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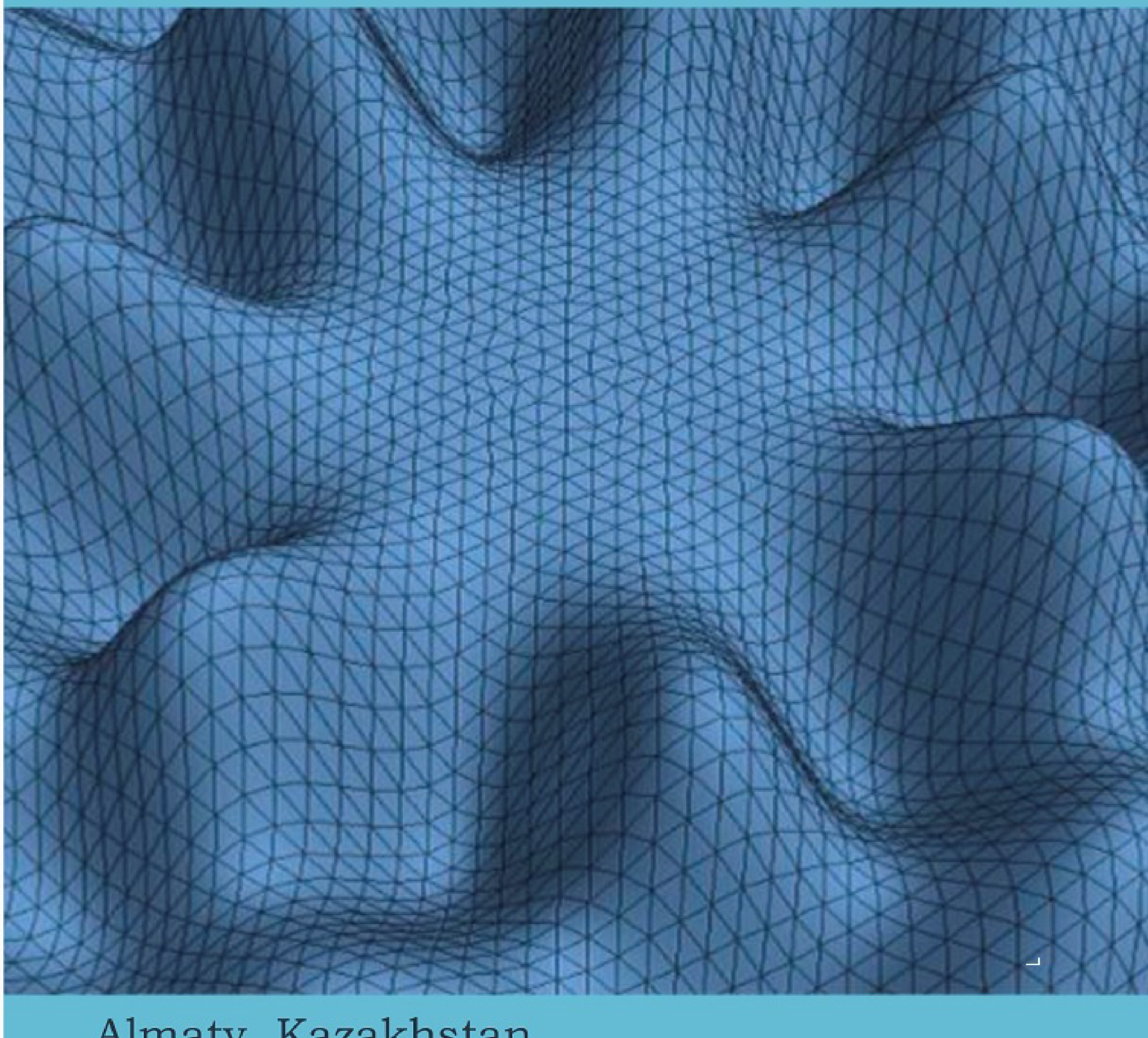

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Green's tensor of subsonic transport boundary value problem for elastic half-space

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Abstract. The first boundary value problem of the theory of elasticity for an elastic half-space at the movement on its surface of subsonic trans loads is considered. The speed of motion is less or more than the speed of distribution of elastic Rayleigh waves. On the basis of the generalized Fourier's transformation the fundamental solution of the task is constructed which describes the dynamics of the massif at the movement of the concentrated force on and along its surface. Based on this, the analytical solution is constructed for arbitrary transport loads distributed over the surface, moving with the pre-Rayleigh and super-Rayleigh velocities. It is shown that when the Rayleigh wave velocity is exceeded, the transport loads generate surface Rayleigh waves. The task is a model for research of the stress-strain state of the massif in the vicinity of road constructions under the action of trans loads moving with high velocities.

Keywords. Isotropic elastic half-space, transport load, first boundary value problem, subsonic speed.

1 Introduction

Trans loads are very widespread in practice. As those we understand the moving loads which form does not change over time, but their position are changing in the environment. Dynamic deformation processes, which arise in the ground under their influence, expand with different speeds, which depend on elastic properties of the medium. In isotropic elastic medium there are two sound speeds of propagation of *dilatation* and *shift* waves. The relation of speed of trans load to the sound velocities significantly influences to the stresses and deformations in the elastic medium. We consider here the subsonic case, when speeds of loads are less then speed of shift waves. This case is a characteristic for trans problems as the speed of the movement of the most modern vehicles is many less then the speeds of elastic

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waves propagation. From trans loads we especially distinguish stationary ones which move in the fixed direction with a constant speed (*transport loads*). This class of loads allows to investigate diffraction processes in isotropic elastic medium in the analytical form.

In papers [1]–[3] the fundamental and generalized solutions of the Lame's equations are constructed and investigated which describe the movement of elastic medium at the action of concentrated on an axis and distributed loading in all range of speeds (subsonic, sound, transonic and supersonic ones). On this basis in [4]–[7] the method of boundary integral equations has been developed for solving the transport BVP in elastic medium with cylindrical boundaries. This class of problems is very important for applications in the field of dynamics of underground constructions, trans tunnels and excavations of deep laying.

However there is a class of model trans tasks (for example, road problems) when loadings move on the surface of a half-space. It is known that there is also sound speed in an elastic half-space with which superficial Rayleigh waves are propagating. The Rayleigh's speed is lower, but is very close to the speed of shift waves [10], [11]. Rayleigh's waves don't create tensions on half-space border, but significantly influence on the tensions and deformations of the massif near a free surface.

For the first time such task was considered and solved for a subsonic pre-Rayleigh case by flat deformation (2D-space) in [9]. Here the analytical solution of this task in 3D-statement is constructed also in a subsonic case, when the speed of subsonic trans load is less or more than the Rayleigh's speed.

2 The statement of transport BVP for elastic half-space

Elastic isotropic medium, with Lame's parameters λ, μ and the density ρ occupies half-space $x_1 > 0$, $n(x) = (-1, 0, 0)$ is a unit vector of the external normal to its boundary $D = \left\{x \in R^3 : x_1 = 0\right\}$. Constants c_1 and c_2 are the velocities of elastic waves propagation [11] (*sonic speeds*):

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_2 = \sqrt{\frac{\mu}{\rho}}, \quad c_2 < c_1.$$

Boundary transport load $P(x, t)$ are moving with a constant subsonic speed $c < c_2 < c_1$) along the axis X_3: $P(x, t) = \mu p_j(x_2, x_3 + ct)e_j$. Components of stress tensor σ_{ij} are connected with medium displacements $u(x, t)$ by Hook's law [11]:

$$\sigma_{ij} = \lambda \operatorname{div} u \, \delta_{ij} + \mu(u_{i,j} + u_{j,i}).$$

For the dynamics problems it is better to write this law in the unitless form:
Hook's law:

$$\frac{\sigma_{ij}}{\mu} = \left(\frac{c_1^2}{c_2^2} - 2\right) \operatorname{div} u \, \delta_{ij} + (u_{i,j} + u_{j,i}). \tag{1}$$

Here and everywhere further on the identical indexes the tensor convolution have been made. Partial derivatives on the corresponding coordinate are designated by the index after comma:

$u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$; $\delta_{ij} = \delta_i^j$ is Kronecker symbol. The stationary movement has been considered that allows to pass into mobile coordinates system connected with transport load. Further we use designations: $x = (x_1, x_2)$, $z = x_3 + ct$.

It is supposed that components of the load allow the Fourier's transformation, i.e. they are representable in the form of Fourier's integrals:

$$P_j(x_2, z) = \sigma_{j1}(0, x_2, z) = \tfrac{\mu}{(2\pi)^2} \int\limits_{R^2} \bar{p}_n(\eta, \varsigma) \exp(-i(x_2\eta + \zeta, z)) d\eta d\varsigma,$$
$$\bar{p}_n(\eta, \varsigma) = \int\limits_{R^2} p_n(x_2, z) \exp(i(x_2\eta + z\varsigma)) dx_2 dz. \tag{2}$$

The Lame's equations for displacements of elastic half-space in mobile coordinates system have the form [1]:

$$\left(\left(M_1^{-2} - M_2^{-2}\right) \frac{\partial^2}{\partial x_i \partial x_j} + \left(M_2^{-2}\Delta - (\partial_z)^2\right) \delta_i^j \right) u_j = 0. \tag{3}$$

We denote this operator by $L_{ij}(\partial_1, \partial_2, \partial_z)$. Here two Mach's numbers are introduced:

$$M_1 = c/c_1, \quad M_2 = c/c_2,$$

which characterize the velocity of transport load in relation to the sound speeds of elastic waves. Here and everywhere there is tensor convolution over repealed indexes.

Eqs. (3) were studied in [2], [3]. There are three cases: subsonic ($c < c_2$), transonic ($c_2 < c < c_1$), supersonic ($c > c_1$) and two sonic cases ($c = c_2$, $c = c_1$). In the first case ($M_1 < 1$, $M_2 < 1$) the system (3) is elliptic, in the second one ($M_1 < 1$, $M_2 > 1$) it has the mixed elliptic-hyperbolic type. In supersonic case ($M_1 > 1$, $M_2 > 1$) this system is strong hyperbolic. By sonic speeds it is mixed parabolic-elliptic if $M_1 < 1$, $M_2 = 1$, and it is hyperbolic-parabolic if $M_1 = 1$, $M_2 > 1$.

By sonic and supersonic velocities the shock waves can appear in elastic medium. There are the next conditions on the jumps on their fronts F:

$$[u_j]_F = 0 \qquad \Rightarrow \quad h_z \left[u_{i,j}\right]_F = h_j \left[u_{i,z}\right]_F;$$
$$h_j \left[\sigma_{ij}\right]_F = \rho c^2 h_z \left[u_{i,z}\right]_F, \quad i, j = 1, 2, 3. \tag{4}$$

Here $h(x_1, x_2, z) = (h_1, h_2, h_3 \triangleq h_z)$ is a wave vector, $\|h\| = 1$. It is perpendicular to the front F in the direction of wave propagation.

The continuity of elastic medium gives the first condition. The second condition is continuity of tangent derivatives at the front of a wave; it is consequence from the first one. The third formula is the law of momentum conservation on waves fronts.

Here we consider the subsonic case. It is required to find the solution of the BVP which must satisfy *the attenuation condition on infinity*:

$$u \to 0 \quad \text{by} \quad x_1 \to +\infty \quad \text{or} \quad z \to \pm\infty. \tag{5}$$

Also we will enter some additional *radiation conditions* later by construction of the BVP solution.

3 Green's tensor of transport BVP

To solve the problem, we use the methods of distribution theory [12]. At first we construct the Green's tensor Π_j^k of the boundary value problem in a moving coordinate system. For its determination we have the following boundary value problem. Find the tensor solution of homogeneous motion equations:

$$\left(\left(M_1^{-2} - M_2^{-2}\right)\frac{\partial^2}{\partial x_i \partial x_j} + \left(M_2^{-2}\Delta - \frac{\partial^2}{\partial z^2}\right)\delta_i^j\right)\Pi_j^k = 0, \quad i,j,k = 1,2,3, \tag{6}$$

in the region $x_1 > 0$, which must satisfy the attenuation condition at infinity:

$$\Pi_j^k(x,z) \to 0 \quad for \ \ \|(x,z)\| \to \infty. \tag{7}$$

Corresponding stress tensor Σ_{jk}^m, which are calculated by using Hook's law (2), has the form:

$$\begin{gathered}\Sigma_{jk}^m = \alpha\,\Pi_l^m,_l\,\delta_{jk} + (\Pi_j^m,_k + \Pi_k^m,_j) = S_{jk}^l(\partial_1,\partial_2,\partial_z)\Pi_l^m(x_1,x_2,z),\\ S_{jk}^l = \alpha\delta_{jk}\partial_l + (\delta_{jl}\partial_k + \delta_{lk}\partial_j).\end{gathered} \tag{8}$$

Theorem. *The solution of the boundary value problem can be represented in the following integral form*

$$u_j(x_1,x_2,z) = \int_{R^q}^{\infty} \Pi_j^n(x_1, x_2 - y_2, z - y_3)p_n(y_2,y_3)\,dy_2dy_3, \quad j = 1,2,3, \tag{9}$$

where tensor Π_j^n *must satisfy to following singular conditions on the free surface for* $x_1 = 0$:

$$\Sigma_{i1}^m = \alpha\,\Pi_k^m,_k\,\delta_{i1} + \ (\Pi_i^m,_1 + \Pi_1^m,_i) = \delta_i^m\delta(x_2)\delta(z), \quad i,m,k = 1,2,3. \tag{10}$$

where $\delta(x_j)$ *is generalized Dirac function,* $\alpha = \frac{\lambda}{\mu} = \left(\frac{c_1^2}{c_2^2} - 2\right) = \left(\frac{M_2^2}{M_1^2} - 2\right)$.

Proof. Indeed, by virtue of (1), (10) and the convolution properties we have on the boundary of the half-space:

$$\int_{R^q}^{\infty} \Sigma_{j1}^m(0, x_2 - y_2, z - y_3)p_m(y_2,y_3)\,dy_2dy_3 = \delta_j^m\delta(x_2)\delta(z) * p_m(x_2,z) = p_j(x_2,z).$$

Here, on the right, there is a functional convolution along the half-space boundary and a tensor convolution by the index m. The displacements (9) satisfy the Lame homogeneous transport equations (3) in the half-space:

$$L_i^j(\partial_1, \partial_2, \partial_z)u_j = \int\limits_{R^q}^{\infty} p_n(y_2, y_3) L_i^j(\partial_1, \partial_2, \partial_z)\Pi_j^n(x_1, x_2 - y_2, z - y_3)\, dy_2 dy_3 = 0$$

in view of (6) and of the invariance of these equations with respect to the shift at the boundary of the half-space.

This tensor $\Pi(x, z)$ gives possibility to use formula (9) for determination of displacements in a half-space for any load on its surface. Stresses at any point of the elastic half-space on an area with a normal n are determined by the formula

$$\begin{aligned} S(x_1, x_2, z, n) &= \sigma_{jk}(x_1, x_2, z) n_j e_k \\ &= \mu\, e_k n_j \int\limits_{R^q}^{\infty} \Sigma_{kj}^l(x_1, x_2 - y_2, z - y_3) p_l(y_2, y_3)\, dy_2 dy_3. \end{aligned} \qquad (11)$$

Thus, the definition of the fundamental displacement tensor determines the solution of the problem.

We construct the tensor $\Pi(x, z)$ using scalar and vector elastic Lame's potentials.

4 Statement of the transport BVP for Lame's potentials

The displacements of the elastic medium can be represented in terms of scalar and vector Lame's potentials [1], [11]:

$$u = grad\varphi + rot\psi. \qquad (12)$$

Since three components of the displacements are determined through four potential components, vector potential is usually associated with Gaussian or Lorentz gauge. Here it is convenient to use representation:

$$\psi = \psi_1 e_3 + rot\,(\psi_2 e_3)\,,$$

which uniquely links three components of displacements with three potentials. If the displacements satisfy the homogeneous Lame equations, then potentials satisfy d'Alembert's wave equation with the corresponding velocity:

$$\begin{aligned} c_1^2 \Delta\varphi - \tfrac{\partial^2 \varphi}{\partial t^2} = 0, \\ c_2^2 \Delta\psi_k - \tfrac{\partial^2 \psi_k}{\partial t^2} = 0, \quad k = 1, 2, \end{aligned} \qquad (13)$$

where Δ is a Laplace operator. In the moving coordinate system these equations are transformed to the form:

$$\Delta\varphi - M_1^2 \frac{\partial^2\varphi}{\partial z^2} = 0,$$
$$\Delta\psi_k - M_2^2 \frac{\partial^2\psi_k}{\partial z^2} = 0, \quad k = 1, 2. \tag{14}$$

To construct a tensor Π_j^i, we use similar potentials. Namely, we represent it in the form:

$$\Pi_k^m(x_1, x_2, z) = D_{kn}(\partial_1, \partial_2, \partial_z)\Phi_n^m$$
$$= \partial_k\Phi_1^m + e_{ki3}\partial_i\Phi_2^m + e_{kjl}e_{li3}\partial_j\partial_i\Phi_3^m,$$
$$D_{k1}(\partial_1, \partial_2, \partial_z) = \partial_k, \tag{15}$$
$$D_{k2}(\partial_1, \partial_2, \partial_z) = e_{ki3}\partial_i,$$
$$D_{k3}(\partial_1, \partial_2, \partial_z) = e_{kjl}e_{li3}\partial_i\partial_j.$$

Here $i, j, k, l, m = 1, 2, 3,$ e_{ijk} is a Levi-Civita pseudotensor. The first potential describes the gradient component of the displacements field, and the other two potentials describe the rotor (solenoidal) components. The potentials satisfy the *transport wave equations*:

$$\Delta\Phi_j^m - M_j^2 \frac{\partial^2\Phi_j^m}{\partial z^2} = 0, \quad j = 1, 2, 3. \tag{16}$$

We call them *fundamental potentials*. To calculate them we use boundary conditions: by $x_1 = 0$

$$\alpha\, \Pi_k^m,_k\, \delta_{i1} + (\Pi_i^m,_1 + \Pi_1^m,_i) = \delta_i^m \delta(x_2)\delta(z),$$

where

$$\Pi_k^m,_k = \Delta\Phi_1^m + e_{ki3}\partial_k\partial_i\Phi_2^m + e_{kjl}e_{li3}\partial_k\partial_i\partial_j\Phi_3^m,$$

$$\Pi_i^m,_1 = \partial_i\partial_1\Phi_1^m + e_{ik3}\partial_k\partial_1\Phi_2^m + e_{ijl}e_{lk3}\partial_k\partial_j\partial_1\Phi_3^m,$$

$$\Pi_1^m,_i = \partial_i\partial_1\Phi_1^m + e_{1k3}\partial_k\partial_i\Phi_2^m + e_{1jl}e_{lk3}\partial_k\partial_j\partial_i\Phi_3^m.$$

We can write it in the form:

$$\mathrm{B}_{in}(\partial_1, \partial_2, \partial_z)\Phi_n^m = \delta_i^m\delta(x_2)\delta(z), \quad n, m = 1, 2, 3, \tag{17}$$

where

$$\mathrm{B}_{in}\Phi_n^m = [2\partial_i\partial_1\Phi_1^m + \partial_k\{(e_{ik3}\partial_1 + e_{1k3}\partial_i)\Phi_2^m + \partial_j(e_{ijl}e_{lk3}\partial_1 + e_{1jl}e_{lk3}\partial_i)\Phi_3^m\}]$$

$$+\alpha[\Delta\Phi_1^m + e_{kj3}\partial_k\partial_j\Phi_2^m + e_{kjl}e_{ls3}\partial_k\partial_s\partial_j\Phi_3^m]\delta_{i1} \Rightarrow$$

$$\mathrm{B}_{in}(\partial_1,\partial_2,\partial_z)\Phi_n^m(\alpha\delta_{i1}\Delta + 2\partial_1\partial_i)\Phi_1^m + \partial_k(\alpha\delta_{i1}e_{kj3}\partial_j + e_{ik3}\partial_1 + e_{1k3}\partial_i)\Phi_2^m$$

$$+\partial_k\partial_j\{\alpha\delta_{i1}e_{kjl}e_{ls3}\partial_s + (e_{ijl}e_{lk3}\partial_1 + e_{1jl}e_{lk3}\partial_i)\}\Phi_3^m$$

$$= \left(\alpha M_1^2\delta_{i1}\partial_z\partial_z + 2\partial_1\partial_i\right)\Phi_1^m + \partial_k(\alpha\delta_{i1}e_{kj3}\partial_j + e_{ik3}\partial_1 + e_{1k3}\partial_i)\Phi_2^m$$

$$+\partial_k\partial_j\{\alpha\delta_{i1}e_{kjl}e_{ls3}\partial_s + (e_{ijl}e_{lk3}\partial_1 + e_{1jl}e_{lk3}\partial_i)\}\Phi_3^m.$$

This implies

$$\mathrm{B}_{i1}(\partial_1,\partial_2,\partial_z) = \left(\alpha M_1^2\delta_{i1}\partial_z\partial_z + 2\partial_1\partial_i\right),$$

$$\mathrm{B}_{i2}(\partial_1,\partial_2,\partial_z) = \partial_k(\alpha\delta_{i1}e_{kj3}\partial_j + e_{ik3}\partial_1 + e_{1k3}\partial_i),$$

$$\mathrm{B}_{i3}(\partial_1,\partial_2,\partial_z) = \partial_k\partial_j\{\alpha\delta_{i1}e_{kjl}e_{ls3}\partial_s + (e_{ijl}e_{lk3}\partial_1 + e_{1jl}e_{lk3}\partial_i)\}.$$

Using the properties of the permutation of the indices of the Levi-Civita tensor and the formula for its convolution:

$$e_{lij}e_{lkm} = \delta_{ik}\delta_{jm} - \delta_{im}\delta_{kj},$$

these operators can be greatly simplified:

$$\mathrm{B}_{11}(\partial_1,\partial_2,\partial_z) = \left(\alpha M_1^2\partial_z^2 + 2\partial_1^2\right),$$

$$\mathrm{B}_{21}(\partial_1,\partial_2,\partial_z) = 2\partial_1\partial_2, \quad \mathrm{B}_{31}(\partial_1,\partial_2,\partial_z) = 2\partial_1\partial_3,$$

$$\mathrm{B}_{12}(\partial_1,\partial_2,\partial_z) = \partial_k(\alpha e_{kj3}\partial_j + e_{1k3}\partial_1 + e_{1k3}\partial_1) = (\alpha e_{kj3}\partial_k\partial_j + 2\partial_1\partial_2)$$

$$= \alpha(e_{123}\partial_1\partial_2 + e_{213}\partial_2\partial_1) + 2\partial_1\partial_2 = 2\partial_1\partial_2,$$

$$\mathrm{B}_{22}(\partial_1,\partial_2,\partial_z) = \partial_k(\alpha\delta_{21}e_{kj3}\partial_j + e_{2k3}\partial_1 + e_{1k3}\partial_2)$$

$$= (e_{213}\partial_1\partial_1 + e_{123}\partial_2\partial_2) = \partial_2\partial_2 - \partial_1\partial_1,$$

$$\mathrm{B}_{32}(\partial_1,\partial_2,\partial_z) = \partial_k(e_{3k3}\partial_1 + e_{1k3}\partial_3) = \partial_2\partial_3,$$

$$\mathrm{B}_{13}(\partial_1,\partial_2,\partial_z)=\partial_k\partial_j\left\{\alpha e_{kjl}e_{lm3}\partial_m+(e_{1jl}e_{lk3}\partial_1+e_{1jl}e_{lk3}\partial_1)\right\}$$

$$=\alpha\left(\delta_{km}\delta_{j3}-\delta_{kj}\delta_{m3}\right)\partial_k\partial_j\partial_m+\left(\delta_{1k}\delta_{j3}-\delta_{13}\delta_{jk}\right)\partial_1\partial_k\partial_j$$

$$+\left(\delta_{1k}\delta_{j3}-\delta_{13}\delta_{jk}\right)\partial_1\partial_k\partial_j=\alpha\left(\partial_3\partial_m\partial_m-\partial_3\partial_j\partial_j\right)+2\partial_1\partial_1\partial_3=2\partial_1\partial_1\partial_3,$$

$$\mathrm{B}_{23}(\partial_1,\partial_2,\partial_z)=e_{2jl}e_{lk3}\partial_1\partial_k\partial_j+e_{1jl}e_{lk3}\partial_2\partial_k\partial_j$$

$$=\left(\delta_{2k}\delta_{j3}-\delta_{23}\delta_{jk}\right)\partial_1\partial_k\partial_j+\left(\delta_{1k}\delta_{j3}-\delta_{13}\delta_{jk}\right)\partial_2\partial_k\partial_j=2\partial_1\partial_2\partial_3,$$

$$\mathrm{B}_{33}(\partial_1,\partial_2,\partial_z)=e_{2jl}e_{lk3}\partial_1\partial_k\partial_j+e_{1jl}e_{lk3}\partial_2\partial_k\partial_j$$

$$=\left(\delta_{2k}\delta_{j3}-\delta_{23}\delta_{kj}\right)\partial_1\partial_k\partial_j+\left(\delta_{1k}\delta_{j3}-\delta_{13}\delta_{jk}\right)\partial_2\partial_k\partial_j=2\partial_1\partial_2\partial_3.$$

As a result, we get:

$$\begin{gathered}\mathrm{B}_{11}=\left(\alpha M_1^2\partial_z\partial_z+2\partial_1^2\right),\ B_{12}=2\partial_1\partial_2,\quad B_{13}=2\partial_1^2\partial_3,\\ \mathrm{B}_{21}(\partial_1,\partial_2,\partial_z)=2\partial_1\partial_2,\quad \mathrm{B}_{22}(\partial_1,\partial_2,\partial_z)=\partial_2\partial_2-\partial_1\partial_1,\\ \mathrm{B}_{23}(\partial_1,\partial_2,\partial_z)=2\partial_1\partial_2\partial_3,\ \ \mathrm{B}_{31}(\partial_1,\partial_2,\partial_z)=2\partial_1\partial_3,\\ \mathrm{B}_{32}(\partial_1,\partial_2,\partial_z)=2\partial_3\partial_2,\quad \mathrm{B}_{33}(\partial_1,\partial_2,\partial_z)=2\partial_1\partial_2\partial_3.\end{gathered}\tag{18}$$

Thus the problem of constructing the transformants of the unknown tensors reduces to determining the Lame potentials which satisfy equations (14), the boundary conditions on the free surface and the damping conditions at infinity:

$$\Phi_j^k\to 0\quad by\ \ \|(x,z)\|\to\infty,\tag{19}$$

and certain radiation conditions which we will write later.

5 Determination of Fourier transforms of fundamental potentials

To construct the solution, we use the Fourier transform of the potentials with respect to x_2, z. In the space of Fourier transforms, they correspond to variables η,ζ. Their Fourier transforms are defined by the relations:

$$\bar{\Phi}^m=\int\limits_{R^2}\Phi^m(x,z)\exp(i\eta x_2+i\zeta z)dzdx_2,$$

$$\Phi^m=\frac{1}{4\pi^2}\int\limits_{R^2}\bar{\Phi}^m(x,\eta,\varsigma)\exp(-i\eta x_2-i\zeta z)d\varsigma d\eta.\tag{20}$$

In the space of Fourier transforms the equations for the potentials (14) have the form:

$$\frac{d^2\,\overset{\rightharpoonup m}{\bar{\Phi}_j}}{dx_1^2}-\eta^2\bar{\Phi}_j^m-\alpha_j^2\zeta^2\bar{\Phi}_j^m=0,\quad \alpha_j=\sqrt{1-M_j^2},\quad j=1,2,3. \tag{21}$$

The expression under the radical is positive, because we consider the subsonic case. The boundary conditions are transformed to the form:

$$\mathrm{B}_{ik}(\partial_1,-i\eta,-i\varsigma)\bar{\Phi}_k^m(x_1,\eta,\zeta)=\delta_i^m\quad \text{by}\quad x_1=0. \tag{22}$$

Conditions for damping at infinity are: for $\forall\eta,\zeta$

$$\bar{\Phi}_k^m(x_1,\eta,\zeta)\to 0\quad \text{by}\quad x_1\to\infty. \tag{23}$$

By these conditions the solution of Eq. (22) has the form:

$$\bar{\Phi}_j^k=\phi_j^k(\eta,\varsigma)\exp\left(-x_1\sqrt{\eta^2+\alpha_j^2\zeta^2}\right),\quad Re\sqrt{\eta^2+\alpha_j^2\zeta^2}\geq 0. \tag{24}$$

Functions $\phi_j^k(\eta,\varsigma)$ are determined from boundary conditions (22):

$$\sum_{j=1}^{3}\mathrm{B}_{in}(-\sqrt{\eta^2+\alpha_j^2\zeta^2},-i\eta,-i\zeta)\phi_n^m=\delta_i^m,\quad k=1,2,3. \tag{25}$$

Thus for each fixed m, we have the linear system of three equations for determination φ_k^m from which we find

$$\varphi_j^m=\frac{\Delta_j^m(\eta,\zeta)}{\Delta(\eta,\zeta)}. \tag{26}$$

Here Δ_j^m is corresponding to algebraic complement, and the denominator is equal to

$$\Delta(\eta,\zeta)=\det\{\mathrm{B}_{kj}(-\sqrt{\eta^2+\alpha_j^2\zeta^2},-i\eta,-i\zeta)\}.$$

This is Rayleigh's determinant. In this case it has the form:

$$\Delta=4\nu^2\sqrt{\nu^2-M_1^2\zeta^2}\sqrt{\nu^2-M_2^2\zeta^2}-(2\nu^2-M_2^2\zeta^2)^2,\quad \nu^2=\zeta^2+\eta^2.$$

The properties of Rayleigh's determinant are known. For transport problems, it was well studied in [1]. In particular,

$$\Delta(\eta,\zeta)=0$$

by

$$\eta=\eta_R^{\pm}(\varsigma)=\pm|\zeta|\sqrt{M_R^2-1}\quad\Leftrightarrow\quad \varsigma=\zeta_R^{\pm}(\eta)=\pm\frac{|\eta|}{\sqrt{M_R^2-1}}, \tag{27}$$

where $M_R = c/c_R$, c_R is the velocity of Rayleigh surface wave, which is subsonic ($c_R < c_2$). It can be determined from the equation:

$$4\sqrt{1-m_1^2}\sqrt{1-m_2^2} - (2-m_2^2)^2 = 0, \quad m_j = c_R/c_j. \tag{28}$$

Formulas (24), (26) formally resolve the problem in the potentials. However, in order to reconstruct the originals, it is necessary to investigate the properties of the transformants – integrand functions, i.e., in (20), which essentially depend on the speed of a transport load.

6 Restoration of originals by pre-Rayleigh speeds

From (15) we get

$$\bar{\Pi}_k^m = D_{kn}(\partial_1, -i\eta, -i\varsigma)\bar{\Phi}_n^m(x_1,\eta,\varsigma)$$

$$= \tfrac{\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)} D_{kn}(\partial_1, -i\eta, -i\varsigma)\exp\left(-x_1\sqrt{\eta^2+\alpha_n\varsigma^2}\right) \Rightarrow \tag{29}$$

$$\bar{\Pi}_k^m = \tfrac{\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)} D_{kn}(-\sqrt{\eta^2+\alpha_n\varsigma^2}, -i\eta, -i\varsigma)\exp\left(-x_1\sqrt{\eta^2+\alpha_n\varsigma^2}\right),$$

$$\bar{\Pi}_k^m(x_1,\eta,\zeta) = D_{kn}(-\sqrt{\eta^2+\alpha_j^2\zeta^2}, -i\eta, -i\zeta)\phi_n^m(x_1,\eta,\zeta)\exp(-x_1\sqrt{\eta^2+\alpha_n^2\zeta^2}). \tag{30}$$

Using the inverse Fourier transform, we obtain

$$(2\pi)^2\Pi_k^m(x_1,x_2,z) = \textstyle\int_{R^2}\bar{\Pi}_k^m(x_1,\eta,\zeta)\exp(-i(\eta x_2+\zeta z))d\varsigma d\eta$$

$$= \textstyle\int_{R^2} D_{kn}(-\sqrt{\eta^2+\alpha_j^2\zeta^2}, i\eta, i\zeta)\phi_n^m(\eta,\zeta)\exp(-x_1\sqrt{\eta^2+\alpha_j^2\zeta^2} - i\eta x_2 - i\zeta z)d\varsigma d\eta \tag{31}$$

$$= \textstyle\int_{R^2} \frac{D_{kn}(-\sqrt{\eta^2+\alpha_j^2\zeta^2}, i\eta, i\zeta)\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)}\exp(-x_1\sqrt{\eta^2+\alpha_j^2\zeta^2} - i\eta x_2 - i\zeta z)d\varsigma d\eta.$$

Let us calculate the fundamental stresses and their transformants. For this, we use formulas (11), from which we obtain

$$\Sigma_{jk}^m = \lambda\,\Pi_l^m{}_{,l}\,\delta_{jk} + \mu\,(\Pi_j^m{}_{,k} + \Pi_k^m{}_{,j}) = S_{jk}^l(\partial_1,\partial_2,\partial_z)\Pi_l^m$$

$$= S_{jk}^l(\partial_1,\partial_2,\partial_z)D_{ln}(\partial_1,\partial_2,\partial_z)\Phi_n^m(x_1,x_2,z) = T_{jkn}(\partial_1,\partial_2,\partial_z)\Phi_n^m(x_1,x_2,z), \tag{32}$$

$$T_{jkn} = S_{jk}^l(\partial_1,\partial_2,\partial_z)D_{ln}(\partial_1,\partial_2,\partial_z).$$

Hence we get

$$\bar{\Sigma}_{jk}^m = T_{jkn}(-\sqrt{\eta^2+\alpha_n\varsigma^2}, -i\eta, -i\varsigma)\hat{\Phi}_n^m(x_1,\eta,\varsigma)$$

$$= T_{jkn}(-\sqrt{\eta^2+\alpha_n\varsigma^2}, -i\eta, -i\varsigma)\tfrac{\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)}\exp\left(-x_1\sqrt{\eta^2+\alpha_n\varsigma^2}\right).$$

The original of the stress tensor at any point $(x,\, z)$ is calculated by using formula

$$\Sigma_{jk}^{m}(x_1,x_2,z)=(2\pi)^{-2}\int\limits_{R^2}\bar{\Sigma}_{jk}^{m}(x_1,\eta,\zeta)\exp(-i(\eta x_2+\zeta z))d\varsigma d\eta. \tag{33}$$

For $c<c_R$ determinant $\Delta(\eta,\zeta)\neq 0$ for any real ζ,η. That is, at the pre-Rayleigh velocities all the integrands are continuous and tend exponentially to zero when (η,ζ) tends to infinity. Therefore, the integrals exist and satisfy the damping conditions at infinity.

When $x_1=0$, $(x_2,z)\neq(0,0)$, the integrands in (31) and (33) are also continuous and integrable, since they are oscillating and have the order of damping not lower $O\left((\eta^2+\varsigma^2)^{-1}\right)$.

7 Determination of displacements and stresses at pre-Rayleigh speeds of transport load

To calculate the displacements of the medium for arbitrary transport load, we find the Fourier transform of the displacements. According to (9) and to the convolution properties we get

$$\bar{u}_j(x_1,\eta,\zeta)=\mathrm{F}_{x_2,z}[u_j(x_1,x_2,z)]=\bar{\Pi}_j^{n}(x_1,\eta,\varsigma)\bar{p}_n(\eta,\varsigma). \tag{34}$$

Substituting it in (30), we have

$$\bar{u}_k(x_1,\eta,\zeta)=\frac{\bar{p}_m(\eta,\varsigma)\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)}D_{kn}(-\sqrt{\eta^2+\alpha_n\varsigma^2},-i\eta,-i\varsigma)\exp\left(-x_1\sqrt{\eta^2+\alpha_n\varsigma^2}\right).$$

Returning to the original, we obtain formulas for calculating the displacements at pre-Rayleigh speeds:

$$u_k(x_1,x_2,z)=\frac{1}{4\pi^2}\iint_{R^2}\bar{u}_k(x_1,\eta,\zeta)\exp(-i(x_2\eta+z\varsigma))d\eta d\varsigma.$$

To determine stresses, we use formula (11), which for the Fourier transforms has the form:

$$\sigma_{kj}(x_1,x_2,z)=\frac{1}{4\pi^2}\int\limits_{R^2}\bar{\Sigma}_{kj}^{n}(x_1,\eta,\zeta)\bar{p}_n(\eta,\varsigma)\,\exp(-i(x_2\eta+z\varsigma))d\eta d\zeta.$$

At pre-Rayleigh velocities in formulas (33) and (34), all the integrands are continuous and tend exponentially to zero when $x_1\to\infty$. Therefore, the integrals exist and satisfy the damping conditions at infinity. The asymptotic behavior of displacements at infinity is determined by the asymptotic of the transport load on the surface of the half-space.

8 Construction of Green's tensor at super-Rayleigh speed

If the subsonic speed c is more than the Rayleigh speed c_R: $c_R<c<c_2$, then for constructing the solution we transform contour of integration in the $\varepsilon-$vicinity of the point

$\zeta_R(\eta)$ at any fixed η by moving along the circle of radius ε in upper half-plane of complex ζ $(z>0)$ and in under half-plane $(z<0)$ to get under sign of integral the waves, which tend to zero by $|z|\to\infty$. If $\varepsilon\to 0$, then, using the theorem on residue of complex analysis, we get the Green's tensor in the form:

$$4\pi^2\,\Pi_k^m(x_1,x_2,z)$$

$$=\int\limits_{-\infty}^{\infty}\left\{\text{V.P.}\int\limits_{-\infty}^{\infty}\sum_{j=1}^{3} d_{kj}^m(\eta,\zeta)\exp(-x_1\sqrt{\eta^2+\alpha_j^2\zeta^2}-i\zeta z)d\varsigma\right\}e^{-i\eta x_2}d\eta \tag{35}$$

$$-i\pi sgnz\sum_{\pm}\int\limits_{-\infty}^{\infty}\sum_{j=1}^{3} Rd_{kj}^m(\eta,\zeta)\exp\left(-x_1\,|\eta|\,\sqrt{\frac{M_R^2-M_j^2}{M_R^2-1}}\right)e^{-i(\eta x_2+z\zeta_R^{\pm}(\eta))}d\eta,$$

where

$$d_{kj}^m(\eta,\zeta)=D_{kn}(-\sqrt{\eta^2+\alpha_j^2\zeta^2},i\eta,i\zeta)\frac{\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)},$$

$$Rd_{kj}^m(\eta,\zeta)=D_{kn}\left(-|\eta|\sqrt{\frac{M_R^2-M_j^2}{M_R^2-1}},i\eta,i\zeta_R^{\pm}\right)\frac{\Delta_n^m(\eta,\zeta_R^{\pm})}{\Delta_\zeta(\eta,\zeta_R^{\pm}(\eta))}.$$

Here, to calculate the Value Principle integral we can use the formulae:

$$\text{V.P.}\int\limits_{-\infty}^{\infty}D_{kn}(-\sqrt{\eta^2+\alpha_j^2\zeta^2},i\eta,i\zeta)\tfrac{\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)}\exp(-x_1\sqrt{\eta^2+\alpha_j^2\zeta^2}-i\zeta z)d\varsigma$$

$$=\int\limits_{0}^{\infty}(\Upsilon_{kn}^m(x_1,z,\eta,\varsigma)+\Upsilon(x_1,z,\eta,-\varsigma))\exp(-x_1\sqrt{\eta^2+\alpha_j^2\zeta^2})d\varsigma,$$

$$\Upsilon_{kn}^m(x_1,z,\eta,\varsigma)=D_{kn}(-\sqrt{\eta^2+\alpha_j^2\zeta^2},i\eta,i\zeta)\tfrac{\Delta_n^m(\eta,\zeta)\,e^{-i\zeta z}}{\Delta(\eta,\zeta)}.$$

The last integral does not have singularities in Rayleigh's points and can be calculated numerically. The second summand in formula (35) describes the surface Rayleigh waves, which are generated by transport load when $c_R<c<c_2$. By $c=c_R$ the stationary solution of this problem does not exist.

9 Conclusion

The solutions of boundary value problems presented here are very useful for applications when assessing the impact of road trans on the environment. This allows to determine the stress-strain state of the rock massif, depending on its elastic properties, the type of the acting load and the speed of the vehicle. This is especially actual now with the development of high-speed road and rail trans, the speed of which can have a devastating impact on the surrounding areas. The obtained solutions allow us to determine the range of possible speeds of movement, taking into account the strength properties of the rock massif and the road surface, which makes it possible to ensure the safety and reliability of operation of modern vehicles.

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Алексеева Л.А. СЕРПІМДІ ЖАРТЫЛАЙ КЕҢІСТІК ҮШІН ДЫБЫСҚА ДЕЙІНГІ КӨЛІКТІК ШЕТТІК ЕСЕПТІҢ ГРИН ТЕНЗОРЫ

Серпімділік теориясының дыбысқа дейінгі көліктік жүктемелер бетімен қозғалыс кезіндегі бірінші шеттік есебі серпімді жартылай кеңістік үшін қарастырылады. Олардың қозғалыс жылдамдығы Рэлэй серпімді беттік толқындарының таралу жылдамдығынан кіші немесе үлкен болады деп болжанады. Есептің фундаменталды шешімі – Грин тензоры Фурье жалпыланған түрлендіруі негізінде тұрғызылды, ол массивтің динамикасын шоғырланған күштің оның бетінің бойымен қозғалысы кезінде сипаттайды. Жартылай кеңістік бетімен таралған кез келген көліктік жүктемелер үшін шеттік есептің аналитикалық шешімі Рэлей жылдамдығынан төмен және Рэлей жылдамдығынан жоғары жылдамдықтар кезінде тұрғызылды. Рэлей толқынының жылдамдығы шамадан тыс артқан кезде, көліктік жүктемелер Рэлей баттік толқындарын тудыратыны көрсетілді. Қарастырылған есеп жол ғимараттарына өте жақын орналасқан жыныс массивінің жоғарғы жылдамдықпен қозғалатын көліктік жүктемелер әсерінен кернеулі-деформацияланған күй жағдайын зерттеу үшін модельдік есеп болып табылады.

Кілттік сөздер. Изотропты серпімді жартылай кеңістік, көліктік жүктеме, бірінші шеттік есеп, дыбысқа дейінгі жылдамдық.

Алексеева Л.А. ТЕНЗОР ГРИНА ДОЗВУКОВОЙ ТРАНСПОРТНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ УПРУГОГО ПОЛУПРОСТРАНСТВА

Рассматривается первая краевая задача теории упругости для упругого полупространства при движении по его поверхности дозвуковых транспортных нагрузок. Предполагается, что скорость их движения меньше или больше скорости распространения упругих поверхностных волны Рэлея. На основе обобщенного преобразования Фурье построено фундаментальное решение задачи – тензор Грина, который описывает динамику массива при движении сосредоточенной силы вдоль его поверхности. Построено аналитическое решение краевой задачи для произвольных транспортных нагрузок, распределенных по поверхности полупространства, при дорелеевских и сверхрелеевских скоростях. Показано, что при превышении скорости волны Рэлея транспортные нагрузки генерируют поверхностные волны Рэлея. Задача является модельной для исследования напряженно-деформированного состояния породного массива в непосредственной близости от дорожных сооружений под действием транспортных нагрузок, движущихся с высокими скоростями.

Ключевые слова. Изотропное упругое полупространство, транспортная нагрузка, первая краевая задача, дозвуковая скорость.

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Unpredictable oscillations of neural networks

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Abstract. The paper considers a new type of oscillations for shunting inhibitory cellular neural networks (SICNNs), unpredictable solutions, which continue the line of periodic, almost periodic, recurrent oscillations. The dynamics admits useful numerical characteristics and can be convenient for analysis of cognitive tasks, artificial intelligence and robotics development. Since the oscillations are robustly related to chaos, the results are advantageous for research of sophisticated dynamics in neuroscience. The existence and stability of an unpredictable solution for SICNN is proved. Numerical example is given to show the feasibility of the obtained results. Results of the paper were announced in [1], [2].

Keywords. Unpredictable oscillations, Shunting inhibitory cellular neural networks, Asymptotical stability.

1 Introduction and preliminaries

In paper [3] deterministic unpredictable functions were introduced as a new type of oscillations. The existence of unpredictable solutions proves Poincaré chaos for a Hopfield type neural networks [4] and the motions admit numerical characteristics, which can be useful for the analysis of neural processes. The description of such functions relies on the dynamics of unpredictable points, which were presented in the study [5]. The research of unpredictable solutions unites the theoretical advantages and challenges which are proper for both oscillations and chaos, and will open up many interesting prospects in neuroscience.

Shunting inhibitory cellular neural networks (SICNNs), which have been introduced by Bouzerdoum and Pinter in [6], play exceptional role in psychophysics, robotics, adaptive pattern recognition, vision and image processing. In the last several decades there have been published many results concerning the dynamics of the neural networks.

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In its original formulation [6], the SICNN model is a two-dimensional grid of processing cells. We will follow the description in the present reseach. Let C_{ij} denote the cell at the (i, j) position of the lattice. Denote by $N_r(i, j)$ the r-neighbourhood of C_{ij}, such that

$$N_r(i, j) = \{C_{kp} : \max(|k - i|, |p - j|) \leq r, 1 \leq k \leq m, 1 \leq l \leq n\},$$

where m and n are fixed natural numbers. In SICNNs, neighbouring cells exert mutual inhibitory interactions of the shunting type. The dynamics of the cell C_{ij} is described by the following nonlinear ordinary differential equation

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} f(x_{kp}(t))x_{ij} + v_{ij}(t), \tag{1}$$

where x_{ij} is the activity of the cell C_{ij}, $v_{ij}(t)$ is the external input to the cell C_{ij}, the constant a_{ij} represents the passive decay rate of the cell activity, $C_{ij}^{kp} \geq 0$ is the connection or coupling strength of postsynaptic activity of the cell C_{kp} transmitted to the cell C_{ij} and the activation $f(x_{kp})$ is a positive continuous function representing the output or firing rate of the cell C_{kp}, $v_{ij}(t)$ is the external input to the cell C_{ij}.

Throughout the paper, $\mathbb{R}$ and $\mathbb{N}$ will stand for the sets of real and natural numbers, respectively. Also, the norm $\|u\|_1 = \sup_{t \in \mathbb{R}} \|u(t)\|$, where $\|u\| = \max_{(i,j)} |u_{ij}|$, $u(t) = (u_{11}, \ldots, u_{1n}, \ldots, u_{m1} \ldots, u_{mn}), t, u_{ij} \in \mathbb{R}, i = 1, 2, ..., m, j = 1, 2, ..., n$, will be used. The following definition is an initial one in our research.

Definition [3]. *A uniformly continuous and bounded function $u : \mathbb{R} \to \mathbb{R}^{m \times n}$ is unpredictable if there exist positive numbers ϵ_0, δ and sequences t_p, s_p both of which diverge to infinity such that $u(t + t_p) \to u(t)$ as $p \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|u(t + t_p) - u(t)\| \geq \epsilon_0$ for each $t \in [s_p - \delta, s_p + \delta]$ and $p \in \mathbb{N}$.*

2 Main result

Let us denote by $\mathcal{B}$ the set of functions $u(t) = (u_{11}, \ldots, u_{1n}, \ldots, u_{m1} \ldots, u_{mn}), t, u_{ij} \in \mathbb{R}, i = 1, 2, ..., m, j = 1, 2, ..., n$, where $m, n \in \mathbb{N}$, such that:

(A1) functions $u(t)$ are uniformly continuous and there exists a positive number H such that $\|u\|_1 < H$ for all $u(t) \in \mathcal{B}$;

(A2) there exists a sequence t_p, $t_p \to \infty$ as $p \to \infty$ such that for each $u(t) \in \mathcal{B}$ the sequence $u(t + t_p)$ uniformly converges to $u(t)$ on each closed and bounded interval of the real axis.

The following conditions will be needed throughout the paper:

(B1) the function $v(t) = (v_{11}, \ldots, v_{1n}, \ldots, v_{m1} \ldots, v_{mn}), t, v_{ij} \in \mathbb{R}, i = 1, 2, ..., m, j = 1, 2, ..., n$, in the system (1) belongs to $\mathcal{B}$ and is unpredictable such that there exist positive numbers $\delta, \epsilon_0 > 0$ and a sequence $s_p \to \infty$ as $p \to \infty$ which satisfy $\|v(t + t_p) - v(t)\| \geq \epsilon_0$ for all $t \in [s_p - \delta, s_p + \delta]$, and $p \in \mathbb{N}$;

(B2) for the rates we assume that $\gamma = \min_{(i,j)} a_{ij} > 0$ and $\bar{\gamma} = \max_{(i,j)} a_{ij}$;

(B3) there exist positive numbers m_{ij} such that $\sup_{t\in\mathbb{R}} |v_{ij}(t)| \le m_{ij}$;

(B4) there exists a positive number m_f such that $\sup_{|s|<H} |f(s)| \le m_f$;

(B5) there exists a positive number L such that $|f(s_1) - f(s_2)| \le L\,|s_1 - s_2|$ for all s_1, s_2, $|s_1| < H, |s_2| < H$;

(B6) $(LH + m_f) \max_{(i,j)} \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp} < \gamma$ for all $i = 1,\dots,m, j = 1,\dots,n$.

Likewise to the result in [7], one can verify that the following assertion is valid.

Lemma 1. *Assume that conditions (B2) to (B4) are valid. A bounded on* $\mathbb{R}$ *function* $y(t) = \{y_{ij}(t)\}$ *is a solution of SICNNs (1) if and only if the following integral equation is satisfied*

$$y_{ij}(t) = -\int_{-\infty}^{t} \mathrm{e}^{-a_{ij}(t-s)}\Big[\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(y_{kl}(s))y_{ij}(s) - v_{ij}(s)\Big]ds. \tag{2}$$

Define on $\mathcal{B}$ the operator Π such that $\Pi u(t) = \{\Pi_{ij}u(t)\}$, $i = 1,\dots,m, j = 1,\dots,n$, where

$$\Pi_{ij}u(t) \equiv -\int_{-\infty}^{t} \mathrm{e}^{-a_{ij}(t-s)}\Big[\sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp} f(u_{kp}(s))u_{ij}(s) - v_{ij}(s)\Big]ds. \tag{3}$$

Lemma 2. *If* $u(t) \in \mathcal{B}$*, then the operator* Π *is invariant in* $\mathcal{B}$*.*

Proof. For the function $u(t) \in \mathcal{B}$, it is not difficult to show that $\Pi u(t)$ satisfies the condition (A1).

Now, let us fix a positive number ϵ and a finite interval $[a, b] \subset \mathbb{R}$. Consider numbers $c < a$ and $\xi > 0$, which satisfy the following inequalities,

$$\frac{2}{\gamma}\Big(\max_{(i,j)} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(m_f H + LH^2) + m_{ij}\Big)e^{-\gamma(a-c)} \le \frac{\epsilon}{2} \tag{4}$$

and

$$\frac{\xi}{\gamma}\Big(\max_{(i,j)} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(m_f + LH) + 1\Big) \le \frac{\epsilon}{2}. \tag{5}$$

We will show that $\|\Pi u(t+t_p) - \Pi u(t)\| < \epsilon$ on $[a,b]$ for sufficiently large p. Let p be a large enough number such that $\|u(t+t_p) - u(t)\| < \xi$ and $\|v(t+t_p) - v(t)\| < \xi$, on $[c,b]$. Then for all $t \in [a,b]$ it is true that

$$|\Pi_{ij}u_{ij}(t+t_n) - \Pi_{ij}u_{ij}(t)| \leq \int_{-\infty}^{t} e^{-\gamma(t-s)}\Big(\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}\Big|f(u_{kl}(s))u_{ij}(s)$$

$$-f(u_{kl}(s+t_n))u_{ij}(s+t_n)\Big| + \Big|(v_{ij}(s+t_n) - v_{ij}(s)\Big|\Big)ds$$

$$\leq \int_{-\infty}^{c} e^{-\gamma(t-s)}\Big(\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}\Big|[f(u_{kl}(s))[u_{ij}(s) - u_{ij}(s+t_n)]$$

$$+[f(u_{kl}(s)) - f(u_{kl}(s+t_n))]u_{ij}(s+t_n)]\Big| + \Big|(v_{ij}(s+t_n) - v_{ij}(s)\Big|\Big)ds$$

$$+\int_{c}^{t} e^{-\gamma(t-s)}\Big(\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}\Big|f(u_{kl}(s))[u_{ij}(s) - u_{ij}(s+t_n)]$$

$$+[f(u_{kl}(s)) - f(u_{kl}(s+t_n))]u_{ij}(s+t_n)\Big| + \Big|(v_{ij}(s+t_n) - v_{ij}(s)\Big|\Big)ds$$

$$\leq \Big(\frac{\max_{(i,j)}\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{\gamma}(m_f 2H + L2HH) + 2m_{ij}\Big)e^{-\gamma(a-c)}$$

$$+\Big(\frac{\xi\max_{(i,j)}\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{\gamma}(m_f + LH) + 1\Big),$$

for all $i = 1,2,...,m, j = 1,2,...,n$. Now inequalities (4) and (5) imply that $\|\Pi u(t+t_n) - \Pi u(t)\| < \epsilon$ for $t \in [a,b]$. Since ϵ is arbitrary small number, the condition (A2) is valid. The lemma is proved. □

Lemma 3. *The operator* Π *is contractive in* $\mathcal{B}$.

Proof. For two functions $\varphi, \psi \in \mathcal{B}$, and fixed $i = 1,2,...,m, j = 1,2,...,n$, we have that

$$|\Pi_{ij}\varphi_{ij}(t) - \Pi_{ij}\psi_{ij}(t)| \leq \int_{-\infty}^{t} e^{-\gamma(t-s)} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}\Big(f(\varphi_{kl}(s))\varphi_{ij}(s) - f(u_{kl}(s))\psi_{ij}(s)\Big)ds$$

$$+\int_{-\infty}^{t} e^{-\gamma(t-s)} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}\Big|f(\varphi_{kl}(s))\psi_{ij}(s) - f(\psi_{kl}(s))\psi_{ij}(s)\Big|ds$$

$$\leq \frac{(LH+m_f)}{\gamma}\max_{(i,j)} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}\,\|\varphi-\psi\|_1 .$$

That is why $\|\Pi\varphi - \Pi\psi\|_1 \leq \dfrac{LH + m_f}{\gamma} \max_{t\in\mathbb{R}} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} \|\varphi - \psi\|_1$. Then condition $(B6)$ implies that the operator Π is contractive in the set $\mathcal{B}$. The lemma is proved. □

Theorem 1. *Suppose that conditions (B1)–(B6) are valid, then the system (1) possesses an unique asymptotically stable unpredictable solution $\omega(t) \in \mathcal{B}$.*

Proof. Let us show that the space $\mathcal{B}$ is complete. Consider a Cauchy sequence $\phi_k(t)$ in $\mathcal{B}$, which converges to a limit function $\phi(t)$ on $\mathbb{R}$. It suffices to show that $\phi(t)$ satisfies condition $(K3)$, since other two conditions can be easily checked. Fix a closed and bounded interval $I \subset \mathbb{R}$. We have that

$$\|\phi(t+t_p) - \phi(t)\| \leq \|\phi(t+t_p) - \phi_k(t+t_p)\| + \|\phi_k(t+t_p) - \phi_k(t)\| + \|\phi_k(t) - \phi(t)\|. \quad (6)$$

Now, one can take sufficiently large p and k such that each term on the right-hand side of (6) is smaller than $\frac{\epsilon}{3}$ for an arbitrary positive ϵ and $t \in I$. The inequality implies that $\|\phi(t+t_p) - \phi(t)\| \leq \epsilon$ on I. That is the sequence $\phi(t+t_p)$ uniformly converges to $\phi(t)$ on I. The completeness of $\mathcal{B}$ is proved. Now, by the contractive mapping theorem, due to Lemmas 2 and 3, there exists a unique solution $\omega(t) \in \mathcal{B}$ of equation (1).

One can find a positive number κ and natural numbers l, k and $j = 1, \ldots, p$, such that:

$$\kappa < \delta, \quad (7)$$

$$\kappa\Big(\frac{1}{2} - (\frac{1}{l} + \frac{2}{k})(\bar{\gamma} + \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp}(m_f + LH))\Big) \geq \frac{3}{2l}, \quad (8)$$

$$|\omega_{ij}(t+s) - \omega_{ij}(t)| < \epsilon_0 \min(\frac{1}{k}, \frac{1}{4l}), \quad t \in \mathbb{R}, |s| < \kappa, \quad (9)$$

for all $i = 1, 2, ..., m, j = 1, 2, ..., n$.

Denote $\Delta = |\omega_{ij}(t_p + s_p) - \omega_{ij}(s_p)|$ and consider two cases: (i) $\Delta < \epsilon_0/l$; (ii) $\Delta \geq \epsilon_0/l$ such that the remaining proof falls naturally into two parts.

(i) From (9) it follows that

$$\|\omega_{ij}(t+s_p) - \omega_{ij}(s_p)\| < \frac{\epsilon_0}{l} + \frac{\epsilon_0}{k} + \frac{\epsilon_0}{k} = \epsilon_0\Big(\frac{1}{l} + \frac{2}{k}\Big), \quad (10)$$

if $t \in [s_p, s_p + \kappa]$. It is true that

$$\omega_{ij}(t+t_p) - \omega_{ij}(t) = \omega(t_p + s_p) - \omega(s_p) - \int_{s_p}^{t} a_{ij}(\omega(s+t_p) - \omega(s))ds$$

$$-\int_{s_p}^{t} \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp}(f(\omega_{kp}(s+t_p))\omega_{ij}(s+t_p) - f(\omega_{kp}(s))\omega_{ij}(s))ds - \int_{s_p}^{t}(v_{ij}(s+t_p) - v_{ij}(s))ds. \quad (11)$$

We obtain from (7)–(8) and (10)–(11) that

$$|\omega_{ij}(t+t_p)-\omega_{ij}(t)| \geq \int_{s_p}^{t} |v_{ij}(s+t_p)-v_{ij}(s)|ds - |\omega_{ij}(t_p+s_p)-\omega_{ij}(s_p)|$$

$$-\int_{s_p}^{t} a_{ij}|\omega_{ij}(s+t_p)-\omega_{ij}(s)|ds - \int_{s_p}^{t} \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp}|f(\omega_{kp}(s+t_p))\omega_{ij}(s+t_p)-f(\omega_{kp}(s))\omega_{ij}(s)|ds$$

$$\geq \epsilon_0\frac{\kappa}{2} - \frac{\epsilon_0}{l} - \epsilon_0\kappa(\frac{1}{l}+\frac{2}{k})(\bar{\gamma} + \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp}(m_f+LH))$$

$$= \epsilon_0\kappa\Big(\frac{1}{2} - (\frac{1}{l}+\frac{2}{k})(\bar{\gamma} + \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp}(m_f+LH))\Big) \geq \frac{3\epsilon_0}{2l},$$

for $t \in [s_p+\frac{\kappa}{2}, s_p+\kappa]$.

ii) For the case $\Delta \geq \epsilon_0/l$, it can be easily found that (9) implies

$$\|\omega(t_p+t)-\omega(t)\| \geq \|\omega(t_p+s_p)-\omega(s_p)\| - \|\omega(s_p)-\omega(t)\| - \|\omega(t_p+t)-\omega(t_p+s_p)\|$$

$$\geq \frac{\epsilon_0}{l} - \frac{\epsilon_0}{4l} - \frac{\epsilon_0}{4l} = \frac{\epsilon_0}{2l},$$

if $t \in [s_p-\kappa, s_p+\kappa]$ and $p \in \mathbb{N}$.

Thus, one can conclude that $\omega(t)$ is the unpredictable solution with $\bar{s}_n = s_n + \frac{3\kappa}{4}, \bar{\delta} = \frac{\kappa}{4}$.

Finally, we will discuss the stability of the unpredictable solution $\omega(t)$. It is true that

$$\omega_{ij}(t) = e^{-a_{ij}(t-t_0)}\omega_{ij}(t_0) - \int_{t_0}^{t} e^{-a_{ij}(t-s)}\Big[\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(\omega_{kl}(s))\omega_{ij}(s) - v_{ij(s)}\Big]ds,$$

$i = 1,\ldots,m, j = 1,\ldots,n.$

Let $z(t) = (z_{11},\ldots,z_{1n},\ldots,z_{m1}\ldots,z_{mn})$ be another solution of the system. One can write

$$z_{ij}(t) = e^{-a_{ij}(t-t_0)}z_{ij}(t_0) - \int_{t_0}^{t} e^{-a_{ij}(t-s)}\Big[\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(z_{kl}(s))z_{ij}(s) - v_{ij(s)}\Big]ds.$$

Making use of the relation

$$z_{ij}(t) - \omega_{ij}(t) = e^{-a_{ij}(t-t_0}(z_{ij}(t_0)-\omega_{ij}(t_0))$$

$$-\int_{t_0}^{t} e^{-a_{ij}(t-s)}\Big[\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(z_{kl}(s))z_{ij}(s) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(\omega_{kl}(s))\omega_{ij}(s)\Big]ds,$$

we obtain that

$$|z_{ij}(t)-\omega_{ij}(t)| \leq e^{-\gamma(t-t_0)}\Big|z_{ij}(t_0)-\omega_{ij}(t_0)\Big| + m_f \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}\int_{t_0}^{t} e^{-\gamma(t-s)}\Big|z_{ij}(s)-\omega_{ij}(s)\Big|ds$$

$$+LH\int_{t_0}^{t} e^{-\gamma(t-s)} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}\Big|z_{kl}(s)-\omega_{kl}(s)\Big|ds,$$

for all $i=1,\ldots,m, j=1,\ldots,n$. Multiply both sides of the last inequality by $e^{\gamma t}$:

$$e^{\gamma t}\left\|z(t)-\omega(t)\right\| \leq \left\|z(t_0)-\omega(t_0)\right\| + (LH+m_f)\max_{(i,j)} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}\int_{t_0}^{t} e^{\gamma s}\left\|z(s)-\omega(s)\right\|ds.$$

Now, applying Gronwall-Bellman Lemma, one can attain that

$$\left\|z(t)-\omega(t)\right\| \leq \left\|z(t_0)-\omega(t_0)\right\| e^{\left((LH+m_f)\max_{(i,j)}\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}-\gamma\right)(t-t_0)}.$$

The last inequality and condition (B6) confirm that the unpredictable solution $\omega(t)$ is uniformly asymptotically stable. The theorem is proved. □

3 Example

Consider the logistic discrete equation

$$\lambda_{i+1} = \mu\lambda_i(1-\lambda_i), \tag{12}$$

with $\mu = 3.92$ [3] . The sequence belongs to the unit interval $[0,1]$. In paper [4] it was proved that equation (12) has an unpredictable solution $\psi_i, i \in \mathbb{Z}$.

Let us construct the solution $\Theta(t)$ of the equation

$$\frac{dv}{dt} = -3v(t) + \Omega(t), \tag{13}$$

where $\Omega(t)$ is a piecewise constant function defined on the real axis through the equation $\Omega(t) = \psi_i$ for $t \in [i, i+1)$, $i \in \mathbb{Z}$. One can check that

$$\Theta(t) = \int_{-\infty}^{t} e^{-3(t-s)}\Omega(s)ds. \tag{14}$$

It is worth noting that $\Theta(t)$ is bounded on the whole real axis such that $\sup_{t\in\mathbb{R}} |\Theta(t)| \leq 1/3$, and is globally exponentially stable. Moreover, the function $\Theta(t)$ is unpredictable [4].

Example. Let us introduce the following SICNNs:

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kp}\in N_1(i,j)} C_{ij}^{kp} f(x_{kp}(t))x_{ij} + v_{ij}(t), \qquad (15)$$

where $i, j = 1, 2, 3,$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 4 & 6 & 2 \\ 1 & 7 & 5 \\ 4 & 8 & 3 \end{pmatrix}, \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} 0.02 & 0.05 & 0 \\ 0.04 & 0.07 & 0.03 \\ 0.06 & 0 & 0.09 \end{pmatrix},$$

and $f(s) = \frac{1}{3}arctan(s)$, $v_{11}(t) = 27\Theta^3(t) + 2$, $v_{12}(t) = 3\Theta(t)$, $v_{13}(t) = -5\Theta(t) + 3$, $v_{21}(t) = 12\Theta(t) + 1$, $v_{22}(t) = 21\Theta^3(t)$, $v_{23}(t) = 19\Theta(t) - 1$, $v_{31}(t) = -8\Theta(t) + 5$, $v_{32}(t) = 6\Theta(t)$, $v_{33}(t) = -19\Theta^3(t)$, $\Theta(t)$ is the unpredictable solution of the system (13). Moreover, by means of Lemma 1.4 and Lemma 1.5 in [8] function $v(t) = \{v_{ij}(t)\}$, $i = 1, 2, 3$, $j = 1, 2, 3$, is unpredictable.

Figure 1 represents the solution $\phi(t)$ of (15) with initial values $\phi_{11}(0) = 0.5211$, $\phi_{12}(0) = 0.1359$, $\phi_{13}(0) = 0.9876$, $\phi_{21}(0) = 3.333$, $\phi_{22}(0) = 0.0444$, $\phi_{23}(0) = 0.5725$, $\phi_{31}(0) = 0.8541$, $\phi_{32}(0) = 0.2278$, $\phi_{33}(0) = -0.0982$. The integral curve approximates the unpredictable solution of the system (15).

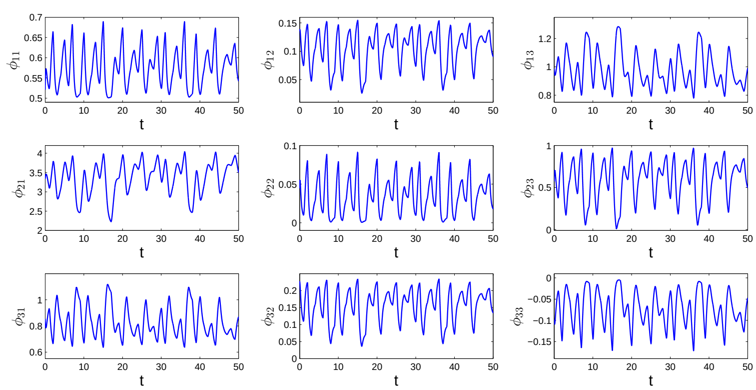

Figure 1 – The coordinates of the solution $\phi(t)$ of SICNNs (15)

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Тлеубергенова М., Сеилова Р., Жаманшин А. НЕЙРОНДЫҚ ЖЕЛІЛЕРДІҢ БОЛЖАП БІЛУГЕ БОЛМАЙТЫН ТЕРБЕЛІСТЕРІ

Мақалада шунттаушы тежегіші бар ұялы нейрондық желілер (ШТҰНЖ) үшін тербелістердің жаңа түрі, периодты, периодты дерлік, рекурренттік тербелістердің шебін жалғастыратын болжап білуге болмайтын шешімдер қарастырылады. Динамикасы пайдалы сипаттамаларға ие болып отыр және когнитивті есептерді, жасанды интеллект пен құлтемір техникасын талдауға ыңғайлы болуы мүмкін. Тербелістер хаоспен тығыз байланысты болғандықтан, нәтижелер нейроғылымдағы күрделі динамиканы зерттеу үшін пайдалы болады. ШТҰНЖ үшін болжап білуге болмайтын шешімнің бар болуы мен орнықтылығы дәлелденген. Алынған нәтижелердің орындалатынын көрсететін мысал келтірілген.

Кілттік сөздер. Болжап білуге болмайтын тербелістер, шунттаушы тежегіші бар ұялы нейрондық желілер, асимптотикалық орнықтылық.

Тлеубергенова М., Сеилова Р., Жаманшин А. НЕПРЕДСКАЗУЕМЫЕ КОЛЕБАНИЯ НЕЙРОННЫХ СЕТЕЙ

В статье рассматривается новый тип колебаний для клеточных нейронных сетей с шунтирующим торможением (КНСШТ), непредсказуемые решения, которые продолжают линию периодических, почти периодических, рекуррентных колебаний. Динамика обладает полезными характеристиками и может быть удобной для анализа когнитивных задач, искусственного интеллекта и развития робототехники. Поскольку колебания тесно связаны с хаосом, результаты полезны для исследования сложной динамики в нейронауке. Доказаны существование и устойчивость непредсказуемого решения для КНСШТ. Приведен пример для того, чтобы показать выполнимость полученных результатов.

Ключевые слова. Непредсказуемые колебания, клеточные нейронные сети с шунтирующим торможением, асимптотическая устойчивость.

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Analyzing variance in central limit theorems

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Abstract. Central limit theorems deal with convergence in distribution of sums of random variables. The usual approach is to normalize the sums to have variance equal to 1. As a result, the limit distribution has variance one. In most papers, existence of the limit of the normalizing factor is postulated and the limit itself is not studied. Here we review some results which focus on the study of the normalizing factor. Applications are indicated.

Keywords. Central limit theorems, convergence in distribution, limit distribution, variance.

1 Introduction

In this paper we review some results concerning central limit theorems (CLTs). The references are by no means comprehensive; in all cases the reader is advised to see the bibliography in the papers we cite. As a point of departure, we use the Lindeberg CLT.

Consider a triangular array $\{X_{nt},\ t=1,...,n,\ n\in N\}$ of random variables defined on the same probability space $(\Omega,\mathcal{F},P)$, having zero mean $EX_{nt}=0$ and variances $\sigma_{nt}^2=EX_{nt}^2$. Then the sums $S_n=\sum_{t=1}^n X_{nt}$ under independence have variances $s_n^2=ES_n^2=\sum_{t=1}^{n}\sigma_{nt}^2$.

Lindeberg theorem [1]. *Let the array* $\{X_{nt}\}$ *be independent and satisfy*

$$\sum_{t=1}^{n}\sigma_{nt}^2=1. \tag{1}$$

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If

$$\lim_{n\to\infty}\sum_{t=1}^{n}\int\limits_{\{|X_{nt}|>\varepsilon\}} X_{nt}^2 dP = 0, \quad \textit{for all} \quad \varepsilon > 0, \tag{2}$$

then S_n *converges in distribution to a standard normal variable* (*with mean* 0 *and variance* $\sigma^2 = 1$).

The main advantage of the Lindeberg theorem, in comparison with previous results, is that it allows for heterogeneity (variances σ_{nt}^2 may be different). Since the publication of this result in 1922 many different developments took place. 1) The independence condition has been relaxed and replaced by various notions of dependence (mixing and linear processes, among others). 2) For (2), weaker versions have been suggested, including the conditional version. 3) Certain applications required the study of expressions that depend on X_{nt} in a nonlinear fashion, quadratic forms $\sum_{s,t=1}^{n} a_{nst}X_{nt}X_{ns}$ being the most important case. There are also results on functionals of stochastic processes where the analytical form of the functional is not specified. 4) Finally, for many CLTs their continuous-time analogues have been obtained, which are called functional CLTs or invariance principles. These have been left out completely in our review.

From the applied point of view, the normalization condition (1) is one of the main obstacles. One can argue that if it is not satisfied, then one can consider S_n/s_n instead of S_n. Convergence in distribution of S_n/s_n can be achieved in this way but the question about the convergence of S_n and asymptotic behavior of s_n remains. It is particularly important to make sure that s_n does not tend to zero or infinity. In the next section we indicate some researches where the behavior of s_n is controlled and the limit $\sigma^2 = \lim_{n\to\infty}\sum_{t=1}^{n}\sigma_{nt}^2$ is found explicitly.

2 Analyzing variance

For the purpose of analyzing s_n, it is convenient to normalize X_{nt} by their standard deviations: $X_{nt} = \sigma_{nt}e_{nt}$. Then S_n becomes

$$S_n = \sum_{t=1}^{n}\sigma_{nt}e_{nt}, \tag{3}$$

where the sigmas are deterministic and e_{nt} are stochastic. In the Lindeberg-Lévy theorem (see [2]) σ_{nt} are of order $n^{-1/2}$ (which we call classical). The following papers are focussed on relaxing the independence condition and maintain the classical order: [3]–[23]. Davidson [24], [25] does not analyze directly s_n but allows variances going to zero or infinity.

In [26] the normalizing factor is classical but the expression for σ^2 is not trivial (see Corollary 1). Let X_j be a linear process

$$X_j = \sum_r c_{j-r}\xi_r, \ \xi_r \text{ are i.i.d. with mean zero and variance 1}, \ \sum_r c_r^2 < \infty. \tag{4}$$

The cumulant $cum(X_{j_1}, ..., X_{j_k})$ is given by $cum(X_{j_1}, ..., X_{j_k}) = d_k \sum c_{j_1-i}...c_{j_k-i}$, where d_k denotes the k-th cumulant of ξ_i. Letting $c(x)$ denote the Fourier transform of the sequence c_j, one finds the k-th cumulant spectral function as $f^{(k)}(x_1, ..., x_{k-1}) = d_k c(x_1)...c(x_{k-1})c(-x_1 - ... - x_{k-1})$. Consider the CLT for $Y_n = \sum_{j=1}^{n} : X_j^{(n)} :$, where $: X_j^{(n)} :$ denotes the Wick power of X_j (it is a polynomial of degree n). Corollary 1 states that $n^{-1/2}Y_n$ converges in law to the normal distribution with mean 0 and variance

$$\sigma^2 = \sum_{G \in \mathfrak{G}_2} \int \prod_{t=1}^{T} f^{(n_t)}(yM^*)dy_1...dy_N.$$

See the definitions of T, $\mathfrak{G}_2$, n_t and M^* in the paper.

Giraitis L. and Taqqu M.S. [27] consider quadratic forms of bivariate Appell polynomials and give σ^2 in terms of these polynomials. Consider quadratic forms

$$Q_N = \sum_{s,t=1}^{N} b(t-s)P_{m,n}(X_t, X_s),$$

where $P_{m,n}(X_t, X_s)$ is a bivariate Appell polynomial of X_t, X_s. Giraitis L. and Taqqu M.S. [27] prove the next theorem:

Theorem. *Suppose*

$$\sum_{l,k,t \in Z} |b(l)b(k)Cov\left(P_{m,n}(X_t, X_{t+l}), P_{m,n}(X_0, X_k)\right)| < \infty.$$

If $b(0) = 0$, *suppose in addition that* $\sum_t |EX_tX_0|^{m+n} < \infty$. *Then* $N^{-1/2}Q_N$ *converges in distribution to a normal variable with mean zero and variance*

$$\sigma^2 = \sum_{l,k,t \in Z} b(l)b(k)Cov\left(P_{m,n}(X_t, X_{t+l}), P_{m,n}(X_0, X_k)\right).$$

Ho H.C. and Sun T.C. [28] in a nonlinear situation (non-instantaneous filter) give σ^2 in terms of the spectral distribution function of a normal stationary process. For a normal stationary process such that $EX_t = 0$ the autocovariances $r_t = EX_nX_{n+t}$ are represented as $r_t = \int_{-\pi}^{\pi} e^{itx}dG(x)$, where $G(x)$ is the spectral distribution function. The process itself is represented as $X_t = \int_{-\pi}^{\pi} e^{itx}Z_G(dx)$, where Z_G is a random Gaussian measure corresponding to $G(x)$. Consider a non-instantaneous filter (a functional) H such that $EH(X_{t_1}, ..., X_{t_d}) = 0$

and $EH(X_{t_1},...,X_{t_d})^2 < \infty$. Put $Y_N = A_N^{-1}\sum_{t=1}^{N} H(X_{t+t_1},...,X_{t+t_d})$. Ho and Sun find conditions for CLT to hold, the normalizing factor A_N being of classical order. Under some conditions they prove that the limits

$$\sigma_j^2 = \lim_{n\to\infty}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\int \exp\left[i(m-n)(x_1+...+x_j)\right]\left|\alpha_j(x_1,...,x_j)\right|^2 dG(x_1)...dG(x_j)$$

exist for each $j \geq k$ and $\sigma^2 = \sum_{j=k}^{\infty}\sigma_j^2 < \infty$ is the variance of the limit normal distribution. The functions α_j arise from Wiener-Ito expansions of $H(X_{t_1},...,X_{t_d})$.

In [29] s_n^2 is related to the spectral density of the innovations of the linear process at zero. For the process in (4) put $S_n = \sum_{k=1}^{n} X_k$, $b_{n,j} = c_{j-1}+...+c_{j-n}$, $b_n^2 = \sum_{j\in Z} b_{n,j}^2$. Under some conditions

$$\lim_{n\to\infty} Var(S_n)/b_n^2 = 2\pi f(0)$$

and the sequence S_n/b_n converges in distribution to $\sqrt{\eta}z$ where z is standard normal and η is defined in terms of innovations ξ_k and independent of z.

To model the behavior of the sigmas in (3), Mynbaev K.T. [30] introduced the L_p-approximability notion. The idea is to represent converging sequences of deterministic vectors with functions of a continuous argument. It is realized as follows. Let $1 \leq p < \infty$. The interpolation operator $\Delta_{np} : R^n \to L_p(0,1)$ is defined by

$$\left(\Delta_{np}w\right)(x) = n^{\frac{1}{p}}\sum_{t=1}^{n} w_t 1_{\left[\frac{t-1}{n},\frac{t}{n}\right)}(x), \quad w \in R^n. \tag{5}$$

If $w_n \in R^n$ for each n and there exists a function $W \in L_p(0,1)$ such that

$$\left\|\Delta_{np}w_n - W\right\|_{L_p(0,1)} \to 0, \quad n \to \infty,$$

then we say that $\{w_n\}$ is L_p-approximable and also that it is L_p-close to W. Suppose, for simplicity, that the e_{nt} in (3) are i.i.d. with mean zero and variance 1. If the sequence $\sigma_n = (\sigma_{n1},...,\sigma_{nn})$ is L_2-close to a function $F \in L_2(0,1)$, then (3) converges in law to a normal variable with variance

$$V = \int_0^1 F^2(x)dx. \tag{6}$$

This result extends to the case when e_{nt} are linear processes with short memory. It would be interesting to obtain something similar in case of processes with long memory.

P.C.B. Phillips and many of his followers use properties of Brownian motion to establish convergence results for regression estimators. Mynbaev K.T. [31] showed that some problems solved using Brownian motion are easier handled applying L_p-approximability.

To state the result from [32] on quadratic forms $Q_n(k_n) = \sum k_{nst} X_s X_t$ we need more notation.

Let A be a compact linear operator in a Hilbert space with a scalar product $(\cdot,\cdot)$. The operator $H = (A^*A)^{\frac{1}{2}}$ is called the modulus of A, here A^* is the adjoint operator of A. The eigenvalues of H, denoted s_i, $i = 1, 2, ...$, and counted with their multiplicity, are called s-numbers of A. U denotes a partially isometric operator that isometrically maps the range $R(A^*)$ onto the range $R(A)$. Then we have the polar representation $A = UH$. Denote by $r(A)$ the dimension of the range $R(A)$ ($r(A) \le \infty$).

Let $\{\phi_j\}$ be an orthonormal system of eigenvectors of H which is complete in $R(H)$. Then, we have the representation

$$Ax = \sum_{i=1}^{r(A)} s_i(x, \phi_i) U\phi_i$$

or, denoting $\psi_i = U\phi_i$,

$$Ax = \sum_{i=1}^{r(A)} s_i(x, \phi_i)\psi_i,$$

where $\{\phi_i\}$ and $\{\psi_i\}$ are orthonormal systems, $H\phi_i = s_i\phi_i$, $\lim\limits_{i\to\infty} s_i = 0$. In particular, when A is selfadjoint, ϕ_i are eigenvectors of A and $s_i = |\lambda_i|$, where λ_i are eigenvalues of A.

Let $K \in L_2\left((0,1)^2\right)$. For each natural n, we define an $(n \times n)$-matrix

$$(\delta_n K)_{ij} = n \int\limits_{\frac{i-1}{n}}^{\frac{i}{n}} \int\limits_{\frac{j-1}{n}}^{\frac{j}{n}} K(s,t)dsdt, \quad 1 \le i, \quad j \le n.$$

We say that the sequence $\{k_n\}$ is L_2-close to K if

$$\left(\sum_{i,j} (k_n - \delta_n K)^2_{ij}\right)^{\frac{1}{2}} = \|k_n - \delta_n K\|_2 \to 0.$$

Unlike the one-dimensional case, where L_2-approximability of $\{\sigma_n\}$ is enough to have convergence in distribution, in the two-dimensional case one has to impose a stronger condition on the rate of approximation. One version of such a condition is

$$\|k_n - \delta_n K\|_2 = o\left(\frac{1}{\sqrt{n}}\right). \tag{7}$$

Define an integral operator by

$$(\mathcal{K}f)(s) = \int\limits_0^1 K(s,t) f(t) dt, \quad f \in L_2(0,1).$$

Theorem [32]. *Let X_j from (4) satisfy $\sum_j |c_j| < \infty$ and let (7) hold. If $\mathcal{K}$ is nuclear, then*

$$Q_n(k_n) \xrightarrow{d} \left(\sum_i c_i\right)^2 \sum_{i\geq 1} s_i u_i^{(1)} u_i^{(2)}, \tag{8}$$

where $\{u_i^{(1)}\}$, $\{u_i^{(2)}\}$ are systems of independent (within a system) standard normals, s_i are s-numbers of $\mathcal{K}$ and

$$cov(u_i^{(1)}, u_j^{(2)}) = (\psi_i, \phi_j) \quad \textit{for all} \quad i,j.$$

If K is symmetric, then $u_i^{(1)} = u_i^{(2)}$ for all i.

For more information about history of these results, see [33], [34] and [32]. Note the difference between the limit in (8), which is not a normal variable, and the above results, where the limit of quadratic forms is normal. This is due to the centering in the above results. Centering requires knowledge of means and may be problematic in applications.

Wu W. and Shao X. [35] prove asymptotic normality of

$$\sum_{1\leq s<t\leq n} a_{nst} X_s X_t / \sigma_n, \text{ where } \sigma_n^2 = \sum_{t=2}^{n} \sum_{j=1}^{t-1} a_{nst}^2,$$

and X_s is a real stationary process with mean zero and finite covariances.

3 Some applications

Here we list a couple of applications that illustrate the following point. With expressions of type (6) and (8) at hand one can study the limit distribution further. We call this analysis at infinity.

[36] initiated the study of regressions with slowly varying regressors. The limit variance matrix of the OLS estimator for such regressions is degenerate. The analysis at infinity comes in very handy, see [37].

The main technical problem with a spatial model $Y_n = \rho W Y_n + X_n\beta + \varepsilon_n$ is that in its reduced form $Y_n = (I - \rho W_n)^{-1}(X_n\beta + \varepsilon_n)$ there is an inverse matrix $(I - \rho W_n)^{-1}$ and one has to deduce the properties of the inverse from the assumptions on W_n. Many researchers have been unable to do that and instead imposed high level conditions involving the inverse. Mynbaev K.T. and Ullah A. [38] and Mynbaev K.T. [39] gave the first derivation of the asymptotic distribution of the OLS estimator for spatial models (without and with exogenous regressors, resp.) that does not rely on high level conditions.

Most of K.T. Mynbaev's contributions are collected in [40]. In particular, for the purely spatial model in Chapter 5 it is shown that the said model violates the habitual notions in several ways:

1. the OLS asymptotics is not normal,
2. the limit of the numerator vector is not normal,
3. the limit of the denominator matrix is not constant,
4. the normalizer is identically 1 (that is, no scaling is necessary) and
5. there is no consistency.

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Мыңбаев Қ.Т., Даркенбаева Г.С. ОРТАЛЫҚ ШЕКТІК ТЕОРЕМАЛАРДАҒЫ ДИСПЕРСИЯЛАРДЫҢ ТАЛДАУЫ

Орталық шектік теоремалар кездейсоқ шамалардың қосындыларын үлестірім бойынша жинақталуымен байланысты. Кәдімгі қолданылатын тәсіл қосындыларды дисперсиясы 1 болатындай етіп қалыптандырудан тұрады. Осының нәтижесінде, шектік үлестірім бірге тең болатын дисперсияны иемденеді. Көптеген жұмыстарда қалыптандыру факторының шегінің бар болуы негіз ретінде алынып, шектің өзі зерттелмеген. Біз мұнда қалыптандыру коэффициентін зерттеуге бағытталған кейбір нәтижелерді қарастырамыз. Олардың қолданыс аясы көрсетілген.

Кілттік сөздер. Орталық шектік теоремалар, үлестірім бойынша жинақталу, шектік үлестірім, дисперсия.

Мынбаев К.Т., Даркенбаева Г.С. АНАЛИЗ ДИСПЕРСИИ В ЦЕНТРАЛЬНЫХ ПРЕДЕЛЬНЫХ ТЕОРЕМАХ

Центральные предельные теоремы связаны со сходимостью по распределению сумм случайных величин. Обычный подход заключается в нормализации сумм так, чтобы иметь дисперсию, равную единице. В результате этого предельное распределение имеет дисперсию, равную единице. Во многих работах существование предела нормализующего фактора постулируется, а сам предел не изучен. Здесь мы рассмотрим некоторые результаты, которые сосредоточены на изучении коэффициента нормализации. Указаны их области применения.

Ключевые слова. Центральные предельные теоремы, сходимость по распределению, предельное распределение, дисперсия.

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Structure of the Hooke law for anisotropic body at plane deformations

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Abstract. The complex form of the Hooke law for anisotropic body is given which made it possible the simplest defining of eigenvectors and eigenvalues of a matrix of elastic modules of an anisotropic body at a plane deformation. The structure of a matrix of elastic parameters and new invariants which play a key role in communication of an intense strained state is defined. It is shown that always one of the found new linear invariants is equal to zero. The relation expressing the mismatch of principal directions of tensors of deformations and tension is received.

Keywords. Anisotropic body, elastic modules, unitary matrix, tensors of deformations and tension.

1 Introduction

In solving various applied and theoretical problems of continuum mechanics of anisotropic elastic body for additional, more complete information on properties the elastic parameters of the Hooke law for anisotropic elastic body is necessary. Therefore, a large number of scientific research are devoted to clarification of regularities of elastic parameters and general structure of the linear Hooke law for non-isotropic elastic mediums. The detailed review of these researches is provided, for example, in [1].

The present work is devoted to the first stage as to the most prime: to research of the general structure of the Hooke law for an anisotropic body and to clarification of those regularities which are not previously investigated using plane deformation (flat stressed state).

The complex form of the Hooke law allows in natural matrix form to define eigenvectors and eigenvalues of the matrix of elastic modules of the anisotropic body. The structure of the matrix of elastic parameters and new linear invariants which play a key role in communication of an intense strained state is defined. It is shown that always one of the found new linear invariants is equal to zero. Own elastic modules and structure of the matrix of elastic modules

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depending on the new found linear invariants are defined by eigenvectors. The ratio expressing mismatch of principal directions of tensors of deformations and tension is received.

2 The main relations

In the system of Cartesian axes $Ox_1x_2x_3$ we will write down the Hooke law for anisotropic linear elastic body [2], [3]:

$$e_{ij} = a_{ij\alpha\beta} \cdot \sigma_{\alpha\beta}, \tag{1}$$

$$\sigma_{ij} = \sigma_{ji},\ e_{ij} = e_{ij},\ a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij},\ i,j,k,l = 1,2,3,$$

where on Greek indices the toting is made, $\sigma_{ij},\ e_{ij}$ are symmetric stress tensors and linear deformation, respectively, and $a_{ij\alpha\beta}$ are elastic modules of pliability. Replacing indices [2] by the rule: $(11) \to (1)$; $(22) \to (2); (33) \to (3); (12) \to (4); (23) \to (5); (13) \to (6)$, we will write down the Hooke law in the developed form:

$$\begin{aligned}
e_{11} &= a_{11}\sigma_{11} + a_{12}\sigma_{22} + a_{13}\sigma_{33} + 2a_{14}\sigma_{12} + 2a_{15}\sigma_{23} + 2a_{16}\sigma_{13},\\
e_{22} &= a_{12}\sigma_{11} + a_{22}\sigma_{22} + a_{23}\sigma_{33} + 2a_{24}\sigma_{12} + 2a_{25}\sigma_{23} + 2a_{26}\sigma_{13},\\
e_{33} &= a_{13}\sigma_{11} + a_{23}\sigma_{22} + a_{33}\sigma_{33} + 2a_{34}\sigma_{12} + 2a_{35}\sigma_{23} + 2a_{36}\sigma_{13},\\
e_{12} &= a_{14}\sigma_{11} + a_{24}\sigma_{22} + a_{34}\sigma_{33} + 2a_{44}\sigma_{12} + 2a_{45}\sigma_{23} + 2a_{46}\sigma_{13},\\
e_{23} &= a_{15}\sigma_{11} + a_{25}\sigma_{22} + a_{35}\sigma_{33} + 2a_{45}\sigma_{12} + 2a_{55}\sigma_{23} + 2a_{56}\sigma_{13},\\
e_{13} &= a_{16}\sigma_{11} + a_{26}\sigma_{22} + a_{36}\sigma_{33} + 2a_{46}\sigma_{12} + 2a_{56}\sigma_{23} + 2a_{66}\sigma_{13}.
\end{aligned} \tag{2}$$

Let us take the axis as the bearing axis Ox_3. Let us enter complex coordinates $z = x_1 + ix_2,\ i^2 = -1$ and complex components of stress tensors and deformations [2], [4]:

$$T_1 = \sigma_{11} + \sigma_{22},\ T_2 = \frac{1}{\sqrt{2}}\{(\sigma_{11} - \sigma_{22}) + 2i\sigma_{12}\},\ T_3 = \sqrt{2}\,(\sigma_{23} - i\sigma_{13}),\ T_5 = \sqrt{2}\sigma_{33},$$

$$\varepsilon_1 = e_{11} + e_{22},\ \varepsilon_2 = \frac{1}{\sqrt{2}}\{(e_{11} - e_{22}) + 2ie_{12}\},\ \varepsilon_3 = \sqrt{2}\,(e_{23} - ie_{13}),\ \varepsilon_5 = \sqrt{2}e_{33}, \tag{3}$$

$$\vec{T} = \left(T_2, \bar{T}_2, T_1, T_3, \bar{T}_3, T_5\right)^T,\ \vec{\varepsilon} = (\varepsilon_2, \bar{\varepsilon}_2, \varepsilon_1, \varepsilon_3, \bar{\varepsilon}_3,\ \varepsilon_5)^T.$$

Then the Hooke law will be registered as:

$$\vec{\varepsilon} = Q\vec{T},\ \vec{T} = Q^{-1}\vec{\varepsilon}, \tag{4}$$

then

$$Q = Q^* = \begin{pmatrix} b & d & c & e & g & n \\ \bar{d} & b & \bar{c} & \bar{g} & \bar{e} & \bar{n} \\ \bar{c} & c & a & j & \bar{j} & i_0 \\ \bar{e} & g & \bar{j} & p & q & m \\ \bar{g} & e & j & \bar{q} & p & \bar{m} \\ \bar{n} & n & i_0 & \bar{m} & m & k \end{pmatrix}, \quad Q^{-1} = Q^{-1} = \begin{pmatrix} B & D & C & E & G & N \\ \bar{D} & B & \bar{C} & \bar{G} & \bar{E} & \bar{N} \\ \bar{C} & C & A & J & \bar{J} & I \\ \bar{E} & G & \bar{J} & P & Q & M \\ \bar{G} & E & J & \bar{Q} & P & \bar{M} \\ \bar{N} & N & I & \bar{M} & M & K \end{pmatrix}. \quad (5)$$

Matrices $Q = Q^*$, $Q^{-1} = Q^{*-1}$ are Hermit and positive definite, as elastic potential

$$P = \frac{1}{2}\sigma_{\alpha\beta}e_{\alpha\beta} = \frac{1}{4}\left\{T_2\bar{\varepsilon}_2 + \bar{T}_2\varepsilon_2 + T_1\varepsilon_1 + T_3\bar{\varepsilon}_3 + \bar{T}_3\varepsilon_3 + T_5\varepsilon_5\right\}$$

$$= \frac{1}{4}\vec{T}^* \cdot \vec{\varepsilon} = \frac{1}{4}\vec{\varepsilon}^*\vec{T} = \frac{1}{4}\vec{T}^*Q\vec{T} = \frac{1}{4}\vec{\varepsilon}^*Q^{-1}\vec{\varepsilon}$$

has positive definite form. Coefficients of the matrix Q are defined as follows:

$$\begin{aligned}
&a = \tfrac{1}{2}(a_{11} + 2a_{12} + a_{22}),\ b = \tfrac{1}{4}(a_{11} - 2a_{12} + a_{22} + 4a_{44}),\\
&i_0 = \tfrac{1}{\sqrt{2}}(a_{13} + a_{23}),\ p = (a_{55} + a_{66}),\ k = a_{33},\\
&c = \tfrac{\sqrt{2}}{4}\{(a_{11} - a_{22}) + 2i(a_{14} + a_{24})\},\\
&d = \tfrac{1}{4}\{(a_{11} - 2a_{12} + a_{22} - 4a_{44}) + 4i(a_{14} - a_{24})\},\\
&e = \tfrac{1}{2}\{(a_{15} - a_{25} - 2a_{46}) + i(a_{16} - a_{26} + 2a_{45})\},\\
&g = \tfrac{1}{2}\{(2a_{46} - a_{25} + a_{15}) + i(2a_{45} + a_{26} - a_{16})\},\\
&j = \tfrac{1}{\sqrt{2}}\{(a_{15} + a_{25}) + i(a_{16} + a_{26})\},\ q = \{(a_{55} - a_{66}) - 2ia_{56}\},\\
&m = \{a_{35} - ia_{36}\},\ n = \tfrac{1}{2}\{(a_{13} - a_{23}) + 2ia_{34}\}.
\end{aligned} \quad (6)$$

Similarly, elements of the inverse matrix Q^{-1} are defined. Apparently from (6), coefficients a, b, i_0, p, k of the matrix are always real numbers.

Let us consider a monocline singoniya (the plane of the elastic symmetry) [2], [3]. Let us put the axis Ox_3 orthogonally to the plane of the elastic symmetry. Then coordinate axes Ox_1, Ox_2 will be in the plane of an elastic symmetry, and elastic modules $a_{15} = a_{25} = a_{35} = a_{45} = a_{16} = a_{26} = a_{36} = a_{46} = 0$, or $e = g = j = m = 0$.

Definition. *Deformation is called flat if all elastic modules, stress tensor and deformations depend only on two coordinates* $x_1,\ x_2$ *and* $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$, *or* $\varepsilon_3 = \bar{\varepsilon}_3 = \varepsilon_5 = 0$. *Let us* *consider the Hooke law (4)–(5) in the conditions of the plane deformation. It follows from the fourth and fifth equations (4) that*

$$0 = pT_3 + q\bar{T}_3,$$
$$0 = q\bar{T}_3 + pT_3. \tag{7}$$

The Q matrix is positive particular, its main minor is of the second order which is determinant of the set of equations (6) $\Delta = p^2 - |q|^2 > 0$. It follows from (6) that $T_3 = \bar{T}_3 = 0$ or $\sigma_{13} = \sigma_{23} = 0$. From the sixth equation (4) we have

$$T_5 = -\frac{1}{k}\left\{\bar{n}T_2 + n\bar{T}_2 + iT_1\right\}. \tag{8}$$

Substituting (6) in the first three equations (4), we will receive:

$$\vec{\varepsilon} = \begin{pmatrix} \varepsilon_2 \\ \bar{\varepsilon}_2 \\ \varepsilon_1 \end{pmatrix} = Q_*\vec{T} = \begin{pmatrix} b_* & d_* & c_* \\ \bar{d}_* & b_* & \bar{c}_* \\ \bar{c}_* & c_* & a_* \end{pmatrix} \begin{pmatrix} T_2 \\ \bar{T}_2 \\ T_1 \end{pmatrix}, \tag{9}$$

$$b_* = b - |n|^2 \Big/ k;\ a_* = a - i_0^2/k;\ d_* = d - n^2/k;\ c_* = c - i_0 n/k.$$

Further asterisks (*) over the elastic modules are lowered. Note that elastic modules (pliability modules) at the plane deformation can also be written down as:

$$\begin{aligned} & a = \tfrac{1}{2}(\beta_{11} + 2\beta_{12} + \beta_{22}),\ b = \tfrac{1}{4}(\beta_{11} - 2\beta_{12} + \beta_{22} + \beta_{44}), \\ & c = \tfrac{\sqrt{2}}{4}\{(\beta_{11} - \beta_{22}) + i(\beta_{14} + \beta_{24})\}, \\ & d = \tfrac{1}{4}\{(\beta_{11} - 2\beta_{12} + \beta_{22} - \beta_{44}) + 2i(\beta_{14} - \beta_{24})\}, \\ & \beta_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}},\ (i,j = 1,2,4), \end{aligned} \tag{10}$$

and elastic potential as

$$P = \frac{1}{2}\sigma_{\alpha\beta}\varepsilon_{\alpha\beta} = \frac{1}{4}\left(T_1\varepsilon_1 + \bar{T}_2\varepsilon_2 + T_2\bar{\varepsilon}_2\right) = \frac{1}{4}T^*QT = \frac{1}{4}\varepsilon^*Q^{-1}\varepsilon,$$
$$\left(T^* = \left(\bar{T}_2, T_2, T_1\right)\right). \tag{11}$$

The matrix Q remains Hermit and positive definite, therefore

$$a, b > 0,\ b > |d|,\ ab > |c|^2.$$

Let the frame $Ox_1'x_2'$ turn out by turning the frame Ox_1x_2 on a corner φ counterclockwise. Then specified elastic parameters and complex components of vectors of tension and deformations $\vec{T}, \vec{\varepsilon}$ are expressed in a new frame through aged as follows [2], [4]:

$$a' = a,\ b' = b,\ c' = ce^{-2i\varphi},\ d' = de^{-4i\varphi}, \tag{12}$$

$$T_2' = T_2 e^{-2i\varphi},\ T_1' = T_1,\ \varepsilon_2' = \varepsilon_2 e^{-2i\varphi},\ \varepsilon_1' = \varepsilon_1.$$

If we introduce matrix of turn V_n, then a ratio of the second line (12) can be written as:

$$\vec{T}' = V_n\vec{T}, \vec{\varepsilon}' = V_n\vec{\varepsilon},\ V_n = \mathrm{diag}(e^{-2i\varphi}, e^{2i\varphi}, 1),\ V_n \cdot V_n^* = E, \tag{13}$$

where E is simple to matrixes, and turn matrix V_n is scalar unitary matrix.

Considering ratios (13), we will receive:

$$\left|\vec{T}'\right|^2 = \left(\vec{T}'^* \cdot \vec{T}'\right) = \left(\vec{T}^* V_n^* \cdot V_n\vec{T}\right) = \left(\vec{T}^* \cdot \vec{T}\right) = \left|\vec{T}\right|^2,\ \left|\vec{\varepsilon}'\right|^2 = |\vec{\varepsilon}|^2. \tag{14}$$

That is, modules of vectors $\vec{T}, \vec{\varepsilon}$ at turn do not change. Writing down the Hooke law in frames $Ox_1'x_2'$ and Ox_1x_2 and, considering (9), (13), we will receive:

$$Q' = V_n Q V_n^*. \tag{15}$$

Let us enter permutation matrix D:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ D \cdot D = E. \tag{16}$$

(9), (13), (16) follows from ratios:

$$\bar{\vec{\varepsilon}} = D\vec{\varepsilon},\ \bar{\vec{T}} = D\vec{T},\ \bar{Q} = DQD. \tag{17}$$

As the matrix Q of elastic constants is Hermite and positive definite, it can be presented in the form [5]:

$$Q = U^*\lambda U, \tag{18}$$

where $U\,(U^*U = UU^* = E)$ is a unitary matrix, and $\lambda = \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3)$ is a scalar matrix, and all eigenvalues of matrix Q, $\lambda_i > 0$ $(i = 1, 2, 3)$.

3 Structure of the matrix U^*

Let us note that U in matrix decomposition (18) is defined ambiguously. The ratio (18) can be written down, for example, in the form

$$Q = U^*\lambda U = U'^*\lambda U' = U^*P^*e^{-i\theta}\lambda e^{i\theta}PU,\ U'^*U' = E,$$

$$U' = e^{i\theta}PU,\ U'^* = U^*P^*e^{-i\theta},\ P^*P = PP^* = E, \tag{19}$$

$$e^{i\theta} = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}e^{i\theta_3}),$$

where P is the unitary matrix, and $e^{i\theta}$ is the scalar matrix with any corners $\theta_1, \theta_2, \theta_3$. It follows from Eq. (19) that

$$P\lambda = \lambda P,\ \lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3). \tag{20}$$

Representing the unitary matrix $U^* = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$ in the form of columns, the ratio (18) can be written down in the form

$$QU^* = U^*\lambda = (\vec{u}_1, \vec{u}_2, \vec{u}_3)\,\lambda = (\lambda_1\vec{u}_1, \lambda_2\vec{u}_2, \lambda_3\vec{u}_3),$$

i.e. the column $\vec{u}_i$ is the eigenvector of Q:

$$Q\vec{u}_i = \lambda_i\vec{u}_i,\ (i = 1, 2, 3). \tag{21}$$

As U^* is the unitary matrix, its columns are orthonormal, i.e. scalar product $(\vec{u}_i^* \cdot \vec{u}_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol.

If the roots $\lambda_1, \lambda_2, \lambda_3$ of characteristic equation

$$|Q - \lambda E| = 0$$

are prime (all different and one rates frequency), then P in (20) is simple ($P = E$). If there are multiple roots, then P (up to permutation matrix) consists of the blocks, standing on the main diagonal, which sizes coincide with rate frequency of roots.

Let us prove the following lemma.

Lemma. *Columns of matrix U^* have the same structure as vectors $\vec{\varepsilon}, \vec{T}$, i.e.*

$$\bar{U}^* = DU^*. \tag{22}$$

Let us write down the ratio (21) for the complex conjugate values

$$\bar{Q}\bar{\vec{u}}_i = \bar{\lambda}_i\bar{\vec{u}}_i. \tag{23}$$

From (16), (17), (23) we receive

$$Q\left(D\bar{\vec{u}}_i\right) = \lambda_i\left(D\bar{\vec{u}}\right),\ (i = 1, 2, 3). \tag{24}$$

That is $(D\bar{\vec{u}}_i)$ as well as $\vec{u}_i$ is the eigenvector, corresponding to eigenvalue λ_i. In the case of simple roots we have $D\bar{\vec{u}}_i = C_i\vec{u}_i$, where C_i is complex constants.

Let us consider i-column of U^*. If $D\bar{\vec{u}}_i = \vec{u}_i$, then i-column of U^* is taken for the basic column. If $D\bar{\vec{u}}_i \neq \vec{u}_i$, then we consider the vector $\vec{x}_i = \vec{u}_i + D\bar{\vec{u}}_i$, which is the eigenvector of Q, i.e. $Q\vec{x}_i = \lambda_i\vec{x}$.

If $\vec{x}_i = \vec{0}$, then $\vec{x}_i' = i\vec{u}_i$ has the property: $D\bar{\vec{x}}_i' = \vec{x}_i'$, and it can be taken for the basic column. If $\vec{x}_i \neq \vec{0}$, then believing $\vec{x}_i' = \vec{x}_i \cdot |\vec{x}_i|^{-1}$, we receive that $\vec{x}_i'$ can be taken for the basic column. Choosing thus $\vec{x}_i'$ are orthonormal. Thus columns of U^* have the same structure as vectors $\vec{\varepsilon}, \vec{T}$ (see (17)).

Now we will consider the case of multiple roots when $\lambda_1 = \lambda_2$, and λ_3 is simple. Let $\vec{u}_1, \vec{u}_2$ be an orthonormal basis, which linear span $G_2(\vec{u}_1, \vec{u}_2)$ is invariant concerning action of operator Q. Then $\vec{x}_1 = \vec{u}_1 + D\bar{\vec{u}}_1$, $\vec{x}_2 = \vec{u}_2 + D\bar{\vec{u}}_2$ are eigenvectors of Q, corresponding to eigenvalue λ_1. Vectors $\vec{x}_1, \vec{x}_2$ have property: $\vec{x}_1 = D\bar{\vec{x}}_1, \vec{x}_2 = D\bar{\vec{x}}_2$.

If $\vec{x}_1 = \vec{x}_2 = \vec{0}$, then we put: $\vec{u}_1' = i\vec{u}_1,\ \vec{u}_2' = i\vec{u}_2$. Then $\vec{u}_1',\ \vec{u}'$ are basic vectors in G_2 and $D\vec{u}_1' = \bar{\vec{u}}_1',\ D\vec{u}_2' = \bar{\vec{u}}_2'$. Besides they are eigenvectors of Q, corresponding to λ_1. Then they can be taken for the first two columns of λ_1 and therefore λ_1 has the structure (22).

If $\vec{x}_1 \neq 0, \vec{x}_2 \neq 0$, then we put: $\vec{x}_1' = \vec{x}_1 |\vec{x}_1|^{-1},\ \vec{x}_2' = \vec{x}_2 |\vec{x}_2|^{-1}$. Then $\vec{x}_1', \vec{x}_2'$ are unit vectors. Let us consider vectors $\vec{z}_1 = (\vec{x}_1' - \vec{x}_2')/\sqrt{2},\ \vec{z}_2 = (\vec{x}_1' + \vec{x}_2')/\sqrt{2}$. They have the properties: $D\vec{z}_1 = \bar{\vec{z}}_1,\ D\vec{z}_2 = \bar{\vec{z}}_2,\ (\vec{z}_1^* \cdot \vec{z}_2) = 0$. Here it is considered that $(\vec{x}_{1*}' \cdot \vec{x}_2') = (\vec{x}_{2*}' \cdot \vec{x}_1')$. Equality $(\vec{z}_1^* \cdot \vec{z}_2) = 0$ means that $\vec{z}_1, \vec{z}_2$ are orthogonal $(\vec{z}_1, \vec{z}_2 \neq \vec{0})$ and linearly independent. Passing to unit vectors $\vec{z}_1' = \vec{z}_1 |\vec{z}_1|^{-1},\ \vec{z}_2' = \vec{z}_2 |\vec{z}_2|^{-1}$ we will receive that U^* has the structure (22).

Now we will consider the case $\vec{z}_1 = 0, \vec{z}_2 \neq 0$, i.e. $\vec{x}_1' = \vec{x}_2'$, or $(\vec{u}_1 - C_0\vec{u}_2) + D(\bar{\vec{u}}_1 - C_0\bar{\vec{u}}_2) = \vec{0},\ C_0 = |\vec{x}_1|\,|\vec{x}_2|^{-1}$. Then the vector $\vec{y} = i(\vec{u}_1 - C_0\vec{u}_2)$ is not zero (since $\vec{u}_1, \vec{u}_2$ are linearly independent), and $\vec{y} = D\bar{\vec{y}}$. Let us consider vectors $\vec{z}_1' = (\vec{y} - \vec{x}_2')/\sqrt{2}$, $\vec{z}_2' = (\vec{y} + \vec{x}_2')/\sqrt{2}$ for which $(\vec{z}_{1*}' \cdot \vec{z}_2') = 0$. If $|\vec{z}_1'| \cdot |\vec{z}_2'| \neq 0$, then $\vec{z}_1', \vec{z}_2'$ are linearly independent and again we obtain that U^* has the structure (16). If $\vec{z}_1' = \vec{0}$, then $\vec{y} = \vec{x}_2'$ or $i\,|\vec{x}_2|\,\vec{u}_1 - (1 + i\,|\vec{x}_1|)\vec{u}_2 = D\bar{\vec{u}}_2$. But then $(\bar{\vec{u}}_2^* D \cdot D\bar{\vec{u}}_2) = 1 = 1 + |\vec{x}_1|^2 + |\vec{x}_2|^2$, and, therefore, $\vec{x}_1 = \vec{x}_2 = \vec{0}$. That contradicts the assumption. Similarly we show that the case $\vec{z}_2 = \vec{0}$ leads to the contradiction $\vec{x}_1 = \vec{x}_2 = \vec{0}$. Thus, at $\vec{x}_1 \neq 0, \vec{x}_2 \neq 0$ the matrix U^* has the structure (22).

Now we will consider the case when $\vec{x}_1 = \vec{0}, \vec{x}_2 \neq \vec{0}$. Then, choosing $\vec{x}_1' = i\vec{u}_1$, we show similarly that U^* has the structure (22). For the case $\vec{x}_2 = \vec{0}, \vec{x}_1 \neq \vec{0}$, we choose $\vec{x}_2' = i\vec{u}_2$.

The case $\lambda_1 = \lambda_2 = \lambda_3$ corresponds to the case of proportionality of stress and deformations tensors (as it is possible to take any unitary matrix as U^* and, in particular, any matrix with the structure (22)).

The lemma is proved.

Let us write expanded form of U, U^*, which we will use further:

$$U = \begin{pmatrix} \bar{u}_{11} & u_{11} & m_1 \\ \bar{u}_{22} & u_{22} & m_2 \\ \bar{u}_{31} & u_{31} & m_3 \end{pmatrix}, \;\; U^* = \begin{pmatrix} u_{11} & u_{22} & u_{31} \\ \bar{u}_{11} & \bar{u}_{22} & \bar{u}_{31} \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{25}$$

$$m_1 = u^0_{13}, \; m_2 = u^0_{23}, \; m_3 = u^0_{33}.$$

Apparently from (25) matrix U has 3 complex components and 3 real non-negative components, so it is described by 9 real components. Besides, the first and second columns of the matrix U are complex conjugated, and the third one is real. From (22) it follows:

$$\bar{U} = UD, \; U^*\bar{U} = \bar{U}^*U = D. \tag{26}$$

As columns of U^* have the same structure as vectors $\vec{T}, \vec{\varepsilon}$, at coordinates axes rotation they are transformed according to (13):

$$U'^* = V_n U^*, \; U' = UV_n^*, \tag{27}$$

and then m_1, m_2, m_3 are invariants which are nonnegative (note, U^* is defined about accuracy to the matrix $e^{i\theta} = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2} e^{i\theta_3})$, see (19)).

Let's make one more important remark. We consider, for example, the matrix $\lambda' = diag(\lambda_1, \lambda_3, \lambda_2)^T = D' diag(\lambda_1, \lambda_2, \lambda_3)^T D'^*$, where the permutation D' has the form:

$$D' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; D'^* \cdot D' = D' \cdot D'^* = E.$$

Then

$$Q = U^*\lambda U = U^*(D'^*\lambda' D')U = U'^*\lambda' U', \; U'^* = U^* D'^*, \; U' = D'U.$$

That is, the second and third columns of U'^* are the perturbation of the second and third columns of U^*, and the scalar matrix λ' stands on the main diagonal in decomposition of Q. Therefore, the eigenvalues of matrix λ can be in any order, for example, as they decrease. At the same time columns of U'^* are the perturbation of columns of U^*. Therefore, the structure of U'^* will be the same (up to columns perturbation of this matrix).

4 The invariants

As columns $\vec{u}_1, \vec{u}_2, \vec{u}_3$ of U^* are the orthonormal basis, it is possible to decompose vectors $\vec{T}, \vec{\varepsilon}$ on this basis:

$$\begin{aligned} \vec{T} &= \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \alpha_3 \vec{u}_3, \\ \vec{\varepsilon} &= \beta_1 \vec{u}_1 + \beta_2 \vec{u}_2 + \beta_3 \vec{u}_3, \end{aligned} \tag{28}$$

where $\vec{\alpha}, \vec{\beta}$ are decomposition of coordinates $\vec{T}, \vec{\varepsilon}$ on the basis $\vec{u}_1, \vec{u}_2, \vec{u}_3$: $\alpha_i = (\vec{u}_i^* \cdot \vec{T})$, $\beta_i = (\vec{u}_i^* \cdot \vec{\varepsilon})$.

Due to the Hooke law (9), ratio (21) and orthonormal $\vec{u}_i$, we have:

$$\varepsilon = \sum_{i=1}^{3} \beta_i \vec{u}_i = QT = Q \sum_{i=1}^{3} \alpha_i \vec{u}_i = \sum_{i=1}^{3} \alpha_i Q \vec{u}_i = \sum_{i=1}^{3} \alpha_i \lambda_i \vec{u}_i.$$

As $\vec{u}_i$ are linearly independent, then

$$\beta_i = \lambda_i \alpha_i, \ \vec{\beta} = \lambda \vec{\alpha}, \ \vec{\beta} = (\beta_1, \beta_2, \beta_3)^T, \ \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T. \tag{29}$$

Let us show that $\vec{\alpha} = \bar{\vec{\alpha}}, \vec{\beta} = \bar{\vec{\beta}}$, i.e. vectors $\vec{\alpha}, \vec{\beta}$ are real. Really, $D\vec{u}_i = \bar{\vec{u}}_i$, then $\bar{\vec{\varepsilon}} = \bar{U}^* \bar{\vec{\beta}} = DU^* \bar{\vec{\beta}} = D\vec{\varepsilon} = DU^* \vec{\beta}$. From here $\vec{\beta} = \bar{\vec{\beta}}$. Similarly we show that $\vec{\alpha} = \bar{\vec{\alpha}}$. At rotation of coordinate axes $\vec{u}_i, \vec{T}$ are transformed under the law $\vec{u}_i' = V_n \vec{u}_i$, $\vec{T}' = V_n \vec{T}$. Then $\vec{u}_i' = V_n \vec{u}_i, \vec{T}' = V_n \vec{T}$. Then $\alpha_i' = (\vec{u}_i'^* \cdot \vec{T}') = (\vec{u}_i'^* V_n^* \cdot V_n \vec{T}) = (\vec{u}_i^* \cdot \vec{T}) = \alpha_i$, and $\vec{\alpha}$ is invariant. It is similarly proved that $\vec{\beta}$ is the invariant. Then it is easy to show that elastic potential P is also the invariant and it is a positive definite quadratic form:

$$P = \frac{1}{4}\left(\vec{\varepsilon}^* \vec{T}\right) = \frac{1}{4}\left(\vec{T}^* \vec{\varepsilon}\right) = \frac{1}{4} \sum_{i=1}^{3} \alpha_i \beta_i = \frac{1}{4} \sum_{i=1}^{3} \lambda_i \alpha_i^2 = \frac{1}{4} \sum_{i=1}^{3} \frac{\beta_i^2}{\lambda_i}. \tag{30}$$

The ratio (28) can be written in a more compact form:

$$\vec{\varepsilon} = U^* \vec{\beta}, \ \vec{T} = U^* \vec{\alpha}, \tag{31}$$

and invariants $\vec{m} = (m_1, m_2, m_3)^T$ are presented in the form:

$$\vec{m} = U\vec{F} = \bar{U}\vec{F}, \quad \vec{F} = (0, 0, 1)^T. \tag{32}$$

From (31), (29) it follows:

$$\begin{aligned} \varepsilon_1 &= m_1 \beta_1 + m_2 \beta_2 + m_3 \beta_3 = m_1 \lambda_1 \alpha_1 + m_2 \lambda_2 \alpha_2 + m_3 \lambda_3 \alpha_3, \\ T_1 &= m_1 \alpha_1 + m_2 \alpha_2 + m_3 \alpha_3 = m_1 \lambda_1^{-1} \beta_1 + m_2 \lambda_2^{-1} \beta_2 + m_3 \lambda_3^{-1} \beta_3. \end{aligned} \tag{33}$$

If we introduce the vector $\vec{L} = \{c, \bar{c}, a\}^T$, then it is easy to receive:

$$\vec{L} = U^* \lambda \vec{m} = U^* \lambda U \vec{F} = Q\vec{F}. \tag{34}$$

From (33), (34) it follows that the invariant a is equal to:

$$a = \lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2. \tag{35}$$

Let us note that the matrix column $\vec{F}' = V_n\vec{F} = V_n^*\vec{F} = \vec{F}$ at turn does not change (the vector $\vec{F}$ represents complex components of a spherical tensor), and the sum $m_1^2+m_2^2+m_3^2 = 1$.

5 Eigenvalues and eigenvectors

The characteristic equation $|Q - \lambda E| = 0$ for calculation of eigenvalues has the following appearance:

$$G(\lambda) = \lambda^3 - J_1\lambda^2 + J_2\lambda - J_3E = 0, \tag{36}$$

where $J_i\,(i = 1, 2, 3)$ is the sum of all main minors of order i of Q and λ is scalar matrix of eigenvalues. For a plane deformation

$$\begin{aligned} &J_1 = trQ = \lambda_1 + \lambda_2 + \lambda_3 = 2b + a, \\ &J_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = b\,(b + 2a) - 2\,|c|^2 - |d|^2, \\ &J_3 = \lambda_1\lambda_2\lambda_3 = \left(d\bar{c}^2 + \bar{d}c^2\right) + a\left(b^2 - |d|^2\right) - 2b\,|c|^2. \end{aligned} \tag{37}$$

Let us multiply (36) at the left on U^*, and at the right on U. Using (18), we obtain Hamilton-Cayley theorem which claims, that the matrix Q satisfies to the characteristic equation:

$$Q^3 - J_1Q^2 + J_2Q - J_3E = 0. \tag{38}$$

Let us consider matrix U^*. We present it in the form:

$$\begin{aligned} U^* &= \begin{pmatrix} |u_{11}|\,e^{i\varphi_1} & |u_{22}|\,e^{i\varphi_2} & |u_{31}|\,e^{i\varphi_3} \\ |u_{11}|\,e^{-i\varphi_1} & |u_{22}|\,e^{-i\varphi_2} & |u_{31}|\,e^{-i\varphi_3} \\ m_1 & m_2 & m_3 \end{pmatrix} = V_3 \cdot U_3^* \\ &= \begin{pmatrix} e^{i\varphi_3} & 0 & 0 \\ 0 & e^{-i\varphi_3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} |u_{11}|\,e^{i\psi_1} & |u_{22}|\,e^{i\psi_2} & |u_{31}| \\ |u_{11}|\,e^{-i\psi_1} & |u_{22}|\,e^{-i\psi_2} & |u_{31}| \\ m_1 & m_2 & m_3 \end{pmatrix}, \end{aligned} \tag{39}$$

where corners $\psi_1 = (\varphi_1 - \varphi_3), \psi_2 = (\varphi_2 - \varphi_3)$ are the invariants in virtue of (12), and $|u_{11}| = \sqrt{\dfrac{1 - m_1^2}{2}}, |u_{22}| = \sqrt{\dfrac{1 - m_2^2}{2}}, |u_{31}| = \sqrt{\dfrac{1 - m_3^2}{2}}$, in virtue of columns normalization. All $m_i(i = 1, 2, 3)$ can not be zero at the same time. Therefore, for example, $m_3 \neq 0$ (in virtue of the remark in Section 2). Separately we consider cases when $m_3 = 1$ and $m_3 \neq 1$.

The first case ($m_3 = 1$). Then $m_1 = m_2 = 0$, $u_{11} = \dfrac{e^{i\varphi}}{\sqrt{2}}, u_{22} = \dfrac{ie^{i\varphi}}{\sqrt{2}}, |u_{13}| = 0$, and we receive from (18)

$$U^* = \begin{pmatrix} e^{i\varphi}/\sqrt{2} & ie^{i\varphi}/\sqrt{2} & 0 \\ e^{-i\varphi}/\sqrt{2} & -ie^{-i\varphi}/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; b = \frac{(\lambda_1+\lambda_2)}{2}, \; d = \frac{ie^{2i\varphi}(\lambda_1-\lambda_2)}{2}, \tag{40}$$

$$c = 0, \; a = \lambda_3, \; \lambda_{1,2} = b \pm |d|.$$

Let us note that by $d = 0$ we receive an isotropic body, for which $\lambda_1 = \lambda_2 = b, \lambda_3 = a$. The following converse statement is true: if $\lambda_1 = \lambda_2$, then we have the isotropic body ($d = c = 0$).

If $\lambda_1 = \lambda_3 \neq \lambda_2$, then from (40) it follows:

$$a = \lambda_1 = \lambda_3 = b + |d|, \; \lambda_2 = b - |d|. \tag{41}$$

If $\lambda_2 = \lambda_3 \neq \lambda_1$, then from (40) it follows:

$$a = \lambda_2 = \lambda_3 = b - |d|, \; \lambda_1 = b + |d|. \tag{42}$$

For $\lambda_1 = \lambda_2 = \lambda_3$, we have a hyper elastic body ($a = b, d = c = 0$) with zero Poisson's coefficient [6]. In this case tensions are proportional to deformations. The matrix U^* can be chosen in any type and, in particular, in the type (40).

The second case ($m_3 \neq 1$). Using conditions of orthogonality of columns and rows of U_3^*, it is easy to receive:

$$e^{i\psi_1} = -\frac{(m_1m_3 + im_2)}{\sqrt{(1-m_1^2)(1-m_3^2)}}, \quad e^{i\psi_2} = -\frac{(m_2m_3 - im_1)}{\sqrt{(1-m_2^2)(1-m_3^2)}}, \tag{43}$$

$$U_3^* = \begin{pmatrix} -\dfrac{(m_1m_3+im_2)}{\sqrt{2(1-m_3^2)}} & -\dfrac{(m_2m_3-im_1)}{\sqrt{2(1-m_3^2)}} & \sqrt{\dfrac{1-m_3^2}{2}} \\ -\dfrac{(m_1m_3-im_2)}{\sqrt{2(1-m_3^2)}} & -\dfrac{(m_2m_3+im_1)}{\sqrt{2(1-m_3^2)}} & \sqrt{\dfrac{1-m_3^2}{2}} \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{44}$$

From the ratios (18), (39)–(44) it follows:

$$\begin{gathered} b = \tfrac{1}{2}\left\{\lambda_1(1-m_1^2) + \lambda_2(1-m_2^2) + \lambda_3(1-m_3^2)\right\}, \quad a = \lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2, \\ d = \tfrac{e^{2i\varphi}}{2(1-m_3^2)}\left\{\lambda_3(1-m_3^2)^2 + \lambda_1(m_1^2m_3^2 - m_2^2) \right. \\ \left. + \lambda_2(m_2^2m_3^2 - m_1^2) + 2im_1m_2m_3(\lambda_1-\lambda_2)\right\}, \\ c = \tfrac{e^{i\varphi}}{\sqrt{2(1-m_3^2)}}\left\{m_3(\lambda_3 - a) - im_1m_2(\lambda_1-\lambda_2)\right\}, \; \varphi = \varphi_3. \end{gathered} \tag{45}$$

The ratios (45) show that elastic modules of Q are expressed over elastic modules $\lambda_1, \lambda_2, \lambda_3$, invariants m_1, m_2, m_3 of U^* and the corner $\varphi = \varphi_3$.

It is well known [2], [3] that at a plane deformation an elastic isotropic body behaves as an orthotropic body. I.e. the frame can be turned on a particular corner so that in a new frame elastic parameters d, c will be real, i.e. $c = |c|$, $d = d_0 = \pm |d|$. But then, from (45) it follows that $m_1 m_2(\lambda_1 - \lambda_2) = 0$.

Let $\lambda_1 \neq \lambda_2$. Then $m_1 m_2 = 0$, and taking into account the remark in Section 2, we can assume that $m_2 = 0$. The ratios (44), (45) for this case have the form:

$$U^* = \begin{pmatrix} -m_3 e^{i\varphi}/\sqrt{2} & e^{i\varphi} i/\sqrt{2} & m_1 e^{i\varphi}/\sqrt{2} \\ -m_3 e^{-i\varphi}/\sqrt{2} & -ie^{-i\varphi}/\sqrt{2} & m_1 e^{-i\varphi}/\sqrt{2} \\ m_1 & 0 & m_3 \end{pmatrix},$$
$$b = \tfrac{1}{2}\left\{\lambda_1 m_3^2 + \lambda_2 + \lambda_3 m_1^2\right\},\ a = \lambda_1 m_1^2 + \lambda_3 m_3^2, \tag{46}$$
$$d = e^{2i\varphi_3}\left\{b - \lambda_2\right\},\quad c = \tfrac{e^{i\varphi_3} m_1 m_3}{\sqrt{2}}(\lambda_3 - \lambda_1),\quad \varphi = \varphi_3.$$

From the fourth ratio of (46) it follows

$$\lambda_2 = b - d_0, (d_0 = \pm |d|). \tag{47}$$

Then from (37), (47) we have:

$$\lambda_1 + \lambda_3 = 2\eta = a + b + d_0,$$
$$\lambda_1 \lambda_3 = a(b + d_0) - 2|c|^2$$

and then:

$$\lambda_{3,1} = \eta \pm D,\quad D = \frac{1}{2}\sqrt{(b + d_0 - a)^2 + 8|c|^2}. \tag{48}$$

Here $m_1 m_3(\lambda_3 - \lambda_1) = \sqrt{2}|c|$, $m_1 m_3 \geq 0$. For the case of simple roots ($\lambda_1 \neq \lambda_2 \neq \lambda_3$), in virtue of (47) from (46) we define:

$$m_1 = \sqrt{\frac{D + (\eta - a)}{2D}},\ m_3 = \sqrt{\frac{D - (\eta - a)}{2D}}. \tag{49}$$

If $\lambda_1 = \lambda_3 \neq \lambda_2$, then from (46) it follows that $m_3 = 1$. This case has been considered earlier.

Now we consider the case when $\lambda_1 = \lambda_2 \neq \lambda_3$. Then ratios (45) have the form:

$$b = \tfrac{1}{2}\left\{(\lambda_1 + \lambda_3) - (\lambda_3 - \lambda_1)m_3^2\right\},\ a = \lambda_1 + (\lambda_3 - \lambda_1)m_3^2,$$
$$d = \tfrac{e^{2i\varphi}}{2}(\lambda_3 - \lambda_1)(1 - m_3^2),\ c = \tfrac{e^{i\varphi} m_3 \sqrt{(1 - m_3^2)}}{\sqrt{2}}(\lambda_3 - \lambda_1). \tag{50}$$

From here we receive

$$|c| = m_3\sqrt{1-m_3^2}(\lambda_3-\lambda_1)\Big/\sqrt{2},\ d_0 = 0,5\cdot(\lambda_3-\lambda_1)(1-m_3^2). \tag{51}$$

But then $\lambda_3 > \lambda_1$ $(m_3 > 0)$, and therefore $d_0 = |d|$. From ratios (50), (51) we find

$$\begin{gathered} |c|^2 = |d|\,(|d|+a-b),\ m_3^2 = \frac{(|d|+a-b)}{(3\,|d|+a-b)} = \frac{|c|^2}{(2\,|d|^2+|c|^2)}, \\ m_1^2+m_2^2 = \frac{2\,|d|}{(3\,|d|+a-b)},\ \lambda_1=\lambda_2=b-|d|\,,\ \lambda_3 = a+2\,|d|. \end{gathered} \tag{52}$$

The first ratio of (52) gives connection between elastic modules due to the fact that λ_1 is a double root, and invariants m_1, m_2 are any numbers but they are connected by the fourth ratio (52). Therefore we can put $m_2 = 0$ (without loss of generality). Thus $m_2 = 0$ for all cases. Further for invariants m_1, m_3 more convenient designations are used:

$$m_1 = \sin\omega,\ m_3 = cos\omega. \tag{53}$$

For $|d| = 0$ from the first ratio (52) it follows that $c = 0$, and then we have the isotropic body. If $c = 0$, then from the same ratio it follows that $d = 0$ (the case of an isotropic body), or $|d| = b-a$. If $|d| = b-a$, then from the second ratio (52) it follows that $m_3^2 = 0$, but this contradicts the assumption $0 < m_3 < 1$.

Let us note that at a plane deformation the transversal isotropic body behaves as an isotropic body [2], [3]. Therefore, all results given above for the isotropic body are true also for the transversal isotropic body.

Thus, we proved the following theorem.

Theorem. *At a plane deformation one of the invariants of U^* is always equal to zero. For $m_3 = 1$ eigenvectors and own elastic modules are defined by ratios (40)–(41). For $m_3 \neq 1$ in case of simple roots ($\lambda_1 \neq \lambda_2 \neq \lambda_3$) vectors, own elastic modules and invariants m_1, m_3 are defined by ratios (46)–(52); for $\lambda_1 = \lambda_2 \neq \lambda_3$ they are defined by ratios (52). Besides, for $m_3 \neq 1$, $\lambda_1 = \lambda_2 \neq \lambda_3$, if one of elastic modules d or c is equal to zero, then an elastic body is isotropic.*

6 Mismatch of tensors of deformations and tensions

At first we consider the case of simple roots. From the ratios (31), (46), (53) it follows

$$\begin{gathered} \beta_1 = -\tfrac{\cos\omega}{\sqrt{2}}\left\{\varepsilon_2 e^{-i\varphi}+\bar{\varepsilon}_2 e^{i\varphi}\right\}+\sin\omega\varepsilon_1,\quad \alpha_1 = -\tfrac{\cos\omega}{\sqrt{2}}\left\{T_2 e^{-i\varphi}+\bar{T}_2 e^{i\varphi}\right\}+\sin\omega T_1, \\ \beta_2 = -\tfrac{i}{\sqrt{2}}\left\{\varepsilon_2 e^{-i\varphi}-\bar{\varepsilon}_2 e^{i\varphi}\right\},\quad \alpha_2 = -\tfrac{i}{\sqrt{2}}\left\{T_2 e^{-i\varphi}-\bar{T}_2 e^{i\varphi}\right\}, \\ \beta_3 = \tfrac{\sin\omega}{\sqrt{2}}\left\{\varepsilon_2 e^{-i\varphi}+\bar{\varepsilon}_2 e^{i\varphi}\right\}+\varepsilon_1\cos\omega,\quad \alpha_3 = \tfrac{\sin\omega}{\sqrt{2}}\left\{T_2 e^{-i\varphi}+\bar{T}_2 e^{i\varphi}\right\}+T_1\cos\omega. \end{gathered} \tag{54}$$

As $\varepsilon_2 = |\varepsilon_2|\, e^{2i\mu_1},\ T_2 = |T_2|\, e^{2i\mu_2}$, where μ_1, μ_2 are the angles between the axis Ox_1 and the first principal directions of tensors of deformations and tensions [7], respectively, the ratio (54) using (29) can be presented in the form:

$$\begin{aligned} &\sqrt{2}\cos\omega\,\{\,|\varepsilon_2|\cos\psi_1 - \lambda_1\,|T_2|\cos\psi_2\}\, - \sin\omega\,\{\,\varepsilon_1 - \lambda_1 T_1\}\, = 0,\\ &\sqrt{2}\sin\omega\,\{\,|\varepsilon_2|\cos\psi_1 - \lambda_3\,|T_2|\cos\psi_2\}\, + \cos\omega\,\{\,\varepsilon_1 - \lambda_3 T_1\}\, = 0,\\ &\sqrt{2}\,\{\,|\varepsilon_2|\sin\psi_1 - \lambda_2\,|T_2|\sin\psi_2\}\, = 0,\\ &\psi_1 = 2\mu_1 - \varphi, \psi_2 = 2\mu_2 - \varphi. \end{aligned} \tag{55}$$

Here corners ψ_1, ψ_2 are invariants in virtue of (12). The third ratio (55) gives

$$\sin(2\mu_1 - \varphi) = \lambda_2 \frac{|T_2|}{|\varepsilon_2|} \sin(2\mu_2 - \varphi). \tag{56}$$

This means that mismatch of principal directions of tensors of deformations and tension is defined both by the eigenvalue λ_2 and the modules of deviators of stress and deformations tensors.

From (56) it follows that for the isotropic body (and the transversal isotropic body, $d = c = 0$) the principal directions of tensors of deformations and tensions coincide, i.e $\mu_1 = \mu_2$.

For the case $c = 0, d \neq 0$ we have $\omega = 0,\ (m_3 = 1)$, and then

$$\begin{gathered} |\varepsilon_2|\cos\psi_1 = \lambda_1\,|T_2|\cos\psi_2, \quad |\varepsilon_2|\sin\psi_1 = \lambda_2\,|T_2|\sin\psi_2, \quad \varepsilon_1 = \lambda_3 T_1,\\ tg\psi_1 = \frac{\lambda_2}{\lambda_1} tg\psi_2. \end{gathered} \tag{57}$$

In the case of the existing plain-stress state ($\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$) and an elastic symmetry planes all reasoning are carried out similarly to the case of plane deformation. In the resulting expressions it is necessary to replace the coefficients of Q to the coefficients of Q^{-1}, λ_i, on λ_i^{-1} and ε_i on T_i and otherwise.

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Мартынов Н.И., Рамазанова М.А. ЖАЗЫҚ ДЕФОРМАЦИЯ КЕЗІНДЕГІ АНИЗОТРОПТЫ ДЕНЕ ҮШІН ГУК ЗАҢЫНЫҢ ҚҰРЫЛЫМЫ

Жазық деформация кезінде анизотропты дененің серпімді модульдері матрицасының меншікті векторларын және меншікті мәндерін барынша жеңіл анықтауға мүмкіндік беретін анизотропты дене үшін Гук заңының кешенді түрі келтірілген. Кернеулі-деформацияланған жағдаймен байланысты негізгі рөл атқаратын серпімді параметрлер матрицасының құрылымы және жаңа сызықты инварианттар анықталды. Табылған жаңа сызықты инварианттардың ішінде біреуі әрдайым нөл болатыны көрсетілді. Кернеулер мен деформациялар тензорларының басты бағыттарының үйлесімсіздігін білдіретін арақатынас алынды.

Кілттік сөздер. Анизотропты дене, серпімділік модульдері, унитарлық матрица, деформациялар мен кернеулердің тензорлары.

Мартынов Н.И., Рамазанова М.А. СТРУКТУРА ЗАКОНА ГУКА АНИЗОТРОПНОГО ТЕЛА ПРИ ПЛОСКОЙ ДЕФОРМАЦИИ

Приведена комплексная форма закона Гука для анизотропного тела, позволившая наиболее просто определить собственные вектора и собственные значения матрицы упругих модулей анизотропного тела при плоской деформации. Определена структура матрицы упругих параметров и новые линейные инварианты, которые играют ключевую роль в связи напряженно-деформированного состояния. Показано, что всегда один из найденных новых линейных инвариантов равен нулю. Получено соотношение, выражающее рассогласованность главных направлений тензоров деформаций и напряжений.

Ключевые слова. Анизотропное тело, модули упругости, унитарная матрица, тензоры деформаций и напряжений.

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On the integral perturbation of the boundary condition of one problem that does not have a basic property

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Abstract. It is well known that the system of eigenfunctions of an operator given by a formally self-adjoint differential expression, with arbitrary self-adjoint boundary conditions providing a discrete spectrum, forms an orthonormal basis. In many papers, the question on saving basis properties under some (weak in a certain sense) perturbation of the initial operator has been investigated. For the case of an arbitrary ordinary differential operator, when unperturbed boundary conditions are strongly regular, the question of the stability of the basis property of root vectors under their integral perturbation is positively solved in papers of A.A. Shkalikov. In a series of our previous papers, we have considered the question of constructing a characteristic determinant and of the stability of the basis property of root vectors under the integral perturbation of one of the boundary conditions. Almost all possible types of the boundary conditions that are regular but not strongly regular have been considered. Moreover, it was required that the system of root functions of the unperturbed problem possesses the basis property. In this paper we consider a spectral problem for a multiple differentiation operator with an integral perturbation of boundary conditions of one type which are regular, but not strongly regular. The unperturbed problem has an asymptotically simple spectrum, and its system of eigenfunctions does not form a basis in L_2. We construct a characteristic determinant of the spectral problem with an integral perturbation of boundary conditions. It is shown that the set of kernels of the integral perturbation, under which the absence of the basis properties of the system of root functions persists, is dense in L_2.

Keywords. ordinary differential operator, boundary value problem, integral perturbation of boundary condition, eigenvalues, eigenfunctions, basis property, characteristic determinant

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1 Introduction and statement of the problem

It is well known that the system of eigenfunctions of an operator given by a formally self-adjoint differential expression, with arbitrary self-adjoint boundary conditions providing a discrete spectrum, forms an orthonormal basis. In many papers, the question on saving basis properties under some (weak in a certain sense) perturbation of the initial operator has been investigated. For the case of an arbitrary ordinary differential operator, when unperturbed boundary conditions are strongly regular, the question of the stability of the basis property of root vectors under their integral perturbation is positively solved in papers of A.A. Shkalikov.

In a series of our previous papers, we have considered the question of constructing a characteristic determinant and of the stability of the basis property of root vectors under the integral perturbation of one of the boundary conditions. Almost all possible types of the boundary conditions that are regular but not strongly regular have been considered. Moreover, it was required that the system of root functions of the unperturbed problem possesses the basis property.

In this paper we consider a spectral problem for a multiple differentiation operator with an integral perturbation of boundary conditions of one type which are regular, but not strongly regular. The unperturbed problem has an asymptotically simple spectrum, and its system of eigenfunctions does not form a basis in L_2. We construct the characteristic determinant of the spectral problem with an integral perturbation of the boundary conditions. It is shown that the set of kernels of the integral perturbation, under which the absence of the basis properties of the system of root functions persists, is dense in L_2.

The question of persisting the basis properties under some (weak in definite sense) perturbation of the original operator was investigated in many works. For example, the analogous question for the case of a self-adjoint original operator was investigated in [1]–[3], and for a non-selfadjoint operator in [4]–[6].

In the present paper we consider the spectral problem:

$$l\left(u\right)\equiv-u''\left(x\right)=\lambda u\left(x\right),\quad 0<x<1, \tag{1}$$

$$U_1\left(u\right)\equiv u'(0)-u'(1)-\alpha u(1)=0, \tag{2}$$

$$U_2\left(u\right)\equiv u\left(0\right)=0, \tag{3}$$

which is close to investigations in [1], [4], [7]. Here $\alpha<0$ is an arbitrary negative number. The case of a positive parameter α was considered in our paper [8].

Let $\mathcal{L}_1$ be an operator in $L_2\left(0,1\right)$ given by expression (1) and by "perturbed" boundary conditions:

$$U_1\left(u\right)=\int_0^1\overline{p\left(x\right)}u\left(x\right)dx,\quad U_2\left(u\right)=0,\quad \text{where } p\left(x\right)\in L_2\left(0,1\right). \tag{4}$$

By $\mathcal{L}_0$ we denote the unperturbed operator (case $p\left(x\right)=0$).

In our previous papers [6], [7], [9], [15], [16], [17], [18] we considered different variants of the integral perturbation of boundary conditions. In these papers, under the assumption that the unperturbed operator $\mathcal{L}_0$ had the system of eigen- and associated functions (EAF) forming the Riesz basis in $L_2(0,1)$, we constructed the characteristic determinant of the spectral problem for the operator $\mathcal{L}_1$. On the basis of the obtained formula we concluded on stability or instability of the Riesz basis properties of EAF of the problem under the integral perturbation of the boundary condition. In [9] the questions of stability of the basis properties of root vectors of the spectral problem, where $\alpha = 0$, and with the integral perturbation of the second boundary condition, were investigated. Further development of these results was published in [10]–[13]. A review of the results we obtained in this direction can be found in our work [14].

As follows from [4], the system of root vectors of the spectral problem (1), (4) forms the Riesz basis with brackets in $L_2(0,1)$ for any choice of $p(x) \in L_2(0,1)$. However even for $p(x) = 0$ (i.e., in case of the perturbed problem) the system of root vectors of the problem does not form the basis [19] in $L_2(0,1)$. Therefore, the direct using the methods of our previous papers is impossible. We use a special auxiliary system for constructing the characteristic determinant.

2 Constructing a basis from eigenfunctions of the operator $\mathcal{L}_0$

The boundary conditions in (1)–(3) are regular but not strongly regular. The system of root functions of the operator $\mathcal{L}_0$ is a complete system but does not form even an ordinary basis in $L_2(0,1)$ [19]. However, as shown in [20], on the basis of these eigenfunctions one can construct the basis allowing to apply the method of separating variables for solving initial-boundary value problems with the boundary condition (2).

In this section we introduce results from [20] and make additional calculations which are necessary for our further work. The spectral problem (1)–(3) is easily reduced to the characteristic determinant of the problem

$$\Delta_0(\lambda) = \sqrt{\lambda}\left(1 - \cos\sqrt{\lambda}\right) - \alpha \sin\sqrt{\lambda} = 0. \tag{5}$$

Therefore the problem has two series of eigenvalues

$$\lambda_k^{(1)} = (2\pi k)^2,\ k = 1,2,\dots\ , \quad \lambda_k^{(2)} = (2\beta_k)^2,\ k = 0,1,2,\dots\ .$$

Here β_k are roots of the equation

$$\mathrm{tg}\beta = \alpha/2\beta, \quad \beta > 0\ . \tag{6}$$

They are positive and satisfy the inequalities

$$\pi k - \pi/2 < \beta_k < \pi k,\ k = 0,1,2,\dots\ .$$

Two-sided estimates

$$\frac{|\alpha|}{2\pi k}\left(1-\frac{1}{2\pi k}\right) < \delta_k < \frac{|\alpha|}{2\pi k}\left(1+\frac{1}{2\pi k}\right) \tag{7}$$

hold for the difference $\delta_k = \pi k - \beta_k$ for large enough k.

The eigenfunctions of (1)–(3) have the form

$$y_k^{(1)}(x) = \sin(2\pi k x),\ k = 1,2,\dots, \quad y_k^{(2)}(x) = \sin(2\beta_k x),\ k = 0,1,2,\dots .$$

This system is almost normalized but does not form even an ordinary basis in $L_2(0,1)$. However, as shown in [20], the auxiliary system

$$y_0(x) = y_0^{(2)}(x)\,(2\beta_0)^{-1},\ \ y_{2k}(x) = y_k^{(1)}(x),$$

$$y_{2k-1}(x) = \left(y_k^{(2)}(x) - y_k^{(1)}(x)\right)(2\delta_k)^{-1},\ \ k=1,2,\dots ,$$

constructed from this system, already forms the Riesz basis in $L_2(0,1)$. The system

$$v_0(x) = 2\beta_0 v_0^{(2)}(x),$$

$$v_{2k}(x) = v_k^{(2)}(x) + v_k^{(1)}(x),\ \ v_{2k-1}(x) = 2\delta_k v_k^{(2)}(x),\ k = 1,2,\dots$$

is biorthogonal to the auxiliary system. This system is constructed from eigenfunctions of the problem

$$v_k^{(1)}(x) = C_k^{(1)}\left(\cos(2\pi k x) - \frac{\alpha}{2\pi k}\sin(2\pi k x)\right),\ k = 1,2,\dots ,$$

$$v_k^{(2)}(x) = C_k^{(2)}\left(\cos(2\beta_k x) + \frac{\alpha}{2\beta_k}\sin(2\beta_k x)\right),\ k = 0,1,2,\dots$$

adjoint to (1)–(3). The constants $C_k^{(j)}$ are chosen from the orthogonality relation $(y_k^{(j)}, v_k^{(j)}) = 1, j = 1,2$. It is evident that the system $\{v_k(x)\}$ forms the Riesz basis in $L_2(0,1)$.

By direct calculation it is easy to make sure that

$$C_k^{(1)} = -\frac{4\pi k}{\alpha}, \quad C_k^{(2)} = \frac{4\pi k}{\alpha} + O\left(\frac{1}{k}\right). \tag{8}$$

It is easy to see that $\left\|y_k^{(1)}\right\|\left\|v_k^{(1)}\right\| = 1 + \frac{2\pi k}{|\alpha|}$. Therefore $\lim_{k\to\infty}\left\|y_k^{(1)}\right\|\left\|v_k^{(1)}\right\| = \infty$. That is, the necessary condition of the basis property does not hold. Due to this reason, the systems $\{y_k^{(1)},\ y_k^{(2)}\}$ and $\{v_k^{(1)},\ v_k^{(2)}\}$ do not form the unconditional basis in $L_2(0,1)$.

3 Characteristic determinant of the spectral problem (1), (4)

Representing a general solution to equation (1) by the formula

$$u(x,\lambda) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x,$$

and satisfying it to the boundary conditions (4), we get that $C_1 = 0$ and

$$C_2 \left[\sqrt{\lambda}\left(1-\cos\sqrt{\lambda}\right) - \alpha \sin\sqrt{\lambda} - \int_0^1 \overline{p(x)} \sin\sqrt{\lambda}x \, dx\right] = 0.$$

Therefore the characteristic determinant of (1), (4) has the form

$$\Delta_1(\lambda) = \sqrt{\lambda}\left(1-\cos\sqrt{\lambda}\right) - \lambda \sin\sqrt{\lambda} - \int_0^1 \overline{p(x)} \sin\sqrt{\lambda}x \, dx. \quad (9)$$

It is easy to see that the characteristic determinant of the unperturbed problem (1)–(3) is obtained here for $p(x) = 0$. As in (5), we denote it by

$$\Delta_0(\lambda) = \sqrt{\lambda}\left(1-\cos\sqrt{\lambda}\right) - \alpha \sin\sqrt{\lambda}.$$

By virtue of Section 2, we represent the function $p(x)$ in the form of Fourier series with respect to the auxiliary system $\{v_k(x)\}$:

$$p(x) = a_0 v_0(x) + \sum_{k=1}^{\infty} [a_k v_{2k}(x) + b_k v_{2k-1}(x)]. \quad (10)$$

Using (10), we find more convenient representation of the determinant $\Delta_1(\lambda)$. For this, firstly we calculate the integral belonging to (9).

By simple calculation we show that the following inequalities take place:

$$\int_0^1 \overline{v_0(x)} \sin\sqrt{\lambda}x \, dx = 2\beta_0 C_0^{(2)} \int_0^1 \left(\cos(2\beta_0 x) + \frac{\alpha}{2\beta_0}\sin(2\beta_0 x)\right) \sin\sqrt{\lambda}x \, dx$$

$$= \frac{2\beta_0 C_0^{(2)}}{\lambda - (2\beta_0)^2} \left\{\sqrt{\lambda}\left(1 - \cos\sqrt{\lambda}\cos(2\beta_0) - \frac{\alpha}{2\beta_0}\sin(2\beta_0)\cos\sqrt{\lambda}\right)\right\} \quad (11)$$

$$+ \frac{2\beta_0 C_0^{(2)}}{\lambda - (2\beta_0)^2} \left\{\sin\sqrt{\lambda}\,[\alpha\cos(2\beta_0) - (2\beta_0)\sin(2\beta_0)]\right\}.$$

From (5) we obtain that $2\beta_0(1-\cos(2\beta_0)) = \alpha \sin(2\beta_0)$. Therefore, in the first summand of (11) inside the round brackets we have:

$$\left(1 - \cos\sqrt{\lambda}\cos(2\beta_0) - \frac{\alpha}{2\beta_0}\sin(2\beta_0)\cos\sqrt{\lambda}\right) = 1 - \cos\sqrt{\lambda}.$$

Using (6), we find that

$$\sin\left(2\beta_0\right) = \frac{2\mathrm{tg}\left(\beta_0\right)}{1+\mathrm{tg}^2\left(\beta_0\right)} = \frac{4\alpha\beta_0}{\left(2\beta_0\right)^2+\alpha^2}, \cos\left(2\beta_0\right) = \frac{1-\mathrm{tg}^2\left(\beta_0\right)}{1+\mathrm{tg}^2\left(\beta_0\right)} = \frac{\left(2\beta_0\right)^2-\alpha^2}{\left(2\beta_0\right)^2+\alpha^2}.$$

Therefore, in the second summand of (11) inside the square brackets we will have:

$$\left[\alpha\cos\left(2\beta_0\right) - \left(2\beta_0\right)\sin\left(2\beta_0\right)\ \right] = \left[\alpha\frac{\left(2\beta_0\right)^2-\alpha^2}{\left(2\beta_0\right)^2+\alpha^2} - \left(2\beta_0\right)\frac{4\alpha\beta_0}{\left(2\beta_0\right)^2+\alpha^2}\right] = -\alpha.$$

Finally we obtain:

$$\int_0^1 \overline{v_0\left(x\right)}\sin\sqrt{\lambda}x\ dx \tag{12}$$

$$= \frac{2\beta_0 C_0^{(2)}}{\lambda-\left(2\beta_0\right)^2}\left\{\sqrt{\lambda}\left(1-\cos\sqrt{\lambda}\ \right) - \alpha\sin\sqrt{\lambda}\ \right\} = \frac{2\beta_0 C_0^{(2)}}{\lambda-\left(2\beta_0\right)^2}\Delta_0\left(\lambda\right).$$

Analogously, we calculate the integral

$$\int_0^1 \overline{v_{2k-1}\left(x\right)}\sin\sqrt{\lambda}x\ dx == 2\delta_k C_k^{(2)}\frac{1}{\lambda-\left(2\beta_k\right)^2}\Delta_0\left(\lambda\right). \tag{13}$$

Further we have

$$\int_0^1 \overline{v_{2k}\left(x\right)}\sin\sqrt{\lambda}x\ dx = \int_0^1 \left(v_k^{(2)}\left(x\right)+v_k^{(1)}\left(x\right)\right)\sin\sqrt{\lambda}x\ dx$$

$$= C_k^{(2)}\frac{1}{\lambda-\left(2\beta_k\right)^2}\Delta_0\left(\lambda\right) + C_k^{(1)}\frac{1}{\lambda-\left(2\pi k\right)^2}\Delta_0\left(\lambda\right).$$

And so, we finally obtain

$$\int_0^1 \overline{p\left(x\right)}\sin\sqrt{\lambda}x\ dx = \Delta_0\left(\lambda\right)A\left(\lambda\right),$$

$$A\left(\lambda\right) = \frac{2a_0\beta_0 C_0^{(2)}}{\lambda-\left(2\beta_0\right)^2} + \sum_{k=1}^{\infty}\left[a_k\left(\frac{C_k^{(2)}}{\lambda-\left(2\beta_k\right)^2} + \frac{C_k^{(1)}}{\lambda-\left(2\pi k\right)^2}\right) + \frac{2b_k\delta_k C_k^{(2)}}{\lambda-\left(2\beta_k\right)^2}\right]. \tag{14}$$

The convergence of the obtained numerical series for $\lambda \neq \left(2\beta_k\right)^2$ and $\lambda \neq \left(2\pi k\right)^2$ is provided by asymptotic behaviors (7) and (8). From these formulas it follows that the round brackets inside the sign of sum can not be opened because it can lead to a divergence of the obtained series.

In representation (14) the function $A\left(\lambda\right)$ has poles at $\lambda = \left(2\beta_k\right)^2$ and $\lambda = \left(2\pi k\right)^2$. But at the same points the function $\Delta_0\left(\lambda\right)$ has zeros. So the function $\Delta_0\left(\lambda\right)A\left(\lambda\right)$ is an entire analytic function of the variable λ.

Now we substitute all the calculations into (9). Let us formulate the obtained result in the form of a theorem.

Theorem 1. *The characteristic determinant of problem* (1), (4) *with the perturbed boundary conditions can be represented in the form*

$$\Delta_1(\lambda) = \Delta_0(\lambda)(1 - A(\lambda)), \tag{15}$$

where $\Delta_0(\lambda)$ *is a characteristic determinant of unperturbed problem* (1)–(3), $A(\lambda)$ *is given by* (14), *in which* a_k *and* b_k *are the Fourier coefficients of biorthogonal expansion* (10) *of the function* $p(x)$ *with respect to the auxiliary system* $\{v_k(x)\}$.

Let us note that earlier the basis properties of the system of root functions of the unperturbed problem has been necessarily required for constructing the characteristic determinant in all previous works. The principal difference of the present paper is that the characteristic determinant (15) is constructed without such a requirement.

4 The case of a simple form of the characteristic determinant (15)

The case of a simple form of the characteristic determinant (15) takes place when $p(x)$ is represented in the form (10) with the finite second sum. That is, when there is a number N such that $a_k = 0$ and $b_k = 0$ for all $k > N$. In this case, formula (15) takes the form

$$\Delta_1(\lambda) = \Delta_0(\lambda)\Bigg(1 - a_0\frac{2\beta_0 C_0^{(2)}}{\lambda - (2\beta_0)^2}$$

$$-\sum_{k=1}^{N}\left[a_k\left(C_k^{(2)}\frac{1}{\lambda - (2\beta_k)^2} + C_k^{(1)}\frac{1}{\lambda - (2\pi k)^2}\right) + b_k\frac{2\beta_k C_k^{(2)}}{\lambda - (2\beta_k)^2}\right]\Bigg). \tag{16}$$

On the basis of this particular case of formula (15), one can readily prove the following theorem.

Theorem 2. *For any prescribed numbers, a complex number* $\widehat{\lambda}$ *and a natural one* $\widehat{m}$, *there always exists a function* $p(x)$ *such that* $\widehat{\lambda}$ *is an eigenvalue of problem* (1), (4) *of multiplicity* $\widehat{m}$.

From the analysis of formula (16) it is also easy to see that $\Delta_1(\lambda_k^{(1)}) = \Delta_1(\lambda_k^{(2)}) = 0$ for all $k > N$. Hence all the eigenvalues $\lambda_k^{(1)}, \lambda_k^{(2)}, k > N$ of the unperturbed problem (1)–(3) are eigenvalues of the perturbed problem (1), (4). Also it is not difficult to see that the multiplicity of the eigenvalues $\lambda_k^{(1)}, \lambda_k^{(2)}, k > N$ is also preserved.

Moreover from the biorthogonality condition of the system of eigenfunctions $\{y_k^{(1)}(x), y_k^{(2)}(x)\}$ and $\{v_k^{(1)}(x), v_k^{(2)}(x)\}$ of the adjoint problems it follows that in this case

$$\int_0^1 \overline{p(x)}\, y_k^{(j)}(x)\,dx = 0, \;\; j = 1, 2, \;\; k > N.$$

So, eigenfunctions $\{y_k^{(1)}(x), y_k^{(2)}(x)\}$ of problem (1)–(3) for $k > N$ satisfy the boundary conditions (4) and hence, are eigenfunctions of problem (1), (4). Thus in this case the system of eigenfunctions of problem (1), (4) and the system of eigenfunctions of problem (1)–(3) (not forming a basis) coincide except for a finite number of first terms. Consequently, the system of eigenfunctions of problem (1), (4) also is not a basis in $L_2(0,1)$.

By the Riesz basis property in $L_2(0,1)$ of the system $\{v_k(x)\}$, the set of functions $p(x)$, represented by finite sums of (10) is dense in $L_2(0,1)$. Hence the following statement is proved.

Theorem 3. *The set of all functions $p \in L_2(0,1)$, for which the system of eigenfunctions of problem* (1), (4) *is not a basis in $L_2(0,1)$, is dense in $L_2(0,1)$.*

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Садыбеков М.А., Иманбаев Н.С. БАЗИСТІЛІК ҚАСИЕТІН ИЕЛЕНБЕГЕН БІР ЕСЕПТІҢ ШЕКАРАЛЫҚ ШАРТЫНЫҢ ИНТЕГРАЛДЫҚ АУЫТҚУЫ ЖӘЙЛІ

Кез-келген өзіне-өзі түйіндес шекаралық шарттармен және өзіне-өзі түйіндес формальді дифференциалдық амалмен берілген, спектрі дискретті болатын оператордың меншікті функцияларының жүйесінің ортонормаланған базис құрайтындығы белгілі жәй. Көптеген жұмыстарда бастапқы берілген оператордың қандай да бір әлсіз (белгілі мағынада) ауытқуы кезінде оның базистілік қасиеттерінің сақталуы мәселесі зерттелген. Ауытқымаған шекаралық шарттары күшейтілген регулярлы болған жағдайдағы жәй дифференциалдық оператор үшін осы шарттардың интегралдық ауытқуы кезіндегі түбірлік функциялардың базистілік қасиеттерінің орнықтылығы туралы мәселе А.А. Шкаликовтың жұмыстарында оң шешімін тапқан. Біздің бұрын жарияланған бірқатар жұмыстарымызда шекаралық шарттардың арасында біреуі интегралдық ауытқығандағы характеристикалық анықтауышты құру мен түбірлік функциялардың базистілік қасиеттерінің орнықтылығын анықтау сұрақтары зерттелген болатын. Әрі регулярлы, бірақ күшейтілмеген регулярлы шекаралық шарттардың мүмкін болатын типтері түгелдей дерлік қарастырылды. Бұл мақалада бір типтегі интегралдық ауытқуы бар, регулярлы, бірақ күшейтілген регулярлы емес болатын шекаралық шарттармен берілген екі еселі дифференциалдау оператор үшін спектралдық есеп қарастырылады. Ауытқымаған есеп асимптоталық тұрғыдан қарапайым спектрді иеленіп, ал оған сәйкес меншікті функцияларының жүйесі L_2 кеңістігінде базис құрмайды. Шекаралық шарттардың біреуіне интегралдық ауытқу жасағандағы спектралдық есептің характеристикалық анықтауышы құрылған. Түбірлік функциялардың жүйесінің базистілік қасиетінің жоқтығы сақталатын интегралдық ауытқулардың өзектерінің жиыны L_2 кеңістігінде тығыз болатындығы көрсетілген.

Кілттік сөздер. Жәй дифференциалдық оператор, шеттік шарттар, шекаралық шарттың интегралдық ауытқуы, меншікті мәндер, меншікті функциялар, базистілік, характеристикалық анықтауыш.

Садыбеков М.А., Иманбаев Н.С. ОБ ИНТЕГРАЛЬНОМ ВОЗМУЩЕНИИ ГРАНИЧНОГО УСЛОВИЯ ОДНОЙ ЗАДАЧИ, НЕ ОБЛАДАЮЩЕЙ СВОЙСТВОМ БАЗИСНОСТИ

Хорошо известно, что система собственных функций оператора, заданного формально самосопряженным дифференциальным выражением, с произвольными самосопряженными граничными условиями, дающими дискретный спектр, образует ортонормированный базис. Во многих работах исследовался вопрос о сохранении базисных свойств при некотором слабом (в определенном смысле) возмущении исходного оператора. Для случая произвольного обыкновенного дифференциального оператора, когда невозмущенные граничные условия усиленно регулярны, вопрос об устойчивости свойства базисности корневых векторов при их интегральном возмущении положительно решен в работах А.А. Шкаликова. В серии наших предыдущих работ мы рассмотрели вопрос о построении характеристического определителя и устойчивости свойства базисности корневых векторов при интегральном возмущении одного из граничных условий. Были рассмотрены почти все возможные типы граничных условий, которые являются регулярными, но не усиленно регулярными. В данной работе рассматривается спектральная задача для оператора двукратного дифференцирования с интегральным возмущением граничных условий одного типа, которые являются регулярными, но не усиленно регулярными. Невозмущенная задача имеет асимптотически простой спектр, а ее система собственных функций не образует базиса в L_2. Построен характеристический определитель спектральной задачи с интегральным возмущением одного граничного условия. Показано, что множество ядер интегрального возмущения, при которых сохраняется отсутствие базисных свойств системы корневых функций, плотно в L_2.

Ключевые слова. обыкновенный дифференциальный оператор, краевая задача, интегральное возмущение граничного условия, собственные значения, собственные функции, базисность, характеристический определитель.

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On two-point initial boundary value problem for fourth order partial differential equations

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Abstract. A two-point initial boundary value problem for fourth order partial differential equations is studied. We consider the existence of classical solutions to the initial two-point boundary value problem for the fourth order partial differential equations and offer the methods for finding its approximate solutions. Sufficient conditions for the existence and uniqueness of a classical solution to the two-point initial boundary value problem for the fourth order partial differential equations are set. We first introduce a new unknown function twice: we reduce the problem considered to the equivalent problem consisting of a nonlocal problem for a system of second order hyperbolic equations with integral relations, and then to the equivalent problem consisting of a two-point boundary value problem for a system of first order differential equations. We offer the algorithm for finding the approximate solution to the problem considered and prove its convergence.

Keywords. Fourth order partial differential equations, two-point initial boundary value problem, nonlocal problem, system of second order hyperbolic equations, first order differential equations, solvability, algorithm.

1 Introduction

In recent years the theory of nonlocal boundary value problems for hyperbolic equations are drawn by great attention. This is of practical importance, as well as for their new mathematical content, which often has no analogues in classical mathematical physics [1]–[3]. One of the important classes of such problems are the initial two-point boundary value problem for fourth order partial differential equations. Over the past decades, the theory of the initial-boundary value problems for the fourth order partial differential equations of hyperbolic type, has been intensively developed in works of many mathematicians [4]–[8]. To

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study the initial two-point boundary value problem for differential equations of hyperbolic type it is very important for solving theoretical and practical problems [9]. From this point of view, the paper is devoted to actual problem of mathematical physics. The methods and results from [10]–[14] will be developed for the two-point initial boundary value problems for the fourth order partial differential equations. Based on them, the conditions for solvability of considered boundary value problems are obtained, and the ways for finding their solutions are offered. Results of this paper are announced at the International Conference "Actual Problems of Analysis, Differential Equations and Algebra" (EMJ-2019), dedicated to the 10th anniversary of the Eurasian Mathematical Journal [15].

2 Statement of the problem

In the present paper, on the domain $\Omega = [0,T]\times[0,\omega]$ we consider the following two-point initial boundary value problem for the system of fourth order partial differential equations:

$$\frac{\partial^4 u}{\partial t\partial x^3} = A_1(t,x)\frac{\partial^3 u}{\partial x^3} + A_2(t,x)\frac{\partial^3 u}{\partial t\partial x^2} + A_3(t,x)\frac{\partial^2 u}{\partial x^2} + A_4(t,x)\frac{\partial^2 u}{\partial t\partial x}$$

$$+A_5(t,x)\frac{\partial u}{\partial x} + A_6(t,x)\frac{\partial u}{\partial t} + A_7(t,x)u + f(t,x), \tag{1}$$

$$u(t,0) = \psi_1(t), \quad t\in[0,T], \tag{2}$$

$$\frac{\partial u(t,x)}{\partial x}\Big|_{x=0} = \psi_2(t), \quad t\in[0,T], \tag{3}$$

$$\frac{\partial^2 u(t,x)}{\partial x^2}\Big|_{x=0} = \psi_3(t), \quad t\in[0,T], \tag{4}$$

$$P(x)u(0,x) + S(x)u(T,x) = \varphi(x), \quad x\in[0,\omega], \tag{5}$$

where $u(t,x) = col(u_1(t,x),...,u_n(t,x))$ is unknown function, $(n\times n)$-matrices $A_i(t,x)$, $(i=\overline{1,7})$, and n-vector–function $f(t,x)$ are continuous on Ω; $(n\times n)$-matrices $P(x), S(x)$ and n-vector–function $\varphi(x)$ are continuously three times differentiable on $[0,\omega]$; n-vector–functions $\psi_1(t), \psi_2(t)$ and $\psi_3(t)$ are continuously differentiable on [0,T].

The compatibility conditions are valid:

$$P(0)\psi_1(0) + S(0)\psi_1(T) = \varphi(0), \; P'(0)\psi_1(0) + P(0)\psi_2(0) + S'(0)\psi_1(T) + S(0)\psi_2(T) = \varphi'(0),$$

$$P''(0)\psi_1(0) + 2P'(0)\psi_2(0) + P(0)\psi_3(0) + S''(0)\psi_1(T) + 2S'(0)\psi_2(T) + S(0)\psi_3(T) = \varphi''(0).$$

Let $C(\Omega,\mathbb{R}^n)$ be the space of continuous on Ω vector–functions $u(t,x)$ with the norm $||u||_0 = \max_{(t,x)\in\Omega}||u(t,x)||$, $||u(t,x)|| = \max_{i=\overline{1,n}}|u_i(t,x)|$.

A function $u(t,x)\in C(\Omega,\mathbb{R}^n)$ having partial derivatives

$$\frac{\partial u(t,x)}{\partial x}\in C(\Omega,\mathbb{R}^n), \; \frac{\partial u(t,x)}{\partial t}\in C(\Omega,\mathbb{R}^n), \; \frac{\partial^2 u(t,x)}{\partial x^2}\in C(\Omega,\mathbb{R}^n), \; \frac{\partial^2 u(t,x)}{\partial t\partial x}\in C(\Omega,\mathbb{R}^n),$$

$$\frac{\partial^3 u(t,x)}{\partial x^3} \in C(\Omega,\mathbb{R}^n), \quad \frac{\partial^3 u(t,x)}{\partial t\partial x^2} \in C(\Omega,\mathbb{R}^n), \quad \frac{\partial^4 u(t,x)}{\partial t\partial x^3} \in C(\Omega,\mathbb{R}^n),$$

is called a classical solution to the problem (1)–(5) if it satisfies the system (1) for all $(t,x)\in\Omega$, and the initial and the boundary conditions (2)–(5).

Using the properties of initial data and differentiating the two-point condition (5) three times with respect to x, we obtain:

$$P'''(x)u(0,x) + 3P''(x)\frac{\partial u(0,x)}{\partial x} + 3P'(x)\frac{\partial^2 u(0,x)}{\partial x^2} + P(x)\frac{\partial^3 u(0,x)}{\partial x^3} + S'''(x)u(T,x)$$

$$+3S''(x)\frac{\partial u(T,x)}{\partial x} + 3S'(x)\frac{\partial^2 u(T,x)}{\partial x^2} + S(x)\frac{\partial^3 u(T,x)}{\partial x^3} = \dddot{\varphi}(x), \quad x\in[0,\omega]. \tag{6}$$

3 Reduction to the equivalent family of two-point boundary value problems for a system of ordinary differential equations with integral relations

First, we introduce new unknown functions

$$v(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2}, \quad v_1(t,x) = \frac{\partial u(t,x)}{\partial x}$$

and rewrite the problem (1)–(5) in the following from

$$\frac{\partial^2 v}{\partial t\partial x} = A_1(t,x)\frac{\partial v}{\partial x} + A_2(t,x)\frac{\partial v}{\partial t} + A_3(t,x)v + F(t,x,u,v_1) + f(t,x), \tag{7}$$

$$v(t,0) = \psi_3(t), \quad t\in[0,T], \tag{8}$$

$$3P'(x)v(0,x) + P(x)\frac{\partial v(0,x)}{\partial x} + 3S'(x)v(T,x) + S(x)\frac{\partial v(T,x)}{\partial x} = D(x,u,v_1), \quad x\in[0,\omega], \tag{9}$$

$$v_1(t,x) = \psi_2(t) + \int_0^x v(t,\xi)\,d\xi, \tag{10}$$

$$u(t,x) = \psi_1(t) + \psi_2(t)x + \int_0^x\int_0^\xi v(t,\xi_1)\,d\xi_1 d\xi, \tag{11}$$

where

$$F(t,x,u,v_1) = A_4(t,x)\frac{\partial v_1}{\partial t} + A_5(t,x)v_1 + A_6(t,x)\frac{\partial u}{\partial t} + A_7(t,x)u,$$

$$D(x,u,v_1) = \dddot{\varphi}(x) - [P'''(x)u(0,x) + 3P''(x)v_1(0,x) + S'''(x)u(T,x) + 3S''(x)v_1(T,x)].$$

A solution to the problem (7)–(11) is a triple of functions $\{v(t,x), v_1(t,x), u(t,x)\}$, where the n-function $v(t,x)\in C(\Omega,\mathbb{R}^n)$ has partial derivatives $\dfrac{\partial v(t,x)}{\partial x}\in C(\Omega,\mathbb{R}^n)$, $\dfrac{\partial v(t,x)}{\partial t}\in C(\Omega,\mathbb{R}^n)$, $\dfrac{\partial^2 v(t,x)}{\partial x\partial t}\in C(\Omega,\mathbb{R}^n)$, the n-function $v_1(t,x)\in C(\Omega,\mathbb{R}^n)$ with

$\dfrac{\partial v_1(t,x)}{\partial t} \in C(\Omega,\mathbb{R}^n)$, the n-function $u(t,x) \in C(\Omega,\mathbb{R}^n)$ with $\dfrac{\partial u(t,x)}{\partial t} \in C(\Omega,\mathbb{R}^n)$, if it satisfies the system of hyperbolic equations (7) for all $(t,x)\in\Omega$, the boundary condition (8) for all $t\in[0,T]$, the nonlocal condition (9) for all $x\in[0,\omega]$ and the integral relations (10), (11).

Here the functions $v_1(t,x)$ and $u(t,x)$ are connected with the function $v(t,x)$ by the integral conditions (10) and (11), respectively.

Conditions (2) and (3) are included in the integral relations (11) and (10).

The problems (1)–(5) and (7)–(11) are equivalent.

Differentiating relations (10) and (11) by t for partial derivatives $\dfrac{\partial v_1(t,x)}{\partial t}$ and $\dfrac{\partial u(t,x)}{\partial t}$ we obtain the following equalities:

$$\frac{\partial v_1(t,x)}{\partial t} = \dot\psi_2(t) + \int\limits_0^x \frac{\partial v(t,\xi)}{\partial t} d\xi, \quad \frac{\partial u(t,x)}{\partial t} = \dot\psi_1(t) + \dot\psi_2(t)x + \int_0^x \int_0^\xi \frac{\partial v(t,\xi_1)}{\partial t} d\xi_1 d\xi. \tag{12}$$

At fixed $v_1(t,x)$ and $u(t,x)$ the problem (7)–(9) is a two-point boundary value problem for the system of second order hyperbolic equations with respect to $v(t,x)$ on Ω. Integral relations (10) and (11) allow us to determine the unknown functions $v_1(t,x)$ and $u(t,x)$. From (12) we define the partial derivatives $\dfrac{\partial v_1(t,x)}{\partial t}$ and $\dfrac{\partial u(t,x)}{\partial t}$.

Two-point and multi-point boundary value problems for a system of second order hyperbolic equations were studied in [10-14]. Sufficient conditions for the unique solvability of these problems are established in terms of the initial data by the method of introducing functional parameters [10].

Then, second, we introduce new unknown functions $\dfrac{\partial v}{\partial x} = V(t,x)$, $\dfrac{\partial v}{\partial t} = W(t,x)$.

We reduce the problem (7)–(11) to the following equivalent problem:

$$\frac{\partial V}{\partial t} = A_1(t,x)V + A_2(t,x)W(t,x) + A_3(t,x)v(t,x) + F(t,x,u,v_1) + f(t,x), \tag{13}$$

$$P(x)V(0,x) + S(x)V(T,x) = D_1(x,u,v_1,v), \quad x\in[0,\omega], \tag{14}$$

$$v(t,x) = \psi_3(t) + \int_0^x V(t,\xi)\,d\xi, \quad W(t,x) = \dot\psi_3(t) + \int_0^x \frac{\partial V(t,\xi)}{\partial t}\,d\xi, \tag{15}$$

$$v_1(t,x) = \psi_2(t) + \int_0^x v(t,\xi)\,d\xi, \tag{16}$$

$$u(t,x) = \psi_1(t) + \psi_2(t)x + \int_0^x \int_0^\xi v(t,\xi_1)\,d\xi_1 d\xi, \tag{17}$$

where $D_1(x,u,v_1,v) = D(x,u,v_1) + 3P'(x)v(0,x) + 3S'(x)v(T,x)$.

In the problem (13)–(17), the condition (8) is taken into account in relations (14).

A solution to the problem (13)–(17) is the five functions $\{V(t,x), W(t,x), v(t,x), v_1(t,x), u(t,x)\}$, if they satisfy the system of differential equations (13) for all $(t,x) \in \Omega$, the two-point condition (14) for all $x \in [0,\omega]$ and the integral relations (15)–(17) for all $(t,x) \in \Omega$.

Using the fundamental matrix of the differential equation

$$\frac{\partial V}{\partial t} = A_1(t,x)V, \tag{18}$$

we present a solution to the problem (13), (14).

Let $X(t,x)$ be the fundamental matrix of the system (18), and $X(0,x) = I$, where I is the identity matrix of the dimension n.

Consider the two-point boundary value problem

$$\frac{\partial V}{\partial t} = A_1(t,x)V + g(t,x), \tag{19}$$

$$P(x)V(0,x) + S(x)V(T,x) = \Phi(x), \quad x \in [0,\omega], \tag{20}$$

where $g(t,x) \in C(\Omega, \mathbb{R}^n)$, the n-vector function $\Phi(x)$ is continuous on $[0,\omega]$.

The solution to the system (18) can be written as

$$V(t,x) = X(t,x)V(0,x) + X(t,x)\int_0^t X^{-1}(\tau,x)g(\tau,x)d\tau. \tag{21}$$

Substituting it into the condition (20) for $t = T$, we obtain

$$P(x)V(0,x) + S(x)X(T,x)V(0,x) + S(x)X(T,x)\int_0^T X^{-1}(\tau,x)g(\tau,x)d\tau = \Phi(x).$$

From here we have

$$[P(x) + S(x)X(T,x)]V(0,x) = \Phi(x) - S(x)X(T,x)\int_0^T X^{-1}(\tau,x)g(\tau,x)d\tau.$$

To uniquely determine the function $V(0,x)$, we assume that $det[P(x) + S(x)X(T,x)] \neq 0$ for all $x \in [0,\omega]$. We obtain

$$V(0,x) = [P(x) + S(x)X(T,x)]^{-1}\Big\{\Phi(x) - S(x)X(T,x)\int_0^T X^{-1}(\tau,x)g(\tau,x)d\tau\Big\}. \tag{22}$$

Then the solution to the problem (19), (20) has the following form

$$V(t,x) = X(t,x)[P(x) + S(x)X(T,x)]^{-1}\Big\{\Phi(x)$$

$$-S(x)X(T,x)\int_0^T X^{-1}(\tau,x)g(\tau,x)d\tau\Big\} + X(t,x)\int_0^t X^{-1}(\tau,x)g(\tau,x)d\tau. \quad (23)$$

The following estimate holds for the function $V(t,x)$:

$$\max\Big(\max_{t\in[0,T]}\|V(t,x)\|, \max_{t\in[0,T]}\Big\|\frac{\partial V(t,x)}{\partial t}\Big\|\Big) \le \tilde{K}\max\big(\max_{t\in[0,T]}\|g(t,x)\|, \|\Phi(x)\|\big),$$

where the constant $\tilde{K}$ is calculated using the fundamental matrix $X(t,x)$, the inverse matrix $[P(x)+S(x)X(T,x)]^{-1}$, matrices $A_1(t,x)$, $P(x)$, $S(x)$ and T.

Theorem 1. *Let*

1) $X(t,x)$ be the fundamental matrix of differential equation $\frac{\partial V}{\partial t} = A_1(t,x)V$;

2) $(n\times n)$-matrix $P(x)+S(x)X(T,x)$ is invertible for all $x\in[0,\omega]$.

Then the two-point boundary value problem (19), (20) has a unique solution $V^*(t,x)$ represented by (23).

3 Algorithm and unique solvability of the problem (1)–(5)

For fixed $W(t,x)$, $v(t,x)$, $v_1(t,x)$ and $u(t,x)$ the unknown function $V(t,x)$ will be found from two-point boundary value problem for the system of differential equations (13), (14). The unknown functions $W(t,x)$ and $v(t,x)$ will be determined from integral relations (15) by $V(t,x)$ and its partial derivative $\frac{\partial V(t,x)}{\partial t}$. And, using $v(t,x)$, we define the unknown functions $v_1(t,x)$ and $u(t,x)$ through integral relations (16), (17). Since $V(t,x)$, $W(t,x)$, $v(t,x)$, $v_1(t,x)$ and $u(t,x)$ are unknown, to find a solution to the problem (13)–(17) we use an iterative method. Therefore, the solution of the problem (13)–(17) is found as the limits of the sequences $\{V^{(k)}(t,x)\}, \{W^{(k)}(t,x)\}, \{v^{(k)}(t,x)\}, \{{v_1}^{(k)}(t,x)\}, \{u^{(k)}(t,x)\}$, $k=0,1,2,...$, defined by the following algorithm:

0-step: 1) setting $v^{(0)}(t,x)=\psi_3(t)$, $W^{(0)}(t,x)=\dot{\psi}_3(t)$ in integral relations (15) and (16), we obtain

$$v_1^{(0)}(t,x)=\psi_2(t)+\psi_3(t)x, \quad \frac{\partial v_1^{(0)}(t,x)}{\partial t}=\dot{\psi}_2(t)+\dot{\psi}_3(t)x,$$

$$u^{(0)}(t,x)=\psi_1(t)+\psi_2(t)x+\psi_3(t)\frac{x^2}{2}, \quad \frac{\partial u^{(0)}(t,x)}{\partial t}=\dot{\psi}_1(t)+\dot{\psi}_2(t)x+\dot{\psi}_3(t)\frac{x^2}{2}$$

for all $(t,x)\in\Omega$;

2) then, we suppose $v(t,x)=v^{(0)}(t,x)$, $W(t,x)=W^{(0)}(t,x)$, $v_1(t,x)=v_1^{(0)}(t,x)$, $u(t,x)=u^{(0)}(t,x)$ in the right-hand sides of the system (13) and condition (14). From the following two-point boundary value problem

$$\frac{\partial V}{\partial t}=A_1(t,x)V+A_2(t,x)W^{(0)}(t,x)+A_3(t,x)v^{(0)}(t,x)+F(t,x,u^{(0)},{v_1}^{(0)})+f(t,x), \quad (24)$$

$$P(x)V(0,x)+S(x)V(T,x)=D_1\big(x,u^{(0)},v_1^{(0)},v^{(0)}\big),\ x\in[0,\omega], \tag{25}$$

we find the initial approximation $V^{(0)}(t,x)$ and its derivative $\dfrac{\partial V^{(0)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$.

1-step: 1) From integral relations (15) for $V(t,x)=V^{(0)}(t,x)$ and $\dfrac{\partial V(t,x)}{\partial t}=\dfrac{\partial V^{(0)}(t,x)}{\partial t}$, we find the functions $v^{(1)}(t,x)$ and $W^{(1)}(t,x)$:

$$v^{(1)}(t,x)=\psi_3(t)+\int_0^x V^{(0)}(t,\xi)\,d\xi,\quad W^{(1)}(t,x)=\dot\psi_3(t)+\int_0^x \frac{\partial V^{(0)}(t,\xi)}{\partial t}\,d\xi$$

for all $(t,x)\in\Omega$.

Setting $v(t,x)=v^{(1)}(t,x),\ \ W(t,x)=W^{(1)}(t,x)$ in integral relations (16) and (17), we obtain

$$v_1{}^{(1)}(t,x)=\psi_2(t)+\int_0^x v^{(1)}(t,\xi)\,d\xi,\quad \frac{\partial v_1{}^{(1)}(t,x)}{\partial t}=\dot\psi_2(t)+\int_0^x \frac{\partial v^{(1)}(t,\xi)}{\partial t}\,d\xi,$$

$$u^{(1)}(t,x)=\psi_1(t)+\psi_2(t)x+\int_0^x\int_0^\xi v^{(1)}(t,\xi_1)\,d\xi_1 d\xi,$$

$$\frac{\partial u^{(1)}(t,x)}{\partial t}=\dot\psi_1(t)+\dot\psi_2(t)x+\int_0^x\int_0^\xi \frac{\partial v^{(1)}(t,\xi_1)}{\partial t}\,d\xi_1 d\xi$$

for all $(t,x)\in\Omega$;

2) then, we suppose $v(t,x)=v^{(1)}(t,x)$, $W(t,x)=W^{(1)}(t,x)$, $v_1(t,x)=v_1^{(1)}(t,x)$, $u(t,x)=u^{(1)}(t,x)$ in the right-hand sides of the system (13) and condition (14). From the following two-point boundary value problem

$$\frac{\partial V}{\partial t}=A_1(t,x)V+A_2(t,x)W^{(1)}(t,x)+A_3(t,x)v^{(1)}(t,x)+F(t,x,u^{(1)},v_1{}^{(1)})+f(t,x), \tag{26}$$

$$P(x)V(0,x)+S(x)V(T,x)=D_1(x,u^{(1)},v_1^{(1)},v^{(1)}),\quad x\in[0,\omega], \tag{27}$$

we find the first approximation $V^{(1)}(t,x)$ and its derivative $\dfrac{\partial V^{(1)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$.

And so on.

k-step: 1) From integral relations (15) for $V(t,x)=V^{(k-1)}(t,x)$ and $\dfrac{\partial V(t,x)}{\partial t}=\dfrac{\partial V^{(k-1)}(t,x)}{\partial t}$, we find the functions $v^{(k)}(t,x)$ and $W^{(k)}(t,x)$:

$$v^{(k)}(t,x)=\psi_3(t)+\int_0^x V^{(k-1)}(t,\xi)\,d\xi,\quad W^{(k)}(t,x)=\dot\psi_3(t)+\int_0^x \frac{\partial V^{(k-1)}(t,\xi)}{\partial t}\,d\xi$$

for all $(t,x)\in\Omega$.

Setting $v(t,x)=v^{(k)}(t,x)$, $W(t,x)=W^{(k)}(t,x)$ in integral relations (16) and (17), we obtain

$$v_1{}^{(k)}(t,x)=\psi_2(t)+\int_0^x v^{(k)}(t,\xi)\,d\xi,\ \ \frac{\partial v_1{}^{(k)}(t,x)}{\partial t}=\dot{\psi}_2(t)+\int_0^x\frac{\partial v^{(k)}(t,\xi)}{\partial t}\,d\xi,$$

$$u^{(k)}(t,x)=\psi_1(t)+\psi_2(t)x+\int_0^x\int_0^\xi v^{(k)}(t,\xi_1)\,d\xi_1 d\xi,$$

$$\frac{\partial u^{(k)}(t,x)}{\partial t}=\dot{\psi}_1(t)+\dot{\psi}_2(t)x+\int_0^x\int_0^\xi\frac{\partial v^{(k)}(t,\xi_1)}{\partial t}\,d\xi_1 d\xi$$

for all $(t,x)\in\Omega$;

2) then, we suppose $v(t,x)=v^{(k)}(t,x)$, $W(t,x)=W^{(k)}(t,x)$, $v_1(t,x)=v_1^{(k)}(t,x)$, $u(t,x)=u^{(k)}(t,x)$ in the right-hand sides of the system (13) and condition (14). From the following two-point boundary value problem

$$\frac{\partial V}{\partial t}=A_1(t,x)V+A_2(t,x)W^{(k)}(t,x)+A_3(t,x)v^{(k)}(t,x)+F(t,x,u^{(k)},v_1{}^{(k)})+f(t,x), \quad (28)$$

$$P(x)V(0,x)+S(x)V(T,x)=D_1(x,u^{(k)},v_1^{(k)},v^{(k)}),\ x\in[0,\omega], \quad (29)$$

we find the k-th approximation $V^{(k)}(t,x)$ and its derivative $\dfrac{\partial V^{(k)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$. Here $k=1,2,3,\ldots$.

So, the method of introducing additional functions divides the process of finding unknown functions into two parts: 1) from the two-point boundary value problems for the system of differential equations (13), (14) we find the unknown function $V(t,x)$ $\Big($and its derivative $\frac{\partial V(t,x)}{\partial t}\Big)$; 2) From integral relations (15)–(17) we find the functions $W(t,x)$, $v(t,x)$, $v_1(t,x)$ and $u(t,x)$ $\big($and also their partial derivatives $\frac{\partial v_1(t,x)}{\partial t}$ and $\frac{\partial u(t,x)}{\partial t}\big)$.

The following statement gives conditions for the convergence of the proposed algorithm and the unique solvability of problem (1)–(5) in terms of the initial data.

Theorem 2. *Suppose that*

i) $(n\times n)$-matrices $A_i(t,x)$, $i=\overline{1,7}$, and n-vector–function $f(t,x)$ are continuous on Ω;

ii) $(n\times n)$-matrices $P(x),S(x)$ and n-vector–function $\varphi(x)$ are continuously three times differentiable on $[0,\omega]$;

iii) n-vector–functions $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ are continuously differentiable on $[0,T]$;

iv) $(n\times n)$-matrix $P(x)+S(x)X(T,x)$ is invertible for all $x\in[0,\omega]$.

Then the two-point initial boundary value problem for the system of fourth order partial differential equations (1)–(5) has a unique classical solution $u^*(t,x)$.

Proof. By using the iterative method proposed above we find estimates of the sequences of functions

$$\|v_1^{(k)}(t,x)\| \leq \|\psi_2(t)\| + \int_0^x \|v^{(k)}(t,\xi)\| d\xi, \tag{30}$$

$$\Big|\Big|\frac{\partial v_1^{(k)}(t,x)}{\partial t}\Big|\Big| \leq \|\dot{\psi}_2(t)\| + \int_0^x \Big|\Big|\frac{\partial v^{(k)}(t,\xi)}{\partial t}\Big|\Big| d\xi, \tag{31}$$

$$\|u^{(k)}(t,x)\|$$

$$\leq \|\psi_1(t)\| + \int_0^x \|v_1^{(k)}(t,\xi)\| d\xi \leq \|\psi_1(t)\| + x\|\psi_2(t)\| + \int_0^x \int_0^\xi \|v^{(k)}(t,\xi_1)\| d\xi_1 d\xi, \tag{32}$$

$$\Big|\Big|\frac{\partial u^{(k)}(t,x)}{\partial t}\Big|\Big| \leq \|\dot{\psi}_1(t)\| + \int_0^x \Big|\Big|\frac{\partial v_1^{(k)}(t,\xi)}{\partial t}\Big|\Big| d\xi$$

$$\leq \|\dot{\psi}_1(t)\| + x\|\dot{\psi}_2(t)\| + \int_0^x \int_0^\xi \Big|\Big|\frac{\partial v^{(k)}(t,\xi_1)}{\partial t}\Big|\Big| d\xi_1 d\xi. \tag{33}$$

From inequalities (30)–(33), we obtain

$$\max\Big(\|v_1^{(k)}(t,x)\|, \|u^{(k)}(t,x)\|, \Big|\Big|\frac{\partial v_1^{(k)}(t,x)}{\partial t}\Big|\Big|, \Big|\Big|\frac{\partial u^{(k)}(t,x)}{\partial t}\Big|\Big|\Big)$$

$$\leq (1+x)\max\big(\|\psi_1(t)\|, \|\dot{\psi}_1(t)\|, \|\psi_2(t)\|, \|\dot{\psi}_2(t)\|\big)$$

$$+\max(1,x)\int_0^x \max\Big(\|v^{(k)}(t,\xi)\|, ||W^{(k)}(t,\xi)||\Big) d\xi. \tag{34}$$

For the functions $V^{(k)}(t,x)$ and $\dfrac{\partial V^{(k)}(t,x)}{\partial t}$ we have the following estimate

$$\max\Big(\max_{t\in[0,T]} \|V^{(k)}(t,x)\|, \max_{t\in[0,T]} \Big|\Big|\frac{\partial V^{(k)}(t,x)}{\partial t}\Big|\Big|\Big)$$

$$\leq \hat{K}\max\Big(\max_{t\in[0,T]} \|v^{(k)}(t,x)\|, \max_{t\in[0,T]} \|W^{(k)}(t,x)\|, \max_{t\in[0,T]} \|v_1^{(k)}(t,x)\|, \max_{t\in[0,T]} \|u^{(k)}(t,x)\|,$$

$$\max_{t\in[0,T]} \Big|\Big|\frac{\partial v_1^{(k)}(t,x)}{\partial t}\Big|\Big|, \max_{t\in[0,T]} \Big|\Big|\frac{\partial u^{(k)}(t,x)}{\partial t}\Big|\Big|, \max_{t\in[0,T]} \|f(t,x)\|, \|\dddot{\varphi}(x)\Big), \tag{35}$$

where $\hat{K} = \tilde{K}\max\Big(\max_{i=\overline{2,7}} ||A_i||_0 + 1,$

$$\max_{x\in[0,\omega]} [||P'''(x)|| + 3||P''(x)|| + 3||P'(x)|| + ||S'''(x)|| + 3||S''(x)|| + 3||S'(x)||] + 1\Big).$$

Introduce the notation

$$\alpha^{(k)}(x) = \max\Big(\max_{t\in[0,T]} \|V^{(k+1)}(t,x) - V^{(k)}(t,x)\|, \max_{t\in[0,T]}\Big\|\frac{\partial V^{(k+1)}(t,x)}{\partial t} - \frac{\partial V^{(k)}(t,x)}{\partial t}\Big\|\Big),$$

$$\beta^{(k)}(x) = \max\Big(\max_{t\in[0,T]} \|v^{(k+1)}(t,x) - v^{(k)}(t,x)\|, \max_{t\in[0,T]} \|W^{(k+1)}(t,x) - W^{(k)}(t,x)\|\Big),$$

$$\gamma^{(k)}(x) = \max\Big(\max_{t\in[0,T]} \|v_1^{(k+1)}(t,x) - v_1^{(k)}(t,x)\|, \max_{t\in[0,T]} \|u^{(k+1)}(t,x) - u^{(k)}(t,x)\|\Big),$$

$$\theta^{(k)}(x) = \max\Big(\max_{t\in[0,T]}\Big\|\frac{\partial v_1^{(k+1)}(t,x)}{\partial t} - \frac{\partial v_1^{(k)}(t,x)}{\partial t}\Big\|, \max_{t\in[0,T]}\Big\|\frac{\partial u^{(k+1)}(t,x)}{\partial t} - \frac{\partial u^{(k)}(t,x)}{\partial t}\Big\|\Big).$$

Then, similarly to (30)–(35), we obtain

$$\max\{\gamma^{(k)}(x), \theta^{(k)}(x)\} \le \max(1,x)\int_0^x \beta^{(k)}(\xi)d\xi, \tag{36}$$

$$\alpha^{(k)}(x) \le \hat{K}\max\Big(\beta^{(k)}(x), \gamma^{(k)}(x), \theta^{(k)}(x)\Big), \tag{37}$$

$$\beta^{(k)}(x) \le \int_0^x \alpha^{(k-1)}(\xi)d\xi. \tag{38}$$

From (37), taking into account (38) and (36), we establish the main inequality

$$\alpha^{(k)}(x) \le \hat{K}\max(1,x,x^2)\int_0^x \alpha^{(k-1)}(\xi)d\xi \tag{39}$$

for all $x \in [0,\omega]$ and $k = 1,2,3,\dots$.

From (39) it follows

$$\alpha^{(k)}(x) \le \frac{(\hat{K}\cdot\max(1,\omega,\omega^2))^k}{k!}\max_{x\in[0,\omega]}\alpha^{(0)}(x). \tag{40}$$

The functional sequence $\{\alpha^{(k)}(x)\}$ converges uniformly to $\alpha^*(x)$ as $k \to \infty$ for all $x \in [0,\omega]$. This means that the functional sequences $\{\beta^{(k)}(x)\}$, $\{\gamma^{(k)}(x)\}$ and $\{\theta^{(k)}(x)\}$ also converge uniformly to $\beta^*(x)$, $\gamma^*(x)$ and as $k \to \infty$ $\theta^*(x)$, respectively for all $x \in [0,\omega]$. So, from here it follows that the functional sequences $\{V^k(t,x)\}$, $\{v^k(t,x)\}$, $\{W^k(t,x)\}$, $\{v_1^k(t,x)\}$ and $\{u^k(t,x)\}$ converge uniformly to $V^*(t,x)$, $v^*(t,x)$, $W^*(t,x)$, $v_1^*(t,x)$ and $u^*(t,x)$ as $k \to \infty$, respectively for all $(t,x) \in \Omega$. And also the functional sequences of partial derivatives $\{\frac{\partial v_1^k(t,x)}{\partial t}\}$, $\{\frac{\partial u^k(t,x)}{\partial t}\}$ converge uniformly to the corresponding limits $\frac{\partial v_1^*(t,x)}{\partial t}$, $\frac{\partial u^*(t,x)}{\partial t}$ as $k \to \infty$ for all $(t,x) \in \Omega$. The function $u^*(t,x)$ is a classical solution to the problem (1)–(5). The uniqueness of the solution to the problem (1)–(5) is proved by the method of contradiction.

Theorem 2 is proved.

The main condition for the unique solvability of the problem (1)–(5) is the unique solvability of two-point boundary value problem for the system of differential equations (19), (20). The criteria of well-posedness to boundary value problem for the system of differential equations with common two-point and integral conditions are established in terms of the initial data in [10].

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Асанова А.Т., Токмурзин Ж.С. ТӨРТІНШІ РЕТТІ ДЕРБЕС ТУЫНДЫЛЫ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ҮШІН ЕКІ НҮКТЕЛІ - БАСТАПҚЫ ШЕТТІК ЕСЕП ТУРАЛЫ

Төртінші ретті дербес туындылы дифференциалдық теңдеулер үшін екі нүктелі - бастапқы шеттік есеп зерттеледі. Біз төртінші ретті дербес туындылы дифференциалдық теңдеулер үшін бастапқы-екі нүктелі шеттік есептің классикалық шешімдерінің бар болуы және оның жуық шешімдерін табуға арналған әдістерді ұсынамыз. Төртінші ретті дербес туындылы дифференциалдық теңдеулер үшін бастапқы-екі нүктелі шеттік есептің классикалық шешімінің бар болуы мен жалғыздығының жеткілікті шарттары тағайындалған. Біз екі мәрте жаңа функциялар енгіземіз және қарастырылып отырған есепті алдымен екінші ретті гиперболалық теңдеулер жүйесі үшін интегралдық қатынастары бар бейлокал есепті, сосын бірінші ретті дифференциалдық теңдеулер жүйесі үшін екі нүктелі шеттік есепті қамтитын пара-пар есепке келтіреміз. Зерттелініп отырған есептің жуық шешімін табу алгоритмі тұрғызылады және оның жинақтылығы дәлелденеді.

Кілттік сөздер. Төртінші ретті дербес туындылы дифференциалдық теңдеулер, екі нүктелі - бастапқы шеттік есеп, бейлокал есеп, екінші ретті гиперболалық теңдеулер жүйесі, бірінші ретті дифференциалдық теңдеулер, шешілімділік, алгоритм.

Асанова А.Т., Токмурзин Ж.С. О ДВУХТОЧЕЧНО-НАЧАЛЬНОЙ КРАЕВОЙ ЗАДАЧЕ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ ЧЕТВЕРТОГО ПОРЯДКА

Исследуется двухточечно-начальная краевая задача для дифференциальных уравнений в частных производных четвертого порядка. Мы рассматриваем существование классических решений двухточечно-начальной краевой задачи для дифференциальных уравнений в частных производных четвертого порядка и предлагаем методы нахождения ее приближенных решений. Установлены достаточные условия существования и единственности классического решения двухточечно-начальной краевой задачи для дифференциальных уравнений в частных производных четвертого порядка. Мы дважды вводим новые неизвестные функции: мы сводим рассмотренную проблему сначала к эквивалентной задаче, состоящей из нелокальной задачи для системы гиперболических уравнений второго порядка с интегральными соотношениями, затем к двухточечной краевой задаче для системы дифференциальных уравнений первого порядка. Предложен алгоритм нахождения приближенного решения исследуемой задачи и доказана его сходимость.

Ключевые слова. Дифференциальные уравнения в частных производных четвертого порядка, двухточечно-начальная краевая задача, нелокальная задача, система гиперболических уравнений второго порядка, дифференциальные уравнения первого порядка, разрешимость, алгоритм.

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Non-local boundary value problems for one class of multidimensional hyperbolic equations

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Abstract. Numerous important physical phenomena in space are modelled as multidimensional hyperbolic equations. This paper proves the solvability of non-local boundary-value problems in a cylindrical domain for multidimensional hyperbolic equations with the wave operator. These problems are generalizations of the mixed problem, the Dirichlet problem, and the Poincaré problem.

Keywords. Multidimensional PDEs, hyperbolic equations, non-local problem, Bessel functions.

1 Introduction

Mathematical modelling of oscillatory processes is a key area of study in mathematical physics. Numerous important physical phenomena in space are modelled as multidimensional hyperbolic equations. For example, the vibration of an elastic string is often modelled as a hyperbolic equation (see [1]). In models of oscillations of elastic membranes in space, considering the deflection of the membrane as a function $u(x,t)$, $x = (x_1, ..., x_m)$, $m \geq 2$, and then applying the Hamilton principle, one obtains a multi-dimensional hyperbolic equation (see [2]). Also, in the mathematical modelling of electromagnetic and wave fields in space ([3]), the key feature is the properties of the medium. If the medium is non-conducting, the analysis leads to a multidimensional hyperbolic equation.

Despite the importance of multidimensional hyperbolic equations for applied work, their mathematical analysis is still a rather under-studied topic, mostly because of the analytical complexity of the multidimensional case. So far, good progress has been made in the analysis of local boundary-value problems for multidimensional hyperbolic equations in a cylindrical domain (see [4]–[8]).

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To the best of our knowledge, the non-local problems for these equations have not yet been subjected to analysis, with the exception of [9], which focuses on the simple wave equation.

This paper shows the existence of solutions of non-local boundary-value problems in a cylindrical domain for the more general multidimensional hyperbolic equations with the wave operator. These problems are generalizations of the mixed problem and of the Dirichlet and Poincaré problems.

2 Setup of the problem and main results

Let D_α be a cylindrical domain of the Euclidean space E_{m+1} of points $(x_1, ..., x_m, t)$, bounded by the cylinder $\Gamma = \{(x,t) : |x| = 1\}$, the planes $t = \alpha > 0$ and $t = 0$, where $|x|$ is the length of the vector $x = (x_1, ..., x_m)$. Let us denote, respectively, with Γ_α, S_α, and S_0 the parts of these surfaces that form the boundary ∂D of the domain D.

We study, in the domain D_α, the following multidimensional hyperbolic equation

$$Lu \equiv \Delta_x u - u_{tt} + \sum_{i=1}^{m} a_i(x,t) u_{x_i} + b(x,t) u_t + c(x,t) u = 0, \tag{1}$$

where Δ_x is the Laplace operator on the variables $x_1, ..., x_m$, $m \geq 2$.

Hereafter, it is useful to switch from the Cartesian coordinates $x_1, ..., x_m, t$ to the spherical ones $r,\ \theta_1, ..., \theta_{m-1},\ t,\ r \geq 0,\ 0 \leq \theta_1 < 2\pi,\ 0 \leq \theta_i \leq \pi,\ i = 2, 3, ..., m-1$.

Let us analyze the following non-local boundary-value problems.

Problem 1. *Find a solution of (1) in the domain D_α belonging to the class $C(\overline{D}_\alpha) \cap C^1(D_\alpha \cup S_0 \cup S_\alpha) \cap C^2(D_\alpha)$, and satisfying the boundary-value conditions*

$$\begin{cases} \beta_1 u(r,\theta,0) = \gamma_1 u(r,\theta,\alpha) + \varphi_1(r,\theta), \\ \beta_2 u_t(r,\theta,0) = \gamma_2 u_t(r,\theta,\alpha) + \varphi_2(r,\theta),\ u\Big|_{\Gamma_\alpha} = \psi(t,\theta). \end{cases} \tag{2}$$

Problem 2. *Find a solution of equation (1) in the domain D_α belonging to the class $C(\overline{D}_\alpha) \cap C^1(D_\alpha \cup S_0) \cap C^2(D_\alpha)$, and satisfying the boundary-value conditions*

$$u(r,\theta,0) = \varphi_1(r,\theta),\ \beta_1 u_t(r,\theta,0) = \gamma_1 u(r,\theta,\alpha) + \varphi_2(r,\theta),\ u\Big|_{\Gamma_\alpha} = \psi(t,\theta), \tag{3}$$

where $\beta_j,\ \gamma_j = const,\ \beta_j^2 + \gamma_j^2 \neq 0,\ j = 1, 2$.

These problems are generalizations of the mixed problem, and of the Dirichlet and Poincaré problems, that have been analyzed in [4]–[8]. Let us also note that the well-posedness of the above problems for the simple multidimensional wave equation has been shown in [9].

Let $\left\{Y_{n,m}^k(\theta)\right\}$ be a system of linearly independent spherical functions of order $n,\ 1 \leq k \leq k_n,\ (m-2)!n!k_n = (n+m-3)!(2n+m-2)$, $\theta = (\theta_1, ..., \theta_{m-1})$, and let $W_2^l(S_0),\ l = 0, 1, ...,$ be the Sobolev spaces.

The following lemmas, that we will use later, were shown in [10].

Lemma 1. *Let $f(r,\theta) \in W_2^l(S_0)$. If $l \geq m-1$, then the series*

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} f_n^k(r) Y_{n,m}^k(\theta), \tag{4}$$

as well as the series obtained through its differentiation of order $p \leq l-m+1$, converge absolutely and uniformly.

Lemma 2. *For $f(r,\theta) \in W_2^l(S_0)$, it is necessary and sufficient that the coefficients of the series (3) satisfy the inequalities*

$$|f_0^1(r)| \leq c_1, \ \ \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} n^{2l}|f_n^k(r)|^2 \leq c_2, \ \ c_1, c_2 = const.$$

Let us denote as $\tilde{a}_{in}^k(r,t)$, $a_{in}^k(r,t)$, $\tilde{b}_n^k(r,t)$, $\tilde{c}_n^k(r,t)$,ρ_n^k, $\bar{\varphi}_{1n}^k(r)$, $\bar{\varphi}_{2n}^k(r)$, $\psi_n^k(t)$, the coefficients of the series (4), respectively, of the functions $a_i(r,\theta,t)\rho(\theta)$, $a_i\frac{x_i}{r}\rho$, $b(r,\theta,t)\rho$, $c(r,\theta,t)\rho$, $\rho(\theta)$, $i = 1,...,m$, $\varphi_1(r,\theta)$, $\varphi_2(r,\theta)$, $\psi(t,\theta)$, whereas $\rho(\theta) \in C^\infty(H)$, and H is a unit sphere in E_m.

Let $a_i(r,\theta,t), \ b(r,\theta,t), \ c(r,\theta,t) \in W_2^l(D_\alpha) \subset C(\bar{D}_\alpha), \ i = 1,...,m, \ l \geq m+1, \ \varphi_1(r,\theta),$ $\varphi_2(r,\theta) \in W_2^p(S_0), \ \psi(t,\theta) \in W_2^p(\Gamma_\alpha), \ l > \frac{3m}{2}$.

Then, the following theorems hold.

Theorem 1. *If the following condition holds*

$$(\beta_1\gamma_2 + \beta_2\gamma_1)\cos\mu_{s,n}\alpha \neq \beta_1\beta_2 + \gamma_1\gamma_2, \ \ s = 1,2,\ldots, \tag{5}$$

then Problem 1 has a solution.

Theorem 2. *If the following relationship holds*

$$\gamma_1 \sin\mu_{s,n}\alpha \neq \mu_{s,n}\beta_1, \ \ s = 1,2,\ldots, \tag{6}$$

then Problem 2 has a solution. Here $\mu_{s,n}$ are positive nulls of the Bessel functions of the first kind $J_{n+\frac{(m-2)}{2}}(z)$.

Proof of Theorem 1. In the spherical coordinates equation (1) has the form

$$Lu \equiv u_{rr} + \frac{m-1}{r}u_r - \frac{\delta u}{r^2} - u_{tt} + \sum_{i=1}^{m} a_i(r,\theta,t)u_{x_i} + b(r,\theta,t)u_t + c(r,\theta,t)u = 0, \tag{7}$$

$$\delta \equiv -\sum_{j=1}^{m-1}\frac{1}{g_j sin^{m-j-1}\theta_j}\frac{\partial}{\partial\theta_j}\left(sin^{m-j-1}\theta_j\frac{\partial}{\partial\theta_j}\right), \ g_1 = 1, \ g_j = (sin\theta_1...sin\theta_{j-1})^2, \ j > 1.$$

It is known [10] that the spectrum of the operator δ consists of eigenvalues $\lambda_n = n(n+m-2)$, $n = 0, 1, ...$, to each of which correspond k_n orthonormalized eigenfunctions $Y^k_{n,m}(\theta)$.

We will look for the solution of Problem 1 in the form of the series

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} \bar{u}^k_n(r,t) Y^k_{n,m}(\theta), \tag{8}$$

where $\overline{u}^k_n(r,t)$ are the functions to be determined.

Substituting (8) into (7), multiplying the obtained expression by $\rho(\theta) \neq 0$, and then integrating over the unit sphere H, we obtain for $\overline{u}^k_n$:

$$\rho^1_0 \bar{u}^1_{0rr} - \rho^1_0 \bar{u}^1_{0tt} + \left(\frac{m-1}{r}\rho^1_0 + \sum_{i=1}^{m} a^1_{i0}\right)\bar{u}^1_{0r} + \tilde{b}^1_0 \bar{u}^1_{0t} + \tilde{c}^1_0 \bar{u}^1_0$$

$$+\sum_{n=1}^{\infty}\sum_{k=1}^{k_n}\left\{\rho^k_n \bar{u}^k_{nrr} - \rho^k_n \bar{u}^k_{ntt} + \left(\frac{m-1}{r}\rho^k_n + \sum_{i=1}^{m} a^k_{in}\right)\bar{u}^k_{nr} + \tilde{b}^k_n \bar{u}^k_{nt}\right.$$

$$\left. + \left[\tilde{c}^k_n - \lambda_n \frac{\rho^k_n}{r^2} + \sum_{i=1}^{m}(\tilde{a}^k_{in-1} - n a^k_{in})\right]\bar{u}^k_n\right\} = 0. \tag{9}$$

Next, let us analyze the infinite system of differential equations

$$\rho^1_0 \bar{u}^1_{0rr} - \rho^1_0 \bar{u}^1_{0tt} + \frac{(m-1)}{r}\rho^1_0 \bar{u}^1_{0r} = 0, \tag{10}$$

$$\rho^k_1 \bar{u}^k_{1rr} - \rho^k_1 \bar{u}^k_{1tt} + \frac{(m-1)}{r}\rho^k_1 \bar{u}^k_{1r} - \frac{\lambda_1}{r^2}\rho^k_1 \bar{u}^k_1 = -\frac{1}{k_1}\left(\sum_{i=1}^{m} a^1_{i0}\bar{u}^1_{0r} + \tilde{b}^1_0 \bar{u}^1_{0t} + \tilde{c}^1_0 \bar{u}^1_0\right), n = 1,$$

$$k = \overline{1, k_1}, \tag{11}$$

$$\rho^k_n \bar{u}^k_{nrr} - \rho^k_n \bar{u}^k_{ntt} + \frac{(m-1)}{r}\rho^k_n \bar{u}^k_{nr} - \frac{\lambda_n}{r^2}\rho^k_n \bar{u}^k_n = -\frac{1}{k_n}\sum_{k=1}^{k_{n-1}}\left\{\sum_{i=1}^{m} a^k_{in-1}\bar{u}^k_{n-1r}\right.$$

$$\left. + \tilde{b}^k_{n-1}\bar{u}^k_{n-1t} + \left[\tilde{c}^k_{n-1} + \sum_{i=1}^{m}(\tilde{a}^k_{in-2} - (n-1)a^k_{in-1})\right]\bar{u}^k_{n-1}\right\},\ k = \overline{1, k_n},\ n = 2, 3, \ldots. \tag{12}$$

Summing equation (11) from 1 to k_1, and equation (12) from 1 to k_n, and finally summing the obtained expressions to (10), we obtain equation (9).

Clearly, if $\left\{\bar{u}^k_n\right\}$, $k = \overline{1, k_n}$, $n = 2, 3, ...$, is the solution of the system (10)–(12), then it is also the solution of equation (9).

It is easy to see that each equation of the system (10)–(12) can be represented in the form

$$\bar{u}^k_{nrr} + \frac{(m-1)}{r}\bar{u}^k_{nr} - \frac{\lambda_n}{r^2}\bar{u}^k_n - \bar{u}^k_{ntt} = f^k_n(r,t), \tag{13}$$

where $f^k_n(r,t)$ are determined from the previous equations of this system, whereby $f^1_0(r,t) \equiv 0$.

Next, from the boundary-value condition (2), taking into account (8) and Lemma 1, we get

$$\beta_1\bar{u}^k_n(r,0) = \gamma_1\bar{u}^k_n(r,\alpha) + \bar{\varphi}^k_{1n}(r),\ \beta_2\bar{u}^k_{nt}(r,0) = \gamma_2\bar{u}^k_{nt}(r,\alpha) + \bar{\varphi}^k_{2n}(r),\ \bar{u}^k_n(1,t) = \psi^k_n(t),$$
$$k = \overline{1,k_n},\ n = 0,1,\dots. \tag{14}$$

In (13)–(14), substituting $\bar{\upsilon}^k_n(r,t) = \bar{u}^k_n(r,t) - \psi^k_n(t)$, we obtain

$$\bar{\upsilon}^k_{nrr} + \frac{(m-1)}{r}\bar{\upsilon}^k_{nr} - \frac{\lambda_n}{r^2}\upsilon^k_n - \bar{\upsilon}^k_{ntt} = \bar{f}^k_n(r,t), \tag{15}$$

$$\beta_1\bar{\upsilon}^k_n(r,0) = \gamma_1\bar{\upsilon}^k_n(r,\alpha) + \varphi^k_{1n}(r), \quad \beta_2\bar{\upsilon}^k_{nt}(r,0) = \gamma_2\bar{\upsilon}^k_{nt}(r,\alpha) + \varphi^k_{2n}(r), \quad \bar{\upsilon}^k_n(1,t) = 0,$$
$$k = \overline{1,k_n}, \quad n = 0,1,\dots, \tag{16}$$

$$\bar{f}^k_n(r,t) = f^k_n(r,t) + \psi^k_{ntt} + \frac{\lambda_n}{r^2}\psi^k_n(t),\ \varphi^k_{1n}(r) = \bar{\varphi}^k_{1n}(r) + \gamma_1\psi^k_n(\alpha) - \beta_1\psi^k_n(0),$$
$$\varphi^k_{2n}(r) = \bar{\varphi}^k_{2n}(r) + \gamma_2\psi^k_{nt}(\alpha) - \beta_2\psi^k_{nt}(0).$$

Then, substituting $\bar{\upsilon}^k_n(r,t) = r^{\frac{(1-m)}{2}}\upsilon^k_n(r,t)$, we can reduce the problem (15), (16) to the following problem

$$L\upsilon^k_n \equiv \upsilon^k_{nrr} - \upsilon^k_{ntt} + \frac{\bar{\lambda}_n}{r^2}\upsilon^k_n = \tilde{f}^k_n(r,t), \tag{17}$$

$$\beta_1\upsilon^k_n(r,0) = \gamma_1\upsilon^k_n(r,\alpha) + \tilde{\varphi}^k_{1n}(r), \quad \beta_2\upsilon^k_{nt}(r,0) = \gamma_2\upsilon^k_{nt}(r,\alpha) + \tilde{\varphi}^k_{2n}(r),$$
$$\upsilon^k_n(1,t) = 0,\ k = \overline{1,k_n},\ n = 0,1,\dots, \tag{18}$$

$$\bar{\lambda}_n = \frac{(m-1)(3-m) - 4\lambda_n}{4}, \quad \tilde{f}^k_n(r,t) = r^{\frac{(m-1)}{2}}\bar{f}^k_n(r,t), \quad \tilde{\varphi}^k_{jn}(r) = r^{\frac{(m-1)}{2}}\varphi^k_{jn}(r),\ j = 1,2.$$

Let us analyze the solution of the problem (17), (18) in the form

$$\upsilon^k_n(r,t) = \sum_{s=1}^{\infty} R_s(r)T_s(t), \tag{19}$$

whereby

$$\tilde{f}_n^k(r,t) = \sum_{s=1}^{\infty} a_{ns}^k(t) R_s(r),\ \tilde{\varphi}_{1n}^k(r) = \sum_{s=1}^{\infty} b_{ns}^k R_s(r),\ \tilde{\varphi}_{2n}^k(r) = \sum_{s=1}^{\infty} e_{ns}^k R_s(r). \quad (20)$$

Substituting (19) into (17), (18), and taking into account (20), we obtain

$$R_{srr} + \left(\frac{\overline{\lambda}_n}{r^2} + \mu\right) R_s = 0,\ 0 < r < 1, \quad (21)$$

$$R_s(1) = 0,\ |R_s(0)| < \infty, \quad (22)$$

$$T_{stt} + \mu T_s(t) = -a_{ns}^k(t),\ 0 < t < \alpha, \quad (23)$$

$$\beta_1 T_s(0) = \gamma_1 T_s(\alpha) + b_{ns}^k,\ \beta_2 T_{st}(0) = \gamma_2 T_{st}(\alpha) + e_{ns}^k. \quad (24)$$

The bounded solution of the problem (21), (22) is ([11])

$$R_s(r) = \sqrt{r} J_\nu(\mu_{s,n} r), \quad (25)$$

where $\nu = n + \frac{(m-2)}{2},\ \mu = \mu_{s,n}^2$.

The general solution of equation (23) can be represented in the form ([10])

$$\begin{gathered} T_{s,n}(t) = c_{1s} \cos \mu_{s,n} t + c_{2s} \sin \mu_{s,n} t + \frac{\cos \mu_{s,n} t}{\mu_{s,n}} \int_0^t a_{ns}^k(\xi) \sin \mu_{s,n} \xi d\xi \\ - \frac{\sin \mu_{s,n} t}{\mu_{s,n}} \int_0^t a_{ns}^k(\xi) \cos \mu_{s,n} \xi d\xi, \end{gathered} \quad (26)$$

where c_{1s}, c_{2s} are arbitrary constants. Satisfying the condition (24), we obtain the system of algebraic equations

$$\begin{cases} (\beta_1 - \gamma_1 \cos \mu_{s,n}\alpha) c_{1s} - \gamma_1 c_{2s} \sin \mu_{s,n}\alpha \\ = \dfrac{\gamma_1}{\mu_{s,n}} [\cos \mu_{s,n}\alpha \int_0^\alpha a_{ns}^k(\xi) \sin \mu_{s,n}\xi d\xi - \sin \mu_{s,n}\alpha \int_0^\alpha a_{ns}^k(\xi) \cos \mu_{s,n}\xi d\xi] + b_{ns}^k, \\ \gamma_2 c_{1s} \sin \mu_{s,n}\alpha + (\beta_2 - \gamma_2 \cos \mu_{s,n}\alpha) c_{2s} \\ = \dfrac{[e_{ns}^k - \gamma_2 (\sin \mu_{s,n}\alpha \int_0^\alpha a_{ns}^k(\xi) \sin \mu_{s,n}\xi d\xi + \cos \mu_{s,n}\alpha \int_0^\alpha a_{ns}^k(\xi) \cos \mu_{s,n}\xi d\xi)]}{\mu_{s,n}}, \end{cases} \quad (27)$$

which has the unique solution if the condition (5) is satisfied.

Substituting (25) into (20), we obtain

$$r^{-\frac{1}{2}}\tilde{f}_n^k(r,t)=\sum_{s=1}^{\infty}a_{ns}^k(t)J_\nu(\mu_{s,n}r),\ r^{-\frac{1}{2}}\tilde{\varphi}_{1n}^k(r)=\sum_{s=1}^{\infty}b_{ns}^kJ_\nu(\mu_{s,n}r),$$
$$r^{-\frac{1}{2}}\tilde{\varphi}_{2n}^k(r)=\sum_{s=1}^{\infty}e_{ns}^kJ_\nu(\mu_{s,n}r),\ 0<r<1. \tag{28}$$

The series (28) is a decomposition into the Fourier-Bessel series ([12]), if

$$a_{ns}^k(t)=2[J_{\nu+1}(\mu_{s,n})]^{-2}\int_0^1\sqrt{\xi}\tilde{f}_n^k(\xi,t)J_\nu(\mu_{s,n}\xi)d\xi,$$
$$b_{ns}^k=2[J_{\nu+1}(\mu_{s,n})]^{-2}\int_0^1\sqrt{\xi}\tilde{\varphi}_{1n}^k(\xi)J_\nu(\mu_{s,n}\xi)d\xi, \tag{29}$$
$$e_{ns}^k=2[J_{\nu+1}(\mu_{s,n})]^{-2}\int_0^1\sqrt{\xi}\tilde{\varphi}_{2n}^k(\xi)J_\nu(\mu_{s,n}\xi)d\xi,$$

where $\mu_{s,n}$, $s=1,2,...$, are positive nulls of the Bessel functions $J_\nu(z)$ ranked in the growing order.

From (25), (26) we obtain the solution of the problem (17), (18) in the form

$$\upsilon_n^k(r,t)=\sum_{s=1}^{\infty}\sqrt{r}T_{s,n}(t)J_{n+\frac{(m-2)}{2}}(\mu_{s,n}r), \tag{30}$$

where $a_{ns}^k(t)$, b_{ns}^k, e_{ns}^k are determined from (29), whereas c_{1s}, c_{2s} are determined from (27).

Hence, first having solved the problem (10), (14) ($n=0$), then the problem (11), (14)($n=1$), etc., we find sequentially all $\upsilon_n^k(r,t)$ from (30), $k=\overline{1,k_n},\ n=0,1,\dots$.

Therefore, in the domain D_α, it holds that

$$\int_H\rho(\theta)LudH=0. \tag{31}$$

Now, let $f(r,\theta,t)=R(r)\rho(\theta)T(t)$, where as $R(r)\in V_0,\ V_0$ is dense in $L_2((0,1))$, $\rho(\theta)\in C^\infty(H)$ is dense in $L_2(H)$, and $T(t)\in V_1,\ V_1$ is dense in $L_2((0,\alpha))$. Then $f(r,\theta,t)\in V$, $V=V_0\otimes H\otimes V_1$ is dense in $L_2(D_\alpha)$ (see [13]).

From this and from (31), it follows that

$$\int_{D_\alpha}f(r,\theta,t)LudD_\alpha=0$$

and

$$Lu = 0,\ \forall(r,\theta,t) \in D_\alpha.$$

Hence, the solution of Problem 1 has the form

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\left[\psi_n^k(t) + r^{\frac{(1-m)}{2}}\upsilon_n^k(r,t)\right]Y_{n,m}^k(\theta), \tag{32}$$

where $\upsilon_n^k(r,t)$ is found from (30).

Taking into account the formula $2J'_\nu(z) = J_{\nu-1}(z) - J_{\nu+1}(z)$ (see [12]), the estimates (see [14], [10])

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + 0\left(\frac{1}{z^{3/2}}\right),\ \nu \geq 0,$$

$$|k_n| \leq c_1 n^{m-2},\ \left|\frac{\partial^q}{\partial\theta_j^q}Y_{n,m}^k(\theta)\right| \leq c_2 n^{\frac{m}{2}-1+q},\ j = \overline{1, m-1},\ q = 0, 1, \dots,$$

the lemmas above, the restrictions on the coefficients of the equation (1) and on the given functions $\varphi_1(r,\theta),\ \varphi_2(r,\theta),\ \psi(t,\theta)$ we can show, as in [6]- [8], that the obtained solution (32) belongs to the class $C(\bar{D}_\alpha) \cap C^1(D_\alpha \cup S_0 \cup S_\alpha) \cap C^2(D_\alpha)$.

Therefore, we have established the solvability of Problem 1.

This completes the proof of Theorem 1.

Proof of Theorem 2. We will look for the solution of the problem (1), (3) in the form (8), where the functions $\bar{u}_n^k(r,t)$ are determined below. Then, analogously to the previous section, the functions $\bar{u}_n^k(r,t)$ satisfy the system of equations (10)–(12).

Next, from the boundary-value condition (3), taking into account (8), we obtain

$$\bar{u}_n^k(r,0) = \overline{\varphi}_{1n}^k(r),\ \beta_1\bar{u}_{nt}^k(r,0) = \gamma_1\overline{u}_n^k(r,\alpha) + \bar{\varphi}_{2n}^k(r),\ \bar{u}_n^k(1,t) = \psi_n^k(t),$$
$$k = \overline{1, k_n},\quad n = 0, 1, \dots. \tag{33}$$

As it was established earlier, each equation of the system (10)–(12) can be represented in the form (13).

Then, substituting $\bar{\upsilon}_n^k(r,t) = \bar{u}_n^k(r,t) - \psi_n^k(t)$, and then letting $\bar{\upsilon}_n^k(r,t) = r^{\frac{(1-m)}{2}}\upsilon_n^k(r,t)$ we reduce the problem (13), (33) to the problems

$$L\upsilon_n^k = \tilde{f}_n^k(r,t), \tag{17}$$

$$\upsilon_n^k(r,0) = \tilde{\varphi}_{1n}^k(r),\ \beta_1\upsilon_{nt}^k(r,0) = \gamma_1\upsilon_n^k(r,\alpha) + \tilde{\varphi}_{2n}^k(r),\ \upsilon_n^k(1,t) = 0,\ k = \overline{1, k_n},\ n = 0, 1, \dots, \tag{34}$$

where $\tilde{\varphi}_{1n}^k(r) = r^{\frac{(m-1)}{2}}(\bar{\varphi}_{1n}^k(r) - \psi_n^k(0)),\ \tilde{\varphi}_{2n}^k(r) = r^{\frac{(m-1)}{2}}(\bar{\varphi}_{2n}^k(r) + \gamma_1\psi_n^k(\alpha) - \beta_1\psi_{nt}^k(0))$.

If we look for the solution of the problem (17), (34) in the form (19), then we obtain the problem (21), (22) and to the problem for (23) with the data

$$T_s(0) = b_{ns}^k, \ \beta_1 T_{st}(0) = \gamma_1 T_s(\alpha) + e_{ns}^k. \tag{35}$$

Satisfying the general solution (26) of equation (23) with the boundary-value condition (35), we obtain

$$\begin{cases} c_{1s} = b_{ns}^k, \\ (\mu_{s,n}\beta_1 - \gamma_1 \operatorname{sh} \mu_{s,n}\alpha) c_{2s} \\ = \gamma_1 b_{ns}^k + \dfrac{\gamma_1}{\mu_{s,n}} (\cos \mu_{s,n}\alpha \int\limits_0^\alpha a_{ns}^k(\xi) \sin \mu_{s,n}\xi d\xi - \sin \mu_{s,n}\alpha \int\limits_0^\alpha a_{ns}^k(\xi) \cos \mu_{s,n}\xi d\xi) + e_{ns}^k, \end{cases} \tag{36}$$

from which the coefficients c_{1s}, c_{2s} are uniquely determined, if the condition (6) is satisfied.

Therefore, from (25), (26) we get the solution of the problem (17), (34) in the form (30), where $a_{ns}^k(t)$, b_{ns}^k, e_{ns}^k are found from (29), whereas c_{1s}, c_{2s} are found from (36).

The rest of the proof of Theorem 2 is completed just like in the case of Theorem 1.

Thus, the solvability of Problem 2 is shown.

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Алдашев С.А. КӨП ӨЛШЕМДІ ГИПЕРБОЛАЛЫҚ ТЕҢДЕУЛЕРДІҢ БІР КЛАСЫ ҮШІН ЛОКАЛДЫ ЕМЕС ШЕТТІК ЕСЕПТЕР

Кеңістіктегі көптеген маңызды физикалық құбылыстар көп өлшемді гиперболалық теңдеулермен моделденеді. Осы мақалада цилиндрлік аймақтағы толқындық операторлы көп өлшемді гиперболалық теңдеулер үшін локалды емес шеттік есептердің шешілімділігі дәлелденеді. Бұл есептер аралас есептің, Дирихле және Пуанкаре есептерінің жалпылауы болып табылады.

Кілттік сөздер. Көп өлшемді дербес туындылы теңдеулер, гиперболалық теңдеулер, локалды емес есеп, Бессель функциялары.

Алдашев С.А. НЕЛОКАЛЬНЫЕ КРАЕВЫЕ ЗАДАЧИ ДЛЯ ОДНОГО КЛАССА МНОГОМЕРНЫХ ГИПЕРБОЛИЧЕСКИХ УРАВНЕНИЙ

Многочисленные важные физические явления в пространстве моделируются многомерными гиперболическими уравнениями. В данной статье доказывается разрешимость нелокальных краевых задач в цилиндрической области для многомерных гиперболических уравнений с волновым оператором. Эти задачи являются обобщением смешанной задачи, задачи Дирихле и задачи Пуанкаре.

Ключевые слова. Многомерные уравнения в частных производных, гиперболические уравнения, нелокальная задача, функции Бесселя.

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Blow-up of solutions for nonlinear pseudo-parabolic Rockland equation on graded Lie groups

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Abstract. In this paper we study blow-up of solutions for the nonlinear pseudo-parabolic equation for Rockland operators on graded Lie groups. Also, we show Fujita type exponent for the pseudo-parabolic Rockland equation.

Keywords. Rockland operator, nonlinear pseudo-parabolic equation, graded Lie group, blow-up, Fujita exponent.

1 Introduction

In the paper we study blowing-up results of the nonlinear pseudo-parabolic equation

$$\begin{cases} u_t + a\mathcal{R}u_t + \mathcal{R}u = |u|^p + f(x,t), \ (x,t) \in \mathbb{G} \times \mathbb{R}_+ := \Omega, \\ u(x,0) = u_0(x) \geq 0, \quad x \in \mathbb{G}, \end{cases} \tag{1}$$

where $\mathcal{R}$ is a Rockland operator on a graded Lie group (see Section 1.1), and $a \geq 0$.

We start by recalling previous results. When $a = 0$ equation (1) restricts to the heat equation case, which was firstly considered by Fujita [1]. Namely, it is showed that if $0 < p < \frac{2}{N}$ then the Cauchy problem

$$\begin{cases} u_t - \Delta_x u = |u|^{1+p}, \ (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \ x \in \mathbb{R}^N, \end{cases} \tag{2}$$

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has blow-up a finite time. In papers [2]–[4], authors showed the blowing-up of solutions to the following initial value problem for the fractional Laplacian $(-\Delta)^s$:

$$\begin{cases} u_t + (-\Delta)^s u = a(x,t)|u|^{1+p}, \ \ (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \ \ x \in \mathbb{R}^N. \end{cases} \tag{3}$$

So, in [5], [6] it is considered non-existence results for parabolic equations on the Heisenberg groups with Kohn-Laplacian. For more information, we refer to [7]–[9] and references therein.

In this paper, we are focused on the nonzero coefficient case ($a > 0$) in equation (1), that is a pseudo-parabolic Rockland equation. As for motivation part, we note that the pseudo-parabolic equations appear in describing the nonlinear dispersive long wave unidirectional propagation [10], creep buckling [11], and the population aggregation [12]. For more information, we refer to the book [13].

The critical Fujita exponent determined as $p^* = 1 + \frac{2}{N}$ for the pseudo-parabolic equation in the Euclidean case was firstly established in the papers [14], [15]. In [16] authors studied the nonexistence of global solutions to the nonlinear pseudo-parabolic equation on the Heisenberg group

$$u_t + (-\Delta_H)^m u_t + (-\Delta_H)^m u = |u|^p, (\eta, t) \in H \times (0,\infty), \tag{4}$$

with the Cauchy data

$$u(\eta, 0) = u_0(\eta), \eta \in H, \tag{5}$$

where $m > 1, p > 1$, Δ_H is the Kohn-Laplace operator on (2×2)-dimensional Heisenberg group H. For more details, the reader is referred to [16] and references therein, [17]–[21].

1.1. Graded Lie groups. Now we give a very brief introduction to graded Lie groups [22]. Recall that $\mathbb{G}$ is a graded Lie group if its Lie algebra $\mathfrak{g}$ admits a gradation as

$$\mathfrak{g} = \bigoplus_{l=1}^{\infty} \mathfrak{g}_l,$$

where $\mathfrak{g}_l$ are vector subspaces of $\mathfrak{g}$ for all $l = 1, 2, ...$, but finitely many equal to $\{0\}$, and satisfying the following inclusion

$$[\mathfrak{g}_l, \mathfrak{g}_{l'}] \subset \mathfrak{g}_{l+l'}, \ \ \forall l, l' \in \mathbb{N}.$$

The group is called stratified if $\mathfrak{g}_1$ generates the whole of $\mathfrak{g}$ through these commutators. Let us fix a basis $\{X_1, ..., X_n\}$ of $\mathfrak{g}$ adapted to the gradation. By the exponential mapping $\exp_{\mathbb{G}} : \mathfrak{g} \to \mathbb{G}$ we get points in $\mathbb{G}$:

$$x = \exp_{\mathbb{G}}(x_1 X_1 + ... + x_n X_n).$$

A family of linear mappings

$$D_r = \mathrm{Exp}(A \ln r) = \sum_{k=0}^{\infty} \frac{1}{k!}(\ln(r)A)^k$$

is a family of dilations of $\mathfrak{g}$. Here A is a diagonalisable linear operator on the Lie algebra $\mathfrak{g}$ with positive eigenvalues. Every D_r is a morphism of $\mathfrak{g}$, i.e., D_r is a linear mapping from the Lie algebra $\mathfrak{g}$ to itself with the property

$$\forall X, Y \in \mathfrak{g}, r > 0, [D_r X, D_r Y] = D_r[X, Y],$$

here $[X, Y] := XY - YX$ is the Lie bracket. One can extend these dilations through the exponential mapping to the group $\mathbb{G}$ by

$$D_r(x) = rx := (r^{\nu_1}x_1, ..., r^{\nu_1}x_n),\ x = (x_1, ..., x_n) \in \mathbb{G}, r > 0,$$

where $\nu_1, ..., \nu_n$ are weights of the dilations. The sum of these weights of the form

$$Q := \mathrm{Tr}A = \nu_1 + \ldots + \nu_n$$

is called the homogeneous dimension of the graded Lie group $\mathbb{G}$. Also recall that the standard Lebesgue measure dx on $\mathbb{R}^n$ is the Haar measure for the graded Lie group $\mathbb{G}$. Also, in this note we denote a homogeneous quasi-norm on $\mathbb{G}$ by $q(x)$, which is a continuous non-negative function

$$\mathbb{G} \ni x \mapsto q(x) \in [0, \infty), \tag{6}$$

with the properties 1) $q(x) = q(x^{-1})\ \forall x \in \mathbb{G}$, 2) $q(\lambda x) = \lambda q(x)$ for all $x \in \mathbb{G}$ and $\lambda > 0$, and 3) $q(x) = 0 \Leftrightarrow x = 0$.

Moreover, the following property will be used in our proofs.

Property 1. *Let $\mathbb{G}$ be a graded Lie group with homogeneous dimension Q, $r > 0$ and dx be a Haar measure. Then, we have*

$$drx = r^Q dx.$$

For more detailed information, see, e.g. the book of Fischer and Ruzhansky [22].

The main object of this paper is equation (1). In this paper we are interested in pseudo-parabolic type equations. Without loss of generality, we study the case when $a = 1$.

2 Main results

In this section, we concern nonexistence of global weak solutions to the following nonlinear pseudo–parabolic equation

$$u_t(x, t) + \mathcal{R}u_t(x, t) + \mathcal{R}u(x, t) = |u(x, t)|^p + f(x, t), \quad (x, t) \in \mathbb{G} \times (0, \infty) := \Omega, \tag{7}$$

under the initial condition

$$u(x,0)=u_0(x), \quad x\in\mathbb{G}, \tag{8}$$

where $\mathcal{R}$ is a Rockland operator of k-th order on the graded Lie group $\mathbb{G}$, that is,

$$\mathcal{R}=\sum_{j=1}^{n}(-1)^{\frac{\nu_0}{\nu_j}}c_jX_j^{2\frac{\nu_0}{\nu_j}}.$$

We denote by $\mathfrak{C}^{k,1}_{x,t}(\Omega)$ the space of test functions φ with a compact support $supp\,\varphi\subset\Omega$ such that $\varphi,\partial_t\varphi$, $\mathcal{R}\varphi$ and $\partial_t\mathcal{R}\varphi$ are continuous functions on Ω with compact supports $supp\,\partial_t\varphi, supp\,\mathcal{R}\varphi, supp\,\partial_t\mathcal{R}\varphi\subset\Omega$.

Definition 1. *We say that u is a global weak solution to the problem* (7)–(8) *on Ω with the initial data $u(\cdot,0)=u_0(\cdot)\in L^1_{loc}(\mathbb{G})$, if $u\in L^p_{loc}(\Omega)$ and satisfies*

$$\int\limits_{\Omega}|u|^p\varphi dxdt+\int\limits_{\mathbb{G}}u_0(x)\varphi(x,0)dx+\int\limits_{\Omega}f\varphi dxdt$$

$$=-\int\limits_{\Omega}u\varphi_tdxdt+\int\limits_{\Omega}u(\mathcal{R}\varphi)_tdxdt-\int\limits_{\Omega}u\mathcal{R}\varphi dxdt+\int\limits_{\mathbb{G}}u_0(x)\mathcal{R}\varphi(x,0)dx \tag{9}$$

for any regular test function φ with $\varphi(\cdot,t)=0$ for large enough t.

For $R>0$, we define

$$\Gamma_R=\{(x,t)\in\Omega:0\leq t\leq R^{\alpha},\ 0\leq q(x)\leq R\}.$$

Theorem 1. *Assume that $\mathcal{R}$ is a Rockland operator of k-th order. Let $u_0\in L^1(\mathbb{G})$ and $f^-\in L^1(\Omega)$, where $f^-=\max\{-f,0\}$. Suppose that*

$$\int\limits_{\mathbb{G}}u_0dx+\lim_{R\to\infty}\inf\int\limits_{\Gamma_R}fdxdt>0. \tag{10}$$

If $1<p\leq p^*=1+\frac{k}{Q}$, then the problem (7)–(8) *does not admit any global weak solution.*

Proof. Suppose that u is a global weak solution to the problem (7)–(8). Then for any regular test function φ, we have

$$\int\limits_{\Omega}|u|^p\varphi dxdt+\int\limits_{\mathbb{G}}u_0(x)\varphi(x,0)dx+\int\limits_{\Omega}f\varphi dxdt$$

$$\leq \int_{\Omega} |u||\varphi_t|dxdt + \int\limits_{\Omega} |u||(\mathcal{R}\varphi)_t|dxdt - \int\limits_{\Omega} |u||\mathcal{R}\varphi|dxdt$$

$$+ \int\limits_{\mathbb{G}} |u_0(x)||\mathcal{R}\varphi(x,0)|dx. \tag{11}$$

Using the ε-Young inequality

$$ab \leq \varepsilon a^p + C(\varepsilon)b^{p'}, \;\; \frac{1}{p} + \frac{1}{p'} = 1, \;\; a, b \geq 0,$$

with parameters p and $p/(p-1)$, we obtain

$$\int\limits_{\Omega} |u||\varphi_t|dxdt \leq \varepsilon \int\limits_{\Omega} |u|^p \varphi dxdt + c_\varepsilon \int\limits_{\Omega} \varphi^{\frac{-1}{p-1}} |\varphi_t|^{\frac{p}{p-1}} dxdt, \tag{12}$$

for some positive constant c_ε.

Similarly, we have

$$\int\limits_{\Omega} |u||(\mathcal{R}\varphi)_t|dxdt \leq \varepsilon \int\limits_{\Omega} |u|^p \varphi dxdt + c_\varepsilon \int\limits_{\Omega} \varphi^{\frac{-1}{p-1}} |(\mathcal{R}\varphi)_t|^{\frac{p}{p-1}} dxdt \tag{13}$$

and

$$\int\limits_{\Omega} |u||\mathcal{R}\varphi|dxdt \leq \varepsilon \int\limits_{\Omega} |u|^p \varphi dtdv + c_\varepsilon \int\limits_{\Omega} \varphi^{\frac{-1}{p-1}} |\mathcal{R}\varphi|^{\frac{p}{p-1}} dtdv. \tag{14}$$

Using (11)–(14), for $\varepsilon > 0$ small enough, we get

$$\int\limits_{\Omega} |u|^p \varphi dxdt + \int\limits_{\Omega} u_0(x)\varphi(x,0)dx + \int\limits_{\Omega} f\varphi dxdt$$

$$\leq C\Big(A_p(\varphi) + B_p(\varphi) + C_p(\varphi) + \int\limits_{\mathbb{G}} |u_0(x)||\mathcal{R}\varphi(x,0)|dx\Big), \tag{15}$$

where

$$A_p(\varphi) = \int\limits_{\Omega} \varphi^{\frac{-1}{p-1}} |\varphi_t|^{\frac{p}{p-1}} dxdt, \tag{16}$$

$$B_p(\varphi) = \int\limits_{\Omega} \varphi^{\frac{-1}{p-1}} |(\mathcal{R}\varphi)_t|^{\frac{p}{p-1}} dxdt, \tag{17}$$

$$C_p(\varphi) = \int\limits_{\Omega} \varphi^{\frac{-1}{p-1}} |\mathcal{R}\varphi|^{\frac{p}{p-1}} dxdt. \tag{18}$$

Let $\Phi_1, \Phi_2 : \mathbb{R}_+ \to [0,1]$ be smooth nonincreasing functions such that

$$\Phi_i(\rho) := \begin{cases} 1, \text{ if } 0 \leq \rho \leq 1, \\ 0, \text{ if } \rho \geq 2, \end{cases} \tag{19}$$

for $i = 1, 2$.

Now, for $R > 0$, let us consider the test function

$$\varphi_R(x,t) = \Phi_1\left(\frac{q(x)}{R}\right)\Phi_2\left(\frac{t}{R^\alpha}\right),$$

for some $\alpha > 0$ to be defined later.

We observe that *supp* φ_R is a subset of

$$\Omega_R = \{(x,t) \in \Omega : 0 \leq t \leq 2R^\alpha,\ 0 \leq q(x) \leq 2R\},$$

while *supp* $\partial_t\varphi_R$, *supp* $\mathcal{R}\varphi_R$ and *supp* $\partial_t\mathcal{R}\varphi_R$ are subsets of

$$\Theta_R = \{(x,t) \in \Omega : R^\alpha \leq t \leq 2R^\alpha,\ R \leq q(x) \leq 2R\},$$

also, we put

$$\Gamma_R = \{(x,t) \in \Omega : 0 \leq t \leq R^\alpha,\ 0 \leq q(x) \leq R\}.$$

It follows that there is a positive constant $C > 0$, independent of R, such that for all $(x,t) \in \Omega_R$, we have

$$|\mathcal{R}_x\varphi_R(t,x)| \leq CR^{-k}\chi(t,x), \tag{20}$$

where $\chi(t,x)$ is a nonnegative function with a compact support in Ω_R, and

$$|\partial_t\mathcal{R}\varphi_R(t,x)| \leq CR^{-k-\alpha}\xi(t,x), \tag{21}$$

where $\xi(t,x)$ is a nonnegative function with a compact support in Ω_R.

Using (20) and (21), we get

$$A_p(\varphi) \leq CR^{\frac{-\alpha p}{p-1}}, \tag{22}$$

$$B_p(\varphi_R) \leq CR^{\frac{-(k+\alpha)p}{p-1}}, \tag{23}$$

$$C_p(\varphi_R) \leq CR^{\frac{-kp}{p-1}}. \tag{24}$$

Let us consider now the change of variables

$$\tilde{t} = R^{-\alpha} t, \quad \tilde{x} = R^{-1} x.$$

Put $\Sigma_R = \{x \in \mathbb{G} : R \le q(x) \le 2R\}$.

Using Property 1, (22), (23) and (24), from (15) we obtain

$$\int_\Omega |u|^p \varphi_R dx dt + \int_\Omega u_0(x)\varphi_R(x,0) dx dt + \int_\Omega f \varphi_R dx dt$$
$$\le C\Big(R^{\lambda_1} + R^{\lambda_2} + R^{\lambda_3} + \int_{\Sigma_R} |u_0(v)||\mathcal{R}\varphi_R(0,v)|dv\Big), \tag{25}$$

where

$$\lambda_1 = Q + \alpha - \frac{\alpha p}{p-1}$$

and

$$\lambda_2 = Q + \alpha - \frac{(k+\alpha)p}{p-1}$$

and

$$\lambda_3 = Q + \alpha - \frac{kp}{p-1}.$$

On the other hand, we have

$$\lim_{R\to\infty} \inf\Big(\int_\Omega |u|^p \varphi_R dx dt + \int_{\mathbb{G}} u_0(x)\varphi_R(x,0) dx + \int_\Omega f \varphi_R dx dt\Big)$$
$$\ge \lim_{R\to\infty} \inf \int_\Omega |u|^p \varphi_R dx dt + \lim_{R\to\infty} \inf \int_\Omega u_0(x)\varphi_R(x,0) dx + \lim_{R\to\infty} \inf \int_\Omega f \varphi_R dx dt.$$

Using the monotone convergence theorem, we get

$$\lim_{R\to\infty} \inf \int_\Omega |u|^p \varphi_R dx dt = \int_\Omega |u|^p dx dt.$$

Since $u_0 \in L^1(\Omega)$, by the dominated convergence theorem, we have

$$\lim_{R\to\infty} \inf \int_{\mathbb{G}} u_0(x)\varphi_R(x,0) dx = \int_{\mathbb{G}} u_0(x) dx.$$

Writing $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$, we have

$$\int_{\Omega} f\varphi_R dxdt = \int_{\Gamma_R} f dxdt + \int_{\Theta_R} f^+\varphi_R dxdt - \int_{\Theta_R} f^-\varphi_R dxdt$$

$$\geq \int_{\Gamma_R} f dxdt - \int_{\Theta_R} f^-\varphi_R dxdt.$$

Since $f^- \in L^1(\Omega)$, by the dominated convergence theorem we have

$$\lim_{R\to\infty} \int_{\Theta_R} f^-\varphi_R dxdt = 0.$$

Then

$$\lim_{R\to\infty} \inf \int_{\Omega} f\varphi_R dxdt \geq \lim_{R\to\infty} \inf \int_{\Gamma_R} f dxdt.$$

Now, we have

$$\lim_{R\to\infty} \inf\Big(\int_{\Omega} |u|^p\varphi_R dxdt + \int_{\Omega} u_0(x)\varphi_R(x,0)dx + \int_{\Omega} f\varphi_R dxdt\Big)$$

$$\geq \int_{\Omega} |u|^p dxdt + \ell,$$

where (10) has the form,

$$\ell = \int_{\Omega} u_0(x)dx + \lim_{R\to\infty} \inf \int_{\Gamma_R} f dxdt > 0.$$

By the definition of the limit inferior, for every $\varepsilon > 0$, there exists $R_0 > 0$ such that

$$\int_{\Omega} |u|^p\varphi_R dxdt + \int_{\Omega} u_0(x)\varphi_R(x,0)dx + \int_{\Omega} f\varphi_R dxdt$$

$$> \lim_{R\to\infty} \inf\Big(\int_{\Omega} |u|^p\varphi_R dxdt + \int_{\Omega} u_0(x)\varphi_R(x,0)dx + \int_{\Omega} f\varphi_R dxdt\Big) - \varepsilon$$

$$\geq \int_{\Omega} |u|^p dxdt + \ell - \varepsilon,$$

for every $R \geq R_0$. Taking $\varepsilon = \ell/2$, we obtain

$$\int_{\Omega} |u|^p \varphi_R dxdt + \int_{\Omega} u_0(x)\varphi_R(x,0)dx + \int_{\Omega} f\varphi_R dxdt$$
$$\geq \int_{\Omega} |u|^p dxdt + \frac{\ell}{2},$$

for every $R \geq R_0$. Then from (25), we have

$$\int_{\Omega} |u|^p dxdt + \frac{\ell}{2} \leq C\Big(R^{\lambda_1} + R^{\lambda_2} + R^{\lambda_3} + \int_{\Sigma_R} |u_0(x)||\mathcal{R}_x\varphi_R(x,0)|dx\Big), \tag{26}$$

for R large enough.

Now, we put $\alpha = k$ and require that $\lambda = \max\{\lambda_1, \lambda_2, \lambda_3\} \leq 0$, which is equivalent to $1 < p \leq 1 + \frac{k}{Q}$. We distinguish two cases.

- Case 1. If $1 < p < 1 + \frac{k}{Q}$.

In this case, letting $R \to \infty$ in (26) and using the dominated convergence theorem, we obtain

$$\int_{\Omega} |u|^p dxdt + \frac{\ell}{2} \leq 0,$$

which is a contradiction with $\ell > 0$.

- Case 2. If $p = 1 + \frac{k}{Q}$.

In this case, from (26), we obtain

$$\int_{\Omega} |u|^p dxdt \leq C < \infty. \tag{27}$$

Using the Hölder inequality with parameters p and $p/(p-1)$ from (11) we obtain

$$\int\limits_{\Omega} |u|^p dxdt + \frac{\ell}{2} \leq C\Big(\int\limits_{\Theta_R} |u|^p \varphi_R dxdt\Big)^{\frac{1}{p}}.$$

Letting $R \to \infty$ in the above inequality and using (27), we obtain

$$\int\limits_{\Omega} |u|^p dxdt + \frac{\ell}{2} = 0.$$

This contradiction completes the proof of the theorem.

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Касымов А. ПСЕВДО-ПАРАБОЛАЛЫҚ РОКЛАНД ТЕҢДЕУІ ҮШІН СЫЗЫҚТЫ ЕМЕС ТЕҢДЕУЛЕРДІҢ ЛИ МЕЖЕЛЕНГЕН ТОПТАРЫНДАҒЫ ШЕШІМДЕРІНІҢ БҰЗЫЛЫМДЫЛЫҒЫ

Бұл жұмыста біз Рокланд операторлары үшін сызықты емес псевдо-параболалық теңдеудің Ли межеленген топтарындағы шешімдерінің бұзылымдығын зерттейміз. Оған қоса, біз псевдо-параболалық Рокланд теңдеуі үшін Фудзита тектес экспонентаны көрсетеміз.

Кілттік сөздер. Рокланд операторы, сызықты емес псевдо-параболалық теңдеу, Ли межеленген тобы, шешімнің бұзылымдығы, Фудзита экспонентасы.

Касымов А. РАЗРУШИМОСТЬ РЕШЕНИЙ НЕЛИНЕЙНЫХ УРАВНЕНИЙ ДЛЯ ПСЕВДО-ПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ РОКЛАНДА НА ГРАДУИРОВАННЫХ ГРУППАХ ЛИ

В настоящей работе мы изучаем разрушимость решений нелинейного псевдопараболического уравнения для операторов Рокланда на градуированных группах Ли. Также мы показываем экспоненту типа Фудзиты для псевдопараболического уравнения Рокланда.

Ключевые слова. Оператор Рокланда, нелинейное псевдо-параболическое уравнение, градуированная группа Ли, разрушимость решения, экспонента Фудзита.

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