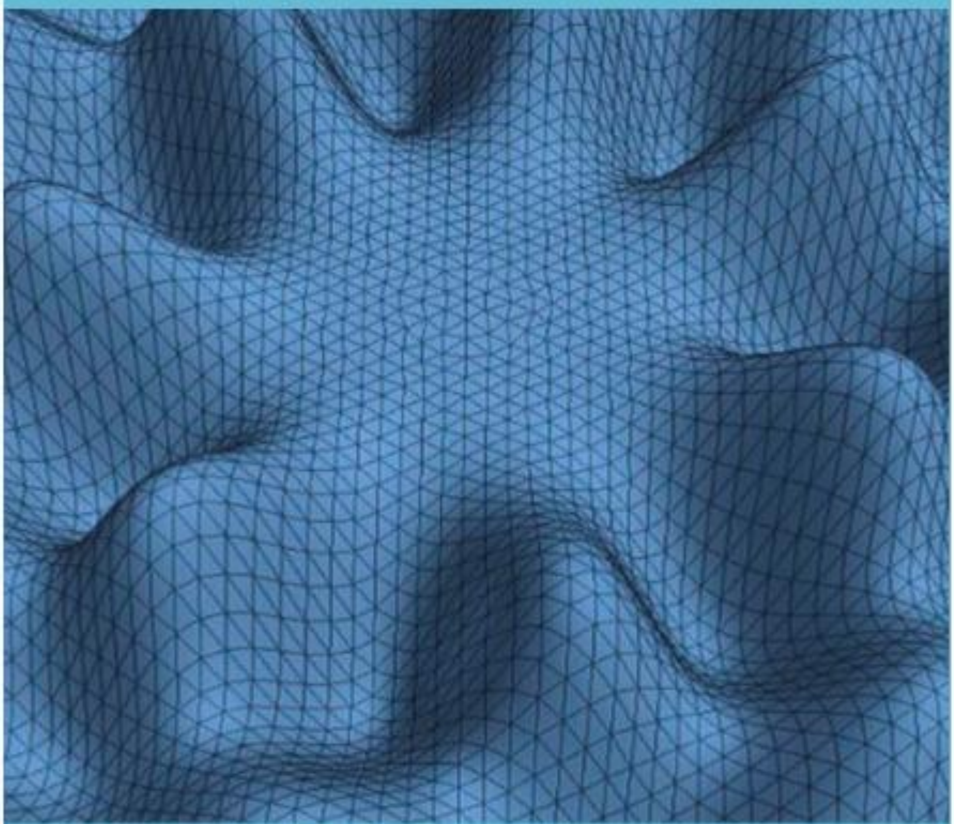

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CONTENTS

19:4 (2019)

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19:4 (2019) 6–20

Bosons and leptons in biquaternionic representation. Periodic system of atoms as simple gamma

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Abstract. Particular monochromatic solutions of biquaternionic wave for the free fields of electro-gravimagnetic charges and currents have been constructed that describe elementary particles as standing monochromatic electro-gravimagnetic waves. Two classes solutions of this biwave equation generated by scalar potentials (pulsars) and vectorial potentials (spinors) are studied. Their asymptotic properties have been researched on the basis of which they are classified on heavy and light elementary particles (bosons and leptons). It is shown that bosons are spherical harmonic pulsars, the mass density of which is determined by their frequency of oscillations. This allows to construct periodic systems of elementary particles on the basis of classical musical scale. In particular, a biquaternionic representation of the hydrogen atom is given and advising it a periodic system, built on the principle of the simple musical scale.

Keywords. Biquaternion, frequency, standing wave, pulsar, spinor, boson, lepton, atom, hydrogen, periodic system, musical scale.

1 Introduction

In [1]–[6], the author developed a biquaternionic model of electro-gravimagnetic field (EGM-field) and electro-gravimagnetic interactions. It is based on biquaternionic representations of Maxwell and Dirac equations (MEq, DEq) and their generalization in biquaternions algebra (GMEq, GDEq). The biquaternionic representation of GMEq expresses the biquaternion of mass charge and EGM current densities through the bigradient of EGM-field intensity. The biquaternionic representation of GDEq defines a transformation of density of EGM-charges (mass-charges) and EGM-currents under the influence of external EGM fields. In particular, in the absence of external fields, on its basis, the biquaternionic wave equation

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of the free field of mass-charges and currents, which is a field analog of the first Newton law, *inertia law* were obtained.

Here the particular monochromatic solutions of this equation are constructed, which describe elementary particles as standing EGM-waves. They can be divided into two classes generated by scalar potentials (*pulsars*) and vectorial potentials (*spinors*). Their asymptotic properties are researched, on the basis of which they are classified on heavy (*bosons*) and light (*leptons*) elementary particles. It is shown that bosons are spherical harmonic pulsars, the mass-charge density of which is determined by their oscillation frequency. This allows us to build periodic system of elementary particles based on the simple harmonic musical scale.

In particular, a biquaternionic representation of hydrogen atom and the corresponding periodic system are built on the principle of a simple gamut.

2 Equation of the free field of charge-current

The equation of the free field of charge-currents has the form of a homogeneous biwave equation:

$$\nabla^{-}\Theta(\tau,x) = (\partial_\tau - i\nabla)\circ(i\rho(\tau,x) + J(\tau,x)) = 0. \tag{1}$$

Here $\Theta(\tau,x)$ is a biquaternion of charge-current (CC), which scalar part $\rho(\tau,x)$ discribes densities of electric and gravimagnetic charges (*EGM-charge*). Vector part $J(\tau,x)$ is the density of electric and gravimagnetic currents (*EGM-currents*):

$$\rho = \frac{1}{\sqrt{\varepsilon}}\rho^E - \frac{i}{\sqrt{\mu}}\rho^H, \quad J = \sqrt{\mu}\, j^E - i\sqrt{\varepsilon}\, j^H.$$

Here $\rho^E(x,t), j^E(x,t)$ are densities of electric charge and current, $\rho^H(x,t)$, $j^H(x,t)$ are densities of gravimagnetic charge and current; ε, μ are constants of electric conductivity and the magnetic permutability of vacuum, $c = 1/\sqrt{\varepsilon\mu}$ is the speed of light ($c = 299792458 \pm 1,2$ m/s), i is imaginary unit.

The action of biquaternionic differential operators ∇^- and ∇^+(*mutual bigradients*) are defined according to the rule of quaternions multiplication:

$$\nabla^{\pm}F(\tau,x) = (\partial_\tau \pm i\nabla)\circ(f(\tau,x) + F(\tau,x))$$

$$= (\partial_\tau f \mp i\,\mathrm{div}\,F) + \{\pm i\mathrm{grad}\,f + \partial_\tau F \pm i\,\mathrm{rot}\,F\}.$$

Energy-pulse Bq of any F-field has the form:

$$\Xi(\tau,x) = W(\tau,x) + i\,P(\tau,x) = 0.5F\circ F^*,$$

where F^* is conjugated Bq:

$$F^* = \bar{f}(\tau,x) - \bar{F}(\tau,x).$$

Here, the line above the symbol means complex conjugation. The scalar part W is the energy density of F-field, and P is an analogue of Pointing vector of electromagnetic field. We will call it by the same name (for details, see [5], [6] about application of differential algebra of biquaternions in electrodynamics).

The scalar part of this equation is well known as

the law of EGM-charge conservation:

$$\partial_\tau \rho + \operatorname{div} J = 0.$$

The vector part describes the connection between charges and currents in absence of external EGM-field. It is

the law of EGM-current motion:

$$\partial_\tau J - \mathrm{i}\,\mathrm{rot} J + \mathrm{grad} \rho = 0.$$

This two laws are a closed hyperbolic system of differential equations for construction of its solutions.

3 Monochromatic EGM-field. Harmonic elementary particles and structures

For monochromatic fields with frequency ω CC-Bq can be represented as

$$\Theta(\tau, x) = \Theta(x, \omega) \exp(-i\omega\tau), \quad \omega > 0.$$

In this case from Eq. (1) we get the equation for complex biquaternionic amplitude (*biamplitude*) $\Theta(x, \omega)$:

$$(\omega + \nabla) \circ (i\rho(x) + J(x)) = 0. \tag{2}$$

Since

$$(\omega + \nabla) \circ (\omega - \nabla) = (\omega - \nabla) \circ (\omega + \nabla) = \omega^2 + \Delta,$$

from here it follows that the biamplitude satisfies to Helmholtz equation:

$$\Delta\Theta + \omega^2\Theta = 0,$$

$\triangle$ is Laplace operator.

Monochromatic solutions of Eq. (1) have the form:

$$\Theta(\tau, x) = \exp(-i\omega\tau)\,(\omega - \nabla) \circ \Big(\psi^0(x, \omega) + \sum_{j=1}^{3} \psi^j(x, \omega) e_j\Big), \tag{3}$$

where the *potentials* ψ^j are any solutions of Helmholtz equation:

$$\Delta\psi + \omega^2\psi = 0,$$

which can be presented as surface integral of the kind

$$\psi^j(x,\omega) = \int\limits_{\|\xi\|=\omega} \phi^j(\xi,\omega) e^{-i(\xi,x)} dS(\xi) \tag{4}$$

for any function ϕ^j summed on the sphere of radius ω.

We consider particular solutions of the Helmholtz equation [7]:

$$\psi_{nm}(x,\omega) = j_n(\omega r) Y_n^m(\vartheta,\varphi), \tag{5}$$

where $j_n(\omega r)$ are spherical Bessel functions of order $n\,(n = 0,1,2...)$, $Y_n^m(\vartheta,\varphi)$ is a spherical harmonic of order n, m $(m = 1,2,...)$:

$$Y_n^m(\vartheta,\varphi) = P_n^m(\cos\vartheta)\exp(im\varphi),$$

$P_n^m(...)$ are associated with Legendre polynomials [8], (r,ϑ,φ) are spherical coordinates.

It is natural to take these solutions to build elementary particles, which can be named *harmonic*. Among them we single out those, generated by the scalar potential, and call them *pulsars*:

$$\begin{aligned}\Theta^0_{nm}(x,\omega) &= (\omega - \nabla)\circ\psi_{nm}(x,\omega)\\ &= \omega\psi_{nm}(x,\omega) - \mathrm{grad}\psi_{nm}(x,\omega),\end{aligned} \tag{6}$$

and which are generated by vectorial potential, we call them *spinors*:

$$\begin{aligned}\Theta^j_{nm}(x,\omega) &= (\omega - \nabla)\circ\psi_{nm}(x,\omega)e_j\\ &= div\ (\psi_{nm}(x,\omega)e_j) + \{\omega\psi_{nm}(x,\omega)e_j - rot\ (\psi_{nm}(x,\omega)e_j)\}.\end{aligned} \tag{7}$$

The latter are polarized in the direction of the coordinate axes, respectively index $j = 1,2,3$.

Using structural biquaternions of an arbitrary form $K(x)$ and biquaternionic operation of convolution (ε_{jlm} is Levi-Civita pseudo-tensor):

$$\begin{aligned}\Theta(x,\omega) * K(x) &= (i\rho + J) * (k + K)\\ &= \left\{i\rho * k - \textstyle\sum_{j=1}^3 (J_j * K_j)\right\} + \left\{i\rho * K + J * k + \textstyle\sum_{j,l,m=1}^3 \varepsilon_{jlm}(J_j * K_l)\, e_m\right\},\end{aligned} \tag{8}$$

we can construct different monochromatic CC-fields:

$$\Theta(x,\omega) = \sum_{j=0}^{3} \Theta^j_{nm}(x,\omega) * K_j(x). \tag{9}$$

The functional convolution for regular function has integral form:

$$\rho(x) * k(x) = \int_{R^3} \rho(y)k(x-y)dy_1dy_2dy_3.$$

Components of convolutions for vector are written in the same way. By virtue of the differentiation property of convolution, it is also a solution of Eq. (2).

Formulas (9) allow us to build various crystal lattices from harmonic elementary particles, if as a structural biquaternion we take *lattices*, which are various shifts of δ-function and others generalized functions.

Here is a simple example of a heterogeneous rectangular grid with variable step (h_l, h_m, h_n) and weights a^{lmn}:

$$K(x) = \sum_{l=0}^{L}\sum_{m=0}^{M}\sum_{n=0}^{N} a^{lmn}\delta(x_1-h_l)\delta(x_2-h_m)\delta(x_3-h_n).$$

It corresponds to such orthotropic crystalline ω-pulsar:

$$\Theta(x,\omega) = \sum_{l=0}^{L}\sum_{m=0}^{M}\sum_{n=0}^{N} a^{lmn}\theta^{\,0}(x_1-lh_l, x_2-mh_m, x_3-nh_n, \omega).$$

Formulas (7)–(9) allow us to build the most diverse monochromatic structures, such as bodies, tissues and threads (about their representation see in more detail in [4]). And their frequency superpositions are generally vast.

4 Elementary spherical harmonic pulsars. Bosons

Among the solutions of the Helmholtz equations (7), only one is spherically symmetric [8]:

$$\psi_{00}(x,\omega) = j_0(\omega r) = \frac{\sin\,\omega r}{\omega r}. \tag{10}$$

Here $j_0(\omega r)$ is spherical Bessel function, $r = \|x\| = \sqrt{x_1^2+x_2^2+x_3^2},\ e_x = x/r$.

The biamplitude of the corresponding pulsar has the form:

$$\begin{aligned}\Theta^0(x,\omega) &= (\omega-\nabla)\circ\psi_{00}(x,\omega) = \omega\tfrac{\sin\omega r}{\omega r} - \operatorname{grad}\tfrac{\sin\omega r}{\omega r}\\ &= \tfrac{\sin\omega r}{r} - \left(\tfrac{\cos\omega r}{r} - \tfrac{\sin\omega r}{\omega r^2}\right)e_x.\end{aligned} \tag{11}$$

Hence, taking into account the fact that $j_0^{'}(z) = -j_1(z)$, we obtain complex amplitudes and amplitudes of CC-field oscillations:

$$i\rho_1^0 + J_1^0 = \omega\left\{j_0(\omega r) + j_1(\omega r)e_x\right\},$$

$$\rho^0 = -i\omega j_0(\omega r), \quad J^0 = \omega j_1(\omega r) e_x,$$
$$\left|\rho^0\right| = \omega |j_0(\omega r)|, \quad \|J^0\| = \omega |j_1(\omega r)|.$$

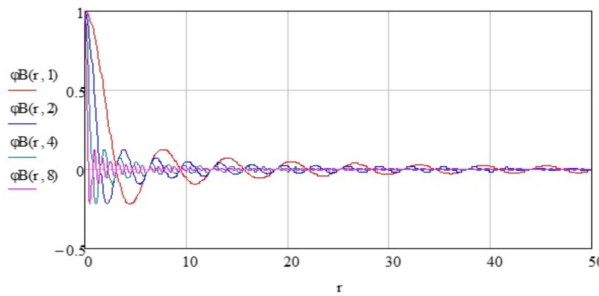

Figure 1 – Bosons scalar potential $\varphi_{00}(r,\omega) : \omega = 1, 2, 4, 8$

Calculating the energy-momentum biquaternion Θ^0:

$$\begin{gathered}\Xi^0(x,\omega) = W^0 + i\,P^0 = 0.5\Theta^0 e^{-i\omega\tau} \circ \left(\Theta^0\right)^* e^{i\omega\tau} \\ = 0,5\omega^2 (j_0^2(\omega r) + j_1^2(\omega r)),\end{gathered} \tag{12}$$

we get that the Pointing vector is zero:

$$P^0(\tau, x) \equiv 0.$$

From these relations and properties of spherical functions it follows that the CC-density decreases with increasing r as r^{-1}, and the vibrations energy decays even faster, like r^{-2}.

Consider the asymptotic behavior of these quantities by $r \to 0$, $\omega = \mathrm{const}$. In so far as

$$j_0(0) = 1, \quad j_1(0) = 0, \tag{13}$$

by $r \to 0$

$$\left|\rho^0\right| \sim \omega, \quad \left\|J^0\right\| \sim 0, \quad W \sim 0,5(\omega^2).$$

And so we have the following .

Properties of spherical harmonic pulsars. At spherical harmonic pulsars in the center at $x = 0$ the GM-charge density is equal to its oscillation frequency ω, the density of the EGM-current is zero, the energy density of the oscillations is $0.5\omega^2$, and Pointing vector is zero everywhere.

Based on these properties of the density of the mass charge, the spherical harmonic pulsars are heavy elementary particles, *bosons*.

In Figures 1–6 you can see their properties for different frequencies depending on r. Figures 1–4 show graphs of changes in the scalar potential and amplitudes of the densities of the EGM charge and the EGM current of bosons and their components in the radial coordinate on which they depend, with successive doubling of frequencies on each graph. With increasing frequency, the density maxima at zero increase, the nodal points thicken, and the decrease along the radius from the center of a boson increases.

Figures 5 and 6 show graphs of changes in the bosons energy density. There are no nodal points. The energy in the center of the boson increases sharply with increasing frequency and decreases faster near its center.

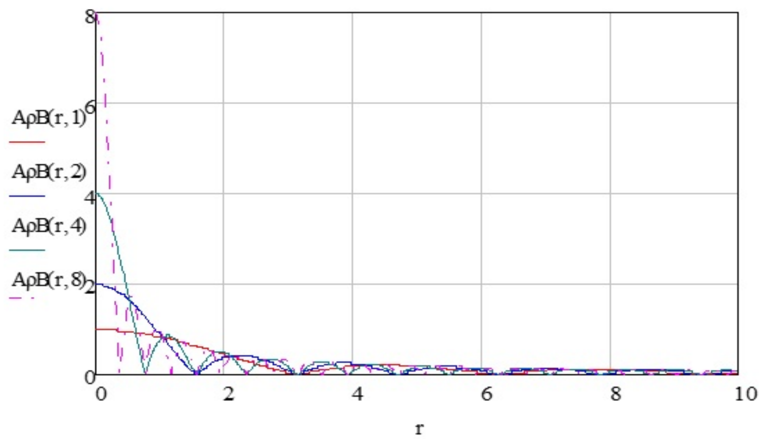

Figure 2 – Amplitude EGM-charge density of bosons $\varphi_{00}(r,\omega) = 1, 2, 4, 8$

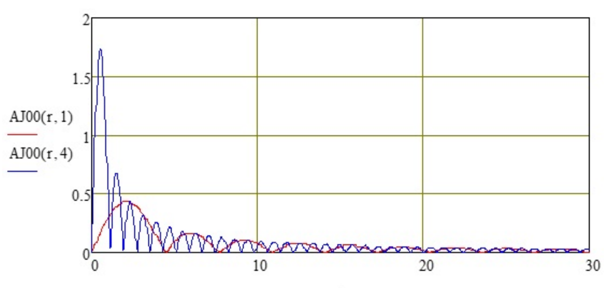

Figure 3 – Amplitude of EGM-current density of bosons: $\omega = 1, 2, 4, 8$

Nonspherical harmonic pulsars (6) for $n > 0$ have zero density at $x = 0$, because [8]

$$j_n(z) = \frac{z^n}{(2n+1)!!}((1 + o(z)) \quad \text{by} \quad z \to 0.$$

They are light elementary particles, *leptons.*

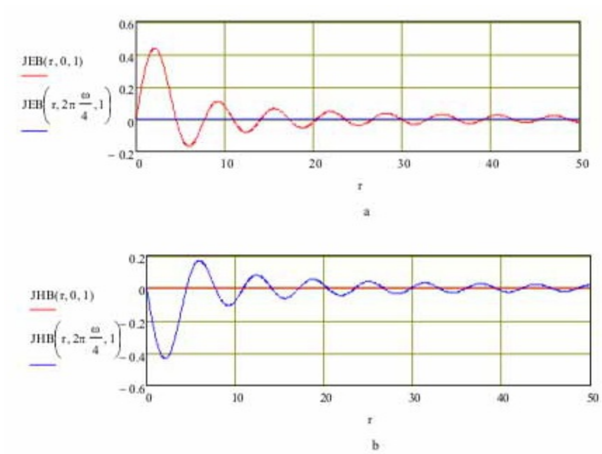

Figure 4 – Amplitude of electric (a) and gravimagnetic (b) density of bosons: $\omega = 1, 2, 4, 8$

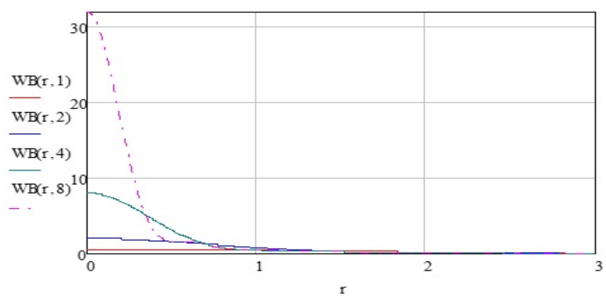

Figure 5 – Energy density of bosons: $\omega = 1, 2, 4, 8$

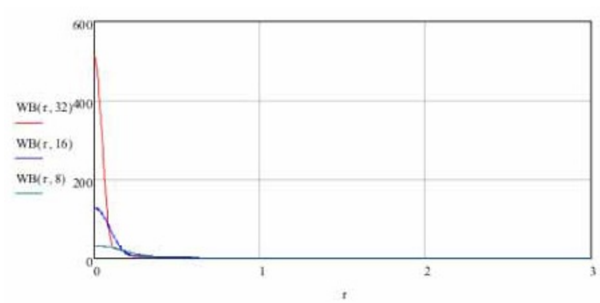

Figure 6 – Energy density of bosons: $\omega = 8, 16, 32$

5 Elementary spherical harmonic spinors. Leptons

At first let us consider a spinor polarized in the direction of X_1 axis:

$$\Theta_1^0(x,t) = i\rho_1^0 + J_1^0 = \Theta_1^0(x,\omega)e^{-i\omega\tau},$$

which biamplitude is

$$\Theta_1^0(x,\omega) = (\omega - \nabla) \circ j_0(\omega r) e_1$$

$$= \operatorname{div}(j_0(\omega r) e_1) + \omega j_0(\omega r) e_1 - \operatorname{rot}(j_0(\omega r) e_1) \tag{14}$$

$$= \omega\{-r_{,1}\, j_1(\omega r) + j_0(\omega r) e_1 + j_1(\omega r)(r_{,3}\, e_2 - r_{,2}\, e_3)\}, \quad r_{,j} = x_j/r.$$

From here follow

$$\rho_1^0 = i\omega j_0(\omega r) r_{,1}\,,$$

$$J_1^0 = \omega\{e_1 j_0(\omega r) + j_1(\omega r)(r_{,3}\, e_2 - r_{,2}\, e_3)\},$$

$$\left|\rho_1^0\right| = \left|\omega j_1(\omega r) r_{,1}\right|, \quad \|J\| = \omega\sqrt{j_0^2(\omega r) + j_1^2(\omega r)\left(r_{,2}^{\,2} + r_{,3}^{\,2}\right)}.$$

Energy-impuls Bq is equal to

$$\Xi_1^0(x,\omega) = 0.5\omega^2 (j_0^2(\omega r) + j_1^2(\omega r)). \tag{15}$$

By $r \to 0$:

$$\left|\rho_1^0\right| \sim 0, \;\; \|J\| \sim \omega,$$

$$W_1^0 \sim 0.5\omega^2 \quad P_1^0 \equiv 0.$$

Following (14), we obtain a biquaternion representation of a spherical spinor polarized along an arbitrary vector e, $|e| = 1$:

$$\Theta_e^0(x,\omega) = (\omega - \nabla) \circ j_0(\omega r) e$$

$$= \operatorname{div}(j_0(\omega r) e) + \omega j_0(\omega r) e - \operatorname{rot}(j_0(\omega r) e) = i\rho_e^0 + J_e^0,$$

where

$$\rho_e^0 = -\omega j_1(\omega r)(e, e_x), \quad J_e^0 = \omega\{j_0(\omega r) e + j_1(\omega r)[e, e_x]\}$$

with the same asymptotic properties. Here (e, e_x), $[e, e_x]$ are scalar and vector productions.

Properties of harmonic spherical spinors. In the center $(x = 0)$ of spherical harmonic spinors the EGM-charge density is zero, the norm of EGM-current density is equal to ω, energy density is equal to $\omega^2/2$, Pointing vector is zero.

Thus, spherical harmonic spinors, in terms of EGM charge density, belong to light elementary particles, that is, *leptons.*

6 Biquaternion representation of elementary hydrogen atom

So, we have shown that among monochromatic solutions of the charge-current free field equations (1), only harmonic spherical pulsars have nonzero density at their center. This suggests that spherical harmonic pulsars can be used to build biquaternionic model of elementary atoms. The simplest atom is hydrogen H. It is known that the spectrum of the hydrogen atom contains a set of frequencies. Denote by ω_0 minimum frequency in its spectrum.

We call an *elementary hydrogen atom* a spherical harmonic pulsar with ω_0 frequency. Its biquaternionic representation has the form:

$$H_0(\tau, x) = \omega_0\{j_0(\omega_0 r) + j_1(\omega_0 r)e_x\}e^{-i\omega_0\tau}. \tag{16}$$

The asymptotic properties of its density at the center of the atom are related to the oscillation frequency:

$$|\rho_{H_0}(x,\tau)| \sim \omega_0, \quad \|J_{H_0}(x,\tau)\| \sim \frac{2}{3}\omega_0^2 r, \quad W_{H_0}(x) \sim 0.5\omega_0^2, \quad r \to 0. \tag{17}$$

The nodes of this *standing wave* with mass density $|\rho_{H_0}|$ are spheres whose radii are determined by a simple trigonometric equation:

$$\sin\omega_0 r_k = 0 \quad \Rightarrow \quad r_k = \frac{\pi k}{\omega_0},\ k = 1, 2, \dots .$$

To determine the nodes of this standing wave by energy density W_{H_0} you need to find the zeros of a more complex equation:

$$\omega_0^2 r_k^2 + \omega_0 r_k \sin 2\omega_0 r_k - \sin^2\omega_0 r_k = 0, \tag{18}$$

where $r_k = \dfrac{z_k}{\omega_0}$, z_k are the roots of transcendental equation

$$f(z) = z^2 + z\sin 2z - \sin^2 z = 0.$$

However, this equation has no real roots (see Figs. 5–6).

In the original space-time

$$exp(-i\omega_0\tau) = exp(-i\omega_0 ct) = exp(-i\varpi_0 t), \quad \varpi_0 = \omega_0 c.$$

Using the representation of complex charges and currents through electric and gravimagnetic charges and currents (1), we obtain the following expressions for its elementary hydrogen atom, electric and gravimagnetic charges, electric and gravimagnetic currents:

$$\rho^E_{H_0}(t,x) = \tfrac{\sqrt{\varepsilon}}{r}\cos\varpi_0 t\sin\tfrac{\varpi_0\|x\|}{c},$$

$$\rho^H_{H_0}(t,x) = \tfrac{\sqrt{\mu}}{r}\sin\varpi_0 t\sin\tfrac{\varpi_0\|x\|}{c},$$

$$J^E_{H_0}(t,x) = \tfrac{1}{\sqrt{\mu}r}\cos\varpi_0 t\left(\cos\tfrac{\varpi_0\|x\|}{c} - \tfrac{c}{\varpi_0 r}\sin\tfrac{\varpi_0\|x\|}{c}\right)e_x,$$

$$J^H_{H_0}(t,x) = \tfrac{1}{\sqrt{\varepsilon}r}\sin\varpi_0 t\left(\cos\tfrac{\varpi_0\|x\|}{c} - \tfrac{c}{\varpi_0 r}\sin\tfrac{\varpi_0\|x\|}{c}\right)e_x.$$

Accordingly, in the initial space-time the biquaternion of the elementary hydrogen has the form:

$$H_0(t,x) = \frac{e^{-i\varpi_0 t}}{r}\left\{\sin(\varpi_0\, r/c) + \left(\cos(\varpi_0 r/c) - \frac{c\sin(\varpi_0 r/c)}{\varpi_0\, r}\right) e_x\right\}.$$

Consequently (17), by $r \to 0$

$$|\rho_{H_0}(x,\tau)| \sim \varpi_0/c, \quad \|J_{H_0}(x,\tau)\| \sim \frac{(\varpi_0/c)^2}{3} r, \quad W_0(x,\tau) \sim \frac{(\varpi_0/c)^2}{2}.$$

7 Periodic system of atoms. Simple gamut

So, in the biquaternionic representation, hydrogen atom is a spherical harmonic standing wave with fix frequency in EGM charges-currents field.

Since the main characteristic of a hydrogen atom is the oscillations frequency, which determines its mass, on its basis it is possible to build the periodic system for atoms according to the musical scale. Indeed, with an increase in the frequency of vibrations, the mass of atom increases.

The musical scale is an *octave* system with frequency doubling for each subsequent octave:

$$\omega_0, 2\omega_0, 4\omega_0, 8\omega_0, 16\omega_0, \ldots .$$

The ratio of vibration frequencies for atoms within the n-th octave:

$$2^{n-1}\omega_0, \ldots, 2^n\omega_0$$

is similar to the ratio of tone frequencies within a musical gamut. Number of notes in the musical scale depends on the type of musical system.

There are many musical scales. They depend on the national characteristics of the musical perception of peoples, the creation of national string musical instruments. Pythagoras gave the earliest mathematical description of the musical scale, studying the spectrum of the string. The most complete description of various musical systems was given by Shilov G.E. in brochure [9]. Here in Table 1 there are two musical scales, which can be taken as basis. In them the ratio of tone frequencies is a rational number. For such tones (*notes*) there is a general period of oscillations, which is determined by the smallest total multiple for the periods of their oscillations. It creates harmonious sound from different notes (*chord*).

For each of them there are substances in nature, that possess the above properties. Which one matches Mendeleev periodic system? This should be the subject of special research for specialists in physical chemistry, spectral properties of substances. Perhaps among them there is no such formation. But a similar musical scale should be. It should contain the frequencies of tones, indicated here. The numerical tones in an octave should increase with the increasing octave numbers, but all similar tones of the previous octave should be present in

Table 1 – Harmonic scale. Simple gamma

Clean structure							
Prima (*ut*)	Major secunda (*re*)	Major tertius (mi)	Quartus (*fa*)	Quintus (*sol*)	Major sextus (*la*)	Major septim (*si*)	Octabas ut
ω	$9\omega/8$	$5\omega/4$	$4\omega/3$	$3\ \omega\ /2$	$5\ \omega\ /3$	$15\ \omega\ /8$	2ω
Pentatonic structure							
Prima	Secunda	Tertius	-	Quintus	Sectus	-	Octabas
ω	$9\omega/8$	$5\omega/4$	-	$3\ \omega\ /2$	$5\ \omega\ /3$	-	2ω

it. This explains the repeatability of the chemical properties of the substances in the columns of Mendeleev periodic system, just as consonant and octave sounds and chords composed of them.

On this basis, atoms can be called *musical elementary particles* with the corresponding names. Hydrogen atom is a note *ut* of the *first natural octaves*.

Accordingly, the biquaternion of the k-th atom in n-th octave has the form:

$$\mathrm{Atom}^{n,k}(t,x) = \frac{e^{-iw_{n,k}t}}{r}\left\{\sin\left(\frac{w_{nk}}{c}r\right) - \left(\cos\left(\frac{w_{nk}}{c}r\right) - \frac{c\sin\left(w_{nk}r/c\right)}{w_{nk}r}\right)e_x\right\}$$

$$= \left(\frac{w_{nk}}{c}\right)\{j_0\left(\frac{w_{nk}}{c}r\right) + j_1\left(\frac{w_{nk}}{c}r\right)\}e^{-iw_{n,k}t}.$$

Here the atomic vibration frequency

$$w_{nk} = 2^n\gamma_k w_0,$$

where γ_k is the number from the k-th column in the table corresponding musical scale.

All above formulas for bosons are true for them with indicating the corresponding vibration frequency. The presented figures 1-6 describe the behavior of the first atoms in the first 6 lines (octaves) of the above table.

8 Conclusion

How many such natural octaves exist? Obviously, no less than the number of rows in the Mendeleev periodic table.

Note that now accepted in classical music twelve-tempered musical system with 12 notes inside octaves cannot be taken, since the ratio of the frequencies of consecutive tones in it is an irrational number ($2^{1/12}$) and there is no the total period of oscillations in the rows of periodic system of atoms.

Similar periodic systems can be constructed for elementary harmonic leptons (spinors and asymmetric pulsars), the addition of which to atoms with the same vibration frequency creates isotopes of these atoms. Moreover, the addition of spinors is apparently associated with the magnetization of a substance. You can build a lot of different isotopes with the same asymptotic density of the EGM charge. Which of them exist in nature? It is also a special question of experimental research.

We also note that this description of atoms is based on the construction of solutions of the free field equations of charge-currents. At the action of external EGM-fields the charges-currents are transformed. Their transformation is described by generalized Dirac equation (see [10]). In particular, static EGM-fields action shifts its vibration spectrum. It should be taken into account in the experimental justification of considered model.

Currently the most common and canonized representations of light and heavy elementary particles and atoms are constructed on the basis of solutions of equations of quantum field theory. The bibliography in this direction is more of half a century and very extensive. Here we use the names for heavy and light particles, adopted from this theory. However, the presented biquaternionic model is completely different, deterministic, based on the definition of real physical characteristics of elementary particles and atoms, not probabilistic.

This paper was presented this year as Keynote report at Int.conf. Quantum mechanics and Nuclear Engineering in Paris [12].

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Алексеева Л.А. БИКВАТЕРНИОНДЫ КӨРІНІСТЕГІ БОЗОНДАР МЕН ЛЕПТОНДАР. АТОМДАРДЫҢ ПЕРИОДТЫҚ ЖҮЙЕСІ ҚАРАПАЙЫМ ГАММА РЕТІНДЕ

Электр-гравимагниттік зарядтар мен тоқтардың еркін өрістері үшін бикватерионды толқындық теңдеудің элементар бөлшектерді монохроматты электр-гравимагнитті толқындар түрінде сипаттайтын дербес монохроматты шешімдері тұрғызылды. Осы битолқындық теңдеудің скалярлық потенциалдар (пульсарлар) және векторлық потенциалдар (спинорлар) тудыратын шешімдерінің екі класы зерттелді. Олардың асимптотикалық қасиеттері зерттелді, соның негізінде олар ауыр және жеңіл элементар бөлшектерге (бозондар мен лептондарға) жіктеледі. Бозондар – сфералық гармоникалық пульсарлар, олардың массалық тығыздығы тербелістерінің жиілігімен анықталатындығы көрсетілген. Бұл элементар бөлшектердің периодтық жүйесін классикалық музыкалық шкала негізінде тұрғызуға мүмкіндік береді. Атап айтқанда, сутегі атомының бикватернионды кейіптемес ұсынылған. Оның негізінде қарапайым атомдардың периодтық жүйесі қарапайым музыкалық шкала қағидаты бойынша тұрғызылды.

Кілттік сөздер. Бикватернион, элементар бөлшек, жиілік, тұрақты ЭГМ-толқын, пульсар, спинор, бозон, лептон, атом, сутегі, музыкалық гамма.

Алексеева Л.А. БОЗОНЫ И ЛЕПТОНЫ В БИКВАТЕРНИОННОМ ПРЕДСТАВЛЕНИИ. ПЕРИОДИЧЕСКАЯ СИСТЕМА АТОМОВ КАК ПРОСТАЯ ГАММА

Построены частные монохроматические решения бикватернионных волновых уравнений для свободных полей электро-гравимагнитных зарядов и токов, которые описывают элементарные частицы, как стоячие монохроматические электро-гравимагнитные волны. Исследованы два класса решений этого биволнового уравнения, порожденные скалярными потенциалами (пульсары) и векторными потенциалами (спиноры). Исследованы их асимптотические свойства, на основании которых они классифицируются на тяжелые и легкие элементарные частицы (бозоны и лептоны). Показано, что бозоны представляют собой сферические гармонические пульсары, массовая плотность которых определяется частотой их колебаний. Это позволяет строить периодические системы элементарных частиц на основе классической музыкальной шкалы. В частности, дано бикватернионное представление атома водорода. На его основе построена периодическая система элементарных атомов, построенная по принципу простой музыкальной гаммы.

Ключевые слова. Бикватернион, элементарная частица, частота, стоячая ЭГМ-волна, пульсар, спинор, бозон, лептон, водород, музыкальная гамма.

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Finite dimensional space chaotification

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Abstract. In this article, we show that a chaotic behavior can be found for sets in finite dimensional space. N-dimensional cube is fixed for discussion. Our approach is based on a recursive division of the set into an infinite number of elements which satisfy specific conditions. By using infinite sequences to the set of points, a chaotic map can be defined on the set. The map is shown to exhibit different types of chaos.

Keywords. Poincarè chaos, Li-Yorke chaos, Devaney chaos, finite dimensional space, cube, chaos generating map, diameter property, separation property.

1 Introduction and preliminaries

Chaos has become a very important concept that is deeply integrated into many, if not most, fields of science such as physics, biology, medicine, engineering, culture, and human activities [1], [2]. The chaotic behavior of some physical and biological properties was formerly attributed to random or stochastic processes or uncontrolled forces [3], [4]. The appearance of chaos in deterministic systems drew the borderline between (deterministic) chaos and stochastic noise. The idea is manifested in the chaotic behavior of simple dynamical systems. However, the of Kolmogorov-Martin-Lofwhich randomness theory still can provide a deeper understanding of the origins of deterministic chaos [1]. The fundamental theoretical framework of chaos was developed in last quarter of the twentieth century. During that period, different types and definitions of chaos were formulated. In general, chaos can be defined as aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions [5]. Devaney [6] and Li-Yorke [7] chaos are the most frequently used types, which are characterized by transitivity, sensitivity, frequent separation and proximality. Another common type occurs through period-doubling cascade which is a sort of route to chaos through local bifurcations [8]–[10]. In papers [11], [12], Poincarè chaos was introduced through the unpredictable point concepts. Further, it was developed to unpredictable functions and sequences.

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Whoever searches in this field can discern from the literature that there is a scientific conception that chaos is everywhere. Realizing such an ideation needs to develope our mathematical tools to conceptualize all manifestation of the phenomenon. Strictly speaking, we should develop simple chaotic mechanisms that has the ability to emulate complex behavior. Investigating the fundamental aspects of multi-dimensional chaotic states is necessary in this direction. Indeed, mathematical modeling of real-world problems shows that real life is very often a multi-dimensional chaos and even chaotic activities in our everyday lives are difficult to describe via low-dimensional systems [14].

In the present paper, we show how to establish multi-dimensional chaos for simple geometrical objects. We focus in the domain structure [13] to construct an invariant set under chaotic map. Our approach is characterized by the simultaneous roles of the domain and the map. In other words, the invariant set is defined as a collection of the domain points such that the map is used to describe the structure of the set by addressing its points utilizing infinite sequence indices. The idea of indexing is essential to prove chaos, however, it is different from that of the symbolic dynamics, since the action of the map is not just a shifting in the string space as much as a transforming of the domain points.

2 The chaotic map

For the sake of comprehension, let us first consider a line segment F. Divide F into 4 parts and denote them by $F_{i_1},\ i_1 = 1, 2, 3, 4$. Divide again each part F_{i_1} into 4 parts and denote them by $F_{i_1 i_2},\ i_2 = 1, 2, 3, 4$. Continue in this procedure such that, at the k-th step of division, each part $F_{i_1 i_2 ... i_{k-1}}$ is divided into 4 parts denoted as $F_{i_1 i_2 ... i_k},\ i_p = 1, 2, 3, 4,\ p = 1, 2, ..., k$. Considering the above simple construction, one can verify the following two properties.

Diameter property: The length of each part $F_{i_1 i_2 ... i_k}$ approaches zero as the number of steps k approaches infinity.

Separation property: There exists a positive number, ε_0, such that for any part $F_{i_1 i_2 ... i_k}$ at any step k, one can find another part $F_{j_1 j_2 ... j_k}$ so that they are separated from each other by a distance of not less than ε_0.

The diameter property implies that an infinite iteration of the procedure would produce infinitely many points that the line F consists of. The points can be represented by $F_{i_1 i_2 ... i_k ...},\ i_p = 1, 2, 3, 4,\ p = 1, 2, \ldots$. Thus, the set F can be defined as the collection of all such points, i.e.,

$$F = \left\{ F_{i_1 i_2 ... i_k ...} \mid i_p = 1, 2, 3, 4,\ p = 1, 2, ... \right\}. \tag{1}$$

Let us now introduce the map $\varphi : F \to F$ defined by

$$\varphi(F_{i_1 i_2 ... i_k ...}) = F_{i_2 i_3 ... i_k ...}, \tag{2}$$

such that for fixed sequence $i_1 i_2 ... i_k$, $\varphi(F_{i_1 i_2 ... i_k}) = F_{i_2 i_3 ... i_k}$ and $\varphi(F_{i_1}) = F$. We call each part $F_{i_2 i_3 ... i_k}$ *a subset of order* k, and the map φ the *chaos generating map*.

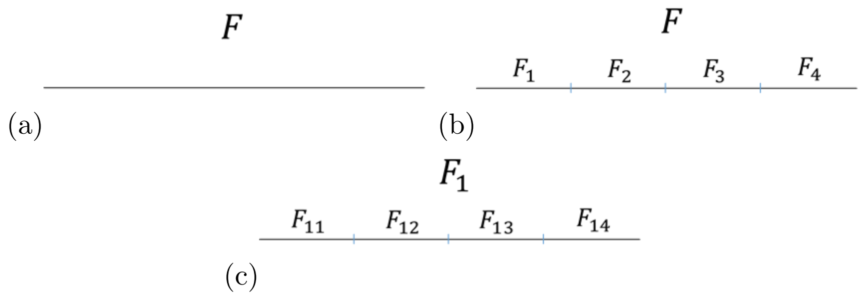

Figure 1 – The division procedure of the line F

Considering the results of Theorems 1, 2 and 3 in the Section 3, one can prove that the chaos generating map is chaotic in the sense of Poincarè, Devaney and Li-Yorke. Thus, we show that a line segment can be a domain for chaos. This simple case is frankly pointed out in [6] for the Devaney chaos of the lositic map $f(x) = 4x(1-x)$ on the interval $[0, 1]$. On the basis of the above construction, more general cases of the chaotic domain can be investigated.

Consider a square (cube) F. Similarly, we divide F into 16 squares (64 cubes) and denote them by $F_{i_1},\ i_1 = 1, 2, ..., 16$ ($F_{i_1},\ i_1 = 1, 2, ..., 64$). Again we divide each square (cube) F_{i_1} into 16 squares (64 cubes) and denote them by $F_{i_1 i_2},\ i_2 = 1, 2, ..., 16$ ($F_{i_1 i_2},\ i_2 = 1, 2, ..., 64$). We continue this procedure such that, at the k-th step of division, each part $F_{i_1 i_2 \ldots i_{k-1}}$ is divided into 16 squares (64 cubes) denoted as $F_{i_1 i_2 \ldots i_k},\ i_p = 1, 2, ..., 16,\ p = 1, 2, ..., k$ ($F_{i_1 i_2 \ldots i_k},\ i_p = 1, 2, ..., 64,\ p = 1, 2, ..., k$). Likewise, the set F can be defined by

$$F = \Big\{ F_{i_1 i_2 \ldots i_k \ldots} \mid i_p = 1, 2, ..., m,\ p = 1, 2, ... \Big\},$$

where m is 16 for the square and 64 for the cube. Again in this cases, the chaos generating map φ is defined for the sets and it can be verified that both the diameter and separation conditions are valid. Therefore Theorems 1, 2 and 3, in the Section 3, are applicable and the chaos generating map is chaotic in the sense of Poincarè, Devaney and Li-Yorke.

For a general case, consider n-dimensional cube F. The first step consists of dividing F into 4^n parts (n-dimensional cube) denoted as $F_{i_1},\ i_1 = 1, 2, ..., 4^n$. In the second step, each part F_{i_1} is again divided into 4^n parts denoted as $F_{i_1 i_2},\ i_2 = 1, 2, ..., 4^n$. Continue this procedure such that, at the k-th step of division, each part $F_{i_1 i_2 \ldots i_{k-1}}$ is divided into 4^n parts denoted as $F_{i_1 i_2 \ldots i_k},\ i_p = 1, 2, ..., 4^n,\ p = 1, 2, ..., k$. Similarly, the validity of the diameter and separation conditions could be substantiated. At the infinite iteration of this process, the cube F can be represented as the collection of all points, $F_{i_1 i_2 \ldots i_k \ldots}$, i.e.,

$$F = \big\{ F_{i_1 i_2 \ldots i_k \ldots} \mid i_p = 1, 2, ..., 4^n,\ p = 1, 2, ... \big\}. \tag{3}$$

Using Theorems 1, 2 and 3, in the Section 3, it can be shown that the chaos generating map defined in (2) is chaotic in the sense of Devaney, Li-Yorke and Poincarè.

3 Chaos for the map φ

In the following theorem, we prove that the map φ defined in (2) possesses three ingredients of Devaney chaos, namely density of periodic points, transitivity and sensitivity [6]. The point $F_{i_1 i_2 i_3 \ldots} \in F$ is periodic with the period n if its index consists of endless repetitions of a block of n terms.

Theorem 1. *If the diameter and separation properties hold, then the similarity map is chaotic in the sense of Devaney.*

Proof. Fix a member $F_{i_1 i_2 \ldots i_n \ldots}$ of F and a positive number ε. Find a natural number k such that $\operatorname{diam}(F_{i_1 i_2 \ldots i_k}) < \varepsilon$ and choose k-periodic element $F_{i_1 i_2 \ldots i_k i_1 i_2 \ldots i_k \ldots}$ of $F_{i_1 i_2 \ldots i_k}$. It is clear that the periodic point is an ε-approximation for the considered member. The density of periodic points is thus proved.

Next, utilizing the diameter property, the transitivity will be proved if we show the existence of element $F_{i_1 i_2 \ldots i_n \ldots}$ of F such that for any subset $F_{i_1 i_2 \ldots i_k}$ there exists a sufficiently large integer p so that $\varphi^p(F_{i_1 i_2 \ldots i_n \ldots}) \in F_{i_1 i_2 \ldots i_k}$. This is true since we can construct the sequence $i_1 i_2 \ldots i_n \ldots$ such that it contains all sequences of the type $i_1 i_2 \ldots i_k$ as blocks.

For sensitivity, fix a point $F_{i_1 i_2 \ldots} \in F$ and an arbitrary positive number ε. Due to the diameter property, there exist an integer k and element $F_{i_1 i_2 \ldots i_k j_{k+1} j_{k+2} \ldots} \neq F_{i_1 i_2 \ldots i_k i_{k+1} i_{k+2} \ldots}$ such that $d(F_{i_1 i_2 \ldots i_k i_{k+1} \ldots}, F_{i_1 i_2 \ldots i_k j_{k+1} j_{k+2} \ldots}) < \varepsilon$. We precise $j_{k+1}, j_{k+2}, \ldots$ such that $d(F_{i_{k+1} i_{k+2} \ldots i_{k+n}}, F_{j_{k+1} j_{k+2} \ldots j_{k+n}}) > \varepsilon_0$, by the separation property. This proves the sensitivity.

For Poincarè chaos, Poisson stable motion is utilized to distinguish the chaotic behavior instead of the periodic motions in Devaney and Li-Yorke types. The existence of infinitely many unpredictable Poisson stable trajectories that lie in a compact set meet all requirements of chaos. Based on this, chaos can be appeared in the dynamics on the quasi-minimal set which is a closure of a Poisson stable trajectory. Therefore, the Poincarè chaos is referred to as the dynamics on the quasi-minimal set of trajectory initiated from unpredictable point. For more details we refer the reader to [11], [12].

Next theorem shows that the Poincarè chaos is valid for the similarity dynamics.

Theorem 2. *If the diameter and separation properties are valid, then the similarity map possesses Poincarè chaos.*

The proof of the last theorem is based on the verification of lemma 3.1 in [12] adopted to the similarity map.

In addition to the Devaney and Poincarè chaos, it can be shown that the Li-Yorke chaos also takes place in the dynamics of the map φ. The proof of the following theorem is similar to that of Theorem 6.35 in [15] for the shift map defined on the space of symbolic sequences.

Theorem 3. *The similarity map is Li-Yorke chaotic if the diameter and separation properties hold.*

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Ахмет М., Аль-Эджали Э.М. АҚЫРЛЫ ӨЛШЕМДІ КЕҢІСТІКТІ ХАОТИФИКАЦИЯЛАУ

Бұл мақалада ақырлы өлшемді кеңістіктегі жиындар үшін хаосты қозғалысты қалай құруға болатындығын көрсетеміз. Зерттеу объектісі ретінде N өлшемді куб қарастырылады. Біздің тәсілдемеміз жиынды белгілі бір шарттарды қанағаттандыратын элементтердің шексіз санына рекурсивті бөлшектеуге негізделген. Осы жиында, берілген нүктелерді индекстеу үшін шексіз тізбектерді пайдалану арқылы, хаостық бейнелеу анықталады. Бұл бейнелеу Пуанкаре, Деваню және Ли-Йорк хаостары сияқты хаостың бірнеше түрлеріне ие болады.

Кілттік сөздер. Пуанкаре хаосы, Ли-Йорк хаосы, Деваню хаосы, ақырлы өлшемді кеңістік, куб, хаосты тудыратын бейнелеу, диаметрдің қасиеті, бөлшектеу қасиеті.

Ахмет М., Аль-Эджали Э.М. ХАОТИФИКАЦИЯ КОНЕЧНОМЕРНОГО ПРОСТРАНСТВА

В этой статье мы показываем, как построить хаотическое движение для множеств в конечномерном пространстве. В качестве объекта исследования рассматривается куб размерности N. Наш подход основан на рекурсивном делении множества на бесконечное число элементов, которые удовлетворяют определенным условиям. Используя бесконечные последовательности для индексации заданных точек, на этом множестве определяется хаотическое отображение. Это отображение имеет разные типы хаоса: Пуанкаре, Ли-Йорка и Деваню.

Ключевые слова. Хаос Пуанкаре, хаос Ли-Йорка, хаос Деваню, конечномерное пространство, куб, отображение порождающее хаос, свойство диаметра, свойство разделения.

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Boundary conditions of volume hyperbolic potential in a domain with curvilinear boundary

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Abstract. A one-dimensional volume hyperbolic potential in a domain with curvilinear boundaries is studied. As a kernel of the hyperbolic potential the fundamental solution of the Cauchy problem is chosen. It is well-known that in this case the volume hyperbolic potential satisfies homogeneous initial conditions. The boundary conditions to which the hyperbolic potential satisfies at lateral boundaries of the domain are constructed. It is shown that the formulated initial-boundary value problem has the unique classical solution.

Keywords. Hyperbolic equation, initial-boundary value problem, boundary condition, hyperbolic potential.

1 Introduction and statement of the problem

In [1], the Riemann-Green method is used to give general solutions of Cauchy problems for a hyperbolic equation in an arbitrary domain. Riemann first engaged in such tasks in the twentieth century [2]. After Riemann, Darboux made a great progress in this area. In these papers, the foundations were laid for representation of solutions of hyperbolic equations in integral form.

The volume elliptic potential is widely used in solving classical problems of Dirichlet, Neumann and other boundary value problems in domains of arbitrary form. But, at the same time, the boundary conditions and the spectral problems of the volume potential have not been researched till the recent time. That is, despite the deep research of the general theory of the volume potential, till the recent time the Newton volume potential

$$u_{NP}(x) = \int\limits_{\Omega} \varepsilon(x-y) f(y) dy$$

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has not been considered as an independent operator being a solution of some boundary value problem. The scientists as Engquist B. and Majda A. [3], Givoli D. [4]–[6], Li J.R., Greengard L. [7], Hagstrom T. [8], Tsynkov S.V. [9], Wu X. and Zhang J. [10] used the foundations of the theory of the boundary value problems for different kinds of the volume potentials for solving various problems of the mathematical physics and numerical calculations.

In the paper of T. Sh. Kal'menov and D. Suragan the boundary conditions of the volume potential u_{NP} for the case of multidimensional Laplace operator were built for the first time [11]. New non-local boundary conditions, which uniquely define the Newton volume potential, have the form

$$\frac{u(x)}{2} - \int\limits_{\partial\Omega} \left(\frac{\partial \varepsilon(x-y)}{\partial n_y} u(y) - \varepsilon(x-y) \frac{\partial u(y)}{\partial n_y} \right) dS_y = 0, \quad x \in \partial\Omega.$$

Despite the complexity of these boundary conditions, they were quite convenient to use. Using these boundary conditions, all eigenvalues and eigenfunctions were constructed for the volume potential in a two-dimensional circle and a three-dimensional ball considered in [12].

The trace of the Newton potential on a boundary surface appeared in Kac's work [13], where he called it as "the principle of not feeling the boundary" and he made the subsequent spectral analysis. This was further expanded in Kac's book [14] (see also Saito [15]) with several further applications to the spectral theory and the asymptotics of the Weyl eigenvalue counting function. For the general background details on potential theory of the time-fractional diffusion equation we refer to [16]–[18].

It is shown in [19] that self-adjoint differential operators are generated by boundary conditions. Further the boundary conditions were constructed for the non-self-adjoint operators. In [20] the initial-boundary value problem for the wave equation in the domain with rectilinear boundaries is considered. In [21] a generalized heat potential for the degenerate (heat) diffusion equation, which satisfies the initial condition with respect to the time variable is considered. In this work the boundary condition for this potential is found. The nonlocal initial boundary value problem for the time-fractional diffusion equation for the Kohn Laplacian and its powers on the Heisenberg group have been recently investigated by Ruzhanksy and Suragan in [22] as well as in [23] for general stratified Lie groups.

The study of the well-posedness of non-local problems for hyperbolic equations with integral conditions is recently an urgent problem. One of the first works in this direction was an article by L.S. Pulkina [24] in which the existence and uniqueness of generalized solution for a second-order hyperbolic equation with integral conditions in a rectangle are proved. In [25] a boundary-value problem for one-dimensional hyperbolic equation with nonlocal initial data in integral form was considered. In a recent paper [26], the problem was considered for a hyperbolic equation with standard initial data and a non-local integral condition of the second kind, which degenerates and turns into the first kind.

In this paper we investigate the problem of constructing boundary conditions for a one-dimensional hyperbolic volume potential in a domain with curvilinear boundary. We show that the solution of the boundary value problem is uniquely determined by the volume potential.

2 Formulation of the problem

Let $Q \subset \mathbb{R}^2$ be a finite domain bounded at the sides by the curves $x = \alpha_1(t)$ and $x = \beta_1(t)$, and bounded above and below by the segments $t = 0,\ 0 < x < 1$ and $t = T,\ x_0 < x < x_1$. Here $T > 0$, $\alpha_1(0) = 0$, $\beta_1(0) = 1$, $\alpha_1(T) = x_0$, $\beta_1(T) = x_1$, $\alpha_1(t) < \beta_1(t)$ (see Fig. 1).

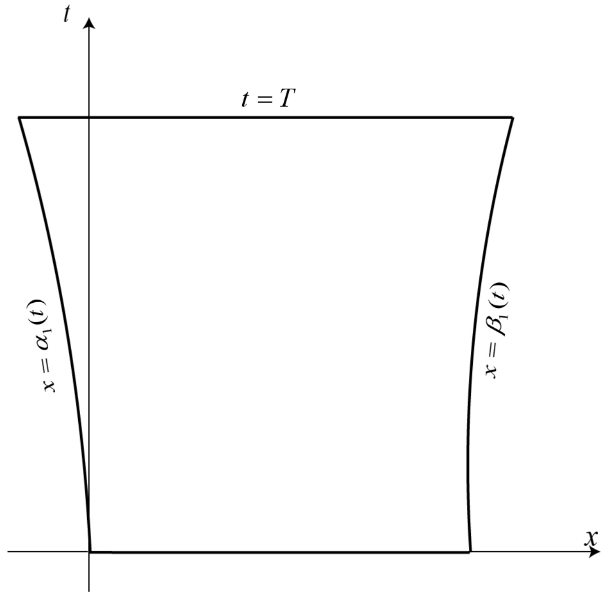

Figure 1 – The domain Q

We consider the following hyperbolic equation

$$Lu \equiv \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + a_1(x,t)\frac{\partial u(x,t)}{\partial x}$$

$$+b_1(x,t)\frac{\partial u(x,t)}{\partial x} + c_1(x,t)u(x,t) = f_1(x,t), (x,t) \in Q, \tag{1}$$

with the initial conditions

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0,\ 0 \le x \le 1, \tag{2}$$

where $a_1, b_1, c_1 \in C^1\left(\overline{Q}\right)$. Additionally, assume that

$$|\alpha_1'(t)| < 1, \quad |\beta_1'(t)| < 1. \tag{3}$$

It is known that for $T > 1/2$ the solution of the hyperbolic equation (1) in Q is reconstructed under the initial conditions (2) not uniquely. For the uniqueness it is necessary to use boundary conditions. We set the task to construct boundary conditions under which (together with the initial conditions) the solution of equation (1) in Q will be uniquely defined in the form of a volume hyperbolic potential (see Eq. (4)). In the case, when $\alpha(t) \equiv 0$, $\beta(t) \equiv 1$ and $a_1, b_1, c_1 = 0$ this problem was considered in [20]. The case where the domain Q remains the same and $a_1, b_1, c_1 = 0$, was considered in our paper [27].

Let $Q_{x,t}$ be a part of Q: $Q_{x,t} = \{(x_1, t_1) \in Q : |x - x_1| < t - t_1\}$. In Q we have the volume hyperbolic potential

$$u(x,t) = -\iint\limits_{\Omega_{x,t}} R_1(x,t;x_1,t_1) f_1(x_1,t_1) dx_1 dt_1, \tag{4}$$

where $R_1(x,t;x_1,t_1)$ is the Riemann-Green function [1], which satisfies the conjugate homogeneous equation

$$L_1^* R_1 \equiv \frac{\partial^2}{\partial x_1 \partial t_1} R(x,t;x_1,t_1) - \frac{\partial}{\partial x_1}\left(a_1(x_1,t_1) R(x,t;x_1,t_1)\right)$$

$$-\frac{\partial}{\partial t_1}\left(R(x,t;x_1,t_1) b_1(x_1,t_1)\right) + c_1(x_1,t_1) R(x,t;x_1,t_1) = 0,\ (x,t) \in \Omega,$$

and the following characteristic equations

$$\frac{\partial R_1(x,t;x_1,t_1)}{\partial x_1} - b_1(x_1,t_1) R_1(x,t;x_1,t_1) = 0, \text{ when } \ x_1 = x,$$

$$\frac{\partial R_1(x,t;x_1,t_1)}{\partial t_1} - a_1(x_1,t_1) R_1(x,t;x_1,t_1) = 0, \text{ when } \ t_1 = t,$$

$$R_1(x,t;x,t) = 1.$$

In the characteristic coordinates $\xi = x + t, \eta = x - t$ equation (1) has the form

$$\frac{\partial^2 u(\xi,\eta)}{\partial \xi \partial \eta} + a \frac{\partial u(\xi,\eta)}{\partial \xi} + b \frac{\partial u(\xi,\eta)}{\partial \eta} + c u(\xi,\eta) = f(\xi,\eta), (\xi,\eta) \in \Omega, \tag{5}$$

and initial conditions (2) has the form

$$u = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} = 0, \text{at } \xi = \eta, 0 \le \eta \le 1, \tag{6}$$

where $a(\xi,\eta), b(\xi,\eta), c(\xi,\eta) \in C^1\left(\overline{\Omega}\right)$ and

$$a(\xi,\eta) = \frac{1}{4}\left(a_1\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) + b_1\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)\right),$$

$$b(\xi,\eta)=\frac{1}{4}\left(a_1\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)-b_1\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)\right),$$

$$c(\xi,\eta)=\frac{1}{4}c_1\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right), f(\xi,\eta)=\frac{1}{4}f_1\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right).$$

Here $\Omega\subset\mathbb{R}^2$ is a domain bounded at the sides by the curves $\xi=\alpha(\eta)$ and $\xi=\beta(\eta)$, and bounded from above and below by the segment $\xi-\eta=0$ and $\xi-\eta=2T$. Here $\alpha(0)=0$, $\beta(1)=1$, $\alpha(\eta)<\beta(\eta)$. From (3) we have

$$-\infty<\alpha'(\eta)<0, \tag{7}$$

$$-\infty<\beta'(\eta)<0. \tag{8}$$

3 Construction of boundary conditions

By $\Omega_{\xi,\eta}$ we denote a part of Ω: $\Omega_{\xi,\eta}=\{(\xi_1,\eta_1)\in\Omega:\xi_1<\xi,\eta_1>\eta\}$. Then the volume potential (4) can be written in the form

$$u(\xi,\eta)=-\iint\limits_{\Omega_{\xi,\eta}}R(\xi,\eta;\xi_1\eta_1)f(\xi_1,\eta_1)d\xi_1d\eta_1, \tag{9}$$

where $R(\xi,\eta;\xi_1,\eta_1)$ is the Riemann-Green function [1], which satisfies the following equations

$$L^*R\equiv\frac{\partial^2}{\partial\xi_1\partial\eta_1}R(\xi,\eta;\xi_1,\eta_1)-\frac{\partial}{\partial\xi_1}(a(\xi_1,\eta_1)R(\xi,\eta;\xi_1,\eta_1))$$

$$-\frac{\partial}{\partial\eta_1}(b(\xi_1,\eta_1)R(\xi,\eta;\xi_1,\eta_1))+c(\xi_1,\eta_1)R(\xi,\eta;\xi_1,\eta_1)=0,\ (\xi,\eta)\in\Omega, \tag{10}$$

$$\frac{\partial R(\xi,\eta;\xi_1,\eta_1)}{\partial\xi_1}-b(\xi_1,\eta_1)R(\xi,\eta;\xi_1,\eta_1)=0,\ \text{when}\ \ \xi_1=\xi, \tag{11}$$

$$\frac{\partial R(\xi,\eta;\xi_1,\eta_1)}{\partial\eta_1}-a(\xi_1,\eta_1)R(\xi,\eta;\xi_1,\eta_1)=0,\ \text{when}\ \ \eta_1=\eta, \tag{12}$$

$$R(\xi,\eta;\xi,\eta)=1. \tag{13}$$

Evidently, for any $f(\xi,\eta)\in C^1(\overline{\Omega})$, the volume potential (9) gives a classical solution of the inhomogeneous hyperbolic equation (5) from the class $u(\xi,\eta)\in C^2(\overline{\Omega})$. Our task is to construct homogeneous boundary conditions on the lateral boundaries $\xi=\alpha(\eta)$ and $\xi=\beta(\eta)$, which the volume potential (9) satisfies for all $f(\xi,\eta)$.

We consider separately various cases of placing $\Omega_{\xi,\eta}$ inside Ω.

Case I

Firstly, we consider a case, when $0 < \eta < \xi < 1$, $(\xi, \eta) \in \Omega$. In this case $\Omega_{\xi,\eta}$ is a triangle that is bounded from above by $\xi_1 = \xi$, is bounded from below by $\eta_1 = \eta$, and is bounded from the right by $\xi_1 = \eta_1$. In this case the domain $\Omega_{\xi,\eta}$ nowhere touches the lateral boundaries Ω (see Figure 2). Therefore there is no need to construct the boundary conditions for the

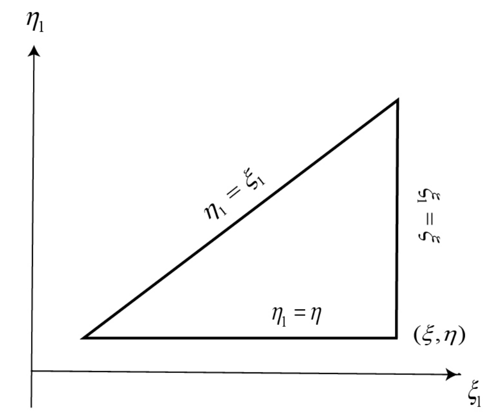

Figure 2 – The domain $\Omega_{\xi,\eta}$ in the case I

hyperbolic volume potential. By the direct calculation it is easy to see that the volume potential (9) satisfies the homogeneous initial conditions (6).

Case II

The case, when $\xi < 1, \eta < 0, (\xi, \eta) \in \Omega$. Let $\xi = \alpha(\eta)$, then $\Omega_{\alpha(\eta),\eta} = \{(\alpha(\eta), \eta) \in \Omega :$ $\eta_1 < \xi_1 < \alpha(\eta)$, at $\eta_1 > 0; \alpha(\eta_1) < \xi_1 < \alpha(\eta)$, at $\eta_1 < 0\}$ is a curvilinear triangle, bounded by $\xi_1 = \alpha(\eta)$ on the right bounded by $\xi_1 = \alpha(\eta_1)$ from below, and bounded by $\xi_1 = \eta_1$ from above (see Fig. 3).

Hereinafter we will use the Green's theorem in a plane [28]: *Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be a domain bounded by C. If L and M are functions of (ξ_1, η_1) defined on an open domain containing D and have continuous partial derivatives there, then*

$$\oint\limits_C (L\, d\xi_1 + M\, d\eta_1) = \iint\limits_D \left(\frac{\partial M}{\partial \xi_1} - \frac{\partial L}{\partial \eta_1} \right) d\xi_1\, d\eta_1,$$

where the left-hand side is a line integral and the right-hand side is a surface integral, and the path of integration along C is anticlockwise.

Applying the Green's theorem in a plane, from (9) we get the following chain of equalities:

$$u\left(\alpha(\eta), \eta\right) = - \iint\limits_{\Omega_{\alpha(\eta),\eta}} R(\alpha(\eta), \eta; \xi_1 \eta_1) f(\xi_1, \eta_1) d\xi_1 d\eta_1$$

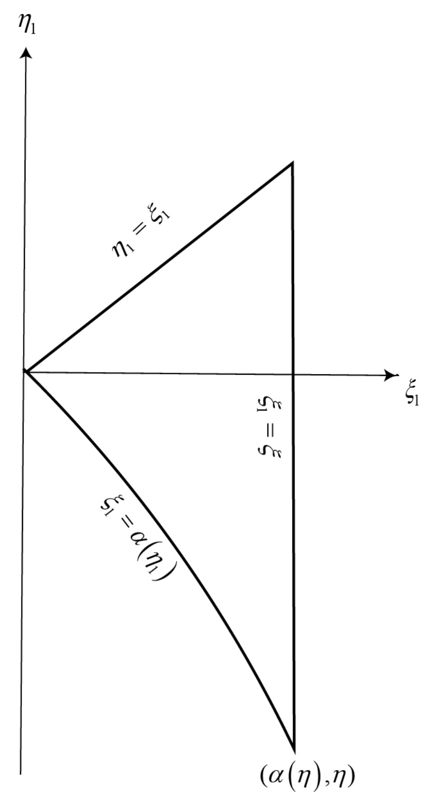

Figure 3 – The domain $\Omega_{\xi,\eta}$ in the case II

$$-\iint\limits_{\Omega_{\alpha(\eta),\eta}} (RLu - uL^*R)\, d\xi_1 d\eta_1$$

$$= -\oint\limits_{\partial\Omega_{\alpha(\eta),\eta}} \left(-bRu + \frac{1}{2}\frac{\partial R}{\partial \xi_1}u - \frac{1}{2}R\frac{\partial u}{\partial \xi_1}\right) d\xi_1 + \left(aRu - \frac{1}{2}\frac{\partial R}{\partial \eta_1}u + \frac{1}{2}R\frac{\partial u}{\partial \eta_1}\right) d\eta_1.$$

Calculating the obtained line integrals, taking into account the initial conditions (6) and conditions (10)–(13), we have

$$I_\alpha u \equiv \int\limits_0^\eta \frac{\partial u(\alpha(\eta_1),\eta_1)}{\partial \eta_1} R(\alpha(\eta),\eta;\alpha(\eta_1),\eta_1)d\eta_1$$

$$+\int\limits_0^\eta u(\alpha(\eta_1),\eta_1)R(\alpha(\eta),\eta;\alpha(\eta_1),\eta_1)\left(a(\alpha(\eta_1),\eta_1) - b(\alpha(\eta_1),\eta_1)\alpha^{'}(\eta_1)\right)d\eta_1$$

$$-\int_0^{\eta} u(\alpha(\eta_1),\eta_1)\frac{\partial R(\alpha(\eta),\eta;\alpha(\eta_1),\eta_1)}{\partial \xi_1}d\eta_1 = 0. \quad (14)$$

Note that (14) is the condition on the boundary $\xi = \alpha(\eta)$, connecting the values of the function u and its derivative on this boundary.

Case III

Consider a case, when $0 < \eta, 1 < \xi, (\xi,\eta) \in \Omega$. Let $\xi = \beta(\eta)$, then $\Omega_{\beta(\eta),\eta} = \{(\beta(\eta),\eta) \in \Omega : \eta_1 < \xi_1 < \beta(\eta_1) \text{ and } \eta_1 > \eta\}$ is a curvilinear triangle, which is bounded from the right by $\xi_1 = \beta(\eta_1)$, is bounded from below by $\eta_1 = \eta$, and is bounded from the left by $\xi_1 = \eta_1$ (see Figure 4).

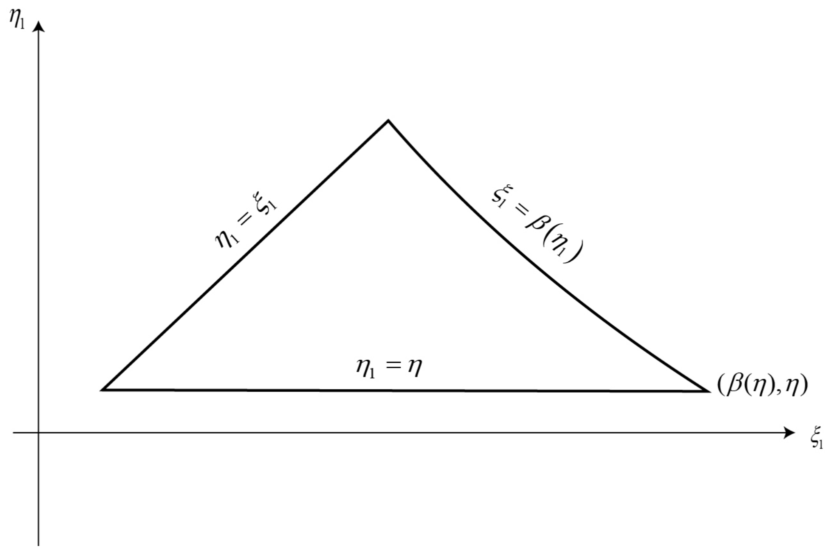

Figure 4 – The domain $\Omega_{\xi,\eta}$ in the case III

Analogously, as in Case II, applying the Green's theorem, from (9) we have the boundary condition

$$I_\beta u \equiv \int_1^{\eta} \frac{\partial u(\beta(\eta_1),\eta_1)}{\partial \xi_1} R(\beta(\eta),\eta;\beta(\eta_1),\eta_1)\beta'(\eta_1)d\eta_1$$

$$-\int_1^{\eta} u(\beta(\eta_1),\eta_1)R(\beta(\eta),\eta;\beta(\eta_1),\eta_1)\left(a(\beta(\eta_1),\eta_1) - b(\beta(\eta_1),\eta_1)\beta'(\eta_1)\right)d\eta_1$$

$$+\int_1^{\eta} u(\beta(\eta_1),\eta_1)\frac{\partial R(\beta(\eta),\eta;\beta(\eta_1),\eta_1)}{\partial \eta_1}d\eta_1 = 0. \quad (15)$$

Note that (15) is the condition on the boundary $\xi = \beta(\eta)$, connecting the values of the function u and its derivative on this boundary.

Case IV

Consider a domain, when $\eta < 0$ and $1 < \xi$. In this case the domain $\Omega_{\xi,\eta}$ is a curvilinear pentagon, bounded by $\xi_1 = \beta(\eta_1)$ from above and $\eta_1 = \xi_1$, bounded by $\xi_1 = \alpha(\eta_1)$ and $\eta_1 = \eta$ from below, bounded by $\xi_1 = \xi$ on the right (see Fig. 5).

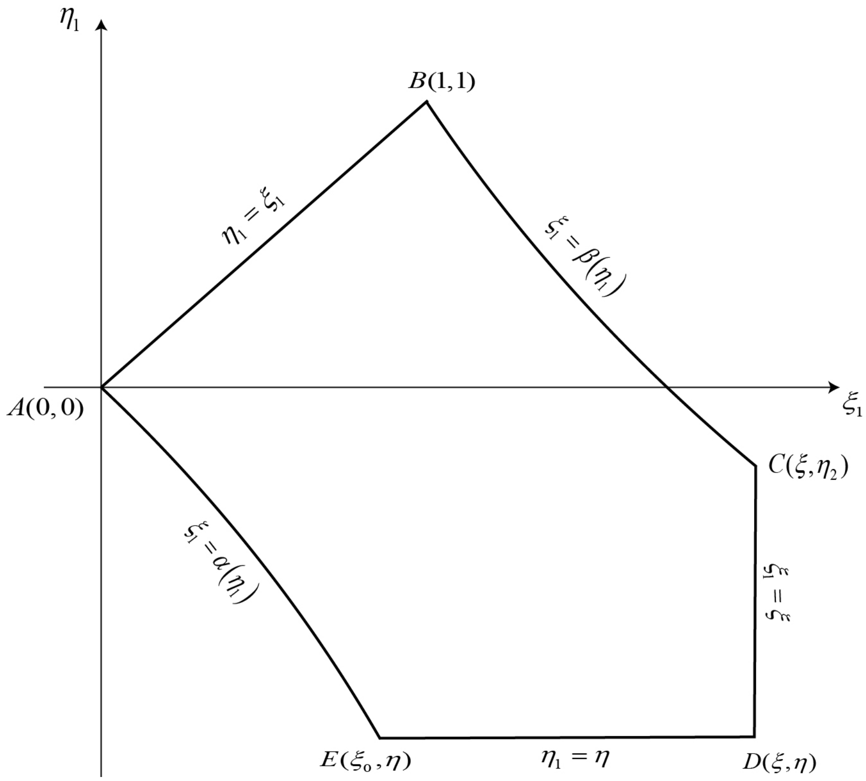

Figure 5 – The domain $\Omega_{\xi,\eta}$ in the case IV

We apply the Green's theorem in a plane for the volume hyperbolic potential

$$u(\xi,\eta) = -\iint\limits_{\Omega_{\xi,\eta}} R(\xi,\eta;\xi_1\eta_1) f(\xi_1,\eta_1) d\xi_1 d\eta_1$$

$$-\iint\limits_{\Omega_{\xi,\eta}} (RLu - uL^*R)\, d\xi_1 d\eta_1$$

$$= -\oint\limits_{\partial\Omega_{\xi,\eta}} \left(-bRu + \frac{1}{2}\frac{\partial R}{\partial \xi_1}u - \frac{1}{2}R\frac{\partial u}{\partial \xi_1}\right)d\xi_1 + \left(aRu - \frac{1}{2}\frac{\partial R}{\partial \eta_1}u + \frac{1}{2}R\frac{\partial u}{\partial \eta_1}\right)d\eta_1$$

$$= \int\limits_{AE} \left(-bRu + \frac{1}{2}\frac{\partial R}{\partial \xi_1}u - \frac{1}{2}R\frac{\partial u}{\partial \xi_1}\right)d\xi_1 + \left(aRu - \frac{1}{2}\frac{\partial R}{\partial \eta_1}u + \frac{1}{2}R\frac{\partial u}{\partial \eta_1}\right)d\eta_1$$

$$+ \int\limits_{ED} \left(-bRu + \frac{1}{2}\frac{\partial R}{\partial \xi_1}u - \frac{1}{2}R\frac{\partial u}{\partial \xi_1}\right)d\xi_1 + \left(aRu - \frac{1}{2}\frac{\partial R}{\partial \eta_1}u + \frac{1}{2}R\frac{\partial u}{\partial \eta_1}\right)d\eta_1$$

$$+ \int\limits_{DC} \left(-bRu + \frac{1}{2}\frac{\partial R}{\partial \xi_1}u - \frac{1}{2}R\frac{\partial u}{\partial \xi_1}\right)d\xi_1 + \left(aRu - \frac{1}{2}\frac{\partial R}{\partial \eta_1}u + \frac{1}{2}R\frac{\partial u}{\partial \eta_1}\right)d\eta_1$$

$$+ \int\limits_{CB} \left(-bRu + \frac{1}{2}\frac{\partial R}{\partial \xi_1}u - \frac{1}{2}R\frac{\partial u}{\partial \xi_1}\right)d\xi_1 + \left(aRu - \frac{1}{2}\frac{\partial R}{\partial \eta_1}u + \frac{1}{2}R\frac{\partial u}{\partial \eta_1}\right)d\eta_1$$

$$+ \int\limits_{BA} \left(-bRu + \frac{1}{2}\frac{\partial R}{\partial \xi_1}u - \frac{1}{2}R\frac{\partial u}{\partial \xi_1}\right)d\xi_1 + \left(aRu - \frac{1}{2}\frac{\partial R}{\partial \eta_1}u + \frac{1}{2}R\frac{\partial u}{\partial \eta_1}\right)d\eta_1.$$

Then we obtain the identity

$$\int\limits_1^{\eta_2} \left(\frac{1}{2}\frac{\partial R(\xi,\eta;\beta(\eta_1),\eta_1)}{\partial \xi_1}u(\beta(\eta_1),\eta_1) - \frac{1}{2}\frac{\partial u(\beta(\eta_1),\eta_1)}{\partial \xi_1}R(\xi,\eta;\beta(\eta_1),\eta_1)\right)\beta^{'}(\eta_1)d\eta_1$$

$$-\int\limits_1^{\eta_2} b(\beta(\eta_1),\eta_1)R(\xi,\eta;\beta(\eta_1),\eta_1)u(\beta(\eta_1),\eta_1)\beta^{'}(\eta_1)d\eta_1$$

$$+\int\limits_1^{\eta_2} \left(\frac{1}{2}\frac{\partial u(\beta(\eta_1),\eta_1)}{\partial \eta_1}R(\xi,\eta;\beta(\eta_1),\eta_1) - \frac{1}{2}\frac{\partial R(\xi,\eta;\beta(\eta_1),\eta_1)}{\partial \eta_1}u(\beta(\eta_1),\eta_1)\right)d\eta_1$$

$$+\int\limits_1^{\eta_2} a(\beta(\eta_1),\eta_1)R(\xi,\eta;\beta(\eta_1),\eta_1)u(\beta(\eta_1),\eta_1)d\eta_1$$

$$-\int\limits_0^{\eta} \left(\frac{1}{2}\frac{\partial R(\xi,\eta;\alpha(\eta_1),\eta_1)}{\partial \xi_1}u(\alpha(\eta_1),\eta_1) - \frac{1}{2}\frac{\partial u(\alpha(\eta_1),\eta_1)}{\partial \xi_1}R(\xi,\eta;\alpha(\eta_1),\eta_1)\alpha^{'}(\eta_1)\right)d\eta_1$$

$$-\int\limits_0^{\eta} b(\alpha(\eta_1),\eta_1)R(\xi,\eta;\alpha(\eta_1),\eta_1)u(\alpha(\eta_1),\eta_1)\alpha^{'}(\eta_1)d\eta_1$$

$$-\int_0^\eta \left(\frac{1}{2}\frac{\partial u(\alpha(\eta_1),\eta_1)}{\partial\eta_1}R(\xi,\eta;\alpha(\eta_1),\eta_1)+\frac{1}{2}\frac{\partial R(\xi,\eta;\alpha(\eta_1),\eta_1)}{\partial\eta_1}u(\alpha(\eta_1),\eta_1)\right)d\eta_1$$

$$+\int_0^\eta a(\alpha(\eta_1),\eta_1)R(\xi,\eta,\alpha(\eta_1),\eta_1)u(\alpha(\eta_1),\eta_1)d\eta_1$$

$$-\frac{1}{2}R(\xi,\eta;\xi,\eta_2)u(\xi,\eta_2)-\frac{1}{2}R(\xi,\eta;\xi_0,\eta)u(\xi_0,\eta)=0, \tag{16}$$

where $(\beta(\eta_2),\eta_2)$ is a point of crossing $\xi_1=\beta(\eta_1)$ and $\xi_1=\xi$, and (ξ_0,η) is a point of crossing $\xi_1=\alpha(\eta_1)$ and $\eta_1=\eta$.

In (16) equating firstly $\xi=\alpha(\eta)$ and then $\xi=\beta(\eta)$, we get two identities

$$J_\alpha u\equiv I_\alpha u \tag{17}$$

$$-\int_1^{\eta_2}\left(\frac{\partial R(\alpha(\eta),\eta;\beta(\eta_1),\eta_1)}{\partial\eta_1}u(\beta(\eta_1),\eta_1)-\frac{\partial u(\beta(\eta_1),\eta_1)}{\partial\xi_1}R(\alpha(\eta),\eta;\beta(\eta_1),\eta_1)\right)d\eta_1$$

$$+\int_1^{\eta_2}\left(a(\beta(\eta_1),\eta_1)-b(\beta(\eta_1),\eta_1)\beta^{'}(\eta_1)\right)R(\alpha(\eta),\eta;\beta(\eta_1),\eta_1)u(\beta(\eta_1),\eta_1)d\eta_1=0,$$

$$J_\beta u\equiv I_\beta u \tag{18}$$

$$+\int_0^{\eta}\left(\frac{\partial R(\beta(\eta),\eta;\alpha(\eta_1),\eta_1)}{\partial\eta_1}u(\alpha(\eta_1),\eta_1)-\frac{\partial u(\alpha(\eta_1),\eta_1)}{\partial\eta_1}R(\beta(\eta),\eta;\alpha(\eta_1),\eta_1)\right)d\eta_1$$

$$+\int_0^{\eta}\left(a(\alpha(\eta_1)),\eta_1)-b(\alpha(\eta_1),\eta_1)\alpha^{'}(\eta_1)\right)R(\beta(\eta),\eta;\alpha(\eta_1),\eta_1)u(\alpha(\eta_1),\eta_1)d\eta_1=0.$$

Note that (17) and (18) are the conditions, connecting the values of the function u and its derivative on the boundaries $\xi=\alpha(\eta)$ and $\xi=\beta(\eta)$.

Thus, the following lemma is proved:

Lemma 1. *The volume hyperbolic potential (9) satisfies the hyperbolic equation (5), the homogeneous initial conditions (6) and the boundary conditions:*

$$\begin{cases} I_\alpha u=0 & at\ \alpha(\eta)\le 1,\\ J_\alpha u=0 & at\ \alpha(\eta)\ge 1,\\ I_\beta u=0 & at\ \eta\ge 0,\\ J_\alpha u=0 & at\ \eta\le 0.\end{cases} \tag{19}$$

Corollary 1. *The volume hyperbolic potential (9) is the solution of the initial-boundary problem (5), (6), (19).*

4 Uniqueness of solution of problem (5), (6), (19)

The constructed in Section 3 boundary conditions (19) will uniquely define the volume hyperbolic potential (9) if the initial-boundary problem (5), (6), (19) has no other solutions except (9).

Lemma 2. *The solution of the initial-boundary problem (5), (6), (19) is unique.*

Proof. As usual, by $u_1(\xi;\eta)$ and $u_2(\xi;\eta)$ we denote two solutions of the initial-boundary problem (5), (6), (19). Then their difference $u(\xi;\eta) = u_1(\xi;\eta) - u_2(\xi;\eta)$ satisfies the homogeneous hyperbolic equation

$$\frac{\partial^2 u(\xi,\eta)}{\partial\xi\partial\eta} + a(\xi,\eta)\frac{\partial u(\xi,\eta)}{\partial\xi} + b(\xi,\eta)\frac{\partial u(\xi,\eta)}{\partial\eta} + c(\xi,\eta)u(\xi,\eta) = 0, (\xi,\eta) \in \Omega, \qquad (20)$$

the homogeneous initial conditions (6) and the boundary conditions (9). We apply the Green's theorem in a plane to the integral

$$0 = -\iint\limits_{\Omega_{\xi,\eta}} R(\xi,\eta;\xi_1\eta_1)\cdot 0 \cdot d\xi_1 d\eta_1 = -\iint\limits_{\Omega_{\xi,\eta}} (RLu - uL^*R)\, d\xi_1 d\eta_1$$

$$= \int_1^{\eta_2} \left(\frac{1}{2}\frac{\partial R(\xi,\eta;\beta(\eta_1),\eta_1)}{\partial\xi_1}u(\beta(\eta_1),\eta_1) - \frac{1}{2}\frac{\partial u(\beta(\eta_1),\eta_1)}{\partial\xi_1}R(\xi,\eta;\beta(\eta_1),\eta_1)\right)\beta^{'}(\eta_1)d\eta_1$$

$$-\int\limits_1^{\eta_2} b(\beta(\eta_1),\eta_1)R(\xi,\eta;\beta(\eta_1),\eta_1)u(\beta(\eta_1),\eta_1)\beta^{'}(\eta_1)d\eta_1$$

$$+\int\limits_1^{\eta_2}\left(\frac{1}{2}\frac{\partial u(\beta(\eta_1),\eta_1)}{\partial\eta_1}R(\xi,\eta;\beta(\eta_1),\eta_1) - \frac{1}{2}\frac{\partial R(\xi,\eta;\beta(\eta_1),\eta_1)}{\partial\eta_1}u(\beta(\eta_1),\eta_1)\right)d\eta_1$$

$$+\int\limits_1^{\eta_2} a(\beta(\eta_1),\eta_1)R(\xi,\eta;\beta(\eta_1),\eta_1)u(\beta(\eta_1),\eta_1)d\eta_1$$

$$-\int\limits_0^{\eta}\left(\frac{1}{2}\frac{\partial R(\xi,\eta;\alpha(\eta_1),\eta_1)}{\partial\xi_1}u(\alpha(\eta_1),\eta_1) - \frac{1}{2}\frac{\partial u(\alpha(\eta_1),\eta_1)}{\partial\xi_1}R(\xi,\eta;\alpha(\eta_1),\eta_1)\alpha^{'}(\eta_1)\right)d\eta_1$$

$$-\int\limits_0^{\eta} b(\alpha(\eta_1),\eta_1)R(\xi,\eta;\alpha(\eta_1),\eta_1)u(\alpha(\eta_1),\eta_1)\alpha^{'}(\eta_1)d\eta_1$$

$$-\int_0^{\eta}\left(\frac{1}{2}\frac{\partial u(\alpha(\eta_1),\eta_1)}{\partial \eta_1}R(\xi,\eta;\alpha(\eta_1),\eta_1)+\frac{1}{2}\frac{\partial R(\xi,\eta;\alpha(\eta_1),\eta_1)}{\partial \eta_1}u(\alpha(\eta_1),\eta_1)\right)d\eta_1$$

$$+\int_0^{\eta} a(\alpha(\eta_1),\eta_1)R(\xi,\eta,\alpha(\eta_1),\eta_1)u(\alpha(\eta_1),\eta_1)d\eta_1$$

$$-\frac{1}{2}R(\xi,\eta;\xi,\eta_2)u(\xi,\eta_2)-\frac{1}{2}R(\xi,\eta;\xi_0,\eta)u(\xi_0,\eta)+u(\xi,\eta)=0. \tag{21}$$

In (21) equating firstly $\xi=\alpha(\eta)$ and then $\xi=\beta(\eta)$, we get two identities:

$$-J_\alpha u+u(\alpha(\eta),\eta)=0, \tag{22}$$

and

$$-J_\beta u+u(\beta(\eta),\eta)=0. \tag{23}$$

Taking into account homogenous boundary conditions (19), from (22), (23) we obtain that

$$u(\alpha(\eta),\eta)=0, \eta<\eta_0, \tag{24}$$

$$u(\beta(\eta),\eta)=0, \eta<0, \tag{25}$$

where $\alpha(\eta_0)=1$. Similarly for the case II and case III we have

$$u(\alpha(\eta),\eta)=0, \eta_0<\eta, \tag{26}$$

$$u(\beta(\eta),\eta)=0, \eta>0. \tag{27}$$

Thus, the function $u(\xi,\eta)$ satisfies the homogeneous hyperbolic equation (5), the homogeneous initial conditions (6) and the boundary conditions (24)–(27), it is the solution of the homogeneous first initial-boundary value problem. By virtue of the uniqueness of its solution we have $u(\xi,\eta)=0$ at $(\xi,\eta)\in\Omega$. Consequently, $u_1(\xi,\eta)=u_2(\xi,\eta)$. Lemma 2 is proved.

5 The statement of main result

Definition 1. *As a classical solution of the initial-boundary problem (5), (6), (9) we call a function $u(\xi,\eta)$ from the class $u(\xi,\eta)\in C^2\left(\overline{\Omega}\right)$ satisfying equation (5) and the initial conditions (6) and the boundary conditions (9).*

Combining the results of Lemma 1 and Lemma 2, we obtain the main result of the paper.

Theorem 1. *Let $f(\xi,\eta)\in C^1\left(\overline{\Omega}\right)$. The volume hyperbolic potential (9) satisfies the hyperbolic equation (5), the homogeneous initial conditions (6) and the boundary conditions (9). Conversely, for any $f(\xi,\eta)\in C^1\left(\overline{\Omega}\right)$ the initial boundary problem (5), (6), (19) has the unique classical solution $u(\xi,\eta)\in C^2\left(\overline{\Omega}\right)$ and this solution is presented in the form of the hyperbolic potential (9).*

Corollary 2. *The boundary conditions (19) together with the initial conditions (6) uniquely determine the volume hyperbolic potential (9), i.e. are boundary conditions of the hyperbolic potential (9).*

6 The case of wave potential

In this section we consider a special case, when $a, b, c = 0$. In this case

$$R(\xi, \eta; \xi_1, \eta_1) = 1.$$

In the case II, from equation (14), substituting $\xi = \alpha(\eta)$ and differentiating with respect to η, and taking into account the initial conditions (6), we have the following condition:

$$\frac{\partial u(\alpha(\eta), \eta_1)}{\partial \eta} = 0, \eta_0 < \eta < 0. \tag{28}$$

In the case III, from equation (15) substituting $\xi = \beta(\eta)$ and differentiating with respect to η, we have the following condition:

$$\frac{\partial u(\beta(\eta), \eta)}{\partial \xi} = 0, 0 < \eta < 1. \tag{29}$$

For the case IV in this special case from (16) we have the next identity

$$-\int\limits_1^{\eta_2} \frac{\partial u(\beta(\eta_1), \eta_1)}{\partial \xi_1} \beta^{'}(\eta_1) d\eta_1 + \int\limits_1^{\eta_2} \frac{\partial u(\beta(\eta_1), \eta_1)}{\partial \eta_1} d\eta_1$$

$$+ \int\limits_0^{\eta} \frac{\partial u(\alpha(\eta_1), \eta_1)}{\partial \xi_1} \alpha^{'}(\eta_1) d\eta_1 - \int\limits_0^{\eta} \frac{\partial u(\alpha(\eta_1), \eta_1)}{\partial \eta_1} d\eta_1 - u(\xi, \eta_2) - u(\xi_0, \eta) = 0. \tag{30}$$

Firstly, in (30) equating $\xi = \alpha(\eta)$ and then $\xi = \beta(\eta)$, and differentiating with respect to η, and taking into account the initial conditions (6), we have the next conditions:

$$-\alpha'(\eta) \frac{\partial u(\beta(\eta_2), \eta_2)}{\partial \xi} = \frac{\partial u(\alpha(\eta), \eta)}{\partial \eta}, \ \eta_0 < \eta < 0, \tag{31}$$

$$-\frac{\partial u(\alpha(\eta), \eta)}{\partial \eta} = \frac{\partial u(\beta(\eta), \eta)}{\partial \xi} \beta'(\eta), \ 0 < \eta < 1. \tag{32}$$

These boundary conditions look more clearly in variables (x, t). In coordinates (x, t) the volume hyperbolic potential is written in the form

$$u(x, t) = -\iint\limits_{\Omega_{x,t}} f_1(x_1, t_1) dx_1 dt_1, \tag{33}$$

hyperbolic equation (1) has the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = f_1(x,t), (x,t) \in Q, \tag{34}$$

and the initial conditions (2) has the form

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0,\ 0 \leq x \leq 1. \tag{35}$$

For the case II, when $x = \alpha_1(t)$ from (28) we have

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}\right)(\alpha_1(t), t) = 0, 0 < t < t_1. \tag{36}$$

For the case III, when $x = \beta_1(t)$ from (29) we have

$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right)(\beta_1(t), t) = 0, 0 < t < t_1. \tag{37}$$

For the case IV from (31), (32), when $x = \alpha_1(t)$ and $x = \beta_1(t)$ we have the next boundary conditions on the left-hand and right-hand sides of the domain $Q_{x,t}$:

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}\right)(\alpha_1(t), t)$$

$$= \frac{1+\alpha_1'(t)}{1-\alpha_1'(t)}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right)(\beta_1(t_2(t)), t_2(t)),\ t_1 < t < T, \tag{38}$$

$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right)(\beta_1(t), t)$$

$$= \frac{1-\beta_1'(t)}{1+\beta_1'(t)}\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}\right)(\alpha_1(t_0(t)), t_0(t))\,,\ t_1 < t < T, \tag{39}$$

where $(\alpha_1(t_0(t)), t_0(t))$ is a point of crossing of the boundary curve $x_1 = \alpha_1(t_1)$ and of the characteristics $x_1 = t_1 - t + \beta_1(t)$; $(\beta_1(t_2(t)), t_2(t))$ is a point of crossing of the boundary curve $x_1 = \beta_1(t_1)$ and of the characteristics $x_1 = t + \alpha_1(t) - t_1$.

The identity (38) holds for $t + \alpha_1(t) > 1$, and the identity (39) holds for $\beta_1(t) - t < 0$.

Both obtained identities (38), (39) connect with each other the traces of solutions on the left-hand and right-hand boundaries of the domain $Q_{x,t}$. Herewith, since $t > t_2(t)$ and $t > t_0(t)$, then the points in which values are taken in the left-hand parts of this identities, are "above" than the points, in which values are taken in the right-hand parts of the identities.

Lemma 3. *The volume wave potential (28) satisfies the wave equation (34), the homogeneous initial conditions (35), the boundary condition on the left-hand boundary of the domain*

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}\right)(\alpha_1(t), t) = 0, \quad at \;\; 0 \le t \le T, \tag{40}$$

and the boundary condition on the right-hand boundary of the domain

$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right)(\beta_1(t), t) = 0, \quad at \;\; 0 \le t \le T. \tag{41}$$

Corollary 3. *The volume wave potential (33) is the solution of the initial boundary value problem (34), (35), (40), (41).*

Boundary conditions (40, (41) have the following physical interpretation. It is well known that the general solution of the homogeneous equation (34), that is, the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{42}$$

is a superposition of two waves

$$u(x,t) = \phi(x+t) + \psi(x-t),$$

one of which ($\phi(x+t)$) extends to the left, and the second ($\psi(x-t)$) extends to the right.

It is easy to see that the boundary condition (44) is "transparent" for the wave going to the left, that is, for the wave of the form $\phi(x+t)$. Similarly, the boundary condition (45) is "transparent" for the wave going to the right, that is, for the wave of the form $\psi(x-t)$. These waves occur at some nonzero initial perturbation

$$u(x,0) = \tau(x), u_t(x,0) = \upsilon(x), \tag{43}$$

given at $t = 0$ on the segment $0 \le x \le 1$. These waves are given by

$$\phi(x+t) = \frac{1}{2}\tau(x+t) + \frac{1}{2}\int\limits_a^{x+t} \upsilon(s)ds, \psi(x-t) = \frac{1}{2}\tau(x-t) + \frac{1}{2}\int\limits_a^{x-t} \upsilon(s)ds.$$

Thus, if we consider the wave process of oscillation of an infinite string described by equation (34) at $-\infty < x < +\infty, t > 0$, with locally inhomogeneous initial conditions (43) (that is, in the case when $supp\{\tau(x)\} \subset [0,1]$ and then to study the behavior of the string at the interval $0 \le x \le 1$ it is sufficient to consider the solutions of the equation (42) only at $0 \le x \le 1, t > 0$ with boundary conditions (40), (41), $supp\{\upsilon(x)\} \subset [0,1]$), then to study the behavior of the string at the interval $0 \leq x \leq 1$ it is sufficient to consider the solutions of the equation (42) only at $0 \leq x \leq 1, t > 0$ with boundary conditions (40), (41).

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Садыбеков М.А., Дербісалы Б.О. ҚИСЫҚ СЫЗЫҚТЫ ШЕКАРАЛЫ ОБЛЫСТАҒЫ КӨЛЕМДІК ГИПЕРБОЛАЛЫҚ ПОТЕНЦИАЛДЫҢ ШЕКАРАЛЫҚ ШАРТТАРЫ

Қисық сызықты шекаралы облыста бір өлшемді көлемдік гиперболалық потенциал зерттелді. Гиперболалық потенциал ядросы ретінде Коши есебінің іргелі шешімі таңдалды. Бұл жағдайда көлемдік гиперболалық потенциал біртекті бастапқы шартты қанағаттандыратыны жақсы белгілі. Облыстың бүйір шекарасында көлемдік гиперболалық потенциалдың қанағаттандыратын шекаралық шарттары құрылды. Тұжырымдалған бастапқы-шеткі есептің жалғыз классикалық шешімі бар екені көрсетілді.

Кілттік сөздер. Гиперболалық теңдеу, бастапқы-шекаралық есеп, шекаралық шарт, гиперболалық потенциал.

Садыбеков М.А., Дербисалы Б.О. ГРАНИЧНЫЕ УСЛОВИЯ ОБЪЕМНОГО ГИПЕРБОЛИЧЕСКОГО ПОТЕНЦИАЛА В ОБЛАСТИ С КРИВОЛИНЕЙНОЙ ГРАНИЦЕЙ

Исследован одномерный объемный гиперболический потенциал в области с криволинейной границей. В качестве ядра гиперболического потенциала выбрано фундаментальное решение задачи Коши. Хорошо известно, что в этом случае объемный гиперболический потенциал удовлетворяет однородному начальному условию. В работе построены граничные условия, которым удовлетворяет гиперболический объемный потенциал на боковых границах области. Показано, что сформулированная начально-краевая задача имеет единственное классическое решение.

Ключевые слова. Уравнение гиперболического типа, начально-краевая задача, краевые условия, гиперболический потенциал.

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A numerical algorithm of solving a quasilinear boundary value problem with parameter for the Duffing equation

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Abstract. For the Duffing equation, a problem of finding a pair consisting of the unknown function and a parameter is considered. The boundary condition of this problem is periodic. For determining the parameter, an additional condition is given as the value of the first coordinate of the solution at the left endpoint of the domain interval. We propose a numerical algorithm for solving the problem under consideration.

Keywords. Duffing equation with parameter, numerical algorithm, Newton's method.

The Duffing equation is used as a mathematical model of many processes in natural sciences. This equation contains numerical parameters which describe impact of various factors to the behavior of studying processes. In order to determine their values, we need to set additional conditions besides the initial and boundary ones. Various problems for differential equations with parameters have been studied and solved in [1]–[8].

In the present paper, we consider the system of quasilinear ordinary differential equations with parameter

$$\frac{dx_1}{dt} = x_2, \quad t \in (0, T), \tag{1}$$

$$\frac{dx_2}{dt} = -\sigma^2 x_1 - \varepsilon\gamma x_1^3 + \mu cos(\omega t) + g(t), \quad t \in (0, T), \tag{2}$$

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subject to the periodic boundary condition

$$x_1(0) = x_1(T), \quad x_1(0) = x_2(T) \tag{3}$$

and additional condition

$$x_1(0) = x_1^0. \tag{4}$$

Here $\varepsilon > 0$, γ, ω, and x_1^0 are given numbers and $g(t)$ is a function continuous on $[0, T]$. System (1), (2) is the Duffing equation expressed via two ordinary differential equations of the first order.

A solution to problem (1)–(4) is a pair $(\mu^*, x^*(t))$, where $\mu^* \in \mathbb{R}^l$ and $x^*(t) = (x_1^*(t), x_2^*(t))$ is a vector function continuous on $[0, T]$ and continuously differentiable on $(0, T)$, satisfying conditions (3), (4) and the system of ordinary differential equations (1), (2) with $\mu = \mu^*$.

The purpose of the present paper is to construct a numerical algorithm for solving problem (1)–(4).

To this end, we reduce this problem to a problem with additional parameters and apply the method proposed in [9], [10]. As additional parameters we choose the values of $x_1(t)$, $x_2(t)$ at the point $t = 0 : \lambda_1 = x_1(0)$, $\lambda_2 = x_2(0)$, and make the substitutions $u_1(t) = x_1(t) - \lambda_1$, $u_2(t) = x_2(t) - \lambda_2$. Then problem (1)–(4) takes the form:

$$\frac{du_1}{dt} = u_2 + \lambda_2, \quad t \in (0, T), \tag{5}$$

$$\frac{du_2}{dt} = -\sigma^2(u_1 + \lambda_1) - \varepsilon\gamma(u_1 + \lambda_1)^3 + \mu cos(\omega t) + g(t), \quad t \in (0, T), \tag{6}$$

$$u_1(0) = 0, \quad u_2(0) = 0, \tag{7}$$

$$u_1(T) = 0, \quad u_2(T) = 0, \tag{8}$$

$$\lambda_1 + u_1(0) = x_1^0. \tag{9}$$

A solution to problem (5)–(9) is a set $(\mu^*, \lambda_1^*, \lambda_2^*, u_1^*(t), u_2^*(t))$, where the functions $u_1^*(t)$, $u_2^*(t)$ satisfy the system of differential equations (5), (6) with $\mu = \mu^*$, $\lambda_1 = \lambda_1^*$, $\lambda_2 = \lambda_2^*$ and conditions (7)–(9). It is clear that if this set is a solution to problem (5)–(9), then a pair $(\mu^*, x^*(t))$ with $x^*(t) = u^*(t) - \lambda^*$ is a solution to problem (1)–(4). From (7) and (9) we get $\lambda_1 = x_1^0$.

Let us choose numbers $\lambda_2^{(0)}$, $\mu^{(0)}$, $\rho_\lambda > 0$, $\rho_\mu > 0$ and assume that the Cauchy problem (5)–(7) has a unique solution $u(t, \lambda, \mu) = (u_1(t, \lambda_1, \lambda_2, \mu), u_2(t, \lambda_1, \lambda_2, \mu))$ for all $\lambda_1 = x_1^0$, $\lambda_2 \in (\lambda_2^{(0)} - \rho_\lambda, \lambda_2^{(0)} + \rho_\lambda)$, and $\mu \in (\mu^{(0)} - \rho_\mu, \mu^{(0)} + \rho_\mu)$. Problem (5)–(9) is solvable if the system of nonlinear algebraic equations

$$u_1(t, x_1^0, \lambda_2, \mu) = 0, \tag{10}$$

$$u_2(t, x_1^0, \lambda_2, \mu) = 0 \tag{11}$$

has a solution $(\lambda_2^*, \mu^*) \in (\lambda_2^{(0)} - \rho_\lambda, \lambda_2^{(0)} + \rho_\lambda) \times (\mu^{(0)} - \rho_\mu, \mu^{(0)} + \rho_\mu)$. We write system (10), (11) in the form

$$Q_*(\lambda_2, \mu) = 0 \tag{12}$$

and solve (12) by Newton's method choosing as initial guess solution a pair $(\lambda_2^{(0)}, \mu^{(0)})$, the centers of the above mentioned intervals.

Taking into consideration that $\varepsilon > 0$ is a small number, we determine $\lambda_2^{(0)}, \mu^{(0)}$ by solving the linear boundary value problem with parameters

$$\frac{du}{dt} = A(t)(u + \lambda) + B(t)\mu + f(t), t \in (0, T), \tag{13}$$

$$u(0) = 0, \tag{14}$$

$$u(T) = 0, \tag{15}$$

$$\lambda_1 = x_1^0, \tag{16}$$

where $A(t) = \begin{pmatrix} 0 & 1 \\ -\sigma^2 & 0 \end{pmatrix}$, $B(t) = \begin{pmatrix} 0 \\ cos\omega t \end{pmatrix}$ and $f(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$.

Denote by $a(P, t)$ a unique solution to the Cauchy problem for the linear ordinary differential equation

$$\frac{dz}{dt} = A(t)z + P(t), \quad t \in [0, T], \quad z(0) = 0, \tag{17}$$

where $P(t)$ is (2×2)-matrix or vector of the dimension 2 continuous on $[0, T]$. Using this solution, we can represent the solution to the Cauchy problem (13), (14) in the form

$$u(t, \lambda, \mu) = a(A, t)\lambda + a(B, t)\mu + a(f, t), \quad t \in [0, T]. \tag{18}$$

Equation (16) and the substitution of the right-hand side of (18) into boundary condition (15) lead to the system of linear algebraic equations in λ_2, μ :

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} x_1^0 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \mu + \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{19}$$

The coefficients and the right-hand side of system (19) are defined by the equalities

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = a(A, T), \quad \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = a(B, T), \quad \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = a(f, T).$$

System (19) is equivalent to the system

$$\alpha_{12}\lambda_2 + \beta_1\mu = -\gamma_1 - \alpha_{11}x_1^0,$$

$$\alpha_{22}\lambda_2 + \beta_2\mu = -\gamma_2 - \alpha_{21}x_1^0.$$

By solving this system, we find $(\lambda_2^{(0)}, \mu^{(0)})$. Further approximations of the solution to the system of nonlinear algebraic equations we find according to the iterative process

$$\begin{pmatrix} \lambda_2^{(k+1)} \\ \mu^{(k+1)} \end{pmatrix} = \begin{pmatrix} \lambda_2^{(k)} \\ \mu^{(k)} \end{pmatrix} - \left(\frac{\partial Q_*(\lambda_2^{(k)}, \mu^{(k)})}{\partial y}\right)^{-1} \cdot Q_*(\lambda_2^{(k)}, \mu^{(k)}),$$

where

$$\frac{\partial Q_*(\lambda_2^{(k)}, \mu^{(k)})}{\partial y} = \begin{pmatrix} \frac{\partial u_1(T,x_1^0,\lambda_2^{(k)},\mu^{(k)})}{\partial \lambda_2} & \frac{\partial u_1(T,x_1^0,\lambda_2^{(k)},\mu^{(k)})}{\partial \mu} \\ \frac{\partial u_2(T,x_1^0,\lambda_2^{(k)},\mu^{(k)})}{\partial \lambda_2} & \frac{\partial u_2(T,x_1^0,\lambda_2^{(k)},\mu^{(k)})}{\partial \mu} \end{pmatrix}, \quad k = 0, 1, 2, \ldots .$$

For the given $\lambda_2^{(k)}$, $\mu^{(k)}$, we solve the Cauchy problem (5)–(7) with $\lambda_2 = \lambda_2^{(k)}$, $\mu = \mu^{(k)}$ and find functions $u_1(t, x_1^0, \lambda_2^{(k)}, \mu^{(k)})$, $u_2(t, x_1^0, \lambda_2^{(k)}, \mu^{(k)})$. Then the value of $Q_*(\lambda_2^{(k)}, \mu^{(k)})$ is equal to the vector $\begin{pmatrix} u_1(T, x_1^0, \lambda_2^{(k)}, \mu^{(k)}) \\ u_2(T, x_1^0, \lambda_2^{(k)}, \mu^{(k)}) \end{pmatrix}$.

In order to determine the elements of the Jacobi matrix $\partial Q_*(\lambda_2^{(k)}, \mu^{(k)})/\partial y$, we use the following equalities:

$$\frac{du_1(t, x_1^0, \lambda_2, \mu)}{dt} = u_2(t, x_1^0, \lambda_2, \mu) + \lambda_2, \quad t \in (0, T), \tag{20}$$

$$\frac{du_2(t, x_1^0, \lambda_2, \mu)}{dt} = -\sigma^2(u_1(t, x_1^0, \lambda_2, \mu) + x_1^0) - \varepsilon\gamma(u_1(t, x_1^0, \lambda_2, \mu) + x_1^0)^3$$
$$+\mu cos(\omega t) + g(t), \quad t \in (0, T), \tag{21}$$

$$u_1(0, x_1^0, \lambda_2, \mu) = 0, \quad u_2(0, x_1^0, \lambda_2, \mu) = 0, \tag{22}$$

which are true for all $(\lambda_2, \mu) \in (\lambda^{(0)} - \rho_\lambda, \lambda^{(0)} + \rho_\lambda) \times (\mu^{(0)} - \rho_\mu, \mu^{(0)} + \rho_\mu)$.

Differentiating both sides of (20), (21), and (22) with respect to λ_2 gives

$$\frac{d}{dt}\left(\frac{\partial u_1(t, x_1^0, \lambda_2, \mu)}{\partial \lambda_2}\right) = \frac{\partial u_2(t, x_1^0, \lambda_2, \mu)}{\partial \lambda_2} + 1, \quad t \in (0, T), \tag{23}$$

$$\frac{d}{dt}\left(\frac{\partial u_2(t, x_1^0, \lambda_2, \mu)}{\partial \lambda_2}\right)$$
$$= -\left\{\sigma^2 + 3\varepsilon\gamma\left[x_1^0 + u_1(t, x_1^0, \lambda_2, \mu)\right]^2\right\}\frac{\partial u_1(t, x_1^0, \lambda_2, \mu)}{\partial \lambda_2}, \quad t \in (0, T), \tag{24}$$

$$\frac{\partial u_1(0,x_1^0,\lambda_2,\mu)}{\partial\lambda_2}=0,\quad \frac{\partial u_2(0,x_1^0,\lambda_2,\mu)}{\partial\lambda_2}=0. \tag{25}$$

Therefore, the functions $v_1^{(k)}(t)=\dfrac{\partial u_1(t,x_1^0,\lambda_2^{(k)},\mu^{(k)})}{\partial\lambda_2}$, $v_2^{(k)}(t)=\dfrac{\partial u_2(t,x_1^0,\lambda_2^{(k)},\mu^{(k)})}{\partial\lambda_2}$ satisfy the Cauchy problem for ordinary differential equations

$$\frac{dv_1}{dt}=v_2+1,\quad t\in[0,T], \tag{26}$$

$$\frac{dv_2}{dt}=-\Big\{\sigma^2+3\varepsilon\gamma\Big[x_1^{(k)}(t)\Big]^2\Big\}v_1,\quad t\in[0,T], \tag{27}$$

$$v_1(0)=0,\quad v_2(0)=0, \tag{28}$$

where $x_1^{(k)}(t)=x_1^0+u_1(t,x_1^0,\lambda_2^{(k)},\mu^{(k)})$.

Similarly, differentiating (20), (21), and (22) with respect to μ, we get that the functions $w_1^{(k)}(t)=\dfrac{\partial u_1(t,x_1^0,\lambda_2^{(k)},\mu^{(k)})}{\partial\mu}$ and $w_2^{(k)}(t)=\dfrac{\partial u_2(t,x_1^0,\lambda_2^{(k)},\mu^{(k)})}{\partial\mu}$ are the solution to the Cauchy problem

$$\frac{dw_1}{dt}=w_2,\quad t\in[0,T], \tag{29}$$

$$\frac{dw_2}{dt}=-\Big\{\sigma^2+3\varepsilon\gamma\Big[x_1^{(k)}(t)\Big]^2\Big\}w_1+cos\omega t,\quad t\in[0,T], \tag{30}$$

$$w_1(0)=0,\quad w_2(0)=0. \tag{31}$$

Thus, if $(v_1^{(k)}(t),v_2^{(k)}(t))$ and $(w_1^{(k)}(t),w_2^{(k)}(t))$ are the solutions to the Cauchy problems (26)–(28) and (29)–(31), respectively, then the elements of the Jacobi matrix are determined by the formula

$$\frac{\partial Q_*(\lambda_2^{(k)},\mu^{(k)})}{\partial y}=\begin{pmatrix} v_1^{(k)}(T) & w_1^{(k)}(T)\\ v_2^{(k)}(T) & w_2^{(k)}(T)\end{pmatrix}. \tag{32}$$

Based on the above results, we propose the following numerical algorithm for solving the quasilinear boundary value problem (1)–(4) with parameters.

Step 1.

(a) Choose the stepsize $h>0: 2Nh=T$ and, using the Runge-Kutta method of the fourth order, find the numerical solution to the Cauchy problem (5)–(8) with $\lambda_1=x_1^0$, $\lambda_2=\lambda_2^{(0)}$, and $\mu=\mu^{(0)}$. So, we have the functions $u_1(\widehat{t},x_1^0,\lambda_2^{(0)},\mu^{(0)})$ and $u_2(\widehat{t},x_1^0,\lambda_2^{(0)},\mu^{(0)})$, where $\widehat{t}=\{0,h,\dots,(2N-1)h,2Nh\}$. Construct the function $x_1^{(0)}(t)=x_1^0+u_1(\widehat{t},x_1^0,\lambda_2^{(0)},\mu^{(0)})$, which is determined for all $t=mh$, $m=0,1,\dots,2N$.

(b) Since the function $x_1^{(0)}(t)$ is known only on the grid $\{0,h,\dots,(2N-1)h,T\}$, to solve the Cauchy problems for linear ordinary differential equations (26)–(28) and (29)–(31), we apply the Runge-Kutta method of the fourth order with the stepsize $h_1 = 2h$. Solving the Cauchy problems

$$\frac{dv_1}{dt} = v_2 + 1, \quad t \in [0,T],$$
$$\frac{dv_2}{dt} = -\Big\{\sigma^2 + 3\varepsilon\gamma\Big[x_1^{(0)}(t)\Big]^2\Big\}v_1, \quad t \in [0,T],$$
$$v_1(0) = 0, \quad v_2(0) = 0,$$

and

$$\frac{dw_1}{dt} = w_2, \quad t \in [0,T],$$
$$\frac{dw_2}{dt} = -\Big\{\sigma^2 + 3\varepsilon\gamma\Big[x_1^{(0)}(t)\Big]^2\Big\}w_1 + cos\omega t, \quad t \in [0,T],$$
$$w_1(0) = 0, \quad w_2(0) = 0,$$

we find the functions $v_1^{(0)}(t)$, $v_2^{(0)}(t)$, and $w_1^{(0)}(t)$, $w_2^{(0)}(t)$ on the grid $\big\{0, 2h, \ldots, 2Nh\big\}$.

(c) Assuming that (2×2)-matrix

$$\frac{\partial Q_*(\lambda_2^{(0)}, \mu^{(0)})}{\partial y} = \begin{pmatrix} v_1^{(0)}(T) & w_1^{(0)}(T) \\ v_2^{(0)}(T) & w_2^{(0)}(T) \end{pmatrix}$$

is invertible, the next approximation of the solution to (12), we determine as follows:

$$\begin{pmatrix} \lambda_2^{(1)} \\ \mu^{(1)} \end{pmatrix} = \begin{pmatrix} \lambda_2^{(0)} \\ \mu^{(0)} \end{pmatrix} - \left(\frac{\partial Q_*(\lambda_2^{(0)}, \mu^{(0)})}{\partial y}\right)^{-1} \cdot Q_*(\lambda_2^{(0)}, \mu^{(0)}).$$

Continuing the process, in the k-th step we find $(\lambda_2^{(k)}, \mu^{(k)})$ and $u_1(t, x_1^0, \lambda_2^{(k)}, \mu^{(k)})$, $u_2(t, x_1^0, \lambda_2^{(k)}, \mu^{(k)})$. The convergence conditions for the iterative process in terms of $Q_*(\lambda_2, \mu)$ and its Jacobi matrix are given in Theorem 4.1 [10, p. 1019].

Example. We consider problem (1)–(4) with $T = 1$, $w = 1$, $\sigma = 2$, $\gamma = 1$, $\epsilon = 0.05$, $g(t) = -16\pi^2\sin(4\pi t) + 4\sin(4\pi t) + 0.05\sin^3(4\pi t) - 2\cos t$. The solution to this problem is the pair $(\mu^*, x^*(t))$, where $\mu^* = 2$, $x^*(t) = \begin{pmatrix} \sin(4\pi t) \\ 4\pi\cos(4\pi t) \end{pmatrix}$.

First, we solve the Cauchy problems for ordinary differential equations by Runge-Kutta method of the fourth order with step size $h = 0.1$.

In order to determine an initial guess solution to system (12), we solve corresponding linear boundary value problem (13)–(15) with parameter and obtain

$$y^{(0)} = \begin{pmatrix} 12.5755281606075435 \\ 1.9992073852266987 \end{pmatrix}.$$

Using this initial approximation, we find the solution to our problem by the algorithm proposed in the paper.

Iteration 1:

$$Q_*(\lambda_2^{(0)},\mu^{(0)}) = \begin{pmatrix} -0.0012409267739448 \\ 0.0039987251271718 \end{pmatrix},$$

$$\frac{\partial Q_*(\lambda_2^{(0)},\mu^{(0)})}{dy} = \begin{pmatrix} 0.4473886530784268 & 0.3166236459146591 \\ -1.4325137911331871 & 0.3176777877070035 \end{pmatrix},$$

$$y^{(1)} = \begin{pmatrix} 12.5783153434482031 \\ 1.9991883493678824 \end{pmatrix}.$$

Iteration 2:

$$Q_*(\lambda_2^{(1)},\mu^{(1)}) = \begin{pmatrix} -0.0000028493576018 \\ -0.0000016787191317 \end{pmatrix},$$

$$\frac{\partial Q_*(\lambda_2^{(1)},\mu^{(1)})}{dy} = \begin{pmatrix} 0.4473887615789600 & 0.3166236635261683 \\ -1.4324996864543631 & 0.3176882131436935 \end{pmatrix},$$

$$y^{(2)} = \begin{pmatrix} 12.5783159707593359 \\ 1.9991964621716516 \end{pmatrix}.$$

After **Iteration 5**, we get:

$$Q_*(\lambda_2^{(4)},\mu^{(4)}) = \begin{pmatrix} 0.0000000000000002 \\ 0.0000000000000036 \end{pmatrix},$$

$$\frac{\partial Q_*(\lambda_2^{(4)},\mu^{(4)})}{dy} = \begin{pmatrix} 0.4473887616392045 & 0.3166236641865047 \\ -1.4324996528324900 & 0.3176882362279510 \end{pmatrix},$$

$$y^{(4)} = \begin{pmatrix} 12.5783159732745986 \\ 1.9991964642523374 \end{pmatrix},\ y^{(5)} = \begin{pmatrix} 12.5783159732746004 \\ 1.9991964642523343 \end{pmatrix}.$$

The comparison with the exact solution to the problem shows that

$$|\lambda_2^{(5)} - \lambda_2^*| \leq 0.0119,$$

$$|\mu_2^{(5)} - \mu_2^*| \leq 0.0008.$$

Next, to reduce the error we increase the number of the partition subintervals by ten times and solve the auxiliary Cauchy problems with step size $h = 0.01$.

Iteration 1:

$$Q_*(\lambda_2^{(0)},\mu^{(0)}) = \begin{pmatrix} -0.0012183459495035 \\ 0.0039110916233056 \end{pmatrix},$$

$$\frac{\partial Q_*(\lambda_2^{(0)},\mu^{(0)})}{dy} = \begin{pmatrix} 0.4464104388179766 & 0.3164970726810607 \\ -1.4330580996129132 & 0.3181099350162241 \end{pmatrix},$$

$$y^{(0)} = \begin{pmatrix} 12.5636425257306055 \\ 1.9999999921366101 \end{pmatrix}, \; y^{(1)} = \begin{pmatrix} 12.5663717215062629 \\ 2.0000000062462897 \end{pmatrix}.$$

Iteration 2:

$$Q_*(\lambda_2^{(1)},\mu^{(1)}) = \begin{pmatrix} 0.0000000000851788 \\ 0.0000000184865454 \end{pmatrix},$$

$$\frac{\partial Q_*(\lambda_2^{(1)},\mu^{(1)})}{dy} = \begin{pmatrix} 0.4464105239728410 & 0.3164970827929271 \\ -1.4330444981906445 & 0.3181199776724043 \end{pmatrix},$$

$$y^{(2)} = \begin{pmatrix} 12.5663717312849208 \\ 1.9999999921846303 \end{pmatrix}.$$

After **Iteration 3,** we obtain

$$Q_*(\lambda_2^{(2)},\mu^{(2)}) = \begin{pmatrix} 0.0000000000000025 \\ -0.0000000000000052 \end{pmatrix},$$

$$\frac{\partial Q_*(\lambda_2^{(2)},\mu^{(2)})}{dy} = \begin{pmatrix} 0.4464105239730589 & 0.3164970827918964 \\ -1.4330444981937203 & 0.3181199776730464 \end{pmatrix},$$

$$y^{(3)} = \begin{pmatrix} 12.5663717312849172 \\ 1.9999999921846281 \end{pmatrix}.$$

Thus, in this case Iteration 3 allows us to reach the proximity of order $O(10^{-8})$.

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Жұмабаев Д.С., Әбіласанов Б., Жұбатқан Ә., Әсетбеков А. ДУФФИНГ ТЕҢДЕУІ ҮШІН КВАЗИСЫЗЫҚТЫ ШЕТТІК ЕСЕПТІ ШЕШУДІҢ САНДЫҚ АЛГОРИТМІ

Дуффинг теңдеуі үшін белгісіз функция мен параметрден тұратын жұпты табу есебі қарастырылады. Бұл есептің шеттік шарты периодты шарт болып табылады. Параметрді анықтау үшін шешімнің бірінші координатасының интервалдың сол жақ шеткі нүктесіндегі мәні қосымша шарт ретінде беріледі. Қарастырылып отырған есепті шешудің сандық алгоритмі ұсынылады.

Кілттік сөздер. Параметрлі Дуффинг теңдеуі, сандық алгоритм, Ньютон әдісі.

Джумабаев Д.С., Абиласанов Б., Жубаткан А., Асетбеков А. ЧИСЛЕННЫЙ АЛГОРИТМ РЕШЕНИЯ КВАЗИЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ УРАВНЕНИЯ ДУФФИНГА

Для уравнения Дуффинга рассматривается задача нахождения пары, состоящей из неизвестной функции и параметра. Краевое условие этой задачи является периодическим. Для определения параметра задано дополнительное условие в качестве значения первой координаты решения в левой точке интервала. Предложен численный алгоритм решения рассматриваемой задачи.

Ключевые слова. Уравнение Дуффинга с параметром, численный алгоритм, метод Ньютона.

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Integral representation of multiperiodic solutions of a linear system in one critical case

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Abstract. The problem of the existence and integral representation of a unique multiperiodic solution of an inhomogeneous linear system with constant coefficients and a differentiation operator D on the direction of the main diagonal of the space of time variables were considered. This problem was solved in non-critical case when all eigenvalues of the matrix of coefficients of the system have non-zero real parts; moreover, the method of studying this case was not suitable for studying a critical case. Thus, critical cases remained open. This proves the existence of a solution to the problem when the matrix of coefficients has several pure imaginary eigenvalues with simple elementary divisors, and a free member of the system has the properties of real analyticity in independent variables that change in the strip of the real axis of the complex plane, and periodicity with rationally incommensurable frequencies. Moreover, the frequencies of the eigenvalues oscillations and excitation forces together satisfy the Diophantine condition of strong incommensurability. The condition for the absence of a nonzero multiperiodic solution of the homogeneous system corresponding to the given system is established. On this basis, the Green-type matrix function is constructed, in terms of which the question of the integral structure and existence of the required unique real analytic multiperiodic solution is solved. When studying the problem of a linear replacement system, it splits into two types of subsystems: a) several similar systems of the second order of critical nature and b) a system of non-critical cases. The problem is solved for these indicated subsystems individually according to the method described above, and then the developed method is described in general form for the original system. In general, the work proposes a new method for studying the problem of the existence and construction of a unique multiperiodic solution of the linear system of equations with constant coefficients and the same differentiation operator D. The method which is applicable in both non-critical and critical cases.

Keywords. Multiperiodic solution, Green's function, differentiation operator, real analytic function, integral representation, critical and non-critical cases.

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1 Introduction

It is known [1]–[6] that if a linear homogeneous θ-periodic system has no periodic solution except zero, then the corresponding inhomogeneous periodic system admits a unique θ-periodic solution $x(\tau)$. Moreover, its integral representation is easily obtained by eliminating the initial given x_0 from the systems of integral equations of solutions $x(\tau)$ and $x(\tau+\theta)$.

In the case of (θ,ω)-periodic in (τ,t) systems, as known [7], the condition for the multiplicity of their solutions is represented by the systems of functional-difference equations with a difference θ, which should be solvable in the space of ω-periodic smooth functions.

Accordingly, in the case, considered in this note as the systems for determining the initial ω-periodic with respect to t function $u(t)$ of (θ,ω)-periodic solution $x(\tau,t)$ we have a linear system of the homological type with a difference θ. From the assumption of the absence of nonzero (θ,ω)-periodic solutions of the corresponding homogeneous system follows that the initial function $u(t)$ is determined from the class of (θ,ω)-periodic zeros $v(\tau,t)$ of the differentiation operator D of the considered system with condition $u(t) = v(0,t)$, where $Dv(\tau,t) = 0$. In this case, by virtue to the rational incommensurability of periods (θ,ω) we have that $v = const$, and therefore $u = c - const$. By virtue of this simplification, we have an integral representation of the unique (θ,ω)-periodic solution of the system in terms of the Green type function, which is described in this paper.

It was required for us that the free member of the system possesses the property of real analyticity in the infinite strip of the real axis, and its frequencies together with the frequency of natural vibrations satisfy the Diophantine condition of their strong incommensurability [8]–[11] in order to justify the integral representation. This allowed us to solve the main problem in terms of Fourier series.

A problem of this type was previously considered only in the non-critical case [12]–[17]. In this case, small dividers do not occur.

This paper is essentially the first in this direction of the theory of multiperiodic solutions of the systems with a vector field differentiation operator, where elements of the methods KAM theory [18]–[24] are used. The study carried out here is a natural continuation of the works [25]–[28].

Thus, we consider the linear system of equations

$$Dx = Ax + f(\tau,t) \tag{1}$$

with respect to the vector-function $x = (x_1,\dots,x_n)$ of variables $\tau \in (-\infty,+\infty) = R$, $t = (t_1,\dots,t_m) \in R\times\dots\times R = R^m$ with the same for all components $Dx = (Dx_1,\dots,Dx_n)$ of operator D of the form

$$D = \frac{\partial}{\partial\tau} + \left\langle e, \frac{\partial}{\partial t}\right\rangle, \tag{2}$$

where $e = (1,\dots,1)$ is m-vector, $\dfrac{\partial}{\partial t} = \left(\dfrac{\partial}{\partial t_1},\dots,\dfrac{\partial}{\partial t_m}\right)$ is a vector operator, $\langle a,b\rangle$ is a scalar product of vectors a and b, $A = [a_{ij}]_1^n$ is a constant matrix, $f(\tau,t) = \big(f_1(\tau,t),\dots,f_n(\tau,t)\big)$ is a vector-function of variables $(\tau,t) \in R \times R^m$.

Suppose that the matrix A has purely imaginary eigenvalues $\lambda_j = \lambda_j(A)$ with simple elementary divisors, or eigenvalues $\lambda_k = \lambda_k(A)$ with nonzero real parts. Therefore, the spectrum σ of eigenvalues λ_j is splitted into the union of two sets σ_0 and σ_1:

$$\begin{aligned}
&\sigma = \sigma_0 \bigcup \sigma_1,\\
&\sigma_0 = \left\{\lambda_j \in \sigma : (\mathrm{Re}\lambda_j = 0) \wedge (\mathrm{Im}\lambda_j \neq 0) \wedge (\mathrm{Ied}\lambda_j = 1), j = \overline{1, 2n_0}\right\},\\
&\sigma_1 = \left\{\lambda_k \in \sigma : \mathrm{Re}\lambda_k \neq 0, k = \overline{1, n_1}\right\},
\end{aligned} \tag{3}$$

where Ied is the index of the elementary divisor.

The vector-function $f(\tau,t)$ has the properties of (θ,ω)-periodicity with respect to (τ,t), with the rationally incommensurate periods $(\omega_0 = \theta, \omega_1, \dots, \omega_m) = (\theta,\omega)$, $q_j\omega_j \neq q_k\omega_k$ for sets of integers $q_j, q_k \in Z$ and the real analyticity for $(\tau,t) \in \Pi_\delta \times \Pi_\delta^m = \Pi_\delta^{m+1}$, $\Pi_\delta = \{\tau \in C : |\mathrm{Im}\tau| < \delta\}$, as well as continuity on the closure $\overline{\Pi}_\delta^{m+1}$ strip Π_δ^{m+1}, where $\delta = const > 0$, C is a complex plane. We represent these properties of $f(\tau,t)$ in the form

$$f(\tau+\theta, t+q\omega) = f(\tau,t) \in A^b\left(\overline{\Pi}_\delta^{m+1}\right),\ q \in Z^m, \tag{4}$$

where $A^b\left(\overline{\Pi}_\delta^{m+1}\right)$ is a class of real analytic in Π_δ^{m+1} and continuous on $\overline{\Pi}_\delta^{m+1}$ vector-functions.

We set the task of investigating a problem of the existence of (θ,ω)-periodically with respect to (τ,t) solutions of the system (1) with the differentiation operator (2), which has properties (3) and (4).

In accordance with condition (3), the system (1)–(2) by a non-singular linear change

$$x = Ky, \det K \neq 0, \tag{5}$$

can be reduced to the system

$$Dy = By + \widetilde{f}(\tau,t) \tag{6}$$

with the block-diagonal matrix

$$B = \mathrm{diag}\left(J_1,\dots,J_{n_0},C\right), \tag{7}$$

where the second-order matrices J_j have the form

$$J_j = \begin{pmatrix} 0 & 2\pi\nu_j^0 \\ -2\pi\nu_j^0 & 0 \end{pmatrix} \tag{8}$$

with constants $\nu_j^0 > 0, j = \overline{1,n_0}$, and in $(n_1 \times n_1)$-matrix C all eigenvalues $\lambda_k = \lambda_k(C)$ have non-zero real parts:

$$\mathrm{Re}\lambda_k(C) \neq 0, k = \overline{1,n}, 2n_0 + n_1 = n. \tag{9}$$

It is obvious that

$$\widetilde{f}(\tau,t) = K^{-1}f(\tau,t). \tag{10}$$

It is clear from the relations (7)–(10), that the system (6) decomposes into n_0 second-type subsystems of the same type with respect to $z=(u,v)$ of the form

$$Dz = Jz + h(\tau,t) \tag{11}$$

with matrix J of the form (8) with constant $\widetilde{\nu} = 2\pi\nu^0 > 0$ and vector-function

$$h(\tau,t) = (\varphi(\tau,t), \psi(\tau,t))$$

and into the subsystem n_1-st order

$$Dw = Cw + g(\tau,t), \tag{12}$$

where the functions $\varphi=\varphi(\tau,t), \psi=\psi(\tau,t)$ and $g=g(\tau,t)$ have properties of the form (4).

Thus, the main problem was reduced to its study for systems (11) and (12).

2 Multiperiodic solutions of the second-order of subsystems in the critical case

The homogeneous system

$$Dz = Jz, \tag{13}$$

corresponding to the system (11) has a matricant

$$Z(\tau) = \begin{pmatrix} \cos 2\pi\nu\tau & \sin 2\pi\nu\tau \\ -\sin 2\pi\nu\tau & \cos 2\pi\nu\tau \end{pmatrix}. \tag{14}$$

Obviously, under condition

$$det[Z(\theta) - E] \neq 0 \tag{15}$$

for the matricant (14), the homogeneous system (13) does not have θ-periodic solutions, except for zero, where E is an identity matrix.

Obviously, the variable τ changing on the numerical axis $R=(-\infty,+\infty)$ can be represented in the form

$$\tau = \left[\theta^{-1}\tau\right]\theta + \left\{\theta^{-1}\tau\right\}\theta = s^*(\tau) + s_*(\tau),$$

where θ is a positive number, $[\alpha]$ and $\{\alpha\}$ are the integers and fractional parts of the number, $s^*(\tau) = \left[\theta^{-1}\tau\right]\theta$, $s_*(\tau) = \left\{\theta^{-1}\tau\right\}\theta$ are functions with the domain $\tau \in R$.

The functions $s^*(\tau)$ and $s_*(\tau)$ at $\tau = k\theta, k \in Z$ with a jump equal to θ, moreover, $s^*(\tau)$ is a non-decreasing step function in steps $h=\theta$, $s^*(\tau+\theta) = s^*(\tau)+\theta$, $s_*(\tau+\theta) = s_*(\tau)$ are non-negative periodic functions of period θ, limited by the number θ.

They are differentiable when $\tau \in [k\theta, k\theta+\theta), k \in Z$, moreover $\dfrac{d}{d\tau}s^*(\tau) = 0$, $\dfrac{d}{d\tau}s_*(\tau) = 1$ and at the breakpoints $\tau = k\theta$ their one-sided derivatives are equal to each other:

$$\left.\frac{ds^*(\tau)}{d\tau}\right|_{\tau=k\theta+0} = \left.\frac{ds^*(\tau)}{d\tau}\right|_{\tau=k\theta-0} = 0,\ \left.\frac{ds_*(\tau)}{d\tau}\right|_{\tau=k\theta+0} = \left.\frac{ds_*(\tau)}{d\tau}\right|_{\tau=k\theta-0} = 1.$$

So the derivatives of these functions at the breakpoints have a removable singularity, therefore, with this in mind, we can assume that they are continuously differentiable and put

$$\frac{d}{d\tau}s^*(\tau) = 0, \tau \in R \text{ and } \frac{d}{d\tau}s_*(\tau) = 1, \tau \in R.$$

Under condition (15), we define matrix

$$G_0(\tau, s) = \begin{cases} \left[Z^{-1}(\tau+\theta) - Z^{-1}(\tau)\right]^{-1} Z^{-1}(s+\theta), s^*(\tau) - \theta \le s < \tau, \\ \left[Z^{-1}(\tau+\theta) - Z^{-1}(\tau)\right]^{-1} Z^{-1}(s),\ \tau \le s \le s^*(\tau), \end{cases} \tag{16}$$

where $s^*(\tau) = \left[\theta^{-1}\tau\right]\theta$ is a step function with step θ and derivative $\dfrac{d}{d\tau}s^*(\tau) = 0,\ \tau \in R$, since for integer $k \in Z$ one-sided derivatives are $\dfrac{ds^*(k\theta-0)}{d\tau} = \dfrac{ds^*(k\theta+0)}{d\tau} = 0$.

It is easy to verify that the matrix (16) has the properties

$$\frac{\partial}{\partial\tau}G_0(\tau, s) = JG_0(\tau, s), \tau \ne s, \tag{17}$$

$$G_0(\tau, \tau+0) - G_0(\tau, \tau-0) = E, \tag{18}$$

$$G_0(\tau+\theta, s+\theta) = G_0(\tau, s). \tag{19}$$

The matrix (16) with properties (17)–(19) can be called the Green's matrix of set problem of a multiperiodic solution.

The matricant (14) can be represented in the form

$$Z(\tau) = \Gamma_+ e^{2\pi i\nu^0\tau} + \Gamma_- e^{-2\pi i\nu^0\tau}, \tag{20}$$

by using Euler's formulas, where Γ_+ and Γ_- are matrix coefficients of the form

$$\Gamma_+ = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \Gamma_- = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}. \tag{21}$$

We have

$$Z^{-1}(\tau) = Z(-\tau) = \Gamma_+ e^{-2\pi i\nu^0\tau} + \Gamma_- e^{2\pi i\nu^0\tau}, \tag{22}$$

by representation (20).

We obtain from the condition (4) the Fourier series expansion for the function $h(\tau,t)$ in the form

$$h(\tau,t)=\sum_{(k_0,k)} h_{(k_0,k)} \exp\left[2\pi i\left(k_0\nu_0\tau+\langle k,\nu t\rangle\right)\right], \tag{23}$$

where $\nu_0=\theta^{-1}=\omega_0^{-1},\ \nu_1=\omega_1^{-1},\dots,\nu_m=\omega_m^{-1},\ \nu=(\nu_1,\dots,\nu_m),\ k=(k_1,\dots,k_m),$ $\nu t=(\nu_1t_1,\dots,\nu_mt_m),\ \langle k,\nu t\rangle=k_1\nu_1t_1+\ldots+k_m\nu_mt_m,\ k_j\in Z, j=\overline{0,m},\ h_{(k_0,k)}$ are the Fourier coefficients that satisfy the estimate

$$\begin{aligned}&\left|h_{(k_0,k)}\right|\le\|h\|_\delta\, e^{-2\pi\delta(|k_0|+|k|)}, |k|=|k_1|+\ldots+|k_m|,\\ &\|h\|_\delta=\sup_{\overline{\Pi}_\delta^{m+1}}|h(\tau,t)|, |h|=max\left\{|\varphi(\tau,t)|,|\psi(\tau,t)|\right\}, h=(\varphi,\psi).\end{aligned}\tag{24}$$

Further, the norms $\|x\|_\delta$ and $\|X(\tau,t)\|_\delta$ of the vector-function $x(\tau,t)=(x_1(\tau,t),\dots,x_n(\tau,t))$ and the matrix function $X(\tau,t)=[x_{ij}(\tau,t)]_1^n$ are determined by the well-known formulas:

$$\begin{aligned}&\|x\|_\delta=\sup_{\overline{\Pi}_\delta^{m+1}}|x(\tau,t)|, |x(\tau,t)|=\max_{1\le j\le n}|x_j(\tau,t)|,\\ &\|X\|_\delta=\sup_{\overline{\Pi}_\delta^{m+1}}|X(\tau,t)|, |X(\tau,t)|=\max_{1\le j\le n}\sum_{k=1}^n|x_{jk}(\tau,t)|.\end{aligned}$$

Allowing for the term-by-term integration of the expansion of the integrand based on (20)–(23) for $\overline{\Pi}_{\delta/2}\times\overline{\Pi}^m_{\delta/2}$ we calculate an integral of the form

$$I(\tau,t)=\int\limits_{s^*(\tau)}^{\tau} Z^{-1}(s)h(s,t-e\tau+es)\,ds=\int\limits_{s^*(\tau)}^{\tau}\left\{\Gamma_+e^{-2\pi i\nu^0 s}+\Gamma_-e^{2\pi i\nu^0 s}\right\}$$

$$\times\sum_{(k_0,k)}h_{(k_0,k)}e^{2\pi i(k_0\nu_0 s+\langle k,\nu(t-e\tau+es)\rangle)}\,ds=\int\limits_{s^*(\tau)}^{\tau}\sum_{(k_0,k)}\Gamma_+h_{(k_0,k)}e^{2\pi i\langle k,\nu(t-e\tau)\rangle}$$

$$\times e^{2\pi i\left(-\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)s}\,ds+\int\limits_{s^*(\tau)}^{\tau}\sum_{(k_0,k)}\Gamma_-h_{(k_0,k)}e^{2\pi i\langle k,\nu(t-e\tau)\rangle}e^{2\pi i\left(\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)s}\,ds$$

$$=\sum_{(k_0,k)}\Gamma_+h_{(k_0,k)}\left[2\pi i\left(-\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)\right]^{-1}e^{2\pi i\langle k,\nu(t-e\tau)\rangle}\left\{e^{2\pi i\left(-\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)\tau}\right.$$

$$\left.-\,e^{2\pi i\left(-\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)s^*(\tau)}\right\}+\sum_{(k_0,k)}\Gamma_-h_{(k_0,k)}\left[2\pi i\left(\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)\right]^{-1}$$

$$\times e^{2\pi i\langle k,\nu(t-e\tau)\rangle}\left\{e^{2\pi i\left(\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)\tau}-e^{2\pi i\left(\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)s^*(\tau)}\right\}$$

$$=\sum_{(k_0,k)}\Gamma_+h_{(k_0,k)}\left[-\nu^0+k_0\nu_0+\langle k,\nu\rangle\right]^{-1}e^{2\pi i\left(-\nu^0\tau+k_0\nu_0\tau+\langle k,\nu t\rangle\right)}$$

$$\times\left\{1-e^{-2\pi i\left(-\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)s_*(\tau)}\right\}+\sum_{(k_0,k)}\Gamma_-h_{(k_0,k)}\left[\nu^0+k_0\nu_0+\langle k,\nu\rangle\right]^{-1}$$

$$\times e^{2\pi i\left(\nu^0\tau+k_0\nu_0\tau+\langle k,\nu t\rangle\right)}\left\{1-e^{-2\pi i\left(\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)s_*(\tau)}\right\}$$

or

$$I(\tau,t)=\sum_{(k_0,k)}\Gamma_+h_{(k_0,k)}\left[-\nu^0+k_0\nu_0+\langle k,\nu\rangle\right]^{-1}e^{2\pi i\left(-\nu^0\tau+k_0\nu_0\tau+\langle k,\nu t\rangle\right)}$$

$$\times\left\{1-e^{-2\pi i\left(-\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)s_*(\tau)}\right\}+\sum_{(k_0,k)}\Gamma_-h_{(k_0,k)}\left[\nu^0+k_0\nu_0+\langle k,\nu\rangle\right]^{-1} \tag{25}$$

$$\times e^{2\pi i\left(\nu^0\tau+k_0\nu_0\tau+\langle k,\nu t\rangle\right)}\left\{1-e^{-2\pi i\left(\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)s_*(\tau)}\right\}.$$

By virtue of the property $Z(\tau+\theta)=Z(\tau)Z(\theta)$ of the matricant (14) from (25), we easily get the integral

$$\widetilde{I}(\tau,t)=\int\limits_{s^*(\tau)-\theta}^{\tau}Z^{-1}(s+\theta)h(s,t-e\tau+es)\,ds=\int\limits_{s^*(\tau)-\theta}^{\tau}Z^{-1}(\theta)Z^{-1}(s)$$

$$\times h(s,t-e\tau+es)\,ds=\sum_{(k_0,k)}Z^{-1}(\theta)\Gamma_+h_{(k_0,k)}\left[2\pi i\left(-\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)\right]^{-1}$$

$$\times e^{2\pi i\left(-\nu^0\tau+k_0\nu_0\tau+\langle k,\nu t\rangle\right)}\left\{1-e^{-2\pi i\left(-\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)(\theta+s_*(\tau))}\right\}$$

$$+\sum_{(k_0,k)}Z^{-1}(\theta)\Gamma_-h_{(k_0,k)}\left[2\pi i\left(\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)\right]^{-1}e^{2\pi i\left(\nu^0\tau+k_0\nu_0\tau+\langle k,\nu t\rangle\right)}$$

$$\times\left\{1-e^{-2\pi i\left(\nu^0+k_0\nu_0+\langle k,\nu\rangle\right)(\theta+s_*(\tau))}\right\}. \tag{26}$$

We assume that the condition of strong rational incommensurability of frequencies $\nu^0,\nu_0,\nu_1,\ldots,\nu_m$ of the form

$$\left|\nu^0+k_0\nu_0+\langle k,\nu\rangle\right|^{-1}\le\frac{c}{2\pi}\left(|k_0|+|k|\right)^{\gamma} \tag{27}$$

with constants $c>0$ and $\gamma\ge m+2$, is satisfied to ensure the legality of the term-by-term integration of series (25), (26) and prove the convergence of the series obtained.

It is obvious that $|\Gamma_{\pm}| = 1$, $|Z^{-1}(\theta)| \leq 2$ and

$$|\exp\left[-2\pi i\left(\pm\nu^0 + k_0\nu_0 + \langle k, \nu\rangle\right)(\alpha + s_*(\tau))\right]| \leq 1,$$

where $\alpha = 0$ or $\alpha = \theta$.

Then, by virtue of the condition (24) for the Fourier coefficients of a real analytic vector-function $h(\tau,t)$ and the condition (27) for the frequencies ν^0 of matricant and $\nu_0, \nu_1, \ldots, \nu_m$ of vector-functions $h(\tau,t)$, for members $I_{(k_0,k)}(\tau,t)$ of the series (25) and (26) we have an estimate of the form

$$\begin{aligned}\left\|I_{(k_0,k)}(\tau,t)\right\|_{\delta/2} &\leq 4\cdot\frac{c}{2\pi}\left(|k_0|+|k|\right)^{\gamma}\|h\|_{\delta}e^{-2\pi\delta(|k_0|+|k|)}e^{\pi\delta(\nu^0+|k_0|+|k|)}\\ &= \frac{2c}{\pi}e^{\pi\nu^0\delta}\left(|k_0|+|k|\right)^{\gamma}e^{-\pi\delta(|k_0|+|k|)}\|h\|_{\delta}\\ &\leq \frac{2c}{\pi}\left(\frac{2\gamma}{\pi\delta e}\right)^{\gamma}e^{\pi\delta\nu^0}e^{-\frac{\pi\delta}{2}(|k_0|+|k|)}\|h\|_{\delta} = a\|h\|_{\delta}e^{-\alpha(|k_0|+|k|)},\end{aligned} \tag{28}$$

where $a = \dfrac{2c}{\pi}\left(\dfrac{2\gamma}{\pi\delta e}\right)^{\gamma}e^{\pi\delta\nu^0}$ and $\alpha = \dfrac{\pi\delta}{2}$.

Note that here the inequality

$$(|k_0|+|k|)^{\gamma}e^{-\frac{\pi\delta}{2}(|k_0|+|k|)} \leq \left(\frac{2\gamma}{\pi\delta e}\right)^{\gamma}$$

is taken into account in the evaluation.

It is clear from the estimate (28) that the series (25) and (26) converge uniformly for $(\tau,t) \in \overline{\Pi}_{\delta/2}\times\overline{\Pi}^m_{\delta/2}$, and their sums are real-analytic and are estimated at the norm by $b = 8^{m+1}\dfrac{a}{\delta^{m+1}}\|h\|_{\delta}, 0<\delta<1$.

We have

$$I(\tau+\theta, t+q\omega) = Z^{-1}(\theta)I(\tau,t) \in A^b\left(\overline{\Pi}_{\delta/2}\times\overline{\Pi}^m_{\delta/2}\right),\ q\in Z^m,$$

$$\|I\|_{\delta/2} \leq b = \frac{8^{m+1}a}{\delta^{m+1}}\|h\|_{\delta}, 0<\delta<1, \tag{29}$$

on the basis of the properties of the matricant $Z(\tau+\theta) = Z(\tau)Z(\theta)$.

The integral $\widetilde{I}(\tau,t)$ also has property (29).

Thus, the following lemma is proved.

Lemma 1. *Under the conditions* (15), (27) *and*

$$h(\tau+\theta, t+q\omega) = h(\tau, t) \in A^b\left(\overline{\Pi}_\delta \times \overline{\Pi}_\delta^m\right), q \in Z^m, \tag{30}$$

the integral $I(\tau, t)$ *defined by* (25) *and the matricant* (14) *have the property* (29).

Under the conditions of Lemma 1, the integral relation $\widetilde{I}(\tau, t)$ defined by the expression (26) also has property (29).

Now we are in a position to prove the following theorem.

Theorem 1. *Under the conditions* (15), (27) *and* (30), *the system* (11) *admits a unique* (θ, ω)*-periodic solution*

$$z^*(\tau+\theta, t+q\omega) = z^*(\tau, t) \in A^b\left(\overline{\Pi}_{\delta/2} \times \overline{\Pi}_{\delta/2}^m\right), q \in Z^m, \tag{31}$$

which is representable in the Green's function by the relation

$$z^*(\tau, t) = \int\limits_{s^*(\tau)-\theta}^{s^*(\tau)} G_0(\tau, s) h(s, t - e\tau + es)\, ds \tag{32}$$

and satisfies the estimate

$$\|z^*\|_{\delta/2} \le \frac{8^{m+1} 2\Delta a}{\delta^{m+1}} \|h\|_\delta, \tag{33}$$

where Δ *is a positive constant that bounds with respect to normal* $\theta^0 = \dfrac{1}{\nu^0}$*-periodic matrix* $Y(\tau) = \left[Z^{-1}(\tau+\theta) - Z^{-1}(\tau)\right]^{-1}$, *i.e.* $\|Y\|_\delta \le \Delta$.

Proof. We represent (32) in the form

$$z^*(\tau, t) = \int\limits_{s^*(\tau)-\theta}^{\tau} G_0(\tau, s) h(s, t - e\tau + es)\, ds + \int\limits_{\tau}^{s^*(\tau)} G_0(\tau, s) h(s, t - e\tau + es)\, ds$$

$$= Y(\tau)\widetilde{I}(\tau, t) - Y(\tau) I(\tau, t). \tag{34}$$

We obtain the analyticity and boundedness properties (33) from (34), by virtue of Lemma 1.

Differentiating (34), by virtue of the properties (17) and (18) of the Green's function, we see that (32) really satisfies the system (11), that is, the expression (32) is an integral representation of the required solution. The periodicity of $z^*(\tau, t)$ with period θ in τ is proved by replacing s by $s+\theta$ and using property (19), taking into account the analogical property

$h(\tau,t)$. Its periodicity by t with period ω follows directly from ω-periodicity of $h(\tau,t)$ in t. Therefore, property (31) is completely proved.

Uniqueness follows from the condition (15).

Theorem 1 is completely proved.

3 Multiperiodic solution of the subsystem in non-critical case

Now we investigate the question of the existence of (θ,ω)-periodic solution of the system (12) according to the method described above.

We consider in accordance with this, the homogeneous system

$$Dw = Cw, \tag{35}$$

which has a matricant $W(\tau)=\exp[C\tau]$ having properties

$$DW(\tau) = CW(\tau), W(0) = E \tag{36}$$

with a unit $(n_1\times n_1)$-matrix E and satisfying conditions

$$|W(\tau,t)| \le ae^{-\alpha|\tau|}, \tag{37}$$

$$det|W(\theta)-E| \ne 0 \tag{38}$$

with some constants $a\ge 1$ and $\alpha>0$ by virtue of the condition (9).

Analogically to constructing matrix (16), we introduce the Green's matrix

$$G_1(\tau,s) = \begin{cases} \left[W^{-1}(\tau+\theta)-W^{-1}(\tau)\right]^{-1}W^{-1}(s+\theta), s^*(\tau)-\theta\le s<\tau, \\ \left[W^{-1}(\tau+\theta)-W^{-1}(\tau)\right]^{-1}W^{-1}(s),\ \tau\le s\le s^*(\tau), \end{cases} \tag{39}$$

and the problem of multiperiodic solutions of the system (12), which has the properties

$$DG_1(\tau,s) = CG_1(\tau,s), \tau\ne s, \tag{40}$$

$$G_1(\tau,\tau+0)-G_1(\tau,\tau-0) = E, \tag{41}$$

$$G_1(\tau+\theta,s+\theta) = G_1(\tau,s). \tag{42}$$

These properties easily follow from relations (35)–(38).

Theorem 2. *Under the conditions* (9) *and*

$$g(\tau+\theta,t+q\omega) = g(\tau,t) \in A^b\left(\overline{\Pi}_\delta\times\overline{\Pi}_\delta\right), q\in Z^m, \tag{43}$$

the system (12) *allows the unique* (θ,ω)*-periodic solution* $w^*(\tau,t)$ *with the integral representation of the form*

$$w^*(\tau,t) = \int\limits_{s^*(\tau)-\theta}^{s^*(\tau)} G_1(\tau,s)g(s,t-e\tau+es)\,ds \in A^b\left(\overline{\Pi}_\delta \times \overline{\Pi}_\delta^m\right) \tag{44}$$

satisfying the estimate

$$\|w^*\|_\delta \le \frac{a}{\alpha}\|g\|_\delta. \tag{45}$$

The proof is carried out similarly to the proof of Theorem 1 with a change of references to the corresponding formulas.

Indeed, the matricant $W^{-1}(s)$ can be represented as a finite quasi-polynomial of the form

$$W^{-1}(s) = \sum_{j=1}^{n} P_j(s)e^{-\lambda_j s}, \tag{46}$$

where $P_j(s)$ are matrix polynomials with respect to of s degree n_j, $\lambda_j = 2\pi(\alpha_j + i\beta_j)$ are eigenvalues of the matrix C with non-zero real parts $\mathrm{Re}\lambda_j = 2\pi\alpha_j \neq 0$.

The vector-function $g(\tau,t)$ can be expanded to the Fourier series by virtue of the condition (43), moreover,

$$g(s,t-e\tau+es) = \sum_{(k_0,k)} g_{(k_0,k)}e^{2\pi i\langle k,\nu(t-e\tau)\rangle}e^{2\pi i[k_0\nu_0+\langle k,\nu\rangle]s} \tag{47}$$

with the Fourier coefficients $g_{(k_0,k)}$ satisfying the inequality

$$|g_{(k_0,k)}| \le \|g\|_\delta e^{2\pi\delta[|k_0|+|k|]} \tag{48}$$

for $(\tau,t) \in \overline{\Pi}_\delta \times \overline{\Pi}_\delta^m$.

Then, when integrating over s the product of the matrix (46) and the vector-function (47), represented as

$$W^{-1}(s)g(s,t-e\tau+es) = \sum_{j=1}^{n}\sum_{(k_0,k)} P_j(s)$$
$$\times e^{2\pi[-\alpha_j+i(-\beta_j+k_0\nu_0+\langle k,\nu\rangle)]s} \cdot g_{(k_0,k)} \cdot e^{2\pi i\langle k,\nu(t-e\tau)\rangle} \tag{49}$$

factors of the form $[-\alpha_j + i\,(-\beta_j + k_0\nu_0 + \langle k,\nu\rangle)]^{-\gamma_j}$ with natural exponents γ_j appeared on the coefficients of quasi polynomials, which do not belong to the category of small denominators, since

$$[-\alpha_j + i\,(-\beta_j + k_0\nu_0 + \langle k,\nu\rangle)] \ge |\alpha_j|, j = \overline{1,n}. \tag{50}$$

Therefore, the integral of the vector-function of the form

$$I(\tau,t)=\int\limits_{s^*(\tau)}^{\tau} W^{-1}(s)g(s,t-e\tau+es)\,ds \tag{51}$$

by virtue of the relations (46)–(50) for $(\tau,t)\in\overline{\Pi}_\delta\times\overline{\Pi}_\delta^m$ is real analytic, and therefore the vector-function $w^*(\tau,t)\in A^b\left(\overline{\Pi}_\delta\times\overline{\Pi}_\delta^m\right)$. We see by the direct verification based on (40), (41) and (43), that the expression (44) is a solution to the system (12). It is easy to prove the (θ,ω)-periodicity of the solution (41) by virtue of the condition (43) and the property (42).

The estimate (45) follows from inequality (37). The uniqueness is a consequence of the condition (38). Theorem 2 is completely proved.

4 Multiperiodic solution of the main system

Now, applying Theorem 1 to the subsystems

$$Dz_j=J_jz_j+h_j(\tau,t),j=\overline{1,n_0}, \tag{52}$$

with matrices of the form (8) with constants ν_j^0, $j=\overline{1,n_0}$, satisfying the condition (27) for $\nu^0=\nu_j^0$, we establish the existence of the unique (θ,ω)-periodic solution $z^*(\tau,t)$ of each subsystem (52) with a free term $h_j(\tau,t)=h(\tau,t)$ satisfying condition (30).

Thus, Theorems 1 and 2 imply the existence of the unique (θ,ω)-periodic solution $x^*(\tau,t)$ of the basic system (1) under conditions (3), (4), (27) for $\nu^0=\nu_j^0,j=\overline{1,n_0}$, and (9).

The equivalence of the main problem of the existence of the multiperiodic solution of the system (1) with analogous problems for subsystems of the types (11) and (12) allows us to extend the developed methodology to the original system in a direct way.

Indeed, a matricant $X(\tau)$ satisfying

$$DX(\tau)=AX(\tau),X(0)=E, \tag{53}$$

by virtue of the conditions (3), (9) and (27) for $\nu^0=\nu_j^0,j=\overline{1,n_0}$, has the property

$$det[X(\theta)-E]\neq 0, \tag{54}$$

where E is an identity matrix of the n-th order.

The inequality (54) allows us to construct the Green's matrix

$$G(\tau,s)=\begin{cases}\left[X^{-1}(\tau+\theta)-X^{-1}(\tau)\right]^{-1}X^{-1}(s+\theta),\ s^*(\tau)-\theta\le s<\tau,\\ \left[X^{-1}(\tau+\theta)-X^{-1}(\tau)\right]^{-1}X^{-1}(s),\ \ \tau\le s\le s^*(\tau),\end{cases} \tag{55}$$

which, by virtue of (53), has the properties

$$DG(\tau,s)=AG(\tau,s),\tau\neq s, \tag{56}$$

$$G(\tau,\tau+0)-G(\tau,\tau-0)=E, \tag{57}$$

$$G(\tau+\theta,s+\theta)=G(\tau,s). \tag{58}$$

The condition (4) and the Green's matrix (55) with properties (56)–(58) give the integral representation

$$x^*(\tau,t)=\int\limits_{s^*(\tau)-\theta}^{s^*(\tau)} G(\tau,s)f(s,t-e\tau+es)\,ds \in A^b\left(\overline{\Pi}_{\delta/2}\times\overline{\Pi}_{\delta/2}\right) \tag{59}$$

of the unique (θ,ω)-periodic solution $x^*(\tau,t)$ of the system (1) satisfying the estimate

$$\|x^*(\tau,t)\|_{\delta/2}\leq d\|f\|_\delta \tag{60}$$

with a constant $d=d\left(c,\gamma,\nu_1^0,...,\nu_{n_0}^0,\lambda_1,...,\lambda_{n_1},\delta\right), 0<\delta<1$.

Thus, the following theorem is proved.

Theorem 3. *Under the conditions* (3) *for the matrix A with eigenvalues $\mu_j = 2\pi i\nu_j^0$, $j=\overline{1,n_0}$, satisfying the conditions* (27) *for $\nu^0=\nu_j^0, j=\overline{1,n_0}$, and $\lambda_j=\lambda_j(C), j=\overline{1,n_1}$, which submit to the condition* (9) *and for the free term $f(\tau,t)$ having the property* (4)*, the system* (1) *has a unique real analytic (θ,ω)-periodic for $(\tau,t)\in\overline{\Pi}_{\delta/2}\times\overline{\Pi}_{\delta/2}^m$ solution $x^*(\tau,t)$ with integral representation* (59) *satisfying the estimate* (60).

In conclusion, we note that the idea of the methodology which is used in this work has broad development prospects for studying the problems of the multiperiodic solutions of linear systems and in other critical cases.

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Сартабанов Ж.А., Омарова Б.Ж. КЕЙБІР СЫНДЫҚ ЖАҒДАЙДА СЫЗЫҚТЫ ЖҮЙЕНІҢ КӨППЕРИОДТЫ ШЕШІМІНІҢ ИНТЕГРАЛДЫҚ БЕЙНЕЛЕНУІ

Тұрақты коэффициентті және уақыт айнымалылары кеңістігінде бас диагонал бағыты бойынша D дифференциалдау операторлы біртекті емес сызықты жүйенің жалғыз көппериодты шешімінің бар болуы мен интегралдық бейнеленуі есебі қарастырылады. Бұл есеп жүйенің коэффициенттері матрицасының барлық меншікті мәндерінің нақты бөліктері нөлден өзгеше болған жағдайда, яғни сындық емес жағдайда шешілген болатын, әрі бұл жағдайда зерттеу әдісі сындық жағдайды зерттеуге жарамсыз болған. Сондықтан сындық жағдайлар әлі де зерттелмеген күйінде қалып отырды. Бұл мақалада коэффициенттер матрицасының жәй элементар бөлгішті бірнеше таза жорымал меншікті мәндері болғанда, ал жүйенің бос мүшесінің комплекс жазықтықтағы нақты ось жолағында өзгеретін тәуелсіз айнымалылары бойынша нақты аналитикалық қасиеті болғанда және меншікті тербеліс пен мәжбүр етуші күштің жиіліктері Диофанттың қатты өлшемдес еместік шартын қанағаттандырғанда, яғни рационал өлшемдес емес жиілікті периодтылығы болғанда есеп шешімінің бар болуы дәлелденеді. Берілген жүйеге сәйкес біртекті жүйенің нөлден өзгеше көппериоды шешімі болмайтындығы шарты анықталды. Соның негізінде ізделінді жалғыз нақты аналитикалық көппериодты шешімнің интегралдық құрылымы мен бар болуы туралы мәселе шешілетін терминде Грин типті матрицалық функция тұрғызылды. Есепті зерттеу кезінде жүйе сызықты алмастыру арқылы екі түрлі ішкі жүйеге ыдырайды: а) сындық сипаттағы екінші ретті бірнеше біртектес жүйелерге және б) сындық емес жағдайдағы жүйе. Қойылған есеп жоғарыда сипатталған әдіс арқылы бұл ішкі жүйелер үшін жеке-жеке шешіледі, одан кейін құрастырылған әдіс жалпы түрде негізгі жүйеге қолданылады. Сонымен мақалада тұрақты коэффициентті және бірдей D дифференциалдау операторлы сызықты теңдеулер жүйесінің жалғыз көппериодты шешімінің бар болуы есебін зерттеудің және оны тұрғызудың жаңа әдісі ұсынылды, Әрі ол сындық емес жағдайда да, сындық жағдайда да қолданылады.

Кілттік сөздер. Көппериодты шешім, Грин функциясы, дифференциалдау операторы, нақты аналитикалық функция, интегралдық бейнелеу, сындық және сындық емес жағдайлар.

Сартабанов Ж.А., Омарова Б.Ж. ИНТЕГРАЛЬНОЕ ПРЕДСТАВЛЕНИЕ МНОГОПЕРИОДИЧЕСКОГО РЕШЕНИЯ ЛИНЕЙНОЙ СИСТЕМЫ В ОДНОМ КРИТИЧЕСКОМ СЛУЧАЕ

Рассматривается задача существования и интегрального представления единственного многопериодического решения неоднородной линейной системы с постоянными коэффициентами и оператором дифференцирования D по направлению главной диагонали пространства временных переменных. Эта задача решена в некритическом случае, когда все собственные значения матрицы коэффициентов системы имеют отличные от нуля действительные части, причем метод исследования этого случая не был пригоден для изучения критического случая. Таким образом, проблема изучения критических случаев оставалась открытой. В данной работе доказывается существование решения задачи, когда матрица коэффициентов имеет несколько чисто мнимых собственных значений с простыми элементарными делителями, а свободный член системы обладает свойствами вещественной аналитичности по независимым переменным, изменяющимися в полосе действительной оси комплексной плоскости и периодичности с рационально несоизмеримыми частотами, причем частоты собственных колебаний и вынуждающей силы вместе удовлетворяют диофантовому условию сильной несоизмеримости. Установлено условие отсутствия ненулевого многопериодического решения однородной системы, соответствующей заданной системе. На этой основе построена матричная функция типа Грина, в терминах которой решается вопрос об интегральной структуре и существовании искомого единственного вещественно аналитического многопериодического решения. При исследовании задачи система линейной заменой расщепляется на подсистемы двух видов: а) несколько однотипных систем второго порядка критического характера и б) систему некритического случая. Поставленная задача решается для этих указанных подсистем в отдельности по описанной выше методике, а затем разработанный метод излагается в общей форме для исходной системы. В целом, в работе предложен новый метод исследования задачи существования и построения единственного многопериодического решения линейной системы уравнений с постоянными коэффициентами и одинаковым оператором дифференцирования D. Данный метод применим как в некритических, так и в критических случаях.

Ключевые слова. Многопериодическое решение, функция Грина, оператор дифференцирования, вещественно аналитическая функция, интегральное представление, критический и некритический случаи.

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Boundary condition of volume potential and its application to inverse problem

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Abstract. In this work, an elliptic potential, that can be represented as an integral operator, is constructed by using the fundamental solution of a linear elliptic equation of the second order. Then, the boundary conditions of this integral operator are found. The inverse problem of finding the density of an elliptic potential is solved.

Keywords. Helmholtz potential, fundamental solution of the Helmholtz equation, potential density, potential boundary condition, inverse problem of finding the density.

1 Introduction

It is known that the continuous distribution of mass and charge in a bounded domain $\Omega \subset R^3$ creates the linear (Newtonian) potential [1] according to the formula

$$u(x) = \int_{\Omega} \varepsilon(x-\xi)\rho(\xi)\, d\xi, \tag{1}$$

where $\varepsilon(x-\xi)$ is a fundamental solution of the Laplace equation

$$-\Delta_x \varepsilon(x) = \delta(x), \quad \varepsilon(x)\,|_{|x|\to\infty} = 0. \tag{2}$$

The task of unambiguously finding the density $\rho(x)$ from the given potential $u(x)$ is incorrect since, together with the necessary smoothness of $u(x)$, an unknown boundary value still requires volume potential conditions, i.e. boundary conditions of integral (1).

In the work [2] (T.Sh. Kal'menov, D. Suragan) the boundary conditions of the volume (Newtonian) potential were found for the first time, and in the work [3] (I.V. Bezmenov), [4]

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(T.Sh. Kal'menov, D. Suragan), the boundary conditions of the Helmholtz potential are studied, while in the work [3] (I.V. Bezmenov) the boundary conditions are given in approximate form, the boundary conditions of the Helmholtz operator are written in explicit form. These boundary conditions have the property that stationary waves arriving at the boundary $\partial\Omega$ from Ω pass through $\partial\Omega$ without reflection, i.e. boundary conditions are transparent.

In this paper, the methods [1] are used to obtain the boundary conditions for the volume potential $u(x)$ defined by the integral

$$u(x) = \int\limits_{\Omega} \varepsilon(x,\xi)\rho(\xi)\,d\xi, \tag{3}$$

where $\varepsilon(x,\xi)$ are fundamental solutions of the elliptic equation

$$L(x,D)\varepsilon(x,\xi) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} \varepsilon(x,\xi) = \delta(x,\xi), \tag{4}$$

$$\varepsilon(x,\xi)|_{|x|\to\infty} \to 0.$$

Using the found boundary conditions of the potential $u(x)$ unequivocally determines density $\rho(x)$ of this potential.

In the domain $\Omega \subset R^n$ with a smooth border $\partial\Omega \in \mathrm{C}^2$ consider the elliptical potential

$$u(x) = \int\limits_{\Omega} \varepsilon(x,\xi)\rho(\xi)d\xi, \tag{5}$$

where $\varepsilon(x,\xi)$ are fundamental solutions of the linear elliptic equation of the second order, i.e.

$$L(x,D)u(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} u(x) + a(x)u = \rho(x), \tag{6}$$

where $\sum\limits_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2,\ \delta > 0,\ |\xi|^2 = \sum\limits_{i,j=1}^{n} \xi_i^2,\ a_{ij}(x) \in C^{1+\alpha}(\bar{\Omega}),\ a(x) \in C^{\alpha}(\bar{\Omega}),\ a(x) \geq 0,\ \varepsilon_n(x,\xi) = \varepsilon_n(\xi,x)$.

Now we give a brief scheme for constructing a fundamental solution to equation (5) according to the scheme proposed by A.V. Bitsadze [5].

Denote by A_{ij} the ratio of the algebraic complement of the elements a_{ij} of the matrix $\|a_{ij}\|$ of highest coefficients of the equation to the determinant $a = \det\|a_{ij}\|$.

We introduce the function:

$$\sigma(x,\xi) = \sum_{i,j,\xi}^{n} a_{ij}(x)(x_i - \xi_i)(x_j - \xi_j),$$

where x, ξ are arbitrary points of the domain Ω.

Assume that the coefficients $a_{ij}(x) \in C^3(\bar{\Omega}), a(x) \in C^1(\bar{\Omega})$. Due to the uniform ellipticity of equation (1), there exist positive constants k_0 and k_1 such that,

$$k_0|x-\xi|^2 \le \sigma(x,\xi) \le k_1\|x-\xi\|^2.$$

For $x \neq \xi$ we define the function

$$\psi\left(x,\xi\right) = \begin{cases} \psi(x,\xi) = \frac{\sigma(x,\xi)}{\sigma_0(\xi)}^{\frac{2-n}{2}}, \ n > 2, \\ -\frac{1}{2\pi\sigma_0(\xi)} \log \sigma, \ n = 2, \end{cases} \tag{7}$$

$\sigma_0 = \omega_n(n-2)(\sqrt{|a(\xi)|})^{-1}$, ω_n is the area of n-dimensional unit sphere.

The function
$$W(x) = \int\limits_{\Omega_0} \psi(x,\xi)\mu(\xi)d\xi,$$

where $\Omega_0 \subset \Omega$ are the subdomains of the domain Ω, we call a generalized potential with the density $\mu(x)$.

Using the jump of the function $\psi(x,\xi)$ it is easy to show that

$$LW(x) = \mu(x) + \int\limits_{\Omega_0} L\psi(x,\xi)\mu(\xi)d\xi, \tag{8}$$

where the second term on the right is a standard improper integral.

We solve the equation in the domain Ω_0

$$Lu = -\sum_{i,j}^{n} \frac{\partial}{\partial x_j} a_{ij}(x) \frac{\partial}{\partial x_i} + a(x)u = f,$$

looking for $u(x)$ in the form of

$$u(x) = \omega(x) + \int\limits_{\Omega_0} \psi(x,\xi)\mu(\xi)d\xi, \tag{9}$$

where $\omega(x)$ is an arbitrary smooth function, and $\mu(x)$ is an unknown function to be determined.

By applying the operator L to ratio (5) taking into account equality (4) we have

$$\mu(x) + \int\limits_{\Omega_0} k(x,\xi)\mu(\xi)d\xi = F(x), \tag{10}$$

where $k(x,\xi) = L\psi(x,\xi),\ F(x) = L\omega(x) + f(x)$.

The integral equation (10) is an integral equation of the second kind, therefore, for small Ω_0 has a unique solution. Consequently, we have

$$\varepsilon(x,y) = \psi(x,y) + \int\limits_{\Omega} \psi(x,\xi)\mu\xi d\xi, \tag{11}$$

where $\psi(x,\xi)$ is a function defined by equality (3), and $\mu(x)$ is a solution of equation (10), where $\omega(x) \equiv W(x), f(x) = 0$.

Let us check that $\varepsilon(x,y)$ is a fundamental solution of equation (7). Suppose that

$$u(x) = \int\limits_{\Omega_0} \varepsilon(x,y)f(y)dy.$$

Then using the jump property of the function $\psi(x,y)$ we have

$$\begin{aligned} Lu = L\int\limits_{\Omega_0} \varepsilon(x,y)f(y)dy = L\int\limits_{\Omega_0}[\psi(x,y) + \int\limits_{\Omega_0} \psi(x,\xi)\mu(\xi)d\xi] \\ = f(x) + \int\limits_{\Omega_0}[L\psi(x,y) - \mu(x) + \int\limits_{\Omega_0} L\psi(x,\xi)\mu(\xi,y)d\xi]f(y)dy. \end{aligned} \tag{12}$$

From here, we choose $\mu(x,y)$ as a solution of the equation

$$L\psi(x,y) + \mu(x,y) + \int\limits_{\Omega_0} L\psi(x,\xi)\mu(\xi,y)d\xi = 0,$$

so

$$Lu = L\int\limits_{\Omega_0} \varepsilon(x,y)f(y)dy = f(x). \tag{13}$$

Thus, the fundamental solution can be represented by formula (11). In this case, the basic properties of the fundamental solution coincide with the properties of the function $\psi(x,y)$, set by formula (7).

2 Problem N

Find the density $\rho(x)$ of the volume elliptic potential (5).

It should be noted that the elliptic potential (5) is an integral equation of the first kind, so in general the inverse problem N is an ill-posed problem.

Theorem 1. *Let $\rho(x) \in L_2(\Omega)$, then the elliptic potential defined by formula* (5) *satisfies the following boundary condition*

$$-\frac{u(x)}{2} + \int\limits_{\partial\Omega} \varepsilon(x,\xi) \sum_{i,j=1}^{n} n_i a_{ij}(\xi) \frac{\partial}{\partial \xi_j} u(\xi) d\xi - \int\limits_{\partial\Omega} u(\xi) \sum_{i,j=1}^{n} n_i a_{ij}(\xi) \frac{\partial}{\partial \xi_j} \varepsilon(x,\xi) d\xi = 0, \tag{14}$$

$x \in \partial\Omega$.

Conversely, if the function $u(x) \in W_2^2(\Omega)$ satisfies equation (6) *and the potential boundary condition* (14)*, then $u(x)$ coincides with the elliptic potential* (5)*.*

Proof. Substituting the differential equation

$$L(\xi, D)u(\xi) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(\xi) \frac{\partial}{\partial x_j} u(\xi) + a(\xi)u,$$

instead of the function $\rho(\xi)$ in equality (5), we get

$$u(x) = \int\limits_{\Omega} \varepsilon(x,\xi) L(\xi,D)u(\xi)d\xi = \int\limits_{\Omega} \varepsilon(x,\xi) \left(-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(\xi) \frac{\partial}{\partial x_j} + a \right) u(\xi)$$

$$= -\int\limits_{\partial\Omega} \varepsilon(x,\xi) \sum_{i,j=1}^{n} n_i a_{ij}(\xi) \frac{\partial}{\partial \xi_j} u(\xi) d\xi + \int\limits_{\partial\Omega} u(\xi) \sum_{i,j=1}^{n} n_j a_{ij}(\xi) \frac{\partial}{\partial \xi_j} \varepsilon(x,\xi) d\xi \tag{15}$$

$$+ \lim_{r \to 0} \int\limits_{U_r(x)} u(\xi) L(\xi,D) \varepsilon(x,\xi) d\xi + \int\limits_{\Omega/U_r(x)} u(\xi) L(\xi,D) \varepsilon(x,\xi) d\xi, \quad x \in \Omega,$$

where $U_r(x) = \{x \in \Omega, \ \| x - \xi \| < r\}$, and n_j – the direction cosines of the boundary $\partial\Omega$. Since when $x \neq \xi$ it is easy to check that $L(\xi, D)\varepsilon(x - \xi) \equiv 0$ and

$$\lim_{x \to \xi} \int\limits_{U_r} u(\xi) L(\xi, D) \varepsilon(x,\xi) d\xi = u(x).$$

With this in mind, from (11) we get

$$u(x) = \int\limits_{\partial\Omega} \left(\frac{\partial \varepsilon(x,\xi)}{\partial n_\xi} u(\xi) - \varepsilon(x,\xi) \frac{\partial u(\xi)}{\partial n_\xi} \right) d\xi + u(x) = 0, \ x \in \Omega. \tag{16}$$

Using the properties of the fundamental solution $\varepsilon(x,\xi)$, we have

$$\lim_{x\to\partial\Omega}\int\limits_{\partial\Omega} u(\xi)\frac{\partial\varepsilon(x,\xi)}{\partial n_\xi}d\xi = -\frac{u(x)}{2} + \int\limits_{\partial\Omega} u(\xi)\frac{\partial\varepsilon(x,\xi)}{\partial n_\xi}, \tag{17}$$

$$\lim_{x\to\partial\Omega}\int\limits_{\partial\Omega} \varepsilon(x,\xi)\frac{\partial u(\xi)}{\partial n_\xi}d\xi = \int\limits_{\partial\Omega} \varepsilon(x,\xi)\frac{\partial u(\xi)}{\partial n_\xi}d\xi. \tag{18}$$

It follows from (15)–(18) that

$$N[u] = -\frac{u(x)}{2} + \int\limits_{\partial\Omega}\left(\frac{\partial\varepsilon(x,\xi)}{\partial n_\xi}u(\xi) - \varepsilon(x,\xi)\frac{\partial u(\xi)}{\partial n_\xi}\right)d\xi = 0,\ \ x\in\partial\Omega. \tag{19}$$

This proves that the elliptic potential (5) satisfies the boundary condition (14).

Conversely, if $u(x)$ is a solution to the second-order elliptic equation (6) and satisfies the potential boundary condition (14), then $u(x)$ coincides with the elliptic potential (5).

Let $\vartheta(x)\in W_2^2(\Omega)$ be an arbitrary regular solution of problem (6) and (14), and let $u(x)$ be an elliptic potential given by formula (5). Let to us denote $\omega(x) = \vartheta(x) - u(x)$. It is obvious that the function $\omega(x)$ satisfies a homogeneous elliptic equation of the second order

$$L(x,D)\omega(x) = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j}\omega(x) + a\omega(x) = 0. \tag{20}$$

By direct calculation, as above, we will make sure that

$$0 = \int\limits_{\Omega}\omega(\xi)L(\xi,D)\varepsilon(x,\xi)d\xi = \omega(x) - \frac{\omega(x)}{2}$$

$$+\int\limits_{\partial\Omega}\left(\frac{\partial\varepsilon(x,\xi)}{\partial n_\xi}\omega(\xi) - \varepsilon(x,\xi)\frac{\partial\omega(\xi)}{\partial n_\xi}\right)d\xi = \omega(x)|_{x\in\partial\Omega} + N[\omega]|_{x\in\partial\Omega} = 0. \tag{21}$$

Since $\omega(x) = \vartheta(x) - u(x)$ and $N[\vartheta]|_{x\in\partial\Omega} = N[u]|_{x\in\partial\Omega} = 0$, then $N[\omega]|_{x\in\partial\Omega} = 0$. With this in mind, it follows from (21) that

$$\omega(x)|_{x\in\partial\Omega} = 0.$$

As $a(x)\ge 0$, due to the uniqueness of the solution to the Dirichlet problem $\omega(x) = \vartheta(x) - u(x)\equiv 0$, that is, $\vartheta(x)$ coincides with the to elliptic potential. Theorem 1 is proved.

It follows from Theorem 1 that for any $\rho(x)\in L_2(\Omega)$, the elliptic potential defined by the formula (5) satisfies the potential boundary condition (14).

Hence it follows

Theorem 2. *The necessary and sufficient condition for the unambiguous solvability of the elliptic potential with respect to the density $\rho(x) \in L_2(\Omega)$ is the condition $u(x) \in W_2^2(\Omega)$ and the fulfillment of the relation*

$$N[u] = -\frac{u(x)}{2} + \int\limits_{\partial\Omega} \left(\frac{\partial \varepsilon(x,\xi)}{\partial n_\xi} u(\xi) - \varepsilon(x,\xi) \frac{\partial u(\xi)}{\partial n_\xi} \right) d\xi = 0, \;\; x \in \partial\Omega. \tag{22}$$

If the condition of Theorem 2, i.e. condition (22)*, is met, the function $\rho(x)$ is defined by the formula*

$$\rho(x) = L(x,D)u(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} u(x) + a(x)u. \tag{23}$$

It should be noted that inverse and ill-posed problems are studied in [6]–[9].

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Кәлменов Т.Ш., Лес А.К. КӨЛЕМДІК ПОТЕНЦИАЛДЫҢ ШЕКАРАЛЫҚ ШАРТЫ ЖӘНЕ ОНЫҢ КЕРІ ЕСЕПКЕ ҚОЛДАНЫСЫ

Бұл жұмыста екінші ретті сызықтық эллипстік теңдеудің іргелі шешімінің көмегімен интегралдық оператор түрінде кейіптелетін эллипстік потенциал тұрғызылды. Одан кейін осы интегралдық оператордың шекаралық шарттары табылды. Эллипстік потенциалдың тығыздығын табудың кері есебі шешілді.

Кілттік сөздер. Гельмгольц потенциалы, Гельмгольц теңдеуінің іргелі шешімі, потенциал тығыздығы, потенциалдың шекаралық шарт, тығыздықты анықтаудың кері есебі.

Кальменов Т.Ш., Лес А.К. ГРАНИЧНОЕ УСЛОВИЕ ОБЪЕМНОГО ПОТЕНЦИАЛА И ЕГО ПРИЛОЖЕНИЕ К ОБРАТНОЙ ЗАДАЧЕ

В работе с помощью фундаментального решения линейного эллиптического уравнения второго порядка построен эллиптический потенциал представимое в виде интегрального оператора. Затем найдены граничные условия этого интегрального оператора. Решена обратная задача нахождения плотности эллиптического потенциала.

Ключевые слова. Потенциал Гельмгольца, фундаментальные решение уравнения Гельмгольца, плотность потенциала, потенциальное граничное условие, обратная задача нахождения плотности.

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On Weyl modules with singular highest weights for algebraic groups of type C_l

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Abstract. In this article, we calculate the composition factors and the submodule structures of some Weyl modules with singular highest weights for simply connected and semisimple algebraic groups of type C_l over an algebraically closed field of characteristic $p \geq h$, where h is the Coxeter number. In particular, we obtain several new examples of simple Weyl modules.

Keywords. Algebraic group, Weyl module, simple Weyl module, singular weight.

1 Introduction

In the singular case, the structures of Weyl modules are very different from the structures of the corresponding Weyl modules with p-regular highest weights. Weyl modules with singular highest weights are studied in [1]–[6]. In [1], all simple Weyl modules for algebraic groups of type A_l are described. Except Weyl modules with highest weights in the initial alcove, they all have singular highest weights. Weyl modules with singular restricted highest weights of rank two algebraic groups were described in [2]. It was proved in [3] that Weyl modules with singular highest weights from alcoves, the main ones adjacent to the initial alcove, are simple. Simple Weyl modules with singular highest weights related to the standard module for semisimple algebraic groups were studied in [4]. The structure of the Weyl modules $V((p-3)(\omega_3+\omega_4))$, $V(2(\omega_3+\omega_4))$ $(p=7)$ with singular highest weights for the algebraic group of type D_4 was calculated in [5, Theorem 2, (d)]. The simplicity of the following Weyl modules with singular highest weights for the algebraic group of type B_4 was proved in [5, Theorem 1, (d)]: $V((p-4)\omega_4)$, $V((p-5)\omega_4)$, $V((p-6)\omega_4)$, $V((p-7)\omega_4)$. In [6], for the algebraic groups of type D_l, the author calculated the structures of all Weyl modules with singular highest weights defined by the dominant elements of the following subsets of

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affine Weyl groups:

$$Y := \{y_{-1} = 1,\, y_0 = s_0,\, y_i = s_0 s_2 \cdots s_{i+1} \,|\, i = 1,\, 2,\, \cdots, l-1\} \cup \{y'_{l-2} = y_{l-3} s_l\};$$

$$Z := \{z_0 = y_0,\, z_i = y_i s_1 \,|\, i = 1,\, 2,\, \cdots, l-1\} \cup \{z'_{l-2} = z_{l-3} s_l\}.$$

In this paper, we give the structure of Weyl modules with singular highest weights for simply connected and semisimple algebraic groups of type C_l ($l > 2$) with highest weights defined by the elements of the following dominant subsets of the affine Weyl group W_p:

$$Y_1 = \{y_{-1} := 1,\, y_0 := s_0,\, y_i := s_0 s_2 \cdots s_{i+1} \,|\, i = 1,\, 2,\, \cdots, l-1\};$$

$$Y_2 = \{y_{l+j-1} := s_0 s_2 s_3 \cdots s_l s_{l-1} \cdots s_{l-j} \,|\, j = 1,\, 2,\, \cdots, l-2\};$$

$$Z_1 = \{z_0 := y_0,\, z_i := y_i s_1 \,|\, i = 1,\, 2,\, \cdots, l-1\};$$

$$Z_2 = \{z_{l+j-1} := y_{l+j-1} s_1 \,|\, j = 1,\, 2,\, \cdots, l-3\}.$$

In the p-regular case, the composition factors and the submodule structures of Weyl modules for the algebraic groups of type C_l were computed in the following cases:

- the group of type $C_2 = B_2$, $p > 0$, for all restricted weights [7], [2];
- the groups of type C_l, $p \geq h$, for all highest weights in

$$\{\lambda_0,\, \lambda_1,\, \cdots,\, \lambda_l \,|\, s_{\beta_i,1} \cdot (\lambda_i) = \lambda_{i-1},\, i = 1,\, 2,\, \cdots,\, l\},$$

where $\lambda_0 \in C_1$,

$$\beta_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{l-1} + \alpha_l,\, i = 1,\, 2,\, \cdots,\, l-2,$$

$\beta_{l-1} = \alpha_1 + \cdots + \alpha_l$ and $\beta_l = 2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l$ [8];

- the groups of type C_l, $p > 0$, for all fundamental weights [9];
- the groups of type $C_l\,(l \leq 4)$, $p = 2$, for all restricted weights [10];
- the group of type C_3, $p = 2$, for all restricted weights [11];
- the group of type C_3, $p = 3$, for all restricted weights [12].

In the main, we use the standard notation and notation introduced in [6]. That is, let R be the irreducible root system of type C_l and let G be a simply connected and semisimple algebraic group with root system R over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq h$, where h is the Coxeter number of R. We assume that $R \subset \mathbb{R}^l$, where $\mathbb{R}$ is the field of real numbers. On $\mathbb{R}^l$ there is the usual euclidian inner product $(\cdot, \cdot)$. This leads to the natural pairing $\langle \cdot, \cdot \rangle : \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ given by $\langle \lambda, \mu \rangle = (\lambda, \mu^\vee)$, where $\mu^\vee = \dfrac{2}{(\mu, \mu)}\mu$. Let R^+ be the set of positive roots and let $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ be the set of simple roots.

Let $T \subseteq G$ be a maximal torus, and let B be the Borel subgroup corresponding to the negative roots. We denote by U the unipotent radical of B. The set $X(T)$ of additive characters for T can be seen as a subset of $\mathbb{R}^l$ with basis $\omega_1, \omega_2, \cdots, \omega_l$ satisfying $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$. The set $X(T)$ also has the following property:

$$X(T) = \{\lambda \in \mathbb{R}^l \,|\, \langle \lambda, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

Any rational G-module can be considered as the direct sum of T-modules:

$$V = \bigoplus_{\lambda \in X(T)} V_\lambda,$$

where $V_\lambda = \{v \in V \,|\, tv = \lambda(t)v, \text{ for all } t \in T\}$. If $V_\lambda \neq 0$ we say that λ is a weight of V. In this case V_λ is called a weight subspace of V.

Let

$$X(T)^+ = \{\lambda \in X(T) \,|\, \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in R^+\}$$

be the set of dominant weights.

We define by

$$[V] = \sum_{\lambda \in X(T)} \dim_k V_\lambda e^\lambda \in \mathbb{Z}(X(T)) = \bigoplus_{\lambda \in X(T)} \mathbb{Z}e^\lambda$$

a formal character of V.

Let $\lambda \in X(T)^+$, and let $H^0(\lambda)$ be the vector space over $\mathbb{K}$ of all regular functions $f : G \to \mathbb{K}$ satisfying:

$$f(bg) = \lambda(b^{-1})f(g), \text{ for all } b \in B, \; g \in G.$$

We define on $H^0(\lambda)$ a G-module structure given by

$$gf(h) = f(hg), \quad f \in H^0(\lambda), \, g, h \in G.$$

And also it is well known that $H^0(\lambda) = Ind_B^G \mathbb{K}_\lambda$, where $\mathbb{K}_\lambda$ is a one-dimensional B-module defined by $\lambda \in X(T)^+$ via the isomorphism $B/U \cong T$. Let $L(\lambda)$ be a maximal semisimple submodule (socle) of $H^0(\lambda)$. Each $L(\lambda)$ is a simple G-module and every simple G-module is isomorphic to $L(\lambda)$ for some $\lambda \in X(T)^+$. Weyl module $V(\lambda)$ with the highest weight $\lambda \in X(T)^+$ is isomorphic to $H^0(-w_0(\lambda))^*$, where w_0 is the maximal element of Weyl group W for R. There is the following Weyl character formula:

$$\chi(\lambda) := [V(\lambda)] = [H^0(\lambda)] = \frac{\sum\limits_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}}{\sum\limits_{w \in W} (-1)^{l(w)} e^{w(\rho)}}.$$

Let V be a G-module. We define *a composition coefficient* $[V : L(\lambda)]$ for $\lambda \in X(T)^+$ such that

$$[V] = \sum_{\lambda \in X(T)^+} [V : L(\lambda)][L(\lambda)].$$

If $[V : L(\lambda)] \neq 0$, then we say that $L(\lambda)$ is a *composition factor of* V.

For $\alpha \in R^+$ and $n \in \mathbb{Z}$ let us define the affine reflections $s_{\alpha,n}$ on $X(T)$ by

$$s_{\alpha,n} \cdot \lambda = \lambda - \langle \lambda + \rho, \alpha \rangle + np\alpha \text{ for all } \lambda \in X(T).$$

Denote by W_p the *affine Weyl group* generated by all $s_{\alpha,n}$ with $\alpha \in R^+$ and $n \in \mathbb{Z}$. Finite Weyl group W of R appears as the subgroup of W_p generated by the reflections $s_{\alpha,0}$ with $\alpha \in R^+$.

Let $\alpha_0 = \omega_2$ be the unique maximal short root of R. We will use the following notation: $s_{\alpha_i,0} := s_i$ for all $i \in \{1, 2, \cdots, l\}$ and $s_0 := s_{\alpha_0,1}$. The set of simple reflections in W is $S = \{s_i \,|\, i = 1, 2, \cdots, l\}$ and the set of *simple affine reflections* in W_p is $S_p = S \cup \{s_0\}$. Then (W_p, S_p) is the Coxeter group of type $\widetilde{C}_l$ with the following defending relations

$$(s_i s_j)^{m_{ij}} = 1, \; s_0^2 = 1, \; (s_0 s_i)^2 = 1 \; (i \neq 2), \; (s_0 s_2)^3 = 1, \tag{1}$$

where $i, j \in \{1, \cdots, l\}$ and

$$m_{ij} = \begin{cases} 1, & i = j; \\ 2, & \text{if } |i - j| > 1; \\ 3, & \text{if } |i - j| = 1 \text{ and } (i,j) \notin \{(l-1,l), \, (l,l-1)\}; \\ 4, & \text{if } (i,j) \in \{(l-1,l), \, (l,l-1)\}. \end{cases}$$

We will also use the affine hyperplanes and the affine alcoves. For $\alpha \in R^+$ and $n \in \mathbb{Z}$ we define the affine hyperplane

$$H_{\alpha,n} = \{v \in \mathbb{R}^l \,|\, \langle v + \rho, \alpha \rangle = np\}.$$

The set of affine alcoves A is defined as the set of connected components of

$$\mathbb{R}^l \setminus \Big(\bigcup_{\alpha \in R_+, \, n \in \mathbb{Z}} H_{\alpha,n} \Big).$$

The initial alcove $C_1 \in A$ is defined by

$$C_1 = \{v \in \mathbb{R}^l \,|\, 0 < \langle v + \rho, \alpha \rangle < p \text{ for all } \alpha \in R^+\}.$$

We denote by $\overline{C}_1$ a closure of C_1.

Let $W_p^+ \subset W_p$ be the set of *dominant elements* defined by

$$W_p^+ = \{w \in W_p \,|\, w \cdot \nu \in X(T)^+ \text{ for any } \nu \in C_1\}.$$

The stabilizer $stab(\lambda)$ of $\lambda \in X(T)$ is the set

$$stab(\lambda) = \{w \in W_p \,|\, w \cdot \lambda = \lambda\}.$$

If $stab(\lambda) \cap S_p = \emptyset$ we say λ is a *regular weight*, otherwise it is called a *singular weight.* Let $\lambda = w \cdot \nu$, where $w \in W_p$ and $\nu \in \overline{C}_1$. It is known that λ is a regular weight if and only if $\nu \in C_1$.

Let $H_0 := H_{\alpha_0,1}$ and $H_i := H_{\alpha_i,0}$ for all $i \in \{1, 2, \cdots, l\}$. Denote by $\nu_{i_1, i_2, \cdots, i_m}$ any element of $\overline{C}_1 \setminus C_1$ satisfying the following conditions:

1) $\nu_{i_1, i_2, \cdots, i_m} \in H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_m}$
2) $i_1, i_2, \cdots, i_m \in \{0, 1, \cdots, l\}$;
3) $i_1 < i_2 < \cdots < i_m$;
4) $m \in \{1, 2, \cdots, l+1\}$.

Let $\nu \in \overline{C}_1 \setminus C_1$. Denote by $\overline{w}_\nu$ the (left) coset of the stabilizer $stab(\nu)$ containing the element $w \in W_p^+$. Then $W_p^+ = \cup_{w \in W_p^+} \overline{w}_\nu$. An action of $\overline{w}_\nu$ on ν is defined by $\overline{w}_\nu \cdot \nu = u \cdot \nu$ for any $u \in \overline{w}_\nu$. If $\overline{w}_\nu \cdot \nu \in X(T)^+$ we say $\overline{w}_\nu$ is *dominant for* ν. Then, up to isomorphism, $\overline{w}_\nu$ defines a simple G-module (respectively, a Weyl module) with highest weight $\overline{w}_\nu \cdot \nu$. We will use notation $\overline{w}$ for the coset $\overline{w}_\nu$ when ν is fixed.

Let $W' \subset W_p^+$ and $\nu \in \overline{C}_1 \setminus C_1$. By definition, put

$$X_\nu^+(W') := \{\overline{w}_\nu \,|\, \overline{w}_\nu \cdot \nu \in X(T)^+ \text{ and } w \in W'\}.$$

We say that $X_\nu^+(W')$ is *the set of dominant elements of* W' *for* ν. In particular,

$$X_\nu^+ := \{\overline{w}_\nu \,|\, \overline{w}_\nu \cdot \nu \in X(T)^+ \text{ and } w \in W_p^+\}$$

is the set of dominant elements of W_p^+ for ν.

The main result of this paper is formulated as a theorem.

Theorem 1. *Let G be the simply connected and semisimple algebraic group of type C_l $(l > 2)$ over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq h$, where h is the Coxeter number. Suppose that $\nu \in \overline{C}_1 \setminus C_1$ and $X_\nu^+(Y_1 \cup Y_2 \cup Z_1 \cup Z_2) \neq \emptyset$. If $\overline{w}_\nu \in X_\nu^+(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)$, then $\chi(\overline{w}_\nu \cdot \nu) = [L(\overline{w}_\nu \cdot \nu)]$ except in the following cases:*

(a) $\chi(\overline{z_{2l-3}} \cdot \nu_0) = [L(\overline{z_{2l-3}} \cdot \nu_0)] + [L(\overline{y_{2l-3}} \cdot \nu_0)]$;

(b) $\chi(\overline{y_i} \cdot \nu_1) = [L(\overline{y_i} \cdot \nu_1)] + [L(\overline{y_{i-1}} \cdot \nu_1)]$, *where* $i \in \{2, 3, \cdots, 2l-3\}$;

(c) $\chi(\overline{z_2} \cdot \nu_2) = [L(\overline{z_2} \cdot \nu_2)] + [L(\overline{y_0} \cdot \nu_2)]$;

(d) $\chi(\overline{z_i} \cdot \nu_2) = [L(\overline{z_i} \cdot \nu_2)] + [L(\overline{z_{i-1}} \cdot \nu_2)]$, *where* $i \in \{3, 4, \cdots, 2l-5\}$;

(e) $\chi(\overline{z_{2l-4}} \cdot \nu_2) = [L(\overline{z_{2l-4}} \cdot \nu_2)] + [L(\overline{z_{2l-3}} \cdot \nu_2)] + [L(\overline{y_{2l-4}} \cdot \nu_2)]$;

(f) $\chi(\overline{z_{i-2}} \cdot \nu_i) = [L(\overline{z_{i-2}} \cdot \nu_i)] + [L(\overline{y_{i-2}} \cdot \nu_i)]$, *where* $i \in \{3, 4, \cdots, l\}$;

(g) $\chi(\overline{z_{2l-i-2}} \cdot \nu_i) = [L(\overline{z_{2l-i-2}} \cdot \nu_i)] + [L(\overline{y_{2l-i-2}} \cdot \nu_i)]$, *where* $i \in \{3, 4, \cdots, l-1\}$.

2 Preliminary results

Let $V(\lambda)$ be Weyl modules with highest weight $\lambda \in X(T)^+$. Then there is a filtration of submodules

$$V(\lambda) = V(\lambda)^0 \supset V(\lambda)^1 \supset V(\lambda)^2 \supset \cdots \tag{2}$$

such that $V(\lambda)/V(\lambda)^1 \cong L(\lambda)$ and

$$\sum_{j>0}[V(\lambda)^j] = \sum_{\alpha\in R^+} \sum_{0<np<\langle\lambda+\rho,\alpha\rangle} \nu_p(np)\chi(s_{\alpha,n} \cdot \lambda), \tag{3}$$

where $\nu_p(m) = \max\{i \in \mathbb{N} \,|\, p^i|m\}$ [13, II.8.19]. The filtration (2) is called the *Jantzen filtration* and the formula (3) is called *Jantzen's sum formula.*

If $\{\varepsilon_i \,|\, i = 1, 2, \cdots, l\}$ is the orthonormal basis of $\mathbb{R}^l$, then the set of positive roots R^+ can be seen as the set

$$\begin{aligned}
&\{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j = \varepsilon_i - \varepsilon_{j+1} \,|\, 1 \le i \le j \le l-1\} \\
&\cup\{\alpha_i + \cdots + \alpha_l = \varepsilon_i + \varepsilon_l \,|\, i = 1, 2, \cdots, l-1\} \\
&\cup\{\alpha_i + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{l-1} + \alpha_l = \varepsilon_i + \varepsilon_{j+1} \,|\, 1 \le i \le j \le l-2\} \\
&\cup\{2\alpha_i + \cdots + 2\alpha_{l-1} + \alpha_l = 2\varepsilon_i,\ \alpha_l = 2\varepsilon_l \,|\, 1 \le i \le l-1\}.
\end{aligned} \tag{4}$$

If $\rho \in X(T)^+$ is the half-sum of the positive roots, then it is easy to prove that

$$\rho = \omega_1 + \omega_2 + \cdots + \omega_l. \tag{5}$$

Between the basis $\{\omega_i \,|\, i = 1, 2, \cdots, l\}$ of $X(T)$ and the orthonormal basis $\{\varepsilon_i \,|\, i = 1, 2, \cdots, l\}$ there is the following relation:

$$\begin{cases} \omega_1 = \varepsilon_1, \\ \omega_2 = \varepsilon_1 + \varepsilon_2, \\ \vdots \\ \omega_{l-1} = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{l-1}, \\ \omega_l = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_l. \end{cases} \tag{6}$$

Let $\mathbb{Z}$ be the set of integers, and $m_1, m_2, \cdots, m_l \in \mathbb{Z}$. If $\lambda = \sum_{i=1}^{l} m_i\omega_i \in X(T)$, then using (4)–(6), we get

$$\langle \lambda + \rho, \alpha \rangle = \begin{cases} m_i + 1, \text{ if } \alpha = \alpha_i, \ i = 1, 2, \cdots, l; \\ \\ m_i + \cdots + m_j + (j - i + 1), \\ \text{if } \alpha = \alpha_i + \cdots + \alpha_j, \ 1 \le i < j = 1, 2, \cdots, l-1; \\ \\ m_i + \cdots + m_{l-1} + 2m_l + l - i + 2, \\ \text{if } \alpha = \alpha_i + \cdots + \alpha_l, \ i = 1, 2, \cdots, l-1; \\ \\ m_i + \cdots + m_j + 2m_{j+1} + \cdots + 2m_l + 2l - i - j + 1, \\ \text{if } \alpha = \alpha_i + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{l-1} + \alpha_l, \ 1 \le i \le j \le l-2; \\ \\ m_i + m_{i+1} + \cdots + m_l + l - i + 1, \\ \text{if } \alpha = 2\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{l-1} + \alpha_l, \ 1 \le i \le l-1. \end{cases} \tag{7}$$

Let $\nu = a_1\omega_1 + a_2\omega_2 + \cdots + a_l\omega_l \in X(T)$, where $a_j \in \mathbb{Z}$ for all $j \in \{1, 2, \cdots, l\}$. We give here two easy lemmas (Lemmas 1 and 2) on relation between $\nu \in X(T)$ and $w \cdot \nu$ for all $w \in Y_1 \cup Y_2 \cup Z_1 \cup Z_2$, for later use.

Lemma 1. *Let $y_i \in Y_1 \cup Y_2$ and $\nu = \sum_{i=1}^{l} a_i\omega_i$, where $a_1, a_2, \cdots, a_l \in \mathbb{Z}$. Then*

(a) $y_0 \cdot \nu = a_1\omega_1 + (p - a_1 - a_2 - 2\sum_{i=3}^{l} a_i - 2l + 1)\omega_2 + \sum_{i=3}^{l} a_i\omega_i$;

(b) for all $i \in \{1, \cdots, l-2\}$,

$$y_i \cdot \nu = (\sum_{j=1}^{i+1} a_j + i)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l} a_j - 2l)\omega_2$$

$$+\sum_{j=3}^{i+1} a_{j-1}\omega_j + (a_{i+1} + a_{i+2} + 1)\omega_{i+2} + \sum_{j=i+3}^{l} a_j\omega_j;$$

(c) $y_{l-1}\cdot\nu=(\sum\limits_{j=1}^{l-1}a_j+2a_l+l)\omega_1+(p-a_1-2\sum\limits_{j=2}^{l}a_j-2l)\omega_2$

$$+\sum_{j=3}^{l-1}a_{j-1}\omega_j+(a_{l-1}+a_l+1)\omega_l;$$

(d) $y_l\cdot\nu=(\sum\limits_{j=1}^{l-2}a_j+2a_{l-1}+2a_l+l+1)\omega_1+(p-a_1-2\sum\limits_{j=2}^{l}a_j-2l)\omega_2$

$$+\sum_{j=3}^{l-2}a_{j-1}\omega_j+(a_{l-2}+a_{l-1}+1)\omega_{l-1}+a_l\omega_l;$$

(e) for all $i\in\{2,3,\cdots,l-3\}$,

$$y_{l+i-1}\cdot\nu=(\sum_{j=1}^{l-i-1}a_j+2\sum_{j=l-i}^{l}a_j+l+i)\omega_1+(p-a_1-2\sum_{j=2}^{l}a_j-2l)\omega_2$$

$$+\sum_{j=3}^{l-i-1}a_{j-1}\omega_j+(a_{l-i-1}+a_{l-i}+1)\omega_{l-i}+\sum_{j=l-i+1}^{l}a_j\omega_j.$$

(f) $y_{2l-3}\cdot\nu=(a_1+2\sum\limits_{j=2}^{l}a_j+2l-2)\omega_1+(p-a_1-a_2-2\sum\limits_{j=3}^{l}a_j-2l+1)\omega_2+\sum\limits_{j=3}^{l}a_j\omega_j.$

Proof. (a) By (7), we have

$$y_0\cdot\nu=\nu-(\langle\nu+\rho,\alpha_0\rangle-p)\alpha_0=\nu+(p-a_1-2\sum_{i=2}^{l}a_i-2l+1)\omega_2$$

$$=a_1\omega_1+(p-a_1-a_2-2\sum_{i=3}^{l}a_i-2l+1)\omega_2+\sum_{i=3}^{l}a_i\omega_i.$$

(b) We use induction on i. According to (7),

$$s_2\cdot\nu=(a_1+a_2+1)\omega_1+(-a_2-2)\omega_2+(a_2+a_3+1)\omega_3+\sum_{i=4}^{l}a_i\omega_i. \qquad (8)$$

Then

$$y_1\cdot\nu=s_2\cdot\nu-(\langle s_2\cdot\nu+\rho,\alpha_0\rangle-p)\alpha_0=s_2\cdot\nu+(p-a_1-a_2-2\sum_{i=3}^{l}a_i-2l+2)\omega_2$$

$$=(a_1+a_2+1)\omega_1+(p-a_1-2\sum_{i=2}^{l}a_i-2l)\omega_2+(a_2+a_3+1)\omega_3+\sum_{i=43}^{l}a_i\omega_i.$$

Therefore, the statement is true for $i = 1$.

Suppose that the statement is true for all $i < t$, where $t \le l-2$. According to (7),

$$s_{t+1}\cdot\nu = \sum_{j=1}^{t-1} a_j\omega_j + (a_t + a_{t+1} + 1)\omega_t + (-a_{t+1} - 2)\omega_{t+1} + (a_{t+1} + a_{t+2} + 1)\omega_{t+2} + \sum_{j=t+3}^{l} a_j\omega_j.$$

By the induction hypothesis,

$$y_{t-1}\cdot\nu = (\sum_{j=1}^{t} a_j + t - 1)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l} a_j - 2l)\omega_2$$

$$+\sum_{j=3}^{t} a_{j-1}\omega_j + (a_t + a_{t+1} + 1)\omega_{t+1} + \sum_{j=t+2}^{l} a_j\omega_j.$$

Then

$$y_t\cdot\nu = y_{t-1}\cdot(s_{t+1}\cdot\nu) = (\sum_{j=1}^{t-1} a_j + (a_t + a_{t+1} + 1) + t - 1)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l} a_j - 2l)\omega_2$$

$$+\sum_{j=3}^{t} a_{j-1}\omega_j + (a_t + a_{t+1} + 1 - a_{t+1} - 2 + 1)\omega_{t+1} + (a_{t+1} + a_{t+2} + 1)\omega_{t+2} + \sum_{j=t+3}^{l} a_j\omega_j$$

$$= (\sum_{j=1}^{t+1} a_j + t)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l} a_j - 2l)\omega_2$$

$$+\sum_{j=3}^{t+1} a_{j-1}\omega_j + (a_{t+1} + a_{t+2} + 1)\omega_{t+2} + \sum_{j=t+3}^{l} a_j\omega_j.$$

Thus, the statement (b) holds for all $i \in \{1,\, 2,\, \cdots,\, l-2\}$.

Other statements can be proved similarly as the previous statement. □

Lemma 2. *Let $z_i \in Z_1 \cup Z_2$ and $\nu = \sum_{i=1}^{l} a_i\omega_i$, where $a_1,\, a_2,\, \cdots,\, a_l \in \mathbb{Z}$. Then*

(a) $z_1\cdot\nu = a_2\omega_1 + (p - a_1 - 2\sum_{i=2}^{l} a_i - 2l)\omega_2 + (a_1 + a_2 + a_3 + 2)\omega_3 + \sum_{i=4}^{l} a_i\omega_i;$

(b) for all $i \in \{2,\, 3,\, \cdots,\, l-2\}$,

$$z_i\cdot\nu = (\sum_{j=2}^{i+1} a_j + i - 1)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l} a_j - 2l)\omega_2 + (a_1 + a_2 + 1)\omega_3$$

$$+\sum_{j=4}^{i+1} a_{j-1}\omega_j + (a_{i+1} + a_{i+2} + 1)\omega_{i+2} + \sum_{j=i+3}^{l} a_j\omega_j;$$

(c) $z_{l-1} \cdot \nu = (\sum_{j=2}^{l-1} a_j + 2a_l + l - 1)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l} a_j - 2l)\omega_2$

$$+(a_1 + a_2 + 1)\omega_3 + \sum_{j=4}^{l-1} a_{j-1}\omega_j + (a_{l-1} + a_l + 1)\omega_l;$$

(d) $z_l \cdot \nu = (\sum_{j=2}^{l-2} a_j + 2a_{l-1} + 2a_l + l)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l} a_j - 2l)\omega_2$

$$+(a_1 + a_2 + 1)\omega_3 + \sum_{j=4}^{l-2} a_{j-1}\omega_j + (a_{l-2} + a_{l-1} + 1)\omega_{l-1} + a_l\omega_l;$$

(e) for all $i \in \{2, 3, \cdots, l-4\}$,

$$z_{l+i-1} \cdot \nu = (\sum_{j=2}^{l-i-1} a_j + 2\sum_{j=l-i}^{l} a_j + l + i - 1)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l} a_j - 2l)\omega_2 + (a_1 + a_2 + 1)\omega_3$$

$$+\sum_{j=4}^{l-i-1} a_{j-1}\omega_j + (a_{l-i-1} + a_{l-i} + 1)\omega_{l-i} + \sum_{j=l-i+1}^{l} a_j\omega_j;$$

(f)

$$z_{2l-4} \cdot \nu = (a_2 + 2\sum_{j=3}^{l} a_j + 2l - 4)\omega_1 + (p - a_1 - 2\sum_{j=2}^{l} a_j - 2l)\omega_2$$

$$+(a_1 + a_2 + a_3 + 2)\omega_3 + \sum_{j=4}^{l} a_j\omega_j.$$

Proof. By the definition, $z_i = y_i s_1$ for all $i \in \{1, 2, \cdots, 2l-4\}$. Then $z_i \cdot \nu = y_i \cdot (s_1 \cdot \nu)$ for all $i \in \{1, 2, \cdots, 2l-4\}$. By (7),

$$s_1 \cdot \nu = (-a_1 - 2)\omega_1 + (a_1 + a_2 + 1)\omega_2 + \sum_{i=3}^{l} a_i\omega_i.$$

Therefore,

$$z_i \cdot \nu = y_i \cdot ((-a_1 - 2)\omega_1 + (a_1 + a_2 + 1)\omega_2 + \sum_{i=3}^{l} a_i\omega_i). \tag{9}$$

Thus, for all $i \in \{1, 2, \cdots, 2l-4\}$, the statement of the lemma for z_i follows from the corresponding statement for y_i of Lemma 1 and from (9). $\square$

Now we find a system of generators of a stabilizer of the elements $\nu_{i_1, i_2, \cdots, i_m} \in \overline{C}_1 \setminus C_1$.

Lemma 3. *Let S_ν be a system of generators of the stabilizer of $\nu \in \overline{C}_1 \setminus C_1$. If $\nu = \nu_{i_1,i_2,\cdots,i_m}$, then $S_\nu = \{s_{i_1},\, s_{i_2},\, \cdots,\, s_{i_m}\}$.*

In particular, if $m = 1$, then $S_{\nu_i} = \{s_i\}$ for all $i \in \{0,\, 1,\, 2,\, \cdots,\, l\}$.

Proof. The generators $s_0,\, s_1,\, \cdots,\, s_l$ of W_p act on ν as follows:

$$s \cdot \nu = \begin{cases} \nu - (\langle \nu + \rho, \alpha_0 \rangle - p)\alpha_0 & \text{if } s = s_0, \\ \\ \nu - \langle \nu + \rho, \alpha_i \rangle \alpha_i & \text{if } i \in \{1,\, 2,\, \cdots,\, l\}. \end{cases} \tag{10}$$

If $i_1 = 0$, then by the definition $\nu_{0,i_2,\cdots,i_m} \in H_0 \cap H_{i_2} \cap \cdots \cap H_{i_m}$. Then

$$\langle \nu_{0,i_2,\cdots,i_m} + \rho, \alpha_0 \rangle = p$$

and

$$\langle \nu_{0,i_2,\cdots,i_m} + \rho, \alpha_i \rangle = 0$$

for all $i \in \{i_2, \cdots, i_m\}$. Therefore, by (10), the condition

$$s \in S_{\nu_{0,i_2,\cdots,i_m}} = \{s \in S_p \,|\, s \cdot \nu_{0,i_2,\cdots,i_m} = \nu_{0,i_2,\cdots,i_m}\}$$

yields $s \in \{s_0,\, s_{i_2},\, \cdots,\, s_{i_m}\} \subset S_p$.

If $i_1 \neq 0$, then by the definition $\nu_{i_1,i_2,\cdots,i_m} \in H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_m}$. Then

$$\langle \nu_{i_1,i_2,\cdots,i_m} + \rho, \alpha_i \rangle = 0$$

for all $i \in \{i_1, i_2, \cdots, i_m\}$. Therefore, by (10), the condition

$$s \in S_{\nu_{i_1,i_2,\cdots,i_m}} = \{s \in S_p \,|\, s \cdot \nu_{i_1,i_2,\cdots,i_m} = \nu_{i_1,i_2,\cdots,i_m}\}$$

yields $s \in \{s_{i_1},\, s_{i_2},\, \cdots,\, s_{i_m}\} \subset S_p$. □

Lemma 4. *Let $w \in Y_1 \cup Y_2$ and $\nu \in \overline{C}_1 \setminus C_1$. Suppose that $w \cdot \nu \in X(T)^+$.*

If $w = 1$, then $\nu \in \{\nu_0\}$.

(b) If $w = y_0$, then $\nu \in \{\nu_2,\, \nu_0\}$.

(c) If $w = y_1$, then $\nu \in \{\nu_1,\, \nu_2,\, \nu_3,\, \nu_{1,3}\}$.

(d) If $w = y_i$, where $i \in \{2,\, 3,\, \cdots,\, l-2\}$, then $\nu \in \{\nu_1,\, \nu_{i+1},\, \nu_{i+2},\, \nu_{1,i+1},\, \nu_{1,i+2}\}$.

(e) If $w = y_{l-1}$, then $\nu \in \{\nu_1,\, \nu_{l-1},\, \nu_l,\, \nu_{1,l-1},\, \nu_{1,l}\}$.

(f) If $w = y_l$, then $\nu \in \{\nu_1,\, \nu_{l-2},\, \nu_{l-1},\, \nu_{1,l-2},\, \nu_{1,l-1}\}$.

(g) If $w = y_{l+i-1}$, where $i \in \{2,\, 3,\, \cdots,\, l-3\}$, then

$$\nu \in \{\nu_1,\, \nu_{l-i-1},\, \nu_{l-i},\, \nu_{1,l-i-1},\, \nu_{1,l-i}\}.$$

(k) If $w = y_{2l-3}$, then $\nu \in \{\nu_1,\, \nu_2,\, \nu_0,\, \nu_{1,2}\}$.

Proof. (a) We prove that $\nu = \nu_{i_1,i_2,\cdots,i_m} \in X(T)^+$ if and only if $m = 1$ and $i_1 = 0$. Indeed, if $m = 1$ and $i_1 = 0$, then by the definition of $\nu_{i_1,i_2,\cdots,i_m}$, $\nu = \nu_0$ and $\nu \in H_0$. Then $\langle \nu_0 + \rho, \alpha_i\rangle \neq 0$ for all $i \in \{1, 2, \cdots, l\}$ and $\langle \nu_0 + \rho, \widetilde{\alpha}\rangle = p$. Therefore $0 < \langle \nu_0 + \rho, \alpha\rangle \leq p$ for all $\alpha \in \Delta$.

Conversely, if $\nu = \nu_{i_1,i_2,\cdots,i_m} \in X(T)^+$, then $\langle \nu_{i_1,i_2,\cdots,i_m} + \rho, \alpha\rangle \neq 0$ for all $\alpha \in R^+$. In particular, $\langle \nu_{i_1,i_2,\cdots,i_m} + \rho, \alpha\rangle \neq 0$ for all $\alpha \in \{\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_m}\}$. Since $\nu_{i_1,i_2,\cdots,i_m} \in \overline{C}_1 \setminus C_1$, then the above condition yields $\nu_{i_1,i_2,\cdots,i_m} \in H_0$. Therefore, by the conditions 1), 3) and 4) of the definition of $\nu_{i_1,i_2,\cdots,i_m}$ we get $m = 1$ and $i_1 = 0$. This implies $\nu = \nu_0$.

(b) Let $\nu = \nu_{i_1,i_2,\cdots,i_m} = \sum\limits_{i=1}^{l} a_i\omega_i$ and $\nu \notin H_0$. Then $\langle \nu_{i_1,i_2,\cdots,i_m} + \rho, \alpha\rangle = 0$ for all $\alpha \in \{\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_m}\}$. This condition yields $a_{i_1} = a_{i_2} = \cdots = a_{i_m} = -1$. Then by the statement (a) of Lemma 1 and by the definition of $\nu_{i_1,i_2,\cdots,i_m}$, $y_0 \cdot \nu \in X(T)^+$ if and only if $m = 1$ and $i_1 = 2$. This implies that $\nu = \nu_2$.

If $\nu \in H_0$, then $i_1 = 0$ and $\langle \nu_{0,i_2,\cdots,i_m} + \rho, \widetilde{\alpha}\rangle = p$. Using (7), we get

$$a_1 + 2\sum_{j=2}^{l} a_j + a_{l-1} + \alpha_l + 2l - 3 = p.$$

Then by the statement (a) of Lemma 1 and by the definition of $\nu_{i_1,i_2,\cdots,i_m}$, $y_0 \cdot \nu \in X(T)^+$ if and only if $m = 1$ and $i_1 = 0$. Therefore, $\nu = \nu_0$.

Other statements can easily be proved similarly as the previous statement. □

For the elements of $Z_1 \cup Z_2$, using Lemma 2, we have the following

Lemma 5. *Let $w \in Z_1 \cup Z_2$ and $\nu \in \overline{C}_1 \setminus C_1$. Suppose that $w \cdot \nu \in X(T)^+$.*

(a) If $w = z_1$, then $\nu \in \{\nu_1, \nu_3, \nu_{1,3}\}$.

(b) If $w = z_2$, then $\nu \in \{\nu_1, \nu_2, \nu_3, \nu_4, \nu_{1,3}, \nu_{1,4}, \nu_{2,4}\}$.

(c) If $w = z_i$, where $i \in \{3, 4, \cdots, l-2\}$, then

$$\nu \in \{\nu_1, \nu_2, \nu_{i+1}, \nu_{i+2}, \nu_{1,i+1}, \nu_{1,i+2}, \nu_{2,i+1}, \nu_{2,i+2}\}.$$

(d) If $w = z_{l-1}$, then $\nu \in \{\nu_1, \nu_2, \nu_{l-1}, \nu_l, \nu_{1,l-1}, \nu_{1,l}, \nu_{2,l-1}, \nu_{2,l}\}$.

(e) If $w = z_l$, then $\nu \in \{\nu_1, \nu_2, \nu_{l-2}, \nu_{l-1}, \nu_{1,l-2}, \nu_{1,l-1}, \nu_{2,l-2}, \nu_{2,l-1}\}$.

(f) If $w = z_{l+i-1}$, where $i \in \{2, 3, \cdots, l-4\}$, then

$$\nu \in \{\nu_1, \nu_2, \nu_{l-i-1}, \nu_{l-i}, \nu_{1,l-i-1}, \nu_{1,l-i}, \nu_{2,l-i-1}, \nu_{2,l-i}\}.$$

(g) If $w = z_{2l-4}$, then $\nu \in \{\nu_1, \nu_2, \nu_3, \nu_{1,2}, \nu_{1,3}, \nu_{2,3}\}$.

By Lemmas 4 and 5, if $X_\nu^+(Y_1 \cup Y_2 \cup Z_1 \cup Z_2) \neq \emptyset$, then

$$\nu \in \{\nu_i \,|\, i = 0, 1, \cdots, l\} \cup \{\nu_{1,i} \,|\, i = 2, 3, \cdots, l\} \cup \{\nu_{2,i} \,|\, i = 3, 4, \cdots, l\}.$$

We calculate the stabilizers of these elements ν.

Lemma 6. *The following statements hold:*

(a) for all $i \in \{0, 1, \cdots, l\}$ $stab(\nu_i) = \{1,\, s_i\}$;

(b) for all $i \in \{3, 4, \cdots, l\}$ $stab(\nu_{1,i}) = \{1,\, s_1,\, s_i,\, s_1 s_i\}$;

(c)for all $i \in \{4, 5, \cdots, l\}$ $stab(\nu_{2,i}) = \{1,\, s_2,\, s_i,\, s_2 s_i\}$;

(d) for all $i \in \{1,\, 2\}$

$$stab(\nu_{i,i+1}) = \{1,\, s_i,\, s_{i+1},\, s_i s_{i+1},\, s_{i+1} s_i,\, s_i s_{i+1} s_i\}.$$

Proof. It follows from the defending relations (1) of the affine Weyl group W_p and Lemma 3. □

Using Lemma 6, we can easily describe $X^+_\nu(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)$ for all ν listed above. Below, we often omit the index ν of the element $\overline{x}_\nu$ when ν is a fixed element of $\overline{C}_1 \setminus C_1$.

Lemma 7. *Suppose that* $\nu \in \{\nu_i \,|\, i = 0,\, 1,\, \cdots,\, l\}$. *Then*

(a) $X^+_{\nu_0}(Y_1 \cup Y_2 \cup Z_1 \cup Z_2) = \{\overline{1},\, \overline{y_{2l-3}}\}$, *where* $\overline{1} = \{1,\, y_0\}$, $\overline{y_{2l-3}} = \{y_{2l-3},\, y_{2l-3}s_0\}$;

(b) $X^+_{\nu_1}(Y_1 \cup Y_2 \cup Z_1 \cup Z_2) = \{\overline{y_i} \,|\, i = 1,\, 2,\, \cdots\, 2l-3\}$, *where* $\overline{y_i} = \{y_i,\, z_i\}$ $(z_{2l-3} = y_{2l-3}s_1)$;

(c) $X^+_{\nu_2}(Y_1 \cup Y_2 \cup Z_1 \cup Z_2) = \{\overline{y_0},\, \overline{y_{2l-4}},\, \overline{z_i} \,|\, i = 2,\, 3,\, \cdots\, 2l-3\}$, *where* $\overline{y_0} = \{y_0,\, y_1\}$, $\overline{y_{2l-4}} = \{y_{2l-4},\, y_{2l-3}\}$ *and* $\overline{z_i} = \{z_i,\, z_i s_2\}$;

(d) for all $i \in \{3,\, 4,\, \cdots,\, l-1\}$

$$X^+_{\nu_i}(Y_1 \cup Y_2 \cup Z_1 \cup Z_2) = \{\overline{y_{i-2}},\, \overline{z_{i-2}},\, \overline{y_{2l-i-2}},\, \overline{z_{2l-i-2}}\},$$

where $\overline{y_{i-2}} = \{y_{i-2},\, y_{i-1}\}$, $\overline{z_{i-2}} = \{z_{i-2},\, z_{i-1}\}$, $\overline{y_{2l-i-2}} = \{y_{2l-i-2},\, y_{2l-i-1}\}$ *and* $\overline{z_{2l-i-2}} = \{z_{2l-i-2},\, z_{2l-i-1}\}$;

(e) $X^+_{\nu_l}(Y_1 \cup Y_2 \cup Z_1 \cup Z_2) = \{\overline{y_{l-2}},\, \overline{z_{l-2}}\}$, *where* $\overline{y_{l-2}} = \{y_{l-2},\, y_{l-1}\}$ *and* $\overline{z_{l-2}} = \{z_{l-2},\, z_{l-1}\}$.

Proof. Let $w \in Y_1 \cup Y_2 \cup Z_1 \cup Z_2$. By the definition,

$$\overline{w}_\nu = \{wx \,|\, x \in stab(\nu)\}. \tag{11}$$

Then, using (11) and Lemmas 4 – 6, we obtain the required statements. □

Lemma 8. *Suppose that* $\nu \in \{\nu_{1,i} \,|\, i = 2,\, 3,\, \cdots,\, l\}$. *Then*

(a) $X^+_{\nu_{1,2}}(Y_1 \cup Y_2 \cup Z_1 \cup Z_2) = \{\overline{y_{2l-4}}\}$, *where*

$$\overline{y_{2l-4}} = \{y_{2l-4},\, y_{2l-3},\, z_{2l-4},\, y_{2l-3}s_1, y_{2l-3}s_1 s_2,\, y_{2l-3}s_1 s_2 s_1\};$$

(b) for all $i \in \{3,\, 4,\, \cdots,\, l-1\}$

$$X^+_{\nu_{1,i}}(Y_1 \cup Y_2 \cup Z_1 \cup Z_2) = \{\overline{y_{i-2}},\, \overline{y_{2l-i-2}}\},$$

where $\overline{y_{i-2}} = \{y_{i-2},\, y_{i-1},\, z_{i-2},\, z_{i-1}\}$ *and*

$$\overline{y_{2l-i-2}} = \{y_{2l-i-2},\, y_{2l-i-1},\, z_{2l-i-2},\, z_{2l-i-1}\};$$

(c) $X^+_{\nu_{1,l}}(Y_1 \cup Y_2 \cup Z_1 \cup Z_2) = \{\overline{y_{l-2}}\}$, *where* $\overline{y_{l-2}} = \{y_{l-2},\, y_{l-1},\, z_{l-2},\, z_{l-1}\}$.

Proof. Follows from (11) and from Lemmas 4–6. □

Lemma 9. *Suppose that* $\nu \in \{\nu_{2,i} \,|\, i = 3, 4, \cdots, l\}$. *Then*

(a) $X^+_{\nu_{2,3}}(Y_1 \cup Y_2 \cup Z_1 \cup Z_2) = \{\overline{z_{2l-5}}\}$, *where*

$$\overline{z_{2l-5}} = \{z_{2l-5},\, z_{2l-4},\, z_{2l-5}s_2,\, z_{2l-5}s_2s_3,\, z_{2l-4}s_2,\, z_{2l-4}s_2s_3\};$$

(b) for all $i \in \{4, 5, \cdots, l-1\}$

$$X^+_{\nu_{2,i}}(Y_1 \cup Y_2 \cup Z_1 \cup Z_2) = \{\overline{z_{i-2}},\, \overline{z_{2l-i-2}}\},$$

where $\overline{z_{i-2}} = \{z_{i-2},\, z_{i-1},\, z_{i-2}s_2,\, z_{i-1}s2, \}$ *and*

$$\overline{z_{2l-i-2}} = \{z_{2l-i-2},\, z_{2l-i-1},\, z_{2l-i-2}s_2,\, z_{2l-i-1}s_2\};$$

(c) $X^+_{\nu_{2,l}}(Y_1 \cup Y_2 \cup Z_1 \cup Z_2) = \{\overline{z_{l-2}}\}$, *where* $\overline{z_{l-2}} = \{z_{l-2},\, z_{l-1},\, z_{l-2}s_2,\, z_{l-1}s_2\}$.

Proof. Follows from (11) and from Lemmas 4–6. □

3 Proof of Theorem 1

Using Lemmas 7 – 9 and the Jantzen's sum formula (3) we can easily prove Theorem 1.
By Lemmas 7 – 9, in the following cases $X^+_\nu(Y_1 \cup Y_2 \cup Z_1 \cup Z_2)$ consists only one coset $\overline{w}$:

- $\nu = \nu_{1,2}$ and $\overline{w} = \overline{y_{2l-4}}$ (Lemma 8, (a)),
- $\nu = \nu_{1,l}$ and $\overline{w} = \overline{y_{l-2}}$ (Lemma 8, (c)),
- $\nu = \nu_{2,3}$ and $\overline{w} = \overline{y_{2l-5}}$ (Lemma 9, (a)),
- $\nu = \nu_{2,l}$ and $\overline{w} = \overline{z_{l-2}}$ (Lemma 9, (c)).

Therefore, in these cases, $\chi(u\cdot\nu) = \chi(\overline{w}\cdot\nu)$ for all $u \in \overline{w}$ and $\chi(x\cdot\nu) = 0$ for all $x \in Y_1 \cup Y_2 \cup Z_1 \cup Z_2 \setminus \overline{w}$.

Hence, by (3), $\sum\limits_{j>0}[V(\overline{w}\cdot\nu)^j] = 0$ for all listed above $\overline{w}$.

Let $i \in \{3, 4, \cdots, l-1\}$. By the statement (b) of Lemma 8, $\chi(w\cdot\nu_{1,i}) = 0$ for all $w \in Y_1 \cup Y_2 \cup Z_1 \cup Z_2$, except in the following cases:

$$\chi(y_{i-2}\cdot\nu_{1,i}) = \chi(z_{i-2}\cdot\nu_{1,i}) = \chi(y_{i-1}\cdot\nu_{1,i}) = \chi(z_{i-1}\cdot\nu_{1,i}) = \chi(\overline{y_{i-2}}\cdot\nu_{1,i}),$$

$$\chi(y_{2l-i-2}\cdot\nu_{1,i}) = \chi(z_{2l-i-2}\cdot\nu_{1,i}) = \chi(y_{2l-i-1}\cdot\nu_{1,i}) = \chi(z_{2l-i-1}\cdot\nu_{1,i}) = \chi(\overline{y_{2l-i-1}}\cdot\nu_{1,i}).$$

Then according to (3), $\sum\limits_{j>0}[V(\overline{y_{i-2}}\cdot\nu_{1,i})^j] = 0$ and $\sum\limits_{j>0}[V(\overline{y_{2l-i-2}}\cdot\nu_{1,i})^j] = 0$ for all $i \in \{3, 4, \cdots, l-1\}$.

Let $i \in \{4, 5, \cdots, l-1\}$. By the statement (b) of Lemma 9, $\chi(w\cdot\nu_{2,i}) = 0$ for all $w \in Y_1 \cup Y_2 \cup Z_1 \cup Z_2$, except in the following cases:

$$\chi(z_{i-2}\cdot\nu_{2,i}) = \chi(z_{i-1}\cdot\nu_{2,i}) = \chi(\overline{z_{i-2}}\cdot\nu_{2,i}),$$

$$\chi(z_{2l-i-2}\cdot\nu_{2,i})=\chi(z_{2l-i-1}\cdot\nu_{2,i})=\chi(\overline{z_{2l-i-1}}\cdot\nu_{2,i}).$$

Then according to (3), $\sum\limits_{j>0}[V(\overline{z_{i-2}}\cdot\nu_{2,i})^j]=0$ and $\sum\limits_{j>0}[V(\overline{z_{2l-i-2}}\cdot\nu_{2,i})^j]=0$ for all $i\in\{4,\,5,\,\cdots,\,l-1\}$.

Arguing as in the above cases, we see that, in the following cases the Jantzen's sum formula also is trivial:

- $\overline{w}\in\{\overline{1},\,\overline{y_{2l-3}}\}\subset X^+_{\nu_0}(Y_1\cup Y_2\cup Z_1\cup Z_2)$ (Lemma 7, (a)),
- $\overline{w}=\overline{y_1}\in X^+_{\nu_1}(Y_1\cup Y_2\cup Z_1\cup Z_2)$ (Lemma 7, (b)),
- $\overline{w}\in\{\overline{y_0},\,\overline{y_{2l-4}}\}\subset X^+_{\nu_2}(Y_1\cup Y_2\cup Z_1\cup Z_2)$ (Lemma 7, (c)),
- $\overline{w}\in\{\overline{y_{i-2}},\,\overline{y_{2l-i-2}}\}\subset X^+_{\nu_i}(Y_1\cup Y_2\cup Z_1\cup Z_2)$, where $i\in\{3,\,4,\,\cdots,\,l-1\}$ (Lemma 7, (d)),
- $\overline{w}=\overline{y_{l-2}}\in X^+_{\nu_l}(Y_1\cup Y_2\cup Z_1\cup Z_2)$ (Lemma 7, (e)).

Therefore, in all these cases $\chi(\overline{w}\cdot\nu)=[L(\overline{w}\cdot\nu)]$.

Thus it remains to prove only the statements (a)–(g) of Theorem 1.

(a) By the statement (a) of Lemma 7, $X^+_{\nu_0}(Y_1\cup Y_2\cup Z_1\cup Z_2)=\{\overline{1},\,\overline{y_{2l-3}},\,\overline{z_{2l-3}}\}$, where $\overline{1}=\{1,\,y_0\}$, $\overline{y_{2l-3}}=\{y_{2l-3}\}$ and $\overline{z_{2l-3}}=\{z_{2l-3}\}$. Therefore, $\chi(w\cdot\nu_0)=0$ for all $w\in Y_1\cup Y_2\cup Z_1\cup Z_2$, except in the following cases:

- $\chi(\nu_0)=\chi(y_0\cdot\nu_0)=\chi(\overline{y_0}\cdot\nu_0)$,
- $\chi(y_{2l-3}\cdot\nu_0)=\chi(\overline{y_{2l-3}}\cdot\nu_0)$ and
- $\chi(z_{2l-3}\cdot\nu_0)=\chi(\overline{z_{2l-3}}\cdot\nu_0)$.

Then, using (3), we get

$$\sum_{j>0}[V(\overline{z_{2l-3}}\cdot\nu_0)^j]=\chi(\overline{y_{2l-3}}\cdot\nu_0).$$

This yields $\chi(\overline{z_{2l-3}}\cdot\nu_0)=[L(\overline{z_{2l-3}}\cdot\nu_0)]+[L(\overline{y_{2l-3}}\cdot\nu_0)]$, since $\chi(\overline{y_{2l-3}}\cdot\nu_0)=[L(\overline{y_{2l-3}}\cdot\nu_0)]$.

(b) By the statement (b) of Lemma 7, $\chi(w\cdot\nu_1)=0$ for all $w\in Y_1\cup Y_2\cup Z_1\cup Z_2$, except in the following cases:

- $\chi(y_i\cdot\nu_1)=\chi(z_i\cdot\nu_1)=\chi(\overline{y_i}\cdot\nu_1)$ for all $i\in\{1,\,2,\,\cdots,\,2l-3\}$.

Then, using the sum formula (3), we have

$$\sum_{j>0}[V(\overline{y_i}\cdot\nu_1)^j]=\sum_{k=2}^{i}(-1)^{i-k}\chi(\overline{y_{k-1}}\cdot\nu_1)\tag{12}$$

for all $i\in\{2,\,3,\,\cdots,\,2l-3\}$. If $i=2$, then by (12),

$$\sum_{j>0}[V(\overline{y}_2\cdot\nu_1)^j]=\chi(\overline{y_1}\cdot\nu_1)=[L(\overline{y_1}\cdot\nu_1)].$$

This implies that $\chi(\overline{y_2}\cdot\nu_1)=[L(\overline{y_2}\cdot\nu_1)]+[L(\overline{y_1}\cdot\nu_1)]$.

Now, suppose that the statement (b) is true for all $i<t$, where $t\leq 2l-3$. Then by (12),

$$\sum_{j>0}[V(\overline{y_t}\cdot\nu_1)^j]=\sum_{k=2}^{i}(-1)^{t-k}([L(\overline{y_{k-1}}\cdot\nu_1)]+[L(\overline{y_{k-2}}\cdot\nu_1)])=[L(\overline{y_{t-1}}\cdot\nu_1)].$$

This yields $\chi(\overline{y_t}\cdot\nu_1) = [L(\overline{y_t}\cdot\nu_1)]+[L(\overline{y_{t-1}}\cdot\nu_1)]$. So, the statement (b) is true for all $i \in \{2,\, 3,\, \cdots\, 2l-3\}$.

(c) By the statement (c) of Lemma 7, $\chi(w \cdot \nu_2) = 0$ for all $w \in Y_1 \cup Y_2 \cup Z_1 \cup Z_2$, except in the following cases:

- $\chi(y_0 \cdot \nu_2) = \chi(y_1 \cdot \nu_2) = \chi(\overline{y_0} \cdot \nu_2)$,
- $\chi(y_{2l-4} \cdot \nu_2) = \chi(y_{2l-3} \cdot \nu_2) = \chi(\overline{y_{2l-4}} \cdot \nu_2)$ and
- $\chi(z_i \cdot \nu_2) = \chi(z_i s_2 \cdot \nu_2) = \chi(\overline{z_i} \cdot \nu_2)$ for all $i \in \{2,\, 3,\, \cdots,\, 2l-4\}$.

Then by (3),

$$\sum_{j>0}[V(\overline{z_i} \cdot \nu_2)^j] = (-1)^i \chi(\overline{y_0} \cdot \nu_2) + \sum_{k=4}^{i+1} (-1)^{i-k+1} \chi(\overline{z_{k-2}} \cdot \nu_2) + \delta(i \geq 2l-4)\chi(\overline{y_i} \cdot \nu_2) \tag{13}$$

for all $i \in \{2,\, 3,\, \cdots,\, 2l-4\}$. If $i = 2$, then using (13), we get

$$\sum_{j>0}[V(\overline{z_2} \cdot \nu_2)^j] = \chi(\overline{y_0} \cdot \nu_2) = [L(\overline{y_0} \cdot \nu_2)].$$

This yields the statement (c).

(d) We use (13) and the induction on i. If $i = 3$, then by the statement (c) of this Theorem 1, (13) yields

$$\sum_{j>0}[V(\overline{z_3} \cdot \nu_2)^j] = -\chi(\overline{y_0} \cdot \nu_2) + \chi(\overline{z_2} \cdot \nu_2) = [L(\overline{z_2} \cdot \nu_2)].$$

Then $\chi(\overline{z_3} \cdot \nu_2) = [L(\overline{z_3} \cdot \nu_2)] + [L(\overline{z_2} \cdot \nu_2)]$.

Now suppose that the statement (d) is true for all $i < t$, where $t \leq 2l-5$. Then by (13),

$$\sum_{j>0}[V(\overline{z_t} \cdot \nu_2)^j] = (-1)^t \chi(\overline{y_0} \cdot \nu_2) + \sum_{k=4}^{t+1} (-1)^{t-k+1} \chi(\overline{z_{k-2}} \cdot \nu_2)$$

$$= (-1)^t [L(\overline{y_0} \cdot \nu_1)] + (-1)^{t-3}([L(\overline{z_2} \cdot \nu_2)] + [L(\overline{y_0} \cdot \nu_2)])$$

$$+ \sum_{k=5}^{t+1} (-1)^{t-k+1}([L(\overline{z_{k-2}} \cdot \nu_2)] + [L(\overline{z_{k-3}} \cdot \nu_2)]) = [L(\overline{z_{t-1}} \cdot \nu_2)].$$

It follows that $\chi(\overline{z_t} \cdot \nu_2) = [L(\overline{z_t} \cdot \nu_2)] + [L(\overline{z_{t-1}} \cdot \nu_2)]$. Therefore, the statement (c) is true for all $i \in \{3,\, 4,\, \cdots\, 2l-5\}$.

(e) By (13),

$$\sum_{j>0}[V(\overline{z_{2l-4}} \cdot \nu_2)^j] = \chi(\overline{y_0} \cdot \nu_2) + \sum_{k=4}^{2l-3} (-1)^{2l-k-3} \chi(\overline{z_{k-2}} \cdot \nu_2) + \chi(\overline{y_{2l-4}}).$$

Using the previous statements (c) and (d) of this Theorem 1, we obtain

$$\sum_{j>0}[V(\overline{z_{2l-4}} \cdot \nu_2)^j] = [L(\overline{y_0} \cdot \nu_2)] + (-1)^{2l-7}([L(\overline{z_2} \cdot \nu_2)] + [L(\overline{y_0} \cdot \nu_2)])$$

$$+\sum_{k=5}^{2l-3}(-1)^{2l-k-3}([L(\overline{z_{k-2}}\cdot\nu_2)]+[L(\overline{z_{k-3}}\cdot\nu_2)])+[L(\overline{y_{2l-4}}\cdot\nu_2)]=[L(\overline{z_{2l-5}}\cdot\nu_2)]+[L(\overline{y_{2l-4}}\cdot\nu_2)].$$

It follows that $\chi(\overline{z_{2l-4}}\cdot\nu_2)=[L(\overline{z_{2l-4}}\cdot\nu_2)]+[L(\overline{z_{2l-5}}\cdot\nu_2)]+[L(\overline{y_{2l-4}}\cdot\nu_2)]$.

(f), (g) Let $i\in\{3,4,\cdots,l-1\}$. By the statement (d) of Lemma 7, $\chi(w\cdot\nu_i)=0$ for all $w\in Y_1\cup Y_2\cup Z_1\cup Z_2$, except in the following cases:

- $\chi(y_{i-2}\cdot\nu_i)=\chi(y_{i-1}\cdot\nu_i)=\chi(\overline{y_{i-2}}\cdot\nu_i)$,
  $\chi(z_{i-2}\cdot\nu_i)=\chi(z_{i-1}\cdot\nu_i)=\chi(\overline{z_{i-2}}\cdot\nu_i)$,
- $\chi(y_{2l-i-2}\cdot\nu_i)=\chi(y_{2l-i-1}\cdot\nu_i)=\chi(\overline{y_{2l-i-2}}\cdot\nu_i)$ and
- $\chi(z_{2l-i-2}\cdot\nu_i)=\chi(z_{2l-i-1}\cdot\nu_i)=\chi(\overline{z_{2l-i-2}}\cdot\nu_i)$.

Then by (3),

$$\sum_{j>0}[V(\overline{z_{i-2}}\cdot\nu_i)^j]=\chi(\overline{y_{i-2}}\cdot\nu_i)\ \text{and}\ \sum_{j>0}[V(\overline{z_{2l-i-2}}\cdot\nu_i)^j]=\chi(\overline{y_{2l-i-2}}\cdot\nu_i)$$

for all $i\in\{3,4,\cdots,l-1\}$. Thus, for all $i\in\{3,4,\cdots,l-1\}$

$$\chi(\overline{z_{i-2}}\cdot\nu_i)=[L(\overline{z_{i-2}}\cdot\nu_i)]+[L(\overline{y_{i-2}}\cdot\nu_i)]$$

and

$$\chi(\overline{z_{2l-i-2}}\cdot\nu_i)=[L(\overline{z_{2l-i-2}}\cdot\nu_i)]+[L(\overline{y_{2l-i-2}}\cdot\nu_i)].$$

Finally, by the statement (e) of Lemma 7, $\chi(w\cdot\nu_l)=0$ for all $w\in Y_1\cup Y_2\cup Z_1\cup Z_2$, except in the following cases:

- $\chi(y_{l-2}\cdot\nu_l)=\chi(y_{l-1}\cdot\nu_l)=\chi(\overline{y_{l-2}}\cdot\nu_l)$ and
- $\chi(z_{l-2}\cdot\nu_l)=\chi(z_{l-1}\cdot\nu_l)=\chi(\overline{z_{l-2}}\cdot\nu_l)$.

Then by (3),

$$\sum_{j>0}[V(\overline{z_{l-2}}\cdot\nu_l)^j]=\chi(\overline{y_{l-2}}\cdot\nu_l).$$

Hence,

$$\chi(\overline{z_{l-2}}\cdot\nu_l)=[L(\overline{z_{l-2}}\cdot\nu_l)]+[L(\overline{y_{l-2}}\cdot\nu_l)].$$

□

Remark 1. *Let $X_\nu(Y_1\cup Y_2\cup Z_1\cup Z_2)\neq\emptyset$. If ν lies in the intersection of two hyperplanes, then by Theorem 1, all Weyl modules with highest weights $\overline{w}\cdot\nu$ with $\overline{w}\in X_\nu(Y_1\cup Y_2\cup Z_1\cup Z_2)$ are simple.*

From the proof of Theorem 1 we immediately obtain the following

Corollary 1. *Let G be a simply connected and semisimple algebraic groups of type C_l and $p\geq h$. Then Weyl modules with the following highest weights are simple:*

(a) ν_0, $\overline{y_{2l-3}}\cdot\nu_0$;

(b) $\overline{y_1}\cdot\nu_1$;

(c) $\overline{y_0}\cdot\nu_2$, $\overline{y_{2l-4}}\cdot\nu_2$;

(d) $\overline{y_{i-1}} \cdot \nu_i$, $\overline{y_{2l-i-1}} \cdot \nu_i$ *where* $i \in \{3, 4, \cdots, l-1\}$;

(e) $\overline{y_{l-2}} \cdot \nu_l$;

(f) $\overline{y_{2l-3}} \cdot \nu_{1,2}$, $\overline{y_{l-2}} \cdot \nu_{1,l}$ *and* $\overline{y_{i-2}} \cdot \nu_{1,i}$, $\overline{y_{2l-i-2}} \cdot \nu_{1,i}$, *where* $i \in \{3, 4, \cdots, l-1\}$;

(g) $\overline{z_{2l-5}} \cdot \nu_{2,3}$, $\overline{z_{l-2}} \cdot \nu_{2,l}$ *and* $\overline{z_{i-2}} \cdot \nu_{2,i}$, $\overline{z_{2l-i-2}} \cdot \nu_{2,i}$, *where* $i \in \{4, 5, \cdots, l-1\}$.

Remark 2. *It is known that, in the restricted region, the differential of each simple G-module is a simple $\mathfrak{g}$-module, where $\mathfrak{g}$ is the Lie algebra of G. In [3] Rudakov proved that*

(a) *if $\nu_0 \in X_1(T)$, then $V(\nu_0)$ is simple ,*

(b) *if $y_0 \cdot \nu \in X_1(T)$ for some $\nu \in \overline{C}_1 \setminus C_1$, then $V(y_0 \cdot \nu)$ is simple.*

Weyl modules $V(\nu_0)$ and $V(\overline{y_0} \cdot \nu_2)$ are satisfy the Rudakov simplicity criterion. In all other cases the highest weights obtained in Corollary 1 don't satisfy the Rudakov simplicity criterion. Therefore, Corollary 1 generalizes the Rudakov simplicity criterion [3, Theorems 1 and 2] for semisimple Lie algebras of type C_l.

Using Lemmas 1 and 2, one can easily describe highest weights of the simple Weyl modules listed in Corollary 1. For example, by the definition, $\overline{z_{l-2}} \cdot \nu_{2,l} = z_{l-2} \cdot \nu_{2,l}$, and $\nu_{2,l}$ satisfies the conditions

$$\langle \nu_{2,l} + \rho, \alpha_2 \rangle = \langle \nu_{2,l} + \rho, \alpha_l \rangle = 0,$$

since $\nu_{2,l} \in H_2 \cap H_l$. If we write $\nu_{2,l} = \sum_{j=1}^{l} a_j \omega_j$, then the above conditions yield $a_2 = a_l = -1$. Then by the statement (b) of Lemma 2,

$$\overline{z_{l-2}} \cdot \nu_{2,l} = z_{l-2} \cdot \nu_{2,l} = (\sum_{j=3}^{l-1} a_j + l - 4)\omega_1 + (p - a_1 - 2\sum_{j=3}^{l-1} a_j - 2l + 4)\omega_2 + a_1\omega_3 + \sum_{j=4}^{l-1} a_{j-1}\omega_j + a_{l-1}\omega_l,$$

where $a_1 + 2\sum_{j=3}^{l-1} a_j + 2l - 5 < p$ and $l \geq 4$.

Corollary 1 gives several new examples of simple Weyl modules for the simple algebraic groups of type C_l.

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Ыбыраев Ш.Ш. C_l ТҮРІНДЕГІ АЛГЕБРАЛЫҚ ТОПТАР ҮШІН СИНГУЛЯРЛЫ АҒА САЛМАҚТЫ ВЕЙЛЬ МОДУЛЬДЕРІ ТУРАЛЫ

Бұл мақалада C_l түріндегі бірбайланысты жартылай жәй алгебралық топтардың сингуляр аға салмақты кейбір Вейль модульдерінің $p \geq h$ сипаттаманың алгебралық тұйық өрісіндегі, мұндағы h – Кокстер саны, композициялық факторлары мен ішкі модульдік құрылымдары есептелді. Атап айтқанда, Вейль жәй модульдерінің кейбір жаңа мысалдары алынды.

Кілттік сөздер. Алгебралық топ, Вейль модулі, Вейль жәй модулі, сингулярлы салмақ.

Ибраев Ш.Ш. О МОДУЛЯХ ВЕЙЛЯ С СИНГУЛЯРНЫМИ СТАРШИМИ ВЕСАМИ ДЛЯ АЛГЕБРАИЧЕСКИХ ГРУПП ТИПА C_l

В данной статье вычислены композиционные факторы и подмодульные структуры некоторых модулей Вейля с сингулярными старшими весами для односвязных полупростых алгебраических групп типа C_l над алгебраически замкнутым полем характеристики $p \geq h$, где h – число Кокстера. В частности, получены некоторые новые примеры простых модулей Вейля.

Ключевые слова. Алгебраическая группа, модуль Вейля, простой модуль Вейля, сингулярный вес.

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Non-existence of uniformly definable family of convex equivalence relations in an 1-type of small ordered theories and maximal number of models

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Abstract. We study the class of small ordered theories which includes the class of small weakly o-minimal theories. We give a condition under which theories from this class have the maximal number of countable pairwise non-isomorphic models.

Keywords. Small ordered theory, number of countable models, omitting types.

1 Introduction

The problem of counting the number of countable pairwise non-isomorphic models of ordered theories was studied by M. Rubin, S. Shelah, L. Mayer, S. Sudoplatov, B. Kulpeshov, S. Moconja, P. Tanovic, and others. The Vaught Conjecture was confirmed for different classes of ordered theories: for o-minimal theories by L. Mayer [1], for quite o-minimal theories by Sudoplatov-Kulpeshov [2], for weakly o-minimal theories of convexity rank one by Alibek-Baizhanov-Kulpeshov-Zambarnaya [3], for binary, stationarily ordered theories by Moconja-Tanovic [4], and for weakly o-minimal theories of finite convexity rank by B. Kulpeshov [5]. In research in this direction an important role plays describing the conditions when a complete theory has the maximal number of countable pairwise non-isomorphic models. In particular, this question was investigated by Alibek-Baizhanov-Zambarnaya in [6], Baizhanov-Baldwin-Zambarnaya in [7], and B. Kulpeshov in [8]. In the work we present a condition for maximality of the number of models for a small theory with a fixed number of convex equivalence relations in an 1-type. That is, for a theory for which a type does not have a uniformly definable family of nested equivalence relations.

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2 Main Part

In the article we consider small countable theories with an $\emptyset$-definable relation of linear order. By Gothic letters ($\mathfrak{M}$, $\mathfrak{N}$, etc.) we denote structures, by capital letters (M, N, etc.) we denote universes of these structures, respectively. By $\mathfrak{N}$ we denote a countable saturated model of the given small theory.

For $A \subseteq B \subseteq N$ (not necessary definable) we use the following notations:

$$A^+ := \{\gamma \in N \mid \text{for all } a \in A, \mathfrak{N} \models a < \gamma\};$$
$$A^- := \{\gamma \in N \mid \text{for all } a \in A, \mathfrak{N} \models \gamma < a\}.$$

We write $A < B$ if for all $a \in A$, $b \in B$ $\mathfrak{N} \models a < b$.

Definition 1. *A set A is said to be* **convex** *in a set B, $A \subseteq B$, if for all $x, y \in A$ and all $z \in B$ $x < z < y$ implies that $z \in A$.*

Let $\Theta(x)$ be an 1-A-formula, then

$$E_\Theta(x,y) := \Theta(x) \wedge \Theta(y) \wedge$$
$$\wedge\Big(x = y \vee \big((x < y \to \forall z(x \leq z \leq y \to \Theta(z))) \wedge (y < x \to \forall z(y \leq z \leq x \to \Theta(z)))\big)\Big)$$

defines an equivalence relation with convex classes on $\Theta(N)$. We call Θ a **zebra-formula** or formula with infinite number of alternations (INA), if there is an infinite number of convex E_Θ-classes. On the set of convex E_Θ-classes there is a linear ordering; if this order contains an infinite discrete chain, then $I(T, \aleph_0) = 2^{\aleph_0}$ [6]. So, in this paper, we assume that there is a natural number n_Θ such that any discrete chain of E_Θ-classes contains at most n_Θ-classes and the order on the set of all E_Θ-classes is dense up to finite discrete chain bounded by n_Θ.

Let Θ and Ψ be two zebra-formulas such that $\models \neg\exists x\big(\Theta(x) \wedge \Psi(x)\big)$, then we say that they are **mutually dense**, if between any two arbitrary E_Θ-class and E_Ψ-class there are both E_Θ-class and an E_Ψ-class.

Definition 2 [9]. *1) The* **convex closure** *of a formula $\varphi(x, \bar{a})$ is the following formula:*

$$\varphi^c(x, \bar{a}) := \exists y_1 \exists y_2 \big(\varphi(y_1, \bar{a}) \wedge \varphi(y_2, \bar{a}) \wedge (y_1 \leq x \leq y_2)\big).$$

2) The **convex closure** *of a type $p(x) \in S_1(A)$ is the following set of formulas*

$$p^c(x) := \{\varphi^c(x, \bar{a}) \mid \varphi(x, \bar{a}) \in p\}.$$

Definition 3 [10], [11]. *Let $\mathfrak{M}$ be a linearly ordered structure, $A \subseteq M$, M be $|A|^+$-saturated, and $p \in S_1(A)$ be non-algebraic.*

1) An A-definable formula $\varphi(x, y)$ is said to be **p–preserving (p-stable)** *if there exist α, γ_1, $\gamma_2 \in p(M)$ such that $p(M) \cap \big(\varphi(M, \alpha) \setminus \{\alpha\}\big) \neq \emptyset$ and $\gamma_1 < \varphi(M, \alpha) < \gamma_2$.*

2) A p-preserving formula $\varphi(x,y)$ is said to be **convex to the right (left)** *if there exists $\alpha \in p(M)$ such that $p(M) \cap \varphi(M,\alpha)$ is convex, α is the left (right) endpoint of the set $\varphi(M,\alpha)$, and $\alpha \in \varphi(M,\alpha)$.*

3) A p-preserving convex to the right (left) formula $\varphi(x,y)$ is **equivalence-generating** *if for any $\alpha \in p(M)$ and any $\beta \in \varphi(M,\alpha) \cap p(M)$ the following holds:*

$$\mathfrak{M} \models \forall x(x \geq \beta \to (\varphi(x,\alpha) \leftrightarrow \varphi(x,\beta))) \ (\mathfrak{M} \models \forall x(x \leq \beta \to (\varphi(x,\alpha) \leftrightarrow \varphi(x,\beta)))).$$

By $CRF(p)$ ($CLF(p)$) we denote the family of all p-preserving convex to the right (left) A-formulas.

Restriction. *Let A and B be subsets of a model of a linearly ordered theory T, A be finite, B be A-definable. We consider 1-types $p \in S_1(A)$ such that*

(i) for every A-formula $E(x,y,\bar{z})$ there is no infinite sequence $\bar{b}_1, \bar{b}_2, ..., \bar{b}_i, ... \in B$ such that for every $i < \omega$, $E(x,y,\bar{b}_i)$ is an equivalence relation on p with classes partitioned into infinitely many of infinite $E(x,y,\bar{b}_{i+1})$-classes;

(ii) the set $\{q \in S_1(A) \mid p^c = q^c\}$ is finite;

(iii) all p-preserving convex to the right formulas are equivalence generating.

Theorem 1. *Let T be a countable complete linearly ordered theory, A be a finite subset of a model of T, and let $p(x) \in S_1(A)$ be a non-algebraic 1-type satisfying the Restriction. If $CRF(p)$ is infinite and has no greatest formula, then T has $2^{\aleph_0}$ countable non-isomorphic models.*

Proof. Since every non-small theory has $2^{\aleph_0}$ countable non-isomorphic models, it remains to prove the case, when the theory T is small. For simplicity we extend our language to $\mathcal{L}(A)$ and work in the theory $T \cup tp(\bar{a})$, where $\bar{a}$ is an enumeration of the set A. Let $\mathfrak{N}$ be a countable saturated model of the small theory T containing the finite set A.

Let $p \in S_1(A)$, then define for an arbitrary p-preserving convex to the right 2-formula φ a relation of equivalence such that any its class is convex in $p(N)$. For $\varphi(x,y) \in CRF(p)$ and for every $\alpha, \beta \in p(M)$ denote by $E_\varphi(\alpha,\beta)$ the formula $\varphi(N,\alpha)^+ = \varphi(N,\beta)^+$. Here, $\varphi(x,y)^+ := \forall z\big(\varphi(z,y) \to z < x\big)$. On the set of all realizations of the type p, $E_\varphi(x,y)$ is a relation of equivalence with convex classes on $p(N)$, but not necessarily on $p^c(N)$.

Since $|\{q \in S_1(A) \mid p^c = q^c\}| < \omega$, then for some $\Theta \in p$, $p = p^c \cup \{\Theta\}$. Then E_φ is an A-definable relation of equivalence with convex E_φ-classes on $\Theta(N)$ for suitable $\Theta \in p$. Thus, for arbitrary $\alpha \in p(N)$, $E_\varphi(N,\alpha) \cap \Theta(N)$ is exactly $A\alpha$-definable convex E_φ-class on $p(N)$ containing α.

We say that two zebra-formulas Θ and Ψ are **mutually attached** if for any E_Θ-class there is a E_Ψ-class such that between these two classes there is no elements. It follows from this definition that if Θ belongs to $p \in S_1(A)$ and $\Psi \in q \in S_1(A)$, then $p^c = q^c$. We say that two zebra-formulas are **attached** if there is a finite sequence of pairwise mutually attached zebra-formulas, started by one of them and ended by second zebra-formula. Consider three A-definable zebra-formulas Θ, Ψ_1, Ψ_2 such that last two are attached, if Θ and Ψ_1 are mutually dense, then Θ and Ψ_2 are mutually dense too.

Note that it is possible to determine an A-definable relation of equivalence on $p^c(N)$, since any A-definable zebra-formula is dense on $p^c(N)$ and consequently, two A-definable dense zebra-formulas on $p^c(N)$ are mutually dense up to discrete finite chain of attached zebra-formulas. Finiteness of chain of attached zebra-formulas to θ follows from the second condition of the Restriction.

Denote by $E^1_\varphi(x,y) := \exists z\big(z < x \wedge z < y \wedge \Theta(z) \wedge x \notin \varphi(N,z)^+ \wedge y \notin \varphi(N,z)^+\big)$. This is a relation of equivalence on $p^c(N)$ and consequently, it is an A-definable relation of equivalence with convex classes on some A-definable convex set $K(N)$, where A-formula $K(x)$ is from p^c.

Denote by $E_{\varphi,\Theta}(N,\alpha) = E_\varphi(N,\alpha) \cap \Theta(N)$.

For $\alpha \models p$ we denote $V_{p(N)}(\alpha) := \{\gamma \in p(N) \mid \text{there exists a formula } \varphi(x,y) \in CRF(p)$ such that $\mathfrak{M} \models \varphi(\alpha,\gamma) \vee \varphi(\gamma,\alpha)\}$. It follows from the definitions that

$$V_{p(N)}(\alpha) = \cup_{\varphi \in CRF(p)} E_{\varphi,\Theta}(N,\alpha).$$

Since there is no greatest equivalence-generating formula, the sets $V_{p(N)}(\alpha)$, $V_{p(N)}(\alpha)^+$, $V_{p(N)}(\alpha)^-$ are not $A\alpha$-definable.

Denote for arbitrary $\alpha \in p^c(N)$ by $V_{p^c(N)}(\alpha) := \cup_{\varphi \in CRF(p)} E^1_\varphi(N,\alpha)$. It follows from the definitions that for any $\alpha \in p(N)$ we have

$$V_{p(N)}(\alpha) \subseteq V_{p^c(N)}(\alpha),\ V_{p(N)}(\alpha)^+ = V_{p^c(N)}(\alpha)^+,\ V_{p(N)}(\alpha)^- = V_{p^c(N)}(\alpha)^-.$$

Denote

$$(V_p(\alpha), V_p(\beta))_{p(N)} := \{\gamma \in p(N) \mid V_{p(N)}(\alpha) < \gamma < V_{p(N)}(\beta)\}.$$

Then

$$(V_p(\alpha), V_p(\beta))_{p(N)} = \{\gamma \in \Theta(N) \mid \text{for every } \varphi \in CRF(p),\\ \mathfrak{N} \models \alpha < \gamma < \beta \wedge \neg E_\varphi(\alpha,\gamma) \wedge \neg E_\varphi(\beta,\gamma)\}.$$

Lemma 1. *Let $\alpha, \beta \in p(N)$ such that $V_{p(N)}(\alpha) < V_{p(N)}(\beta)$. Then for all $\gamma_1, \gamma_2 \in (V_p(\alpha), V_p(\beta))_{p(N)}$, $tp^c(\gamma_1/A\alpha\beta) = tp^c(\gamma_2/A\alpha\beta)$.*

Proof of Lemma 1. Assume that the conclusion of Lemma 1 is not true. This means that there is an $A\alpha\beta$-definable 1-formula $H(x,\alpha,\beta)$ such that $H(N,\alpha,\beta)$ convex in $\Theta(N)$, $H(N,\alpha,\beta) < \neg H(N,\alpha,\beta)$, $H(N,\alpha,\beta) \cup \neg H(N,\alpha,\beta) = \Theta(N)$, $V_{p(N)}(\alpha) < \gamma_1 \in H(N,\alpha,\beta)$, $\gamma_2 \in \neg H(N,\alpha,\beta)$, and $\gamma_2 < V_{p(N)}(\alpha) \subset \neg H(N,\alpha,\beta)$.

Consider $p_1(x,\beta) := tp(\alpha/A\beta) \in S_1(A\beta)$. It follows from the definition that $p \cup (x < V_{p(N)}(\beta)) = p \cup \{x < E_{\varphi,\Theta}(N,\beta) \mid \varphi \in CRF(p)\} \subseteq p_1$ and $p^c_1(N) = p^c(N) \cap V_{p(M)}(\beta)^-$. By the second condition of the Restriction the set $\{q \in S_1(A\beta) \mid p_1^c = q^c\}$ is finite. This means there is an $A\beta$-definable zebra-formula $\Theta_1 \in p_1$ such that $p_1^c \cup \{\Theta_1\} = p_1$. We have $H(x, y, \beta)$ is convex to right p_1-preserving $A\beta$-formula. Notice that $CRF(p) \cup \{H(x, y, \beta)\} \subseteq CRF(p_1)$ because for any $\varphi \in CRF(p)$, $H(N, \alpha, \beta)^+ \subset \varphi(N, \alpha)^+$.

Then by the third condition of the Restriction $A\beta$-definable formula $H(x, \beta)$ defines a relation of equivalence $E_{H(x,\beta)}(x, y, \beta)$ with convex E_H-classes on the set of all realizations of one-type p_1. This is a relation of equivalence with convex classes on $p_1(N)$. Define a new equivalence relation on some convex part of $p(N)$:

$$E^1_{H,\Theta}(x, y, \beta) := \exists z\big(\Theta_1(z) \wedge \Theta(x) \wedge \Theta(y) \wedge z < x \wedge z < y \wedge x \notin H(N, z, \beta)^+ \wedge y \notin H(N, z, \beta)^+\big).$$

Last sentence gives us an $A\beta$-definable relation of equivalence on some part of $p(N)$, namely on the set of elements from $p(N)$ less than $V_{p(N)}(\beta)$. Since $\gamma_1 \notin H(N, \alpha, \beta)^+$, $\mathfrak{N} \models E_{H,\Theta}(\gamma_1, \alpha, \beta)$. Since for $\varphi \in CRF(p)$, $\mathfrak{N} \models \neg E_{\varphi,\Theta}(\gamma_1, \alpha)$, the definable set $E_{H,\Theta}(N, \alpha, \beta)$ is a convex subset of $p(N)$, that contains a densely ordered infinite set of non-definable sets of kinds as $V_{p(N)}(.)$.

Consider the set of 1-formulas $p_1(x, \alpha)$. This is a complete one-type because $\alpha, \beta \in p(N)$. Take an arbitrary realization of $p_1(x, \alpha)$ from $E_{H,\Theta}(N, \alpha, \beta)$ and denote it by α_1. Then we have $E_{H,\Theta}(N, \alpha_1, \alpha) \subset E_{H,\Theta}(N, \alpha, \beta)$. The set $E_{H,\Theta}(N, \alpha, \beta)$, that is convex in $p(N)$, contains an infinite number of convex sets definable by $E_{H,\Theta}(x, y, \alpha)$.

Let f be an A-isomorphism of the countable saturated model $\mathfrak{N}$, generated by an elementary monomorphism such that $f(\beta) = \alpha$, $f(\alpha) = \alpha_1$. The existence of such an elementary monomorphism follows from $p_1(x, y) = tp(\alpha\beta/A) = tp(\alpha_1\alpha/A)$. The isomorphism f generates an infinite sequence $\langle\alpha_n\rangle_{0<n<\omega}$ such that for any $n < \omega$, we have

$$E_{H,\Theta}(N, \alpha_{n+2}, \alpha_{n+1}) \subset E_{H,\Theta}(N, \alpha_{n+1}, \alpha_n).$$

This contradicts to the first condition of the Restriction.

$\square$ Lemma 1

A type q is said to be irrational if $q^c(N)^+$ and $q^c(N)^-$ are both non-definable. By Lemma 1 the type $tp(\gamma_1/A\alpha\beta)$ is irrational, and therefore it is non-principal. A corollary of the proof of Lemma 1 is the following lemma.

Lemma 2. *For every $n < \omega$ and all $\alpha_i \in p(N)$, $i \leq n$, such that $V_{p(N)}(\alpha_i) < V_{p(N)}(\alpha_{i+1})$, $1 \leq i \leq n-1$, for all $\gamma_1, \gamma_2 \in (V_p(\alpha_i), V_p(\alpha_{i+1}))_{p(N)}$, $tp^c(\gamma_1/A\alpha_1, ..., \alpha_n) = tp^c(\gamma_2/A\alpha_1, ..., \alpha_n)$.*

Lemma 2 implies that the type $tp(\gamma_1/A\alpha_1, ..., \alpha_n)$ is non-principal.

Let $\mathfrak{M}$ be an $\aleph_1$-saturated elementary extension of $\mathfrak{N}$. Now for all infinite sequences of zeros and ones, $\tau := \langle\tau_1, \tau_2, ..., \tau_i, ...\rangle_{i<\omega}$, $\tau(i) \in \{0, 1\}$, we will construct countable models $\mathfrak{M}_\tau \prec \mathfrak{M}$ such that for any $\tau_1 \neq \tau_2$, $\mathfrak{M}_{\tau_1} \ncong \mathfrak{M}_{\tau_2}$. Until the end of the proof fix such a sequence τ. We will use Tarski-Vaught criterion in order to show that M_τ is a universe of an elementary substructure of $\mathfrak{M}$. On each step of the construction we will be fixing a set of parameters and promising to realize each satisfiable 1-formula over it. We must keep coming back to the same set of parameters and deal with another formula. So, the different sets of parameters are being attacked in parallel. Recall that we have extended the language to $L(A)$.

Let $B := \{\alpha_{(2i-1,j)} \mid i \in \mathbb{N}, j \in \mathbb{Q}\} \cup \{\alpha_{(2i,1)}, ...\alpha_{(2i,i)} \mid i \in \mathbb{N}, \tau_i = 0\} \cup \{\alpha_{(2i,1)}, \alpha_{(2i,i+1)} \mid i \in \mathbb{N}, \tau_i = 1\}$ be a subset of $p(M)$ such that the sets $V_{p(N)}(\alpha_{(i,j)})$ are disjoint and ordered lexicographically by the indices (i,j). Since the set B is countable, fix an enumeration $B = \{b_1, b_2, ..., b_i, ...\}$.

Lemma 3 [7]. Let D be a finite subset of M. For each satisfiable $(B \cup D)$-formula, $\psi(x, \bar{b}, \bar{d})$, where $\bar{b} = \langle b_1, b_2, ...b_n\rangle \in B$ $(n < \omega)$, and $\bar{d} \in D$, there exists a type $q_\psi \in S_1(B \cup D)$ such that

1) $\psi(x, \bar{b}, \bar{d}) \in q_\psi$;

2) For every $i \geq n$, $q_\psi \upharpoonright (B_i \cup D)$ is principal, where $B_i = \{b_1, b_2, ..., b_i\}$.

Proof of Lemma 3. For $i < \omega$ denote $\bar{b}_i := \langle b_1, b_2, ...b_i\rangle$, and let $\bar{d}'$ be a tuple enumerating the set D. Because the theory T is small, there exists a formula $\psi_0(x, \bar{b}_n, \bar{d}')$ that implies $\psi(x, \bar{b}_n, \bar{d})$ and generates a principal type over $(\{\bar{b}_n\} \cup D)$. In turn there is a principal subformula over $(\{\bar{b}_{n+1}\} \cup D)$ that implies $\psi_0(x, \bar{b}_n, \bar{d}')$. Repeating this procedure, we obtain a consistent infinite decreasing chain of principal over parameters formulas $\psi_i(x, \bar{b}_{n+i}, \bar{d}')$: ... $\subseteq \psi_{i+1}(M, \bar{b}_{n+i+1}, \bar{d}') \subseteq \psi_i(M, \bar{b}_{n+i}, \bar{d}') \subseteq ... \subseteq \psi_0(M, \bar{b}_n, \bar{d}') \subseteq \psi(M, \bar{b}_n, \bar{d})$, where $\mathfrak{M}$ is an arbitrary model of T with $(B \cup D) \subseteq M$. By this we have defined the desired complete type over $(B \cup D)$. $\square$ Lemma 3

Construction of the model M_τ.

Step 1. Denote by Ψ_1 the set of all $\emptyset$-definable unary formulas of $\mathcal{L}(A)$, $\Psi_1 := \{\psi_i^1(x) \mid i < \omega\}$. Choose the least i such that $\mathfrak{M} \models \exists x \psi_i^1(x)$ and $\psi(N) \cap B = \emptyset$. To satisfy the Tarski-Vaught property, we must find a witness for $\psi_i^1(x)$. Since the set B and the formula ψ_i^1 are as in Lemma (consider the set D to be empty), there exists a B-type $q_{\psi_i^1}$ satisfying conditions 1) and 2) of the lemma. And since the model $\mathfrak{M}$ is $\aleph_1$-saturated, this type is realized in $\mathfrak{M}$ by some element, denote it by d_1. Thus, d_1 is principal over $\emptyset$. Denote $D_1 := \{d_1\}$.

Step 2. Choose the least j such that the formula $\psi_j^1(x) \in \Psi_1$ was not considered before with $\mathfrak{N} \models \exists x \psi_j^1(x)$ and such that $\psi(x) \cap (B \cup \{d_1\}) = \emptyset$. We find a special witness for $\psi_j^1(x)$. Apply Lemma to the sets B and $\{d_1\}$, and the formula $\psi_j^1(x)$, to find a realization d_2 of the type $q_{\psi_j^1}$, which exists by the lemma. We can arrange that d_2 is principal over d_1.

Now take b_1 and consider the set of all $(\{b_1\} \cup \{d_1\})$-definable 1-formulas $\Psi_2 := \{\psi_i^2(x, b_1, d_1) \mid i < \omega\}$. Choose the least index i such that the formula $\psi_i^2(x, b_1, d_1) \in \Psi_2$ was not considered previously such that $\mathfrak{M} \models \exists x \psi_i^2(x, b_1, d_1)$ and $\psi(x) \cap (B \cup \{d_1, d_2\}) = \emptyset$, and find a realization d_3 by applying Lemma to B, $\{d_1, d_2\}$, and ψ_i^2.

By the end of step k we will have the following sets:

- Nested sets $D_1 = \{d_1\}$, $D_2 = \{d_1, d_2, d_3\}$, $D_3 = \{d_1, d_2, ..., d_6\}$,...,$D_k = \{d_1, d_2, ..., d_{\frac{(k+1)k}{2}}\}$, where D_i is constructed on step i by adding i new realizations to the set D_{i-1}. It is possible that $d_i = d_j$ for some i and j with $1 \leq i < j \leq \frac{(k+1)k}{2}$.
- The family of all $\emptyset$-definable 1-formulas Ψ_1, and for every m, $2 \leq m \leq k$, a family of $\big(\{\bar{b}_{m-1}\} \cup D_{m-1}\big)$-definable 1-formulas, Ψ_m.

Further we will use the usual notation $\bar{d}_i = \langle d_1, d_2, ..., d_i \rangle$, $i < \omega$.

Step $k+1$**.** Firstly we realize one formula from each of the families we defined earlier. To do this, for each m, $1 \leq m \leq k$, find smallest index i_m such that the formula $\psi^m_{i_m} \in \Psi_m$ was not considered before, and definable set of which in the model $\mathfrak{M}$ is not empty but $\psi(x) \cap (B \cup \{d_1, d_2, ..., d_{\frac{(k+1)k}{2}+m-1}\}) = \emptyset$. Apply Lemma to the sets B and $\{\bar{d}_{\frac{(k+1)k}{2}+m-1}\}$, and the formula $\psi^m_{i_m}$, to find realization $d_{\frac{(k+1)k}{2}+m}$ of the type $q_{\psi^m_{i_m}}$.

Now denote by Ψ_{k+1} the set of all $(\cup\{\bar{b}_k\} \cup D_k)$-definable 1-formulas, and find a smallest index i with $\mathfrak{M} \models \exists x \psi^{k+1}_i(x, \bar{b}_k, \bar{d}_{\frac{(k+1)k}{2}})$ and such that $\psi(x) \cap (B \cup \{d_1, d_2, ..., d_{\frac{(k+1)k}{2}+k+1}\}) = \emptyset$. We choose $d_{\frac{(k+1)k}{2}+k+1}$ as before, as a realization of a type $q_{\psi^{k+1}_i}$, which exists by Lemma applied to the sets B, $\{\bar{d}_{\frac{(k+1)k}{2}+k}\}$, and formula ψ^{k+1}_i. Let D_{k+1} be the set $\{d_1, d_2, ..., d_{\frac{(k+1)k}{2}+k+1}\}$. We can arrange that each new d_i is principal over $\bar{b}_k$ and the d_j's for $j < i$.

Denote $M_\tau := B \cup \bigcup\limits_{i<\omega} D_i$.

The Tarski-Vaught criterion implies that the set M_τ is a universe of an elementary substructure $\mathfrak{M}_\tau$ of $\mathfrak{M}$.

For every $i < \omega$, by choosing d_i to be as in Lemma , the type $tp(d_i/\bar{b}_n, \bar{d}_{i-1})$ is principal for every $n \geq i-1$. From the last statement it easily follows by induction that for every $i < \omega$ and every $n \geq i-1$ the type $tp(\bar{d}_i/\bar{b}_n)$ is principal, and therefore the type $tp(d_i/\bar{b}_n)$ is also principal.

We claim that $p(\mathfrak{M}_\tau) \setminus \bigcup\limits_{b_n \in B} V_{p(M)}(b_n) = \emptyset$. Towards a contradiction suppose that there exists a realization $\delta \in p(\mathfrak{M}_\tau) \setminus \bigcup\limits_{b_n \in B} V_{p(M)}(b_n)$. Since $\delta \notin B$, $\delta \in \bigcup\limits_{i<\omega} D_i$, and for some $k < \omega$, $\delta = d_k$. As we recently showed, the type $tp(d_k/\bar{b}_n)$ is principal for sufficiently large n's. But since all b_n's are $a_{(i,j)}$'s, by Lemma 2 the types $tp(d_k/\bar{b}_n)$ are non-principal. This is a contradiction and we have that $p(\mathfrak{M}_\tau) \setminus \bigcup\limits_{\alpha_{(i,j)} \in B} V_{p(M)}(\alpha_{(i,j)}) = \emptyset$.

Since the number of different infinite sequences τ of zeros and ones equals to $2^{\aleph_0}$, $I(T \cup tp(\bar{a}), \omega) = 2^{\aleph_0}$. As the theory T is small it has at most countably many distinct complete extensions by realizing an n-type $tp(\bar{a})$; consequently, $I(T, \aleph_0) = 2^{\aleph_0}$. $\square$ Theorem 1

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Байжанов Б., Умбетбаев О., Замбарная Т. ШАҒЫН РЕТТЕЛГЕН ТЕОРИЯЛАРДЫҢ 1-ТИПІНДЕГІ ДӨҢЕС ПАРА-ПАРЛЫҚТАР ҚАТЫНАСТАРЫНЫҢ БІРҚАЛЫПТЫ АНЫҚТАЛАТЫН ҮЙІРІНІҢ БОЛМАУЫ ЖӘНЕ МОДЕЛЬДЕРДІҢ МАКСИМАЛДЫ САНЫ

Біз шағын әлсіз о-минималды теориялар класын қамтитын шағын реттелген теориялар класын зерттейміз. Осы класта жататын теориялардың санамалы екеуара изоморфты емес модельдерінің саны максималды болатын шарт беріледі.

Кілттік сөздер. Шағын реттелген теория, санамалы модельдердің саны, типтерді төмендету.

Байжанов Б., Умбетбаев О., Замбарная Т. ОТСУТСТВИЕ РАВНОМЕРНО ОПРЕДЕЛЯЕМОГО СЕМЕЙСТВА ОТНОШЕНИЙ ВЫПУКЛЫХ ЭКВИВАЛЕНТНОСТЕЙ В 1-ТИПЕ МАЛЫХ УПОРЯДОЧЕННЫХ ТЕОРИЙ И МАКСИМАЛЬНОЕ ЧИСЛО МОДЕЛЕЙ

Мы изучаем класс малых упорядоченных теорий, который включает в себя класс малых слабо о-минимальных теорий. Даётся условие, при котором теории из этого класса имеют максимальное число счётных попарно неизоморфных моделей.

Ключевые слова. Малая упорядоченная теория, число счётных моделей, опускание типов.

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