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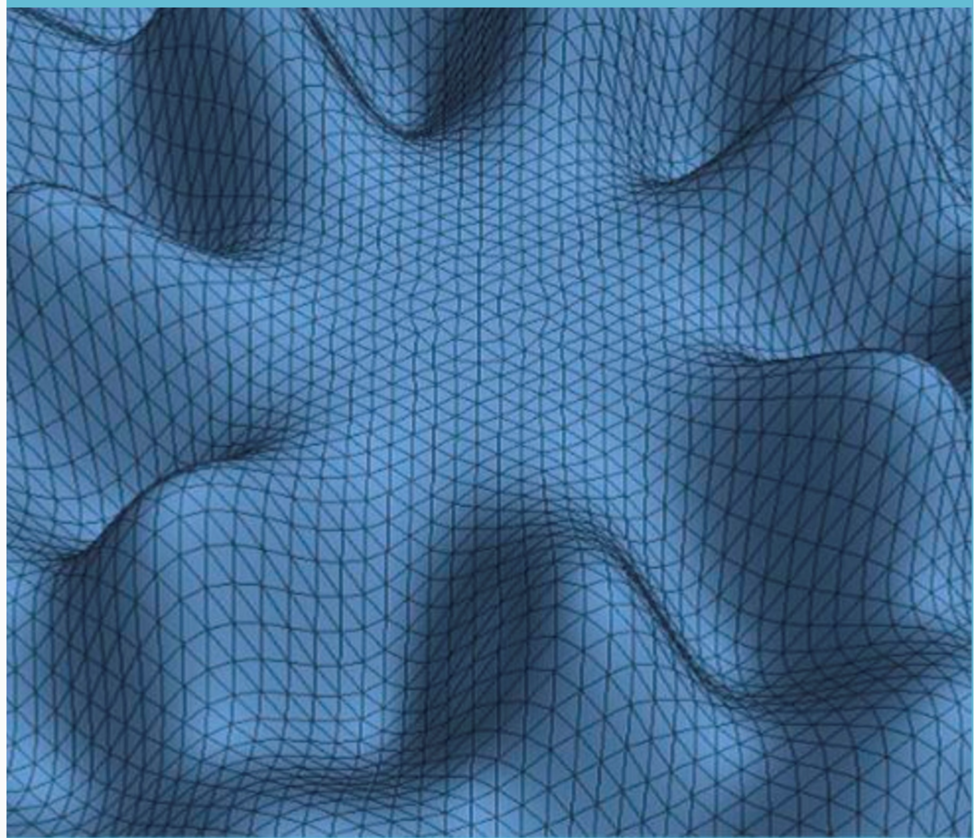

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An algorithm of solving linear boundary value problem for the Fredholm integro-differential equation with impulse effects

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Abstract. Based on parameterization method an algorithm of solving boundary value problem for the Fredholm integro-differential equation with impulse effects is proposed. Numerical implementation of the algorithm is offered.

Keywords. Fredholm integro-differential equation with impulse effects, parameterization method, fourth order Runge-Kutta method, Simpson method.

1 Introduction

In this paper, we study a linear two-point boundary value problem for the Fredholm integro-differential equation with impulse effects at fixed times:

$$\frac{dx}{dt} = A(t)x + \varphi(t)\int_0^T \psi(\tau)x(\tau)d\tau + f(t),\ t \neq \theta_j, j = 1, 2, t \in (0, T),\ x \in R^n, \tag{1}$$

$$B_0x(0) + C_0x(T) = d_0, \quad d_0 \in R^n, \tag{2}$$

$$B_1x(\theta_1 - 0) + C_1x(\theta_1 + 0) = d_1, \quad d_1 \in R^n, \tag{3}$$

$$B_2x(\theta_2 - 0) + C_2x(\theta_2 + 0) = d_2, \quad d_2 \in R^n, \tag{4}$$

where $0 = \theta_0 < \theta_1 < \theta_2 < \theta_3 = T$, $(n \times n)$-matrices $A(t)$, $\varphi(t)$, $\psi(t)$, and n-vector-function $f(t)$ are piecewise continuous on $[0, T]$ with possible discontinuities at the points $t = \theta_j$, $j = 1, 2$.

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Let $PC([0,T],\theta_1,\theta_2,R^n)$ be the space of vector-functions $x(t)$, piecewise continuous on $[0,T]$ with possible discontinuities on $t=\theta_1$, $t=\theta_2$, and the norm be $||x||_1 = \max\limits_{i=\overline{1,3}} \sup\limits_{t\in[\theta_{i-1};\theta_i)} ||x(t)||$.

The solution of the problem (1)–(4) is a piecewise continuously differentiable function $x(t)\in PC([0,T],\theta_1,\theta_2,R^n)$ satisfying integro-differential equation (1), boundary condition (2) and conditions of impulse effects (3), (4).

Integro-differential equations frequently arise in applications being the mathematical models of some processes in physics, biology, chemistry, economy, etc. Their role in the study of processes with aftereffects was noted in monographs [3], [4], and the overview of early works devoted to the initial and boundary value problems for the integro-differential equations was provided as well. Periodic and boundary value problems for impulsive integro-differential equations were studied by numerous authors. For the various aspects of the qualitative theory and approximate methods for the integro-differential equations without or with the impulse effects, and their applications we refer to [1]–[13]. By parameterization method [14] linear boundary value problem for the Fredholm integro-differential equation with impulse effects was studied in [10]. This method is based on the dividing an interval $[0,T]$ into N parts and introducing additional parameters. While applying the method to the problem for impulsive integro-differential equations, the necessity of solving an intermediate problem also arises. The intermediate problem here is a special Cauchy problem for the system of integro-differential equations with parameters. But unlike the intermediate problems of above mentioned methods, the special Cauchy problem is always uniquely solvable for sufficiently small partition step. This property of the intermediate problem allows to establish in [10] the necessary and sufficient conditions for the solvability and the unique solvability of the problem considered.

The goal of this paper is to specify numerical algorithms for finding a solution of the linear boundary value problem for the Fredholm integro-differential equation with impulse effects. To reach the goal we use parameterization method [14].

A partition of an interval $[0,T]$ into 3 parts with the points $\theta_j, j=1,2$, we denote by $\Delta_3(\theta):\ [0,T)=[0,\theta_1)\cup[\theta_1,\theta_2)\cup[\theta_2,T)$. The restriction of the function $x(t)$ to the r-th interval $[\theta_{r-1},\theta_r)$ is denoted by $x_r(t)$, i.e. $x_r(t)=x(t)$, $t\in[\theta_{r-1},\theta_r)$, $r=1,2,3$.

Introducing parameters $\lambda_1=x_1(0)$, $\lambda_2=x_2(\theta_1)$, $\lambda_3=x_3(\theta_2)$ and making the replacement of the function

$$u_1(t)=x_1(t)-\lambda_1,\quad t\in[0,\theta_1),$$

$$u_2(t)=x_2(t)-\lambda_2,\quad t\in[\theta_1,\theta_2),$$

$$u_3(t)=x_3(t)-\lambda_3,\quad t\in[\theta_2,T),$$

we obtain the system of integro-differential equations with parameters:

$$\frac{du_1}{dt} = A(t)[u_1 + \lambda_1] + \varphi(t)\Big[\int\limits_0^{\theta_1} \psi(\tau)[u_1(\tau) + \lambda_1]d\tau$$

$$+ \int\limits_{\theta_1}^{\theta_2} \psi(\tau)[u_2(\tau) + \lambda_2]d\tau + \int\limits_{\theta_2}^{T} \psi(\tau)[u_3(\tau) + \lambda_3]d\tau\Big] + f(t), \quad t \in [0, \theta_1), \tag{5}$$

$$\frac{du_2}{dt} = A(t)[u_2 + \lambda_2] + \varphi(t)\Big[\int\limits_0^{\theta_1} \psi(\tau)[u_1(\tau) + \lambda_1]d\tau$$

$$+ \int\limits_{\theta_1}^{\theta_2} \psi(\tau)[u_2(\tau) + \lambda_2]d\tau + \int\limits_{\theta_2}^{T} \psi(\tau)[u_3(\tau) + \lambda_3]d\tau\Big] + f(t), \quad t \in [\theta_1, \theta_2), \tag{6}$$

$$\frac{du_3}{dt} = A(t)[u_3 + \lambda_3] + \varphi(t)\Big[\int\limits_0^{\theta_1} \psi(\tau)[u_1(\tau) + \lambda_1]d\tau$$

$$+ \int\limits_{\theta_1}^{\theta_2} \psi(\tau)[u_2(\tau) + \lambda_2]d\tau + \int\limits_{\theta_2}^{T} \psi(\tau)[u_3(\tau) + \lambda_3]d\tau\Big] + f(t), \quad t \in [\theta_2, T), \tag{7}$$

initial conditions at the beginning points of subintervals:

$$u_1(0) = 0, \tag{8}$$

$$u_2(\theta_1) = 0, \tag{9}$$

$$u_3(\theta_2) = 0, \tag{10}$$

the boundary condition:

$$B_0\lambda_1 + C_0\lambda_3 + C_0 \lim_{t\to T-0} u_3(t) = d_0, \tag{11}$$

and conditions of impulse effects:

$$B_1\lambda_1 + B_1 \lim_{t\to\theta_1-0} u_1(t) + C_1\lambda_2 = d_1, \tag{12}$$

$$B_2\lambda_2 + B_2 \lim_{t\to\theta_2-0} u_2(t) + C_2\lambda_3 = d_2. \tag{13}$$

The solution to the boundary value problem (5)–(13) is a pair $(\lambda^*, u^*[t])$ with $\lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*) \in R^{3n}$ and $u^*[t] = (u_1^*(t), u_2^*(t), u_3^*(t))$, where the functions $u_1^*(t)$, $u_2^*(t)$, $u_3^*(t)$ are continuous on $[\theta_0, \theta_1)$, $[\theta_1, \theta_2)$, $[\theta_2, \theta_3)$, respectively, satisfy the system of integro-differential equations (5)–(7), initial conditions (8)–(10) and additional conditions (11)–(13) with $\lambda_1 = \lambda_1^*,\ \lambda_2 = \lambda_2^*,\ \lambda_3 = \lambda_3^*$.

The problem (5)–(10) is called the special Cauchy problem for the system of integro-differential equations with parameters.

Using the fundamental matrix $X_r(t)$ of the differential equation $\frac{dx}{dt} = A(t)x$ on $[\theta_{r-1}, \theta_r]$, we reduce the special Cauchy problem for the system of integro-differential equations with parameters (5)–(10) to the equivalent system of integro-differential equations:

$$u_1(t,\lambda) = X(t)\int\limits_0^t X^{-1}(\tau)\Big[A(\tau)\lambda_1 + \varphi(\tau)\Big(\int\limits_0^{\theta_1}\psi(s)[u_1(s)+\lambda_1]ds$$

$$+\int\limits_{\theta_1}^{\theta_2}\psi(s)[u_2(s)+\lambda_2]ds + \int\limits_{\theta_2}^{T}\psi(s)[u_3(s)+\lambda_3]ds\Big) + f(\tau)\Big]d\tau, t \in [0,\theta_1), \tag{14}$$

$$u_2(t,\lambda) = X(t)\int\limits_{\theta_1}^t X^{-1}(\tau)\Big[A(\tau)\lambda_2 + \varphi(\tau)\Big(\int\limits_0^{\theta_1}\psi(s)[u_1(s)+\lambda_1]ds$$

$$+\int\limits_{\theta_1}^{\theta_2}\psi(s)[u_2(s)+\lambda_2]ds + \int\limits_{\theta_2}^{T}\psi(s)[u_3(s)+\lambda_3]ds\Big) + f(\tau)\Big]d\tau,\ t \in [\theta_1,\theta_2), \tag{15}$$

$$u_3(t,\lambda) = X(t)\int\limits_{\theta_2}^t X^{-1}(\tau)\Big[A(\tau)\lambda_3 + \varphi(\tau)\Big(\int\limits_0^{\theta_1}\psi(s)[u_1(s)+\lambda_1]ds$$

$$+\int\limits_{\theta_1}^{\theta_2}\psi(s)[u_2(s)+\lambda_2]ds + \int\limits_{\theta_2}^{T}\psi(s)[u_3(s)+\lambda_3]ds\Big) + f(\tau)\Big]d\tau,\ t \in [\theta_2,T). \tag{16}$$

Further, we consider the auxiliary Cauchy problems for ordinary differential equations on subintervals:

$$\frac{dz}{dt} = A(t)z + P(t), \quad z(0) = 0, \quad t \in [0,\theta_1), \tag{17}$$

$$\frac{dz}{dt} = A(t)z + P(t), \quad z(\theta_1) = 0, \quad t \in [\theta_1,\theta_2), \tag{18}$$

$$\frac{dz}{dt} = A(t)z + P(t), \quad z(\theta_2) = 0, \quad t \in [\theta_2, T), \tag{19}$$

where $P(t)$ is a square matrix or vector of the dimension n, continuous on $[0, \theta_1]$, $[\theta_1, \theta_2]$ or $[\theta_2, T]$.

Their solutions we denote by $a_1(P, t), a_2(P, t)$ and $a_3(P, t)$, respectively. Now, we set

$$\mu = \int_0^{\theta_1} \psi(s)u_1(s)ds + \int_{\theta_1}^{\theta_2} \psi(s)u_2(s)ds + \int_{\theta_2}^{T} \psi(s)u_3(s)ds,$$

$$\hat{\psi}_1 = \int_0^{\theta_1} \psi(s)ds, \qquad \hat{\psi}_2 = \int_{\theta_1}^{\theta_2} \psi(s)ds, \qquad \hat{\psi}_3 = \int_{\theta_2}^{T} \psi(s)ds,$$

and re-write the system of integro-differential equations (14)–(16) as follows:

$$u_1(t, \lambda) = a_1(A, t)\lambda_1 + a_1(\varphi, t)(\mu + \hat{\psi}_1\lambda_1 + \hat{\psi}_2\lambda_2 + \hat{\psi}_3\lambda_3) + a_1(f, t), \quad t \in [0, \theta_1), \tag{20}$$

$$u_2(t, \lambda) = a_2(A, t)\lambda_2 + a_2(\varphi, t)(\mu + \hat{\psi}_1\lambda_1 + \hat{\psi}_2\lambda_2 + \hat{\psi}_3\lambda_3) + a_2(f, t), \quad t \in [\theta_1, \theta_2), \tag{21}$$

$$u_3(t) = a_3(A, t)\lambda_3 + a_3(\varphi, t)(\mu + \hat{\psi}_1\lambda_1 + \hat{\psi}_2\lambda_2 + \hat{\psi}_3\lambda_3) + a_3(f, t), \quad t \in [\theta_2, T). \tag{22}$$

Multiplying both sides of (20)–(22) by $\psi(s)$, integrating on subintervals $[0, \theta_1]$, $[\theta_1, \theta_2]$ and $[\theta_1, T]$, summing up both sides, we obtain the system of linear algebraic equations with respect to μ:

$$\mu = \hat{\psi}_1(A)\lambda_1 + \hat{\psi}_1(\varphi)[\mu + \hat{\psi}_1\lambda_1 + \hat{\psi}_2\lambda_2 + \hat{\psi}_3\lambda_3] + \hat{\psi}_1(f)$$
$$+\hat{\psi}_2(A)\lambda_2 + \hat{\psi}_2(\varphi)[\mu + \hat{\psi}_1\lambda_1 + \hat{\psi}_2\lambda_2 + \hat{\psi}_3\lambda_3] + \hat{\psi}_2(f) + \hat{\psi}_3(A)\lambda_3$$
$$+\hat{\psi}_3(\varphi)[\mu + \hat{\psi}_1\lambda_1 + \hat{\psi}_2\lambda_2 + \hat{\psi}_3\lambda_3] + \hat{\psi}_3(f),$$

where

$$\hat{\psi}_1(P) = \int_0^{\theta_1} \psi(s)a_1(P, s)ds, \quad \hat{\psi}_2(P) = \int_{\theta_1}^{\theta_2} \psi(s)a_2(P, s)ds,$$

$$\hat{\psi}_3(P) = \int_{\theta_2}^{T} \psi(s)a_3(P, s)ds.$$

We re-write this expression in the following form

$$\mu = G(\Delta_3)\mu + D_1(\Delta_3)\lambda_1 + D_2(\Delta_3)\lambda_2 + D_3(\Delta_3)\lambda_3 + g(f, \Delta_3), \tag{23}$$

with $(n \times n)$-matrices

$$G(\Delta_3) = \hat{\psi}_1(\varphi) + \hat{\psi}_2(\varphi) + \hat{\psi}_3(\varphi),$$
$$D_1(\Delta_3) = [\hat{\psi}_1(A) + \hat{\psi}_1(\varphi)\hat{\psi}_1 + \hat{\psi}_2(\varphi)\hat{\psi}_1 + \hat{\psi}_3(\varphi)\hat{\psi}_1],$$
$$D_2(\Delta_3) = [\hat{\psi}_2(A) + \hat{\psi}_1(\varphi)\hat{\psi}_2 + \hat{\psi}_2(\varphi)\hat{\psi}_2 + \hat{\psi}_3(\varphi)\hat{\psi}_2],$$
$$D_3(\Delta_3) = [\hat{\psi}_3(A) + \hat{\psi}_1(\varphi)\hat{\psi}_3 + \hat{\psi}_2(\varphi)\hat{\psi}_3 + \hat{\psi}_3(\varphi)\hat{\psi}_3],$$

and vectors of the dimension n:

$$g(f, \Delta_3) = [\hat{\psi}_1(f) + \hat{\psi}_2(f) + \hat{\psi}_3(f)].$$

We represent the system (23) in the next form:

$$[I - G(\Delta_3)]\mu = D_1(\Delta_3)\lambda_1 + D_2(\Delta_3)\lambda_2 + D_3(\Delta_3)\lambda_3 + g(f, \Delta_3), \tag{24}$$

where I is an identity matrix of the dimention n.

Further, we assume that the matrix $I - G(\Delta_3)$ is invertible and

$$[I - G(\Delta_3)]^{-1} = M(\Delta_3).$$

Then, from (24) we obtain the following expression for determining μ:

$$\mu = M(\Delta_3)D_1(\Delta_3)\lambda_1 + M(\Delta_3)D_2(\Delta_3)\lambda_2$$
$$+M(\Delta_3)D_3(\Delta_3)\lambda_3 + M(\Delta_3)g(f, \Delta_3). \tag{25}$$

The special Cauchy problem (5)–(10) is equivalent to the system of integro-differential equations (14)–(16). By virtue of the kernel degeneracy, this system is equivalent to the system of algebraic equations (25) with respect to $\mu \in R^n$. Substituting the right-hand side of (25) into equations (14)–(16), instead of μ, and taking into account the notation, we get the representation of functions $u_r(t, \lambda)$, $r = \overline{1, 2, 3}$, via λ_j, $j = \overline{1, 3}$:

$$u_1(t, \lambda) = a_1(A, t)\lambda_1 + a_1(\varphi, t)(M(\Delta_3)D_1(\Delta_3)\lambda_1$$
$$+M(\Delta_3)D_2(\Delta_3)\lambda_2 + M(\Delta_3)D_3(\Delta_3)\lambda_3$$
$$+M(\Delta_3)g(f, \Delta_3) + \hat{\psi}_1\lambda_1 + \hat{\psi}_2\lambda_2 + \hat{\psi}_3\lambda_3) + a_1(f, t), \quad t \in [0, \theta_1), \tag{26}$$
$$u_2(t, \lambda) = a_2(A, t)\lambda_2 + a_2(\varphi, t)(M(\Delta_3)D_1(\Delta_3)\lambda_1$$
$$+M(\Delta_3)D_2(\Delta_3)\lambda_2 + M(\Delta_3)D_3(\Delta_3)\lambda_3 + M(\Delta_3)g(f, \Delta_3)$$
$$+\hat{\psi}_1\lambda_1 + \hat{\psi}_2\lambda_2 + \hat{\psi}_3\lambda_3) + a_2(f, t), \quad t \in [\theta_1, \theta_2), \tag{27}$$
$$u_3(t, \lambda) = a_3(A, t)\lambda_3 + a_3(\varphi, t)(M(\Delta_3)D_1(\Delta_3)\lambda_1$$

$$+M(\Delta_3)D_2(\Delta_3)\lambda_2 + M(\Delta_3)D_3(\Delta_3)\lambda_3 + M(\Delta_3)g(f,\Delta_3)$$
$$+\hat{\psi}_1\lambda_1 + \hat{\psi}_2\lambda_2 + \hat{\psi}_3\lambda_3) + a_3(f,t), \quad t \in [\theta_2, T). \tag{28}$$

Substituting the right-hand side of (26)–(28) into the boundary condition (11) and the conditions of impulse actions (12) and (13), we obtain the following system of linear algebraic equations with respect to parameters $\lambda_j,\ j=\overline{1,3}$:

$$B_0\lambda_1 + C_0\lambda_3 + C_0[a_3(A,T)\lambda_3 + a_3(\varphi,T)(M(\Delta_3)D_1(\Delta_3)\lambda_1$$
$$+M(\Delta_3)D_2(\Delta_3)\lambda_2 + M(\Delta_3)D_3(\Delta_3)\lambda_3 + \hat{\psi}_1\lambda_1 + \hat{\psi}_2\lambda_2 + \hat{\psi}_3\lambda_3)]$$
$$= d_0 - C_0[a_3(\varphi,T)M(\Delta_3)g(f,\Delta_3) + a_3(f,T)], \tag{29}$$
$$B_1\lambda_1 + B_1[a_1(A,\theta_1)\lambda_1 + a_1(\varphi,\theta_1)(M(\Delta_3)D_1(\Delta_3)\lambda_1$$
$$+M(\Delta_3)D_2(\Delta_3)\lambda_2 + M(\Delta_3)D_3(\Delta_3)\lambda_3 + \hat{\psi}_1\lambda_1 + \hat{\psi}_2\lambda_2 + \hat{\psi}_3\lambda_3)]$$
$$+C_1\lambda_2 = d_1 - B_1[a_1(\varphi,\theta_1)M(\Delta_3)g(f,\Delta_3) + a_1(f,\theta_1)], \tag{30}$$
$$B_2\lambda_2 + B_2[a_2(A,\theta_2)\lambda_2 + a_2(\varphi,\theta_2)(M(\Delta_3)D_1(\Delta_3)\lambda_1$$
$$+M(\Delta_3)D_2(\Delta_3)\lambda_2 + M(\Delta_3)D_3(\Delta_3)\lambda_3 + \hat{\psi}_1\lambda_1 + \hat{\psi}_2\lambda_2 + \hat{\psi}_3\lambda_3)]$$
$$+C_2\lambda_3 = d_2 - B_2[a_2(\varphi,\theta_2)M(\Delta_3)g(f,\Delta_3) + a_2(f,\theta_2)]. \tag{31}$$

The system (29)–(31) can be written as

$$Q_*(\Delta_3)\lambda = -F_*(\Delta_3), \quad \lambda \in R^{3n}, \tag{32}$$

where

$$Q_*(\Delta_3)\lambda = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix},$$

with

$$Q_{11} = B_0 + C_0[a_3(\varphi,T)(M(\Delta_3)D_1(\Delta_3) + \hat{\psi}_1)],$$
$$Q_{12} = C_0[a_3(\varphi,T)(M(\Delta_3)D_2(\Delta_3) + \hat{\psi}_2)],$$
$$Q_{13} = C_0 + C_0[a_3(A,T) + a_3(\varphi,T)(M(\Delta_3)D_3(\Delta_3) + \hat{\psi}_3)],$$
$$Q_{21} = B_1 + B_1[a_1(A,\theta_1) + a_1(\varphi,\theta_1)(M(\Delta_3)D_1(\Delta_3) + \hat{\psi}_1)],$$
$$Q_{22} = B_1[a_1(\varphi,\theta_1)(M(\Delta_3)D_2(\Delta_3) + \hat{\psi}_2)] + C_1,$$
$$Q_{23} = B_1[a_1(\varphi,\theta_1)(M(\Delta_3)D_3(\Delta_3) + \hat{\psi}_3)],$$
$$Q_{31} = B_2[a_2(\varphi,\theta_2)(M(\Delta_3)D_1(\Delta_3) + \hat{\psi}_1)],$$
$$Q_{32} = B_2 + B_2[a_2(A,\theta_2) + a_2(\varphi,\theta_2)(M(\Delta_3)D_2(\Delta_3) + \hat{\psi}_2)],$$

$$Q_{33} = B_2[a_2(\varphi, \theta_2)(M(\Delta_3)D_3(\Delta_3) + \hat{\psi}_3)] + C_2,$$

and

$$-F_*(\Delta_3) = \begin{pmatrix} -F_1 \\ -F_2 \\ -F_3 \end{pmatrix}$$

with

$$-F_1 = d_0 - C_0[a_3(\varphi, T)M(\Delta_3)g(f, \Delta_3) + a_3(f, T)],$$

$$-F_2 = d_1 - B_1[a_1(\varphi, \theta_1)M(\Delta_3)g(f, \Delta_3) + a_1(f, \theta_1)],$$

$$-F_3 = d_2 - B_2[a_2(\varphi, \theta_2)M(\Delta_3)g(f, \Delta_3) + a_2(f, \theta_2)].$$

The linear boundary value problem for the Fredholm integro-differential equation with impulse effects (1)–(4) is solvable if the system of algebraic equation (32) is solvable [10, p. 1188].

Numerical solution to the problem (1)–(4) we find by the following algorithm.

Step 1. Choose N_1, N_2, N_3 and divide subintervals $[0, \theta_1), [\theta_1, \theta_2)$ and $[\theta_2, T)$ into $2N_1, 2N_2$ and $2N_3$ parts, respectively.

Solving the problem (17)–(19) by fourth order Runge-Kutta method for $P(t) = A(t)$, $P(t) = \varphi(t)$, $P(t) = f(t)$, we obtain $(n \times n)$-matrices $a_i(A, t)$, $a_i(\varphi, t)$, $i = \overline{1, 3}$, and n-vector-function $a_i(f, t)$, $i = \overline{1, 3}$, respectively.

Step 2. Multiply each $(n \times n)$-matrices $a_1(P, t), a_2(P, t)$ and $a_3(P, t)$ to $(n \times n)$-matrix $\psi(t)$, and using Simpson's method, we evaluate the following integrals:

$$\hat{\psi}_1(A) = \int\limits_0^{\theta_1} \psi(s)a_1(A, s)ds, \quad \hat{\psi}_2(A) = \int\limits_{\theta_1}^{\theta_2} \psi(s)a_2(A, s)ds,$$

$$\hat{\psi}_3(A) = \int\limits_{\theta_2}^{T} \psi(s)a_3(A, s)ds,$$

$$\hat{\psi}_1(\varphi) = \int\limits_0^{\theta_1} \psi(s)a_1(\varphi, s)ds, \quad \hat{\psi}_2(\varphi) = \int\limits_{\theta_1}^{\theta_2} \psi(s)a_2(\varphi, s)ds,$$

$$\hat{\psi}_3(\varphi) = \int\limits_{\theta_2}^{T} \psi(s)a_3(\varphi, s)ds, \tag{33}$$

$$\hat{\psi}_1(f)=\int\limits_0^{\theta_1}\psi(s)a_1(f,s)ds,\quad \hat{\psi}_2(f)=\int\limits_{\theta_1}^{\theta_2}\psi(s)a_2(f,s)ds,$$

$$\hat{\psi}_3(f)=\int\limits_{\theta_2}^{T}\psi(s)a_3(f,s)ds,$$

$$\hat{\psi}_1=\int\limits_0^{\theta_1}\psi(s)ds,\quad \hat{\psi}_2=\int\limits_{\theta_1}^{\theta_2}\psi(s)ds,\quad \hat{\psi}_3=\int\limits_{\theta_2}^{T}\psi(s)ds.$$

Summing up the definite integrals (33), we obtain $(n\times n)$-matrices:

$$G(\Delta_3)=\hat{\psi}_1(\varphi)+\hat{\psi}_2(\varphi)+\hat{\psi}_3(\varphi).$$

If the matrix $I-G(\Delta_3)$ is invertible, then we find its inverse matrix and represent it in the form $[I-G(\Delta_3)]^{-1}=M(\Delta_3)$. From the equalities (20)–(22) we define $(n\times n)$-matrices:

$$D_1(\Delta_3)=\hat{\psi}_1(A)+[\hat{\psi}_1(\varphi)+\hat{\psi}_2(\varphi)+\hat{\psi}_3(\varphi)]\cdot\hat{\psi}_1,$$

$$D_2(\Delta_3)=\hat{\psi}_2(A)+[\hat{\psi}_1(\varphi)+\hat{\psi}_2(\varphi)+\hat{\psi}_3(\varphi)]\cdot\hat{\psi}_2,$$

$$D_3(\Delta_3)=\hat{\psi}_3(A)+[\hat{\psi}_1(\varphi)+\hat{\psi}_2(\varphi)+\hat{\psi}_3(\varphi)]\cdot\hat{\psi}_3,$$

and vector of the dimension n:

$$g(f,\Delta_3)=\hat{\psi}_1(f)+\hat{\psi}_2(f)+\hat{\psi}_3(f).$$

STEP 3. Write the system of linear algebraic equations with respect to parameters:

$$Q_*(\Delta_3)\lambda=-F_*(\Delta_3),\quad \lambda\in R^{3n}. \tag{34}$$

Solving the system (34), we find $\lambda^*=(\lambda_1^*,\lambda_2^*,\lambda_3^*)\in R^{3n}$.

STEP 4. By the equalities:

$$\mu^*=M(\Delta_3)D_1(\Delta_3)\lambda_1^*+M(\Delta_3)D_2(\Delta_3)\lambda_2^*+M(\Delta_3)D_3(\Delta_3)\lambda_3^*+M(\Delta_3)g(f,\Delta_3),$$

we find $\mu^*\in R^n$ and then solve the Cauchy problems:

$$\frac{dx}{dt}=A(t)x+E^*(t),\quad x(0)=\lambda_1^*,\quad t\in[0,\theta_1], \tag{35}$$

$$\frac{dx}{dt}=A(t)x+E^*(t),\quad x(\theta_1)=\lambda_2^*,\quad t\in[\theta_1,\theta_2], \tag{36}$$

$$\frac{dx}{dt} = A(t)x + E^*(t), \quad x(\theta_2) = \lambda_3^*, \quad t \in [\theta_2, T], \tag{37}$$

where

$$E^*(t) = \varphi(t)(\mu^* + \hat{\psi}_1\lambda_1^* + \hat{\psi}_2\lambda_2^* + \hat{\psi}_3\lambda_3^*) + f(t).$$

We find that $x_1^*(t)$, $x_2^*(t)$, $x_3^*(t)$ are the numerical solution to the Cauchy problems (35), (36), (37), respectively,

Vector $x^*(t)$ composed by $x_1^*(t)$, $x_2^*(t)$ and $x_3^*(t)$ on the corresponding intervals, is a solution to the problem (1)–(4).

Example. Solve the linear two-point boundary value problem for the two integro-differential equations with impulse effects:

$$\frac{dx}{dt} = A(t)x + \varphi(t)\int\limits_0^1 \psi(\tau)x(\tau)d\tau + f(t), \; t \in (0,1), \; t \neq 0.2, \quad t \neq 0.6, \tag{38}$$

$$Bx(0) + Cx(1) = d_0, \quad d_0 \in R^2, \tag{39}$$

$$B_1x(0.2-0) + C_1x(0.2+0) = d_1, \quad d_1 \in R^2, \tag{40}$$

$$B_2x(0.6-0) + C_2x(0.6+0) = d_2, \quad d_2 \in R^2, \tag{41}$$

where $T = 1$, $\theta_1 = 0.2$, $\theta_2 = 0.6$,

$$A(t) = \begin{pmatrix} 1 & t \\ t^2 & t^3 \end{pmatrix}, \quad \varphi(t) = \begin{pmatrix} 2 & t-1 \\ 3 & t^2+1 \end{pmatrix}, \quad \psi(t) = \begin{pmatrix} t+1 & 2 \\ 1 & t^2-1 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; C_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \; B_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \; C_1 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \; C_2 = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}, \; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$d_0 = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, d_1 = \begin{pmatrix} \frac{-13}{25} \\ \frac{93}{25} \end{pmatrix}, d_2 = \begin{pmatrix} \frac{509}{25} \\ \frac{139}{25} \end{pmatrix},$$

$$f_1(t) = \begin{pmatrix} -\frac{18029\cdot t}{187500} - t\cdot(t+1) - t^2 - \frac{1530721}{187500} \\ -t^2(t^2-1) - \frac{393029\cdot t^2}{187500} - t^3\cdot(t+1) - \frac{843101}{46875} \end{pmatrix},$$

$$f_2(t) = \begin{pmatrix} -\frac{580529\cdot t}{187500} - t\cdot(t^2-2) - \frac{2093221}{187500} \\ 2t - t^3\cdot(t^2-2) - \frac{393029\cdot t^2}{187500} - t^2\cdot(t+3) - \frac{889976}{46875} \end{pmatrix},$$

$$f_3(t) = \begin{pmatrix} -\frac{18029\cdot t}{187500} - t\cdot(t+3) - t^2 - \frac{2093221}{187500} \\ -t^2\cdot(t^2+2) - \frac{393029\cdot t^2}{187500} - t^3\cdot(t+3) - \frac{843101}{46875} \end{pmatrix},$$

with impulse effects at the points $t = 0.2$ and $t = 0.6$.

The exact solution to the problem (38)–(41) has the form:

$$x(t) = \begin{cases} x_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{12}(t) \end{pmatrix} = \begin{pmatrix} t^2-1 \\ t+1 \end{pmatrix}, & t\in[0,0.2); \\ x_2(t) = \begin{pmatrix} x_{21}(t) \\ x_{22}(t) \end{pmatrix} = \begin{pmatrix} t+3 \\ t^2-2 \end{pmatrix}, & t\in[0.2,0.6); \\ x_3(t) = \begin{pmatrix} x_{31}(t) \\ x_{32}(t) \end{pmatrix} = \begin{pmatrix} t^2+2 \\ t+3 \end{pmatrix}, & t\in[0.6,1]. \end{cases}$$

Divide subintervals $[0,0.2),[0.2,0.6)$ and $[0.6,1)$ with step $h = 0.05$. Here (2×2)-matrix $I - G(\Delta_3)$ is invertible and

$$[I-G(\Delta_3)]^{-1} = \begin{pmatrix} -1.049316025 & -0.3119116102 \\ -0.1247687963 & 0.7928199221 \end{pmatrix}.$$

(6×6)-matrix $Q_*(\Delta_3)$ and vector $F_*(\Delta_3)\in R^6$ have the form:

$$Q_*(\Delta_3) = \begin{pmatrix} 1.40550 & 0.45770 & 1.13054 & 1.01511 & -0.02679 & 0.88644 \\ 0.39900 & 1.71426 & 1.19129 & 1.50651 & 1.32230 & 0.45965 \\ 0.71926 & 1.38386 & -1.43662 & -0.39688 & -1.95500 & -1.76272 \\ 1.92849 & 0.54091 & 0.58176 & -0.13811 & -1.87638 & -1.53179 \\ -0.79901 & -0.69781 & 0.82982 & -1.22241 & -1.83742 & 0.77322 \\ -0.34735 & -0.55520 & -0.92495 & -0.14342 & -1.42744 & 0.58381 \end{pmatrix}$$

and $F_*(\Delta_3) = \begin{pmatrix} -3.80829, & -6.95006, & 14.11433, & 9.19795, & -3.59985, & 4.15359 \end{pmatrix}'$.

The solution to the system of linear algebraic equations is $\lambda = -(Q^{-1}\cdot F)$,
$\lambda = (-1, 0.99999, 3.20000, -1.95999, 2.35999, 3.59999)'$.

Using the values λ, we find μ: $\mu = \begin{pmatrix} 0.6468002443 \\ 0.1205816063 \end{pmatrix}$.

There is a numerical solution to the problem (38)–(41) with the proximity $5.7\cdot 10^{-7}$, i.e. $\max \sup\limits_{t\in[0,1]} ||x(t) - x^*(t)|| \le 5.7\cdot 10^{-7}$, where $x^*(t)$ is the numerical solution to the problem (38)–(41).

As we can see, the numerical algorithm proposed is effective and allows us to obtain the numerical solution to the linear boundary value problem for the Fredholm integro-differential equation with impulse effects of higher order accuracy.

Below in Figures 1–3, we give the results obtained by Mathcad 15:

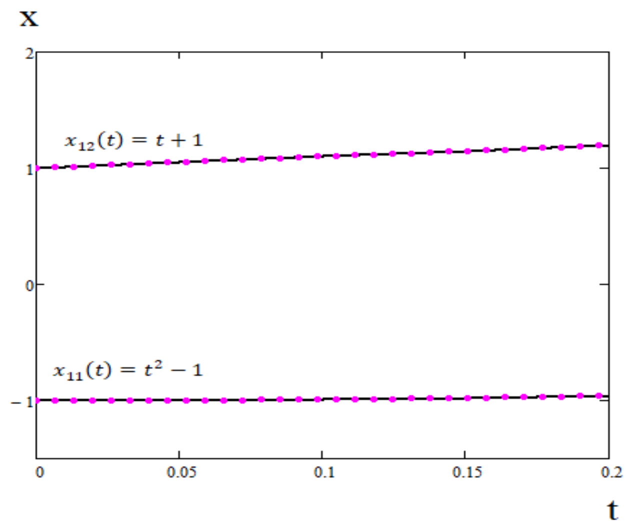

Figure 1 – Graphs of the exact and numerical solutions to the problem (38)-(41) on the interval [0, 0.2]. The blue solid and purple dotted lines correspond to the exact and numerical solutions, respectively

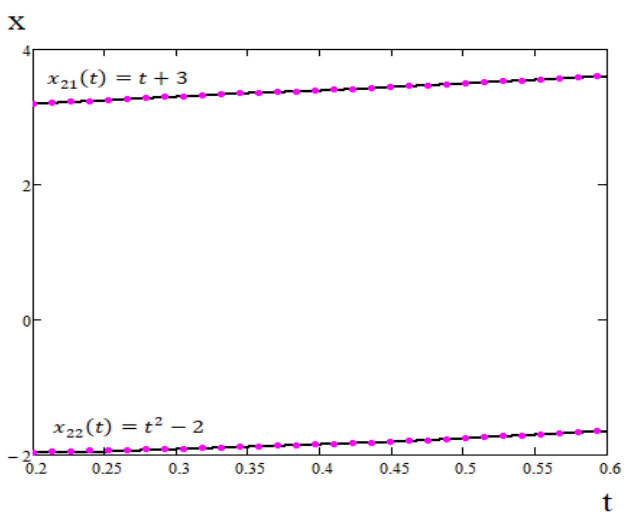

Figure 2 – Graphs of the exact and numerical solutions to the problem (38)-(41) on the interval [0.2, 0.6]

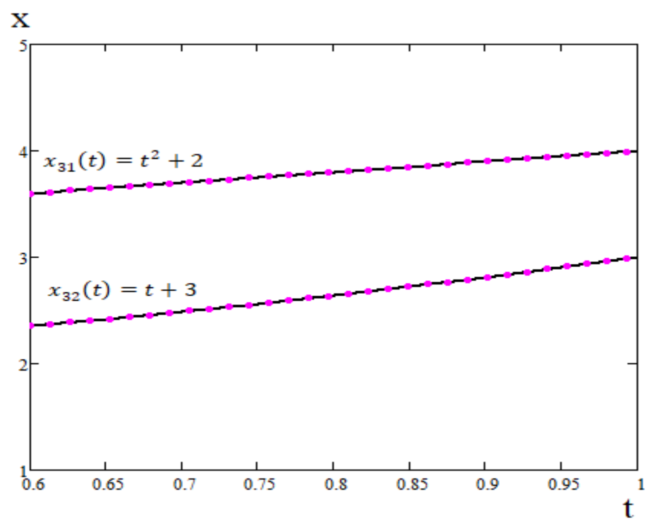

Figure 3 – Graphs of the exact and numerical solutions to the problem (38)-(41) on the interval [0.6, 1]

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Асанова А.Т., Убайда Ж.М. ИМПУЛЬС ӘСЕРЛІ ФРЕДГОЛЬМ ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУІ ҮШІН СЫЗЫҚТЫ ШЕТТІК ЕСЕПТІ ШЕШУ АЛГОРИТМІ

Импульс әсерлі Фредгольм интегралдық-дифференциалдық теңдеуі үшін сызықты шеттік есепті шешудің параметрлеу әдісіне негізделген алгоритмі ұсынылған. Алгоритмнің сандық жүзеге асырылуы келтірілген.

Кілттік сөздер. Импульс әсерлі Фредгольм интегралдық-дифференциальдық теңдеуі, параметрлеу әдісі, төртінші ретті Рунге-Кутта әдісі, Симпсон әдісі.

Асанова А.Т., Убайда Ж.М. АЛГОРИТМ РЕШЕНИЯ ЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ФРЕДГОЛЬМА С ИМПУЛЬСНЫМИ ВОЗДЕЙСТВИЯМИ

На основе метода параметризации предложен алгоритм решения линейной краевой задачи для интегро-дифференциального уравнения Фредгольма с импульсными воздействиями. Представлена численная реализация алгоритма.

Ключевые слова. Интегро-дифференциальное уравнение Фредгольма с импульсными воздействиями, метод параметризации, метод Рунге-Кутта четвертого порядка, метод Симпсона.

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An algorithm of solving nonlinear boundary value problem for the Van der Pol differential equation

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Abstract. Periodical boundary value problem for the Van der Pol differential equation is solved by parametrization's method. Interval is divided into 2 parts, the values of the solution at the left-end points of the subintervals are considered as additional parameters and original problem is reduced to the boundary value problem with parameters. Using solutions to the Cauchy problems for the differential equations with parameters, boundary condition, and the continuity condition at the dividing point, a system of nonlinear algebraic equations with respect to introduced parameters is composed. Explicit form of this system exists in exceptional cases. However, for the given parameters, the values of functions, which present left-hand sides of the system, and their derivatives by parameters, we can find by solving the Cauchy problems for ordinary differential equations on the subintervals. We find solutions to the Cauchy problems by forth order Runge-Kutta method. The solution of the composed system is found by Newton's method.

Keywords. Boundary value problem, Van der Pol equation, parametrization's method, fourth order Runge-Kutta method, Newton's method.

We consider a periodical boundary value problem for the Van der Pol differential equation:

$$\frac{d^2y}{dt^2} = \varepsilon(1-y^2)\frac{dy}{dt} + y - \varepsilon p\cos(\omega t+\alpha) + g(t), t \in (0,T),\ y \in R, \tag{1}$$

$$y(0) = y(T),$$

$$y'(0) = y'(T),$$

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where y is a position coordinate, which is a function of the time t, ω is an angular frequency, ε is a scalar parameter indicating the non-linearity and the strength of damping, $g(t)$ is a function continuous on $[0,T]$.

Differential equation (1) was introduced in 1920 to describe *the oscillation of triode in the electrical circuit* [1]. The Van der Pol equation has a long history of being used in both the *physical* [2] and *biological* [3] sciences.

Boundary value problems for ordinary differential equations have been studied by numerous authors (see [5]– [16] and references cited therein).

By introducing an unknown vector function $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, where $x_1(t) = y(t)$ and $x_2(t) = y'(t)$, we obtain the system of nonlinear ordinary differential equations:

$$\frac{dx_1}{dt} = x_2, \ \ t \in (0,T), \tag{2}$$

$$\frac{dx_2}{dt} = -x_1 + \varepsilon(1 - x_1^2)x_2 - \varepsilon p \cos(\omega t + \alpha) + g(t), \ \ t \in (0,T), \tag{3}$$

with boundary conditions:

$$x_1(0) = x_1(T), \tag{4}$$

$$x_2(0) = x_2(T). \tag{5}$$

Assume that $f_1(t, x_1, x_2) = x_2, f_2(t, x_1, x_2) = -x_1 + \varepsilon(1 - x_1^2)x_2 - \varepsilon p \cos(\omega t + \alpha) + g(t)$ and write down the system of nonlinear ordinary differential equations (2), (3) in the form:

$$\frac{dx}{dt} = f(t,x), \ \ t \in (0,T), \ \ x \in R^2, \ \ ||x|| = \max_{i=1,2} |x_i|.$$

In this paper the periodical boundary value problem for the Van der Pol equation is solved by the method proposed in [4].

Let $C([0,T], R^2)$ be a space of continuous functions $x : [0,T] \to R^2$ with the norm $||x||_1 = \max\limits_{t\in(0,T)} ||x(t)||$. A solution to the problem (2)–(5) is a continuously differentiable on $(0,T)$ function $x(t) \in C([0,T], R^2)$ satisfying the nonlinear differential equations (2), (3) and the periodical boundary conditions (4), (5).

Let Δ_2 be the partition of the interval $[0,T]$ into two parts with the points: $0 = \theta_0 < \theta_1 < \theta_2 = T$.

Denote by $C([0,T], \Delta_2, R^4)$ the space of function systems $x[t] = (x_{(1)}(t), x_{(2)}(t))$, where functions $x_{(r)} : [\theta_{r-1}, \theta_r) \to R^2$ are continuous and have the finite left-sided limits $\lim\limits_{t\to\theta_r-0} x_{(r)}(t)$, $r = \overline{1,2}$, with the norm $||x[\cdot]||_2 = \max\limits_{r=\overline{1,2}} \sup\limits_{t\in[\theta_{r-1},\theta_r)} ||x_{(r)}(t)||$.

Let $x(t)$ be a solution to the problem (2)–(5) and let $x_{(1)}(t)$, $x_{(2)}(t)$ be its restrictions to subintervals $[\theta_0; \theta_1)$, $[\theta_1; \theta_2)$, respectively. Then the system of two functions $x[t] =$

$(x_{(1)}(t),\ x_{(2)}(t))$ belongs to $C([0,T],\Delta_2,R^4)$ and its elements $x_{(1)}(t),\ x_{(2)}(t)$ satisfy the system of nonlinear ordinary differential equations:

$$\frac{dx_{(1)}}{dt} = f(t, x_{(1)}),\ t \in [\theta_0, \theta_1),\ x_{(1)} \in R^2, \tag{6}$$

$$\frac{dx_{(2)}}{dt} = f(t, x_{(2)}),\ t \in [\theta_1, \theta_2),\ x_{(2)} \in R^2, \tag{7}$$

the boundary condition:

$$x_{(1)}(\theta_0) = \lim_{t\to\theta_2-0} x_{(2)}(t), \tag{8}$$

and the continuity condition:

$$\lim_{t\to\theta_1-0} x_{(1)}(t) = x_{(2)}(\theta_1). \tag{9}$$

Introducing parameters $\lambda_{(1)} = x_{(1)}(\theta_0)$, $\lambda_{(2)} = x_{(2)}(\theta_1)$ and making the substitutions $u_{(1)}(t) = x_{(1)}(t) - \lambda_{(1)}$, $u_{(2)}(t) = x_{(2)}(t) - \lambda_{(2)}$ in (6)–(9), we obtain a new system of nonlinear differential equations with parameters:

$$\frac{du_{(1)}}{dt} = f(t, u_{(1)} + \lambda_{(1)}),\ \ t \in [\theta_0, \theta_1), \tag{10}$$

$$\frac{du_{(2)}}{dt} = f(t, u_{(2)} + \lambda_{(2)}),\ \ t \in [\theta_1, \theta_2), \tag{11}$$

initial conditions at the left-end points of subintervals:

$$u_{(1)}(\theta_0) = 0, \tag{12}$$

$$u_{(2)}(\theta_1) = 0, \tag{13}$$

the boundary condition

$$\lambda_{(1)} - \lambda_{(2)} - \lim_{t\to\theta_2-0} u_{(2)}(t) = 0, \tag{14}$$

and the continuity condition

$$\lambda_{(1)} + \lim_{t\to\theta_1-0} u_{(1)}(t) - \lambda_{(2)} = 0. \tag{15}$$

A solution to boundary value problem (10)–(15) is a pair $(\lambda^*, u^*[t])$ with elements $\lambda^* = (\lambda^*_{(1)}, \lambda^*_{(2)}) \in R^4$ and $u^*[t] = (u^*_{(1)}(t),\ u^*_{(2)}(t)) \in C([0,T],\Delta_2,R^4)$, where the functions $u^*_{(1)}(t),\ u^*_{(2)}(t)$ satisfy the system of nonlinear differential equations (10), (11) and additional conditions (14), (15) with $\lambda_{(1)} = \lambda^*_{(1)},\ \lambda_{(2)} = \lambda^*_{(2)}$ and the initial conditions (12), (13).

We suppose that the Cauchy problems with parameters on subintervals (10), (12) and (11), (13) have the unique solutions $u_{(1)}(t, \lambda_{(1)})$ and $u_{(2)}(t, \lambda_{(2)})$, respectively. Substituting corresponding solutions of the Cauchy problems into the boundary and continuity conditions we receive the following system of nonlinear algebraic equations with respect to introduced parameters $\lambda_{(1)},\ \lambda_{(2)}$:

$$\lambda_{(1)} - \lambda_{(2)} - \lim_{t\to\theta_2-0} u_{(2)}(t,\lambda_{(2)}) = 0, \tag{16}$$

$$\lambda_{(1)} + \lim_{t\to\theta_1-0} u_{(1)}(t,\lambda_{(1)}) - \lambda_{(2)} = 0. \tag{17}$$

We rewrite system (16), (17) as follows:

$$Q_*(\Delta_2,\lambda) = 0, \lambda \in R^4. \tag{18}$$

To find λ^*, that satisfies (18), we use Newton's method. Newton's method is an iterative method and requires an initial guess $\lambda^{(0)} \in R^4$. We find it by solving the linear boundary value problem obtained from our boundary value problem by $\varepsilon = 0$:

$$\frac{dx_1}{dt} = x_2, \tag{19}$$

$$\frac{dx_2}{dt} = -x_1 - \varepsilon p\cos(\omega t+\alpha) + g(t), \tag{20}$$

$$x_1(0) = x_1(T), \tag{21}$$

$$x_2(0) = x_2(T). \tag{22}$$

If $x^{(0)}(t) = \begin{pmatrix} x_1^{(0)}(t) \\ x_2^{(0)}(t) \end{pmatrix}$ is the solution to the linear boundary value problem (19)–(22), then the vector $\lambda^{(0)} = \begin{pmatrix} \lambda_{(1)}^{(0)} \\ \lambda_{(2)}^{(0)} \end{pmatrix} \in R^4$ is defined by the equalities $\lambda_{(1)}^{(0)} = \begin{pmatrix} x_1^{(0)}(\theta_0) \\ x_2^{(0)}(\theta_0) \end{pmatrix}$ and $\lambda_{(2)}^{(0)} = \begin{pmatrix} x_1^{(0)}(\theta_1) \\ x_2^{(0)}(\theta_1) \end{pmatrix}$.

In Newton's method the transfer equation has the form:

$$\lambda^{(n+1)} = \lambda^{(n)} + \Delta\lambda^{(n)}, n = 0,1,\dots,$$

where $\Delta\lambda^{(n)}, n = 0,1,...$, is a solution to the system of linear algebraic equations:

$$\frac{\partial Q_*(\Delta_2,\lambda^{(n)})}{\partial\lambda}\Delta\lambda^{(n)} = -Q_*(\Delta_2,\lambda^{(n)}), \tag{23}$$

with the Jacobian matrix

$$\frac{\partial Q_*(\Delta_2,\lambda^{(n)})}{\partial\lambda} = \begin{pmatrix} I & -I - \dfrac{\partial u_{(2)}(\theta_2,\lambda_{(2)}^{(n)})}{\partial\lambda_{(2)}} \\ I + \dfrac{\partial u_{(1)}(\theta_1,\lambda_{(1)}^{(n)})}{\partial\lambda_{(1)}} & -I \end{pmatrix}. \tag{24}$$

To find the value of $Q_*(\Delta_2, \lambda)$ for the given $\lambda = \lambda^{(n)}$, we solve the Cauchy problems (10), (12) and (11), (13) with parameters $\lambda_{(1)} = \lambda_{(1)}^{(n)}$ and $\lambda_{(2)} = \lambda_{(2)}^{(n)}$, respectively. If we denote by $u_{(1)}(t, \lambda_{(1)}^{(n)})$ and $u_{(2)}(t, \lambda_{(2)}^{(n)})$ solutions to problems (10), (12) and (11), (13), then these functions satisfy the following relations:

$$\frac{du_{(1)}(t, \lambda_{(1)}^{(n)})}{dt} = f(t, u_{(1)}(t, \lambda_{(1)}^{(n)}) + \lambda_{(1)}^{(n)}), \quad t \in [\theta_0, \theta_1), \tag{25}$$

$$u_{(1)}(\theta_0, \lambda_{(1)}^{(n)}) = 0, \tag{26}$$

$$\frac{du_{(2)}(t, \lambda_{(2)}^{(n)})}{dt} = f(t, u_{(2)}(t, \lambda_{(2)}^{(n)}) + \lambda_{(2)}^{(n)}), \quad t \in [\theta_1, \theta_2), \tag{27}$$

$$u_{(2)}(\theta_1, \lambda_{(2)}^{(n)}) = 0. \tag{28}$$

In order to determine $\dfrac{\partial u_{(1)}(\theta_1, \lambda_{(1)}^{(n)})}{\partial \lambda_{(1)}}$, we differentiate (25), (26) by $\lambda_{(1)}$:

$$\frac{\partial}{\partial \lambda_{(1)}}\left(\frac{du_{(1)}(t, \lambda_{(1)}^{(n)})}{dt}\right) = f'_x(t, u_{(1)}(t, \lambda_{(1)}^{(n)}) + \lambda_{(1)}^{(n)}) \cdot \left[\frac{\partial u_{(1)}(t, \lambda_{(1)}^{(n)})}{\partial \lambda_{(1)}} + I\right],$$

$$t \in [\theta_0, \theta_1),$$

$$\frac{\partial u_{(1)}(\theta_0, \lambda_{(1)}^{(n)})}{\partial \lambda_{(1)}} = 0.$$

And similarly differentiating (27), (28) by $\lambda_{(2)}$, we get:

$$\frac{\partial}{\partial \lambda_{(2)}}\left(\frac{du_{(2)}(t, \lambda_{(2)}^{(n)})}{dt}\right) = f'_x(t, u_{(2)}(t, \lambda_{(2)}^{(n)}) + \lambda_{(2)}^{(n)}) \cdot \left[\frac{\partial u_{(2)}(t, \lambda_{(2)}^{(n)})}{\partial \lambda_{(2)}} + I\right],$$

$$t \in [\theta_1, \theta_2),$$

$$\frac{\partial u_{(2)}(\theta_1, \lambda_{(2)}^{(n)})}{\partial \lambda_{(2)}} = 0.$$

Thus if we denote by

$$z_{(1)}(t) = \frac{\partial u_{(1)}(t, \lambda_{(1)}^{(n)})}{\partial \lambda_{(1)}}, \quad A_{(1)}^{(n)}(t) = f'_x(t, u_{(1)}(t, \lambda_{(1)}^{(n)}) + \lambda_{(1)}^{(n)}), \ t \in [\theta_0, \theta_1),$$

and

$$z_{(2)}(t) = \frac{\partial u_{(2)}(t, \lambda_{(2)}^{(n)})}{\partial \lambda_{(2)}}, \ \ A_{(2)}^{(n)}(t) = f'_x(t, u_{(2)}(t, \lambda_{(2)}^{(n)}) + \lambda_{(2)}^{(n)}), \ t \in [\theta_1, \theta_2),$$

then matrix functions $z_{(1)}(t)$ and $z_{(2)}(t)$ are the solutions to the Cauchy problems for the matrix linear ordinary differential equations on subintervals:

$$\frac{dz_{(1)}}{dt} = A_{(1)}^{(n)}(t) z_{(1)}(t) + A_{(1)}^{(n)}(t), \ \ t \in [\theta_0, \theta_1), \tag{29}$$

$$z_{(1)}(\theta_0) = 0, \tag{30}$$

$$\frac{dz_{(2)}}{dt} = A_{(2)}^{(n)}(t) z_{(2)}(t) + A_{(2)}^{(n)}(t), \ \ t \in [\theta_1, \theta_2), \tag{31}$$

$$z_{(2)}(\theta_1) = 0, \tag{32}$$

with the (2×2)-matrices

$$A_{(1)}^{(n)}(t) = \begin{pmatrix} 0 & 1 \\ -1 - 2\varepsilon x_{(1)1}^{(n)}(t) x_{(1)2}^{(n)}(t) & \varepsilon(1 - (x_{(1)1}^{(n)}(t))^2) \end{pmatrix}, \ \ t \in [\theta_0, \theta_1),$$

$$A_{(2)}^{(n)}(t) = \begin{pmatrix} 0 & 1 \\ -1 - 2\varepsilon x_{(2)1}^{(n)}(t) x_{(2)2}^{(n)}(t) & \varepsilon(1 - (x_{(2)1}^{(n)}(t))^2) \end{pmatrix}, \ \ t \in [\theta_1, \theta_2),$$

where $x_{(1)}^{(n)}(t) = \lambda_{(1)}^{(n)} + u_{(1)}\big(t, \lambda_{(1)}^{(n)}\big)$ and $x_{(2)}^{(n)}(t) = \lambda_{(2)}^{(n)} + u_{(2)}\big(t, \lambda_{(2)}^{(n)}\big)$.

Description of the algorithm.

Step 1. We solve the Cauchy problems (10), (12) and (11), (13) on the closed subintervals $[\theta_0, \theta_1]$ and $[\theta_1, \theta_2]$, respectively. Using their solutions $u_{(1)}(t, \lambda_{(1)}^{(n)}), u_{(2)}(t, \lambda_{(2)}^{(n)}), \ n = 0, 1, ...$, we find

$$Q_*(\Delta_2, \lambda^{(n)}) = \begin{pmatrix} \lambda_{(1)}^{(n)} - \lambda_{(2)}^{(n)} - u_{(2)}(\theta_2, \lambda_{(2)}^{(n)}) \\ \lambda_{(1)}^{(n)} + u_{(1)}(\theta_1, \lambda_{(1)}^{(n)}) - \lambda_{(2)}^{(n)} \end{pmatrix}, \ n = 0, 1, \dots .$$

Step 2. Compute (2×2)-matrices $A_{(1)}^{(n)}(t), \ A_{(2)}^{(n)}(t), \ n = 0, 1, ...$, for each closed subintervals. Solving (29), (30) and (31), (32) on the closed subintervals $[\theta_0, \theta_1]$ and $[\theta_1, \ \theta_2]$ by forth order Runge-Kutta method, we find $z_{(1)}(\theta_1), \ z_{(2)}(\theta_2)$ and according to formula (24) construct the Jacobian matrix

$$\frac{\partial Q_*(\Delta_2, \lambda^{(n)})}{\partial \lambda} = \begin{pmatrix} I & -I - \dfrac{\partial u_{(2)}(\theta_2, \lambda_{(2)}^{(n)})}{\partial \lambda_{(2)}} \\ I + \dfrac{\partial u_{(1)}(\theta_1, \lambda_{(1)}^{(n)})}{\partial \lambda_{(1)}} & -I \end{pmatrix}, \ n = 0, 1, \dots .$$

Step 3. Solve the system of linear algebraic equations (23):

$$\frac{\partial Q_*(\Delta_2, \lambda^{(n)})}{\partial \lambda}\Delta\lambda^{(n)} = -Q_*(\Delta_2, \lambda^{(n)}), n = 0, 1, \dots,$$

and find $\Delta\lambda^{(n)},\ n = 0, 1, \dots$.

Step 4. Determine vector $\lambda^{(n+1)}$ by the equality $\lambda^{(n+1)} = \lambda^{(n)} + \Delta\lambda^{(n)},\ n = 0, 1, \dots$.

Example. Consider the system of nonlinear ordinary differential equations:

$$\frac{dx_1}{dt} = x_2, \tag{33}$$

$$\frac{dx_2}{dt} = -x_1 + \varepsilon(1 - x_1^2)x_2 - \varepsilon p\cos(\omega t + \alpha) + g(t) \tag{34}$$

with the boundary conditions:

$$x_1(0) = x_1(T), \tag{35}$$

$$x_2(0) = x_2(T). \tag{36}$$

Here $T = 1, \omega = 2\pi, \alpha = 0, p = 1, \varepsilon = 0.5, g(t) = -4\pi^2 cos(2\pi t) + cos(2\pi t) + 2\pi\varepsilon(1 - cos^2(2\pi t))sin(2\pi t) - \varepsilon pcos(\omega t + \alpha)$.

The partition point and the initial guess: $\theta = \frac{1}{2}$, $\lambda^{(0)} = \begin{pmatrix} 0.999999999926279 \\ -0.342961455410546 \\ -0.999999999926273 \\ 0.342961455410548 \end{pmatrix}$, the exact solution $x^*(t) = \begin{pmatrix} cos(2\pi t) \\ -2\pi sin(2\pi t) \end{pmatrix}$.

Iteration 1:

$$\frac{\partial Q_*(\Delta_2, \lambda^{(0)})}{\partial \lambda} = \begin{pmatrix} 1 & 0 & -0.87758256 & -0.47942553 \\ 0 & 1 & 0.47942553 & -0.87758256 \\ 0.87758256 & 0.47942553 & -1 & 0 \\ -0.47942553 & 0.87758256 & 0 & -1 \end{pmatrix},$$

$$Q_*(\Delta_2, \lambda^{(0)}) = \begin{pmatrix} -0.173522158394070 \\ -0.625737662994771 \\ -0.173522158394068 \\ -0.625737662994775 \end{pmatrix},$$

$$\Delta\lambda^{(0)} = \begin{pmatrix} 0.006872550672743 \\ 0.335022572146646 \\ -0.006872550672749 \\ -0.335022572146652 \end{pmatrix}, \lambda^{(1)} = \begin{pmatrix} 1.006872550599022 \\ -0.007938883263900 \\ -1.006872550599022 \\ 0.007938883263896 \end{pmatrix}.$$

Iteration 2:

$$\frac{\partial Q_*(\Delta_2,\lambda^{(1)})}{\partial\lambda} = \begin{pmatrix} 1 & 0 & -0.87758256 & -0.47942553 \\ 0 & 1 & 0.47942553 & -0.87758256 \\ 0.87758256 & 0.47942553 & -1 & 0 \\ -0.47942553 & 0.87758256 & 0 & -1 \end{pmatrix},$$

$$Q_*(\Delta_2,\lambda^{(1)}) = \begin{pmatrix} 0.009710178637406 \\ -0.018336554471911 \\ 0.009710178637405 \\ -0.018336554471907 \end{pmatrix} \cdot 10^{-3},$$

$$\Delta\lambda^{(1)} = \begin{pmatrix} -0.007196134842402 \\ 0.007928569401626 \\ 0.007196134842409 \\ -0.007928569401622 \end{pmatrix} \cdot 10^{-4},\ \lambda^{(2)} = \begin{pmatrix} 0.999676415756619 \\ -0.000010313862274 \\ -0.999676415756612 \\ 0.000010313862274 \end{pmatrix}.$$

In iteration 7, we have:

$$\frac{\partial Q_*(\Delta_2,\lambda^{(2)})}{\partial\lambda} = \begin{pmatrix} 1 & 0 & -0.87758256 & -0.47942553 \\ 0 & 1 & 0.47942553 & -0.87758256 \\ 0.87758256 & 0.47942553 & -1 & 0 \\ -0.47942553 & 0.87758256 & 0 & -1 \end{pmatrix},$$

$$Q_*(\Delta_2,\lambda^{(2)}) = \begin{pmatrix} -0.124380110655409 \\ 0.031595249310507 \\ -0.124380103994071 \\ 0.031595266241408 \end{pmatrix} \cdot 10^{-7},$$

$$\Delta\lambda^{(2)} = \begin{pmatrix} 0.662238688153718 \\ 0.000820966056705 \\ -0.662238323313117 \\ -0.000821011841771 \end{pmatrix} \cdot 10^{-8},\ \lambda^{(3)} = \begin{pmatrix} 0.999999999265294 \\ -0.000000002419266 \\ -0.999999999265277 \\ 0.000000002419262 \end{pmatrix}.$$

In Table 1, we give the numerical solution to the problem (33)–(36) which is obtained by solving ordinary differential equations (6) and (7) with the initial conditions $x(\theta_0) = \lambda^{(7)}_{(1)}$ and $x(\theta_1) = \lambda^{(7)}_{(2)}$, respectively using forth order Runge-Kutta method.

Table 1 – Numerical solution and true error

t	$x_{(1)}(t)$	$\lvert x^*_{(1)}(t) - x_{(1)}(t)\rvert$	$x_{(2)}(t)$	$\lvert x^*_{(2)}(t) - x_{(2)}(t)\rvert$
0	0.9999999992	$0.73470585\cdot10^{-9}$	-0.0000000024	$2.41926569\cdot10^{-9}$
0.1	0.8090169936	$0.70465189\cdot10^{-9}$	-3.6931636619	$0.99506358\cdot10^{-9}$
0.2	0.3090169942	$0.15180812\cdot10^{-9}$	-5.9756643300	$0.54867754\cdot10^{-9}$
0.3	-0.3090169938	$0.54566057\cdot10^{-9}$	-5.9756643332	$3.72425201\cdot10^{-9}$
0.4	-0.8090169932	$1.15572973\cdot10^{-9}$	-3.6931636628	$1.83278814\cdot10^{-9}$
0.5	-0.9999999992	$0.73472261\cdot10^{-9}$	0.0000000024	$2.41926303\cdot10^{-9}$
0.6	-0.8090169936	$0.70466854\cdot10^{-9}$	3.6931636619	$0.99506269\cdot10^{-9}$
0.7	-0.3090169942	$0.15182477\cdot10^{-9}$	5.9756643300	$0.54867754\cdot10^{-9}$
0.8	0.3090169938	$0.54564386\cdot10^{-9}$	5.9756643332	$3.72425112\cdot10^{-9}$
0.9	0.8090169932	$1.15571330\cdot10^{-9}$	3.6931636628	$1.83278325\cdot10^{-9}$
1	0.9999999984	$1.55654689\cdot10^{-9}$	-0.0000000022	$2.21050329\cdot10^{-9}$

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Джумабаев Д.С., Мурсалиев Д. Е., Сергазина А.С., Кенжеева А.А. ВАН ДЕР ПОЛЬ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУІ ҮШІН СЫЗЫҚТЫ ЕМЕС ШЕТТІК ЕСЕПТІ ШЕШУ АЛГОРИТМІ

Ван-дер-Поль дифференциалдық теңдеуі үшін периодты шеттік есеп параметрлеу әдісімен шешіледі. Аралық екі бөлікке бөлінеді, шешімнің ішкі аралықтардың сол жақ шеткі нүктелеріндегі мәндері қосымша параметрлер ретінде қарастырылады, ал берілген есеп параметрлі шеттік есепке келтіріледі. Параметрлі Коши есептерінің шешімдерін, шекаралық шартты және бөлу нүктесіндегі үзіліссіздік шартын қолданып енгізілген параметрлерге қатысты сызықты емес алгебралық теңдеулер жүйесі құрылады. Бұл жүйені айқын түрде сирек жағдайларда ғана жазуға болады. Алайда берілген параметрлер үшін жүйенің сол жақ бөлігіндегі функциялардың мәндерін, және олардың параметрлер бойынша туындыларын жай дифференциалдық теңдеулер үшін ішкі аралықтарда Коши есептерін шешу арқылы таба аламыз. Коши есептерінің шешімдерін төртінші ретті Рунге-Кутта әдісімен табамыз. Құрылған жүйенің шешімі Ньютон әдісімен табылады.

Кілттік сөздер. Шеттік есеп, Ван дер Поль теңдеуі, параметрлеу әдісі, төртінші ретті Рунге-Кутта әді Ньютон әдісі.

Джумабаев Д.С., Мурсалиев Д.Е., Сергазина А.С., Кенжеева А.А. АЛГОРИТМ РЕШЕНИЯ НЕЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВАН ДЕР ПОЛЯ

Периодическая краевая задача для дифференциального уравнения Ван-дер-Поля решается методом параметризации. Интервал делится на 2 части, значения решения в левых конечных точках подинтервалов рассматриваются как дополнительные параметры, а исходная задача сводится к краевой задаче с параметрами. Используя решения задач Коши для дифференциальных уравнений с параметрами, граничное условие и условие непрерывности в точке деления, составляется система нелинейных алгебраических уравнений по введенным параметрам. В явном виде эту систему удается записать в исключительных случаях. Однако для заданных параметров значения функций, которые представляют левые части системы, и их производные по параметрам мы можем найти, решая задачи Коши для обыкновенных дифференциальных уравнений на подинтервалах. Решения задач Коши мы находим методом Рунге-Кутты четвертого порядка. Решение составленной системы находится методом Ньютона.

Ключевые слова. Краевая задача, уравнение Ван дер Поля, метод параметризации, метод Рунге-Кутты четвертого порядка, метод Ньютона.

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About root functions of periodic Sturm-Liouville problem with symmetric potential

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Abstract. In this work, we consider spectral problems for the Sturm - Liouville differential operator $-u''(x)+q(x)u(x)=\lambda u(x)$ on $(0,1)$ with periodic and antiperiodic boundary conditions $u'(0)=\pm u'(1)$, $u(0)=\pm u(1)$. The Riesz basis property of the system of root functions of such problems is proved in the case of a potential $q(x)$ that is summable on an interval, when it satisfies the symmetry condition $q(x)=q(1-x)$.

Keywords. Sturm-Liouville differential operator, boundary value problem, well-posedness, Green's function, eigenfunctions, eigenvalues.

1 Introduction

We consider two spectral problems for the Sturm-Liouville operator with periodic ($\theta=0$) and antiperiodic ($\theta=1$) boundary conditions:

$$L_\theta u \equiv -u''(x)+q(x)u(x)=\lambda u(x), \quad x\in(0,1), \tag{1}$$

$$\begin{cases} U_1(u)\equiv u'(0)-(-1)^\theta u'(1)=0, \\ U_2(u)\equiv u(0)-(-1)^\theta u(1)=0, \quad \theta=0,1. \end{cases} \tag{2}$$

By L_θ we denote a closure in $L_2(0,1)$ of the operator given by the differential expression (1) on a linear manifold of functions $u\in C^2[0,1]$ satisfying the boundary conditions (2).

It is easy to justify that the operator L_θ is a linear operator on $L_2(0,1)$ defined by (1) with the domain

$$D(L_\theta)=\left\{u\in W_2^2(0,1): U_1(u)=0,\ U_2(u)=0\right\}.$$

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For the elements $u \in D(L_\theta)$ we understand the action of the operator $L_\theta u = -u''(x) + q(x)u(x)$ in a sense of almost everywhere on $(0,1)$.

By an eigenvector of the operator L_θ corresponding to an eigenvalue $\lambda_0 \in C$, we mean any non-zero vector $u_0 \in D(L_\theta)$ which satisfies the equation:

$$L_\theta u_0 = \lambda_0 u_0. \tag{3}$$

By an associated vector of the operator L_θ of order m $(m = 1, 2, \ldots)$ corresponding to the same eigenvalue λ_0 and the eigenvector u_0, we mean any function $u_m \in D(L_\theta)$ which satisfies the equation:

$$L_\theta u_m = \lambda_0 u_m + u_{m-1}. \tag{4}$$

The vectors $\{u_0, u_1, \ldots\}$ are called a chain of the eigenvectors and associated vectors of the operator L_θ corresponding to the eigenvalue λ_0.

The eigenvalues of the operator L_θ will be called eigenvalues of the problem (1)–(2). The eigen- and associated vectors of the operator L_θ will be called eigen- and associated functions (EAF) of the problem (1)–(2). The set of all eigen- and associated functions (they are collectively called root functions) corresponding to the same eigenvalue λ_0 forms a root linear manifold. This manifold is called a root subspace.

It is well known that with a real-valued potential $q(x)$, both problems under consideration are self-adjoint. The boundary conditions (2) are Birkhoff regular, but not strongly regular [1, chapter 2]. Therefore, for complex-valued $q(x)$, the EAF system of the problem is complete and minimal in $L_2(0,1)$. The eigenvalues of the problem are asymptotically arranged in pairs. From [2], [3] it follows that two-dimensional subspaces, composed of EAF, corresponding to pairwise close eigenvalues form a Riesz basis in $L_2(0,1)$.

The works [4], [5], [6] are devoted to the study of conditions on $q(x)$, under which EAF of periodic problems form the usual Riesz basis.

In the work [4] for $q(x) \in C^4[0,1],\ q(0) \neq q(1)$ the Riesz basis property of root vectors in $L_2(0,1)$ is proved.

In the work [5] the basis property conditions in $L_2(0,1)$ of EAF systems in terms of the order of decreasing Fourier coefficients of a function

$$q(x) \in W_1^m(0,1), \quad q^{(l)}(0) = q^{(l)}(1),\ l = 0, 1, ..., m-1.$$

are found.

In [6] for the case $q(x) \in W_1^p(0,1),\ q^{(l)}(0) = q^{(l)}(1) = 0,\ l = 0, 1, ..., s-1,\ s \leq p$, it is proved the Riesz basis criterion in $L_2(0,1)$ of EAF system in terms of the order of decreasing Fourier coefficients of the functions

$$q(x), \quad Q(x) = \int_0^x q(t)dt, \quad S(x) = Q^2(x).$$

The main aim of this paper is to justify the Riesz basis property of the EAF system of the periodic and antiperiodic problems (1), (2) with the symmetric potential $q(x) = q(1-x)$.

Note that the symmetry condition of the potential is essential for the spectral properties of boundary value problems. In the work [7] the dependence of the spectrum on the coefficients of the boundary conditions for an even-order differential operator with a certain symmetry of the coefficients of the operator was investigated. There it was first shown that under certain conditions on the coefficients of the equation, the spectrum of the operator does not depend on some coefficients of the boundary condition. In particular, as a result, it was shown that the spectrum of the problem for the equation (1) with boundary conditions

$$u(0) = bu(1), \ u'(0) = u'(1), \tag{5}$$

with the symmetric coefficient $q(x) = q(1-x)$, does not depend on the coefficient $b \neq -1$ of the boundary condition (5) and coincides with the spectrum of the periodic boundary value problem (problems (1), (2) with $\theta = 0$).

In the work [8] it is shown that all Volterra boundary value problems for equation (1) are given by the conditions

$$u(0) = \alpha u(1), \quad u'(0) = -\alpha u'(1)$$

with $\alpha^2 \neq 1$. With $\alpha \neq 0$ the symmetry condition $q(x) = q(1-x)$ is a criterion for the Volterra property of this problem.

The spectral properties of problems with non-reinforced regular boundary conditions are the subject of research by many mathematicians. From recent papers, [9]–[14], we note that some new results are obtained for spectral problems and their applications are given in problems for partial differential equations.

The main result of this paper is formulated as a theorem.

Theorem 1. *If $q(x) \in L_1(0,1)$ and $q(x) = q(1-x)$ for almost all $x \in (0,1)$, then the system of eigen- and associated functions of problem (1), (2) is Riesz basis in $L_2(0,1)$.*

2 On the symmetry of the root functions of Dirichlet and Neumann problems

For the equation (1) consider the Dirichlet problem

$$u(0) = 0, \quad u(1) = 0, \tag{6}$$

and the Neumann problem

$$u'(0) = 0, \quad u'(1) = 0. \tag{7}$$

Lemma 1. *If $q(x) \in L_1(0,1)$ and $q(x) = q(1-x)$, then all eigen- and associated functions of the Dirichlet problem (1), (6) and the Neumann problem (1), (7) possess one of the properties of symmetry:*

$$u(x) = u(1-x) \qquad \text{or} \qquad u(x) = -u(1-x) \qquad \text{for all} \qquad x \in [0,1]. \tag{8}$$

Proof. We will conduct the proof only for the Dirichlet problem. The proof of the Neumann problem is similar. Let λ_k^D be eigenvalues of the Dirichlet problem (1), (6) of multiplicity m_k^D+1, to which there correspond the normalized eigenfunctions $v_{k0}(x)$ and (maybe) chains of adjoined functions $v_{kj}(x),\ j=\overline{1,m_k^D}$:

$$L_D v_{k0}=\lambda_k^D v_{k0}\ ,\quad L_D v_{kj}=\lambda_k^D v_{kj}+v_{kj-1}.$$

Denote

$$v_{kj}^{\pm}=v_{kj}(x)\pm v_{kj}(1-x).$$

It may turn out that $v_{kj}^{+}(x)\equiv 0$ or $v_{kj}^{-}(x)\equiv 0$. But not at the same time. It is obvious that all these functions satisfy one of the symmetry conditions (8).

It is easy to see that the functions $v_{k0}^{+}(x)$ and $v_{k0}^{-}(x)$ are solutions of the Dirichlet problem

$$Lv(x)=\lambda_k^D v(x),\ x\in(0,1);\quad v(0)=0,\ v(1)=0.$$

Those of them that are not identical with zero are eigenfunctions. Since the Dirichlet problem cannot have two (linearly independent) eigenfunctions corresponding to one eigenvalue, there is only one eigenfunction $v_{k0}^{+}(x)$ or $v_{k0}^{-}(x)$. And it satisfies one of the symmetry conditions (8). If λ_k^D is a multiple eigenvalue of the Dirichlet problem, then the corresponding functions v_{kj}^{+} (or v_{kj}^{-}) form a chain of associated to v_{k0}^{+} (respectively to v_{k0}^{-}) functions. Obviously, they have the same symmetry property from (8), as the function v_{k0}^{+} (respectively v_{k0}^{-}).

We show that there is no EAF, that does not possess any of the symmetry properties (8). Consider the system of functions

$$\left\{v_{kj}^{+}(x),\ v_{ni}^{-}(x),\ j=\overline{0,m_k^D},\ i=\overline{0,m_n^D}\right\}_{k,n\in N}. \tag{9}$$

Some of these functions may turn out to be zero, but we do not pay attention to this. We prove that system (9) is complete in $L_2(0,1)$. Indeed, suppose $g(x)\in L_2(0,1)$ is orthogonal to all functions of system (9). Then

$$0=(v_{kj}^{\pm},g)=\int_0^1 v_{kj}^{\pm}(x)\overline{g(x)}dx=\int_0^1\left[v_{kj}(x)\pm v_{kj}(1-x)\right]\overline{g(x)}dx$$

$$=\int_0^1 v_{kj}(x)\overline{g(x)}\pm v_{kj}(1-x)\overline{g(x)}dx=\int_0^1 v_{kj}(x)\overline{g(x)}dx\pm\int_0^1 v_{kj}(1-x)\overline{g(x)}dx.$$

If we let $x\to 1-x$, then note that

$$\int_0^1 v_{kj}(1-x)\overline{g(x)}dx=\int_1^0 v_{kj}(x)\overline{g(1-x)}d(1-x)=\int_0^1 v_{kj}(x)\overline{g(1-x)}dx$$

then

$$0 = \int_0^1 v_{kj}(x)\left[\overline{g(x)} \pm \overline{g(1-x)}\right] dx,\ k \in N,\ j = \overline{0, m_k^D}.$$

Since the system $\left\{v_{kj}(x), j = \overline{0, m_k^D}\right\}_{k \in N}$ is complete in $L_2(0,1)$, then $\overline{g(x)} \pm \overline{g(1-x)} = 0$ $\Rightarrow g(x) = 0$ for all $x \in (0,1)$, which proves the completeness of system (9) in $L_2(0,1)$. The system (9) remains complete when removing identically zero functions from it. All nonzero functions of the system (9) are EAF of the Dirichlet problems (1), (6). Since this system of functions is complete in $L_2(0,1)$, the problem has no other EAF. All elements of the system (9) possess one of the symmetry properties (8). Lemma 1 is proved.

3 Proof of the main theorem

Let $\left\{v_{kj}(x), j = \overline{0, m_k^D}\right\}_{k \in N}$ be a EAF system of Dirichlet problems (1), (6), possessing symmetry property

$$v(x) = (-1)^\theta v(1-x), \qquad \text{for every} \qquad x \in [0,1], \tag{10}$$

and let λ_k^D be their own eigenvalues; and let $\left\{w_{ni}(x), i = \overline{0, m_n^N}\right\}_{n \in N}$ be a EAF system of Neumann problems (1), (7), possessing symmetry property

$$w(x) = -(-1)^\theta w(1-x), \qquad \text{for every} \qquad x \in [0,1], \tag{11}$$

and let λ_n^N be eigenvalues of the Neumann problem corresponding to them.

By direct calculation it is easy to verify that the functions $v_{kj}(x)$ and $w_{ni}(x)$ are the EAF of the original problem (1), (2), corresponding to the eigenvalues λ_k^D and λ_n^N, respectively. If we show that the system

$$\left\{v_{kj}(x),\ w_{ni}(x),\ j = \overline{0, m_k^D},\ i = \overline{0, m_n^N}\right\}_{k,n \in N} \tag{12}$$

is complete in $L_2(0,1)$, then problem (1), (2) has no other EAF.

The space $L_2(0,1)$ is divided into a direct sum of two subspaces: a spaces $L_2^+(0,1)$ of functions, possessing symmetry property (10), and a space $L_2^-(0,1)$ of functions, possessing symmetry property (11). By virtue of the proven Lemma 1 the system $\{v_{kj}(x)\}_{k \in N}$ is complete in $L_2^+(0,1)$, and the system $\{w_{ni}(x)\}_{n \in N}$ is complete in $L_2^-(0,1)$. Therefore, system (12) is complete in $L_2(0,1)$. Therefore, the problem does not have an EAF of other kind.

Thus, the EAF system (12) of the periodic problem (1), (2) consists only of EAF of the Dirichlet problem, possessing symmetry property (10), and of EAF of the Neumann problem with the symmetry property (11). Obviously, the system $\{v_{kj}(x)\}_{k \in N}$ forms the Riesz basis in $L_2^+(0,1)$, and the system $\{w_{ni}(x)\}_{n \in N}$ forms the Riesz basis in $L_2^-(0,1)$. Therefore, the

EAF system of the periodic problem (1), (2) forms the Riesz basis in $L_2(0,1)$. The theorem is proved.

Since the Dirichlet and Neumann problems are strongly regular, they can have only a finite number of associated functions. Therefore, from the course of the proof of the theorem we obtain

Corollary 1. *If $q(x) \in L_1(0,1)$ and $q(x) = q(1-x)$, then the periodic boundary value problems* (1), (2) *may have no more than a finite number of associated functions.*

It is interesting that Lemma 1 has a converse.

Lemma 2. *If $q(x) \in L_1(0,1)$ and all EAF of the periodic problem* (1), (2) *or of the Dirichlet problem* (1), (6) *or of the Neumann problem* (1), (7) *have one of the properties of symmetry* (8)*, then $q(x) = q(1-x)$.*

Proof. Take only the odd EAF $u_{kj}(x)$, that is, having the property of symmetry $u_{kj}(x) + u_{kj}(1-x) = 0$. They satisfy the equation

$$-u''_{kj}(x) + q(x)u_{kj}(x) = \lambda_k u_{kj}(x) + u_{kj-1}(x),\ 0 < x < 1.$$

Integrating it over the interval $0 < x < 1$, we find

$$\int_0^1 q(x)u_{kj}(x)dx = 0.$$

Since the odd EAF $\{u_{kj}(x)\}$ are complete in the subspace of odd functions from $L_2(0,1)$, then $q(x)$ is an even function. Lemma 2 is proved.

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Кәлменов Т.Ш., Қахарман Н., Садыбеков М.А. СИММЕТРИЯЛЫ ПОТЕНЦИАЛДЫ ПЕРИОДТЫ ШТУРМ-ЛИУВИЛЛЬ ЕСЕБІНІҢ ТҮБІРЛІК ФУНКЦИЯЛАРЫ ТУРАЛЫ

Бұл мақалада $(0,1)$ кесіндісінде $u'(0) = \pm u'(1)$, $u(0) = \pm u(1)$ периодты және антипериодты шекаралық шартты $-u''(x) + q(x)u(x) = \lambda u(x)$ Штурм–Лиувилль дифференциалдық операторы үшін спектралды есептер қарастырылған. Қарастырылып отырған есептің аралықта қосындылатын $q(x)$ потенциалы $q(x) = q(1-x)$ симметрия шартын қанағаттандыратын болса, оның түбірлік (меншікті және қосалқы) функциялары Рисс базисі болатыны дәлелденген.

Кілттік сөздер. Лаплас операторы, шеттік есеп, Самарский-Ионкин тектес есеп, қисындылық, Грин функциясы, меншікті функциялар, меншікті мәндер.

Кальменов Т.Ш., Кахарман Н., Садыбеков М.А. О КОРНЕВЫХ ФУНКЦИЙ ПЕРИОДИЧЕСКОЙ ЗАДАЧИ ШТУРМА-ЛИУВИЛЛЯ С СИММЕТРИЧНЫМ ПОТЕНЦИАЛОМ

В этой статье рассматривается спектральные задачи для дифференциального оператора Штурма-Лиувилля $-u''(x)+q(x)u(x)=\lambda u(x)$ на отрезке $(0,1)$ с периодическими и антипериодическими краевыми условиями $u'(0)=\pm u'(1)$, $u(0)=\pm u(1)$. Доказана базисность Рисса системы корневых (собственных и присоединенных) функций рассматриваемых задач в случае суммируемого на интервале потенциала $q(x)$, когда он удовлетворяет условию симметрии $q(x)=q(1-x)$.

Ключевые слова. Оператор Лапласа, краевая задача, задача типа Самарского-Ионкина, Корректность, пункция Грина, собственные функции, собственные значения.

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On solvability of some nonlocal boundary value problems for polyharmonic equation

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Abstract. This work is devoted to the solvability of some non-classical boundary value problems for the polyharmonic equation. These problems generalize the Dirichlet and Neumann problems for the polyharmonic equation. The considered problems are nonlocal boundary value problems of Bitsadze-Samarskii type. The investigated problems are solved by reducing them to the Dirichlet problem and the Neumann type problems. Theorems on the existence and the uniqueness of the problem's solution are proved and exact solvability conditions are received. We obtain necessary and sufficient conditions for the solvability of the Neumann type problem for the polyharmonic equation in the unit ball. By applying Green's functions, as well as the statement of the existence of a solution to the Dirichlet problem, the obtained integral representations for the solutions are constructed.

Keywords. Polyharmonic equation, nonlocal problem, involution, Dirichlet problem, Neumann type problem, uniqueness, existence.

1 Introduction

Nonlocal boundary value problems for elliptic equations in which boundary conditions are given in the form of a connection between the values of the unknown function and its derivatives at various points of the boundary, are called the problems of the Bitsadze-Samarskii type [1]. Numerous applications of the nonlocal boundary value problems for elliptic equations in problems of physics, the engineering, and other branches of the science are described in detail in [2], [3]. Solvability of nonlocal boundary value problems for the elliptic equations is discussed in [4]–[8]. Boundary value problems with involution for elliptic equations of the second and fourth orders, as a special case of nonlocal problems, are considered in [9]–[13].

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Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball, $n \geq 2$, and let $\partial\Omega$ be the unit sphere. For any point $x = (x_1, x_2, \cdots, x_n) \in \Omega$ we consider the point $x^* = Cx$, where C is a real orthogonal matrix $CC^T = E$. Suppose also that there exists a natural $l \in \mathbb{N}$ such that $C^l = E$.

Let $m \geq 1, \alpha_k$ be some real numbers, p take one of the meanings $p = 0$ or $p = 1$, $D_\nu^k = \dfrac{\partial^k}{\partial \nu^k}$, $k \geq 1$, ν be the unit vector of the outward normal to $\partial\Omega$, and let $D_\nu^0 = I$ be the unit operator. In this paper we study the following nonlocal boundary value problem

$$(-\Delta)^m u(x) = f(x), \quad x \in \Omega, \tag{1}$$

$$D_\nu^{k+p} u(x) + \alpha_k D_\nu^{k+p} u(x^*) = g_k(x), x \in \partial\Omega, k = 0, 1, ..., m-1. \tag{2}$$

By a solution of the problem (1), (2) we mean a function $u(x) \in C^{2m}(\Omega) \cap C^{m+p-1}(\bar{\Omega})$ satisfying conditions (1), (2) in the classical sense. In the case $\alpha_k = 0$ when $p = 0$ we obtain the well-known Dirichlet problem [14] and, when $p = 1$ we have the Neumann type problems [15], [16].

2 Auxiliary statements

First we note that if $x \in \Omega$, or $x \in \partial\Omega$, then $x^* = Cx \in \Omega$, or $x^* = Cx \in \partial\Omega$, respectively, since the transformation of the space $\mathbb{R}^n$ by the matrix C preserves the norm $|x^*|^2 = |Cx|^2 = (Cx, Cx) = (C^T Cx, x) = |x|^2$.

The case $x^* = -x$ investigated in [9]–[13] is a particular case of the situation considered here since for $C = -E$ we have $CC^T = -E(-E) = E$ and $l = 2$.

It is obvious that the transformation made by the matrix C can be also a rotation in the space $\mathbb{R}^n$, for example, if $C = C^1_{\varphi_1} C^2_{\varphi_2} \cdots C^{n-2}_{\varphi_{n-2}}$, where

$$C^i_\varphi = \begin{pmatrix} E_i & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & E_{n-i-2} \end{pmatrix},$$

E_i is the unit $i \times i$ matrix and $i = \overline{1, n-2}$. This is so since $C^T = C^{n-2}_{-\varphi_{n-2}} \cdots C^2_{-\varphi_2} C^1_{-\varphi_1}$ and hence $CC^T = C^1_{\varphi_1} C^2_{\varphi_2} \cdots C^{n-2}_{\varphi_{n-2}} C^{n-2}_{-\varphi_{n-2}} \cdots C^2_{-\varphi_2} C^1_{-\varphi_1} = E$.

Consider the operator

$$I_C u(x) = u(Cx) = u(x^*).$$

In view of what has been said above, this operator is defined on functions $u(x), x \in \Omega$. We also consider the operator $\Lambda u = \sum\limits_{i=1}^n x_i u_{x_i}(x)$ that is homogeneous, preserves the polyharmonicity of function $u(x)$, and has the property $D_\nu^m u|_{\partial\Omega} = \Lambda^{[m]} u|_{\partial\Omega}$, where $\Lambda^{[m]} = \Lambda(\Lambda-1)\dots(\Lambda-m+1)$ [15]. Let C^i_{col} and C^i_{row} be the i-th column and i-th row of the matrix C, respectively.

We prove two simple lemmas. Let $u(x)$ be a twice continuously differentiable function in Ω.

Lemma 1. *Operators Λ and I_C are commutative $\Lambda I_C u(x) = I_C \Lambda u(x)$, and also the equality $\nabla I_C = I_C C^T \nabla$ holds, and operators Δ and I_C are also commutative.*

Proof. We can write the operator Λ in the form $\Lambda u = (x, \nabla)u$. Since

$$\frac{\partial}{\partial x_i} I_C u(x) = \frac{\partial}{\partial x_i} u(Cx) = \frac{\partial}{\partial x_i} u((C^1_{row}, x), \ldots, (C^n_{row}, x))$$

$$= \sum_{j=1}^{n} c_{ji} I_C u_{x_j}(x) = (C^i_{col}, I_C \nabla u(x)) = I_C (C^i_{col}, \nabla) u(x), \tag{3}$$

then

$$\Lambda I_C u(x) = \Lambda u(Cx) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} u(Cx) = \sum_{i=1}^{n} x_i \left(C^i_{col}, I_C \nabla u(x)\right)$$

$$= \left(\sum_{i=1}^{n} x_i C^i_{col}, I_C \nabla u(x)\right) = (Cx, I_C \nabla u(x)) = I_C(x, \nabla u(x)) = I_C \Lambda u(x).$$

Further, due to the formula (3), we find

$$\frac{\partial^2}{\partial x_i^2} I_C u(x) = \frac{\partial}{\partial x_i} I_C (C^i_{col}, \nabla) u(x) = I_C (C^i_{col}, \nabla)^2 u(x)$$

and therefore

$$\Delta I_C u(x) = \sum_{i=1}^{n} I_C (C^i_{col}, \nabla)^2 u(x) = I_C \left|\left((C^1_{col}, \nabla), \ldots, (C^n_{col}, \nabla)\right)\right|^2 u(x)$$

$$= I_C \left|C^T \nabla\right|^2 u(x) = I_C (C^T \nabla, C^T \nabla) u(x) = I_C (C C^T \nabla, \nabla) u(x) = I_C \Delta u(x).$$

At last,

$$\nabla I_C u(x) = I_C ((C^1_{col}, \nabla), \ldots, (C^n_{col}, \nabla)) u(x) = I_C (C^T \nabla) u(x).$$

Lemma is proved.

Corollary. *If the function $u(x)$ is polyharmonic in Ω, then the function $u(x^*) = I_C u(x)$ is also polyharmonic in Ω.*

Indeed, due to Lemma 1, $\Delta^m u(x) = 0 \Rightarrow \Delta^m I_C u(x) = I_C \Delta^m u(x) = 0$.

Lemma 2. *The operator $1 + \alpha I_C$, when $(-\alpha)^l \neq 1$ is invertible and the operator*

$$J_\alpha = \frac{1}{1 - (-\alpha)^l} \sum_{k=0}^{l-1} (-\alpha)^k I_C^k \tag{4}$$

is inverse to $1+\alpha I_C$.

Proof. It is easy to see that

$$\left(\sum_{k=0}^{l-1}(-\alpha)^k I_C^k\right)(1+\alpha I_C)u(x)=\left(\sum_{k=0}^{l-1}(-\alpha)^k I_C^k-\sum_{k=1}^{l}(-\alpha)^k I_C^k\right)u(x)$$

$$=\left(E-(-\alpha)^l I_C^l\right)u(x)=(1-(-\alpha)^l)u(x).$$

Thus, if $(-\alpha)^l\neq 1$, then we can divide both sides of the equality by $1-(-\alpha)^l$ and hence the operator J_α is inverse to $1+\alpha I_C$. Lemma is proved.

3 Dirichlet and Neumann type problems

In this section we study the following problem:

$$(-\Delta)^m v(x)=\varphi(x),\quad x\in\Omega, \tag{5}$$

$$D_\nu^{k+p}v(x)|_{\partial\Omega}=\psi_k(x),\quad x\in\partial\Omega, k=0,1,...,m-1, \tag{6}$$

where $p=0$ or $p=1$.

The following statements are true.

Theorem 1 [14]. *Let* $p=0$, $0<\lambda<1$, $\varphi(x)\in C^\lambda(\bar\Omega)$, $\psi_k(x)\in C^{\lambda+m-1-k}(\partial\Omega)$, $k=0,1,...,m-1$. *Then a solution of the problem* (5), (6) *exists, is unique and belong to the class* $C^{\lambda+2m}(\Omega)\cap C^{\lambda+m-1}(\bar\Omega)$.

Theorem 2 [16]. *Let* $p=1$, $\varphi(x)\in C^1(\bar\Omega)$, $\psi_k(x)\in C^k(\partial\Omega)$, $k=0,1,...,m-1$. *Then for the solvability of the problem* (5), (6) *the following condition is necessary and sufficient*

$$\int\limits_{\partial\Omega}\sum_{k=1}^{m}(-1)^{k+1}\binom{2m-k-1}{k-1}(2m-2k-1)!!\psi_k(x)dS_x$$

$$+\int\limits_{\Omega}\frac{\left(|x|^2-1\right)^{m-1}}{(2m-2)!!}\varphi(x)dx=0. \tag{7}$$

If the solution of the problem exists, then it is unique up to a constant.

4 Uniqueness

In this section we investigate the uniqueness of the solution of the problem (1), (2). The following proposition is true.

Theorem 3. *Let* $(-\alpha_k)^l\neq 1$, *and a solution of the problem* (1), (2) *exists. Then*

1) if $p=0$, *then the solution of the problem is unique;*

2) if $p = 1$, then the solution of the problem is unique up to a constant.

Proof. To prove the uniqueness of the solution of problem (1), (2), consider a function $u(x)$ which is a solution of the homogeneous problem (1), (2) (all right-hand sides in the problem are zero). If the problem (1), (2) has at least two solutions, such a function exists. It is clear that $u(x)$ is a polyharmonic function, satisfying the following homogeneous conditions

$$D_\nu^{k+p}u(x) + \alpha_k D_\nu^{k+p}u(x^*)\Big|_{\partial\Omega}$$

$$= \Lambda^{[k+p]}(1+\alpha_k I_C)u(x)\Big|_{\partial\Omega} = 0, \ k = 0,1,...,m-1. \tag{8}$$

Since $(-\alpha_k)^l \neq 1$, then applying the operators J_{α_k} from (4) to the equality (8) and using Lemma 1, we get

$$0 = J_{\alpha_k}\Lambda^{[m+p]}(1+\alpha_k I_C)u(x) = \Lambda^{[m+p]}J_{\alpha_k}(1+\alpha_k I_C)u(x) = \Lambda^{[m+p]}u(x)$$

$$= D_\nu^{k+p}u(x), x \in \partial\Omega,$$

or

$$D_\nu^{k+p}u(x)\Big|_{\partial\Omega} = 0.$$

Therefore, if $u(x)$ is the solution of the homogenous problem (1), (2), then it is also the solution of the homogeneous problem (5), (6). Then, due to uniqueness of the solution of the Dirichlet problem (the case $p = 0$), we obtain the uniqueness of the solution of the problem (1), (2). Similarly, by the statements of Theorem 2, we obtain the remaining statements of this theorem. Theorem is proved.

5 Existence

In this section we present a statement on the existence of the solution of the problem (1), (2).

Theorem 4. *Let $(-\alpha_k)^l \neq 1, k = 0,1,...,m-1$, and $f(x)$, $g_k(x), k = 0,1,...,m-1$, be smooth enough functions. Then*

1) if $p = 0$, then a solution of the problem (1), (2) exists and is unique;

2) if $p = 1$, and $\alpha_k \neq -1, k = 0,1,...,m-1$, then the necessary and sufficient condition for the solvability of the problem (1), (2) has the form

$$\int\limits_{\partial\Omega}\sum_{k=1}^{m}(-1)^{k+1}\begin{pmatrix} 2m-k-1 \\ k-1 \end{pmatrix}\frac{(2m-2k-1)!!}{1+\alpha_k}g_k(x)dS_x$$

$$+\int\limits_{\Omega}\frac{\left(|x|^2-1\right)^{m-1}}{(2m-2)!!}f(x)dx = 0. \tag{9}$$

If the solution exists, then it is unique up to a constant.

Proof. Consider the auxiliary Dirichlet problem

$$(-\Delta)^m v(x) = f(x), \quad x \in \Omega, \tag{10}$$

$$D_\nu^{k+p} v(x) = J_{\alpha_k} g_k(x), \quad x \in \partial\Omega, k = 0, 1, ..., m-1, \tag{11}$$

where the operator J_{α_k} is defined in (4). We check that its solution $v(x)$ is also a solution of the considered problem (1), (2). Indeed, the function $v(x)$ satisfies the equation (1). Applying the operator $1 + \alpha_k I_C$ to the condition (11) and using Lemmas 1 and 2, we get

$$g_k(x) = (1 + \alpha_k I_C) J_{\alpha_k} g_k(x) = (1 + \alpha_k I_C) D_\nu^{k+p} v(x)|_{\partial\Omega}$$
$$= (1 + \alpha_k I_C) \Lambda^{[k+p]} v(x)|_{\partial\Omega} = \Lambda^{[k+p]} (1 + \alpha_k I_C) v(x)|_{\partial\Omega}$$
$$= D_\nu^{k+p} (1 + \alpha_k I_C) v(x)|_{\partial\Omega} = D_\nu^{k+p} v(x) + \alpha_k D_\nu^{k+p} v(x^*)|_{\partial\Omega},$$

where $x \in \partial\Omega$, i.e. the condition (2) holds. So, the function $v(x)$ is the solution of the problem (1), (2), and, if $v(x)$ exists, then the problem (1), (2) is solvable.

The case when the solution of the problem (10), (11) does not exist but $u(x)$ exists, is impossible. Indeed, let $u(x)$ be a solution of the equation (10). Applying the operator J_{α_k} to the condition (2) and using Lemmas 1 and 2, we have

$$J_{\alpha_k} g_k(x) = J_{\alpha_k} (D_\nu^{k+p} u(x) + \alpha_k D_\nu^{k+p} u(x^*))|_{\partial\Omega}$$
$$= J_{\alpha_k} D_\nu^{k+p} (1 + \alpha_k I_C) u(x)|_{\partial\Omega} = J_{\alpha_k} \Lambda^{[k+p]} (1 + \alpha_k I_C) u(x)|_{\partial\Omega}$$
$$= \Lambda^{[k+p]} J_{\alpha_k} (1 + \alpha_k I_C) u(x)|_{\partial\Omega} = \Lambda^{[k+p]} u(x) = D_\nu^{k+p} u(x)|_{\partial\Omega},$$

where $x \in \partial\Omega$, i.e. condition (11) holds. Hence, $u(x)$ is the solution of the problem (10), (11), which contradicts to the assumption. Problems (1), (2) and (10), (11) are solvable simultaneously. Smoothness of the functions $J_{\alpha_k} g_k(x)$ and $g_k(x)$ are the same.

Using Theorems 1 and 2, we can find the solvability conditions of the problem (10), (11). Obviously these conditions will be the solvability conditions of the problem (1), (2).

1) Let $p = 0$. In this case, by Theorem 1, for any functions on the right-hand sides of the problem with a given smoothness its solution exists and is unique.

2) Let $p = 1$. In this case, by Theorem 2, the necessary and sufficient solvability condition of the problem (10), (11) is the integral equality

$$\int\limits_{\partial\Omega} \sum_{k=1}^{m} (-1)^{k+1} \begin{pmatrix} 2m-k-1 \\ k-1 \end{pmatrix} (2m-2k-1)!! J_{\alpha_k} g_k(x) dS_x$$
$$+ \int\limits_{\Omega} \frac{\left(|x|^2 - 1\right)^{m-1}}{(2m-2)!!} f(x) dx = 0. \tag{12}$$

Let us transform the integral on the right-hand side of (12).

Lemma 3. *Let the function $\varphi(x)$ be continuous on $\partial\Omega$ and C be an orthogonal matrix, then*

$$\int\limits_{\partial\Omega} \varphi(Cx)\, dS_x = \int\limits_{\partial\Omega} \varphi(x)\, dS_x.$$

Proof. Let the function $w(x)$ be a solution of the Dirichlet problem for the Laplace equation in Ω with condition $w(x)|_{\partial\Omega} = \varphi(x)$, $x \in \partial\Omega$. Then the function $w(Cx)$ is a solution of the Dirichlet problem for the Laplace equation in Ω with the condition $w(Cx)|_{\partial\Omega} = \varphi(Cx)$, $x \in \partial\Omega$. Therefore, due to the Poisson's formula, we have

$$\int\limits_{\partial\Omega} \varphi(Cx)\, dS_x = \int\limits_{\partial\Omega} w(Cx)\, dS_x = \omega_n w(0) = \int\limits_{\partial\Omega} \varphi(x)\, dS_x,$$

where ω_n is the area of the unit sphere. Lemma is proved.

Using Lemma 3, the condition $\alpha_k \neq -1$, and taking into account that the natural degree of the orthogonal matrix is an orthogonal matrix as well, we find

$$\int\limits_{\partial\Omega} J_{\alpha_k} g_k(x)\, dS_x = \frac{1}{1-(-\alpha_k)^l} \sum_{q=0}^{l-1} (-\alpha_k)^q \int\limits_{\partial\Omega} I_C^q g_k(x)\; dS_x$$

$$= \frac{1}{1-(-\alpha_k)^l} \sum_{q=0}^{l-1} (-\alpha_k)^q \int\limits_{\partial\Omega} g_k(C^q x)\, dS_x = \frac{1}{1-(-\alpha_k)^l} \sum_{q=0}^{l-1} (-\alpha_k)^q \int\limits_{\partial\Omega} g_k(x)\, dS_x$$

$$= \frac{(1+\alpha_k)}{(1+\alpha_k)(1-(-\alpha-k)^l)} \sum_{q=0}^{l-1} (-\alpha_k)^q \int\limits_{\partial\Omega} g_k(x)\, dS_x = \int\limits_{\partial\Omega} \frac{g_k(x)}{1+\alpha_k}\, dS_x.$$

This implies that the condition (12) can be transformed to the form (9). Theorem is proved. □

6 Representation of the solution

In this section we give a method of constructing solutions of the problem (1), (2) with homogeneous boundary conditions.

Theorem 5. *Let $g_k(x) = 0, k = 0, 1, ..., m-1$. Then*

1) if $p = 0$, then the solution of the problem (1), (2) can be represented in the form

$$u(x) = \int\limits_{\Omega} G_D(x, y) f(y)\, dy,$$

where $G_D(x,y)$ is the Green's function of the Dirichlet problem for the polyharmonic equation (1) *in* Ω*;*

2) if $p=1$ and (9) *holds, then the solution of the problem* (1), (2) *can be represented in the form*

$$u(x)=\int_0^1 \frac{v(sx)}{s}\,ds+C, \tag{13}$$

where C is an arbitrary constant and $v(x)$ is a solution of the following Dirichlet problem

$$\begin{gathered}(-\Delta)^m v(x)=(\Lambda+2m)\,f(x),\ x\in\Omega;\\ D_\nu^k v(x)\Big|_{\partial\Omega}=0, k=0,1,...,m-1,\ v(0)=0.\end{gathered} \tag{14}$$

Proof. The auxiliary problem (10), (11), whose solution coincides with the solution of the problem (1), (2) (see the proof of Theorem 3), with the help of properties of the operator Λ takes the form

$$\begin{gathered}(-\Delta)^m v(x)=f(x),\ x\in\Omega,\\ \Lambda^{[k+p]}v(x)|_{\partial\Omega}=0, k=0,1,...,m-1.\end{gathered}$$

1) Let $p=0$, then in this case the auxiliary problem is the Dirichlet problem and its solution coincides with the solution of the problem (1), (2) (see. [17]):

$$v(x)=u(x)=\int_\Omega G_D(x,y)f(y)\,dy. \tag{15}$$

2) Let $p=1$. Boundary conditions for the auxiliary problem take the form

$$\Lambda^{[k+1]}v(x)|_{\partial\Omega}\equiv\frac{\partial^k v(x)}{\partial\nu^k}|_{\partial\Omega}=0, k=0,1,...,m-1.$$

Let us apply the operator $\Lambda+2m$ to the polyharmonic equation of the problem. Due to the equality $\Delta^k\Lambda u=(\Lambda+2k)\Delta^k u$ and denoting $w=\Lambda v$, for $w(x)$, we get the following Dirichlet problem (14):

$$\begin{gathered}(-\Delta)^m w(x)=(\Lambda+2m)f(x),\ x\in\Omega,\\ w(x)|_{\partial\Omega}=0,\ \Lambda^{[k]}w(x)|_{\partial\Omega}=0,\ k=1,2,...,m-1.\end{gathered}$$

By the formula (15) we find

$$w(x)=\int_\Omega G_D(x,y)(\Lambda+4)f(y)\,dy.$$

As in [16] equation $w = \Lambda v$ in the class of smooth functions $v(x)$ has a solution only if $w(0) = 0$, and this solution can be written in the form

$$u(x) = \int_0^1 \frac{w(sx)}{s}\, ds + C.$$

Theorem is proved.

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Карачик В.В., Турметов Б.Х. ПОЛИГАРМОНИЯЛЫҚ ТЕҢДЕУ ҮШІН КЕЙБІР БЕЙЛОКАЛ ШЕТТІК ЕСЕПТЕРДІҢ ШЕШІЛІМДІЛІГІ ЖӘЙЛІ

Бұл жұмыс полигармониялық теңдеу үшін кейбір классикалық емес шеттік есептердің шешілімдігі мәселесіне арналған. Бұл есептер полигармониялық теңдеу үшін Дирихле және Нейман есептерін жалпылайды. Қарастырылатын есептер Бицадзе-Самарский тектес бейлокал шеттік есептер болып табылады. Зерттелетін есептер оларды Дирихле есебіне және Нейман түріндегі есепке келтіру арқылы шешіледі. Есептің шешімінің бар және жалғыз болуы туралы теоремалар дәлелденген. Бірлік шарда полигармониялық теңдеу үшін Нейман түріндегі шеттік есептің шешілімділігінің қажетті және жеткілікті шарттары анықталған. Грин функцияларын қолдана отырып, сондай-ақ Дирихле есебінің шешімінің бар болуы туралы тұжырымды пайдалана отырып, қарастырылған есептердің шешімдері үшін интегралдық кейіптемелер алынған.

Кілттік сөздер. Полигармониялық теңдеу, бейлокал есеп, инволюция, Дирихле есебі, Нейман түріндегі есеп, жалғыздық, бар болу.

Карачик В.В., Турметов Б.Х. О РАЗРЕШИМОСТИ НЕКОТОРЫХ НЕЛОКАЛЬНЫХ КРАЕВЫХ ЗАДАЧ ДЛЯ ПОЛИГАРМОНИЧЕСКОГО УРАВНЕНИЯ

Данная работа посвящена вопросам разрешимости некоторых неклассических краевых задач для полигармонического уравнения. Эти задачи обобщают задачи Дирихле и Неймана для полигармонического уравнения. Рассматриваемые задачи являются нелокальными краевыми задачами типа Бицадзе-Самарского. Исследуемые задачи решаются путем сведения их к задаче Дирихле и задаче типа Неймана. Доказаны теоремы о существовании и единственности решения задачи. Получены необходимые и достаточные условия разрешимости задачи типа Неймана для нелокального полигармонического уравнения в единичном шаре. Применяя функции Грина, а также утверждение о существовании решения задачи Дирихле, получены интегральные представления для решений рассматриваемых задач.

Ключевые слова. Полигармоническое уравнение, нелокальная задача, инволюция, задача Дирихле, задача типа Неймана, единственность, существование.

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Reversed Hardy-Littlewood-Sobolev inequality on homogeneous Lie groups

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Abstract. In this short note, we prove the reversed Hardy-Littlewood-Sobolev inequality on homogeneous Lie groups. Proof of this inequality is based on reversed Young's inequality and reversed Marcienkiewicz interpolation theorem.

Keywords. Hardy-Littlewood-Sobolev inequality, Reversed Hardy-Littlewood-Sobolev inequality, fractional integral, homogeneous Lie group.

1 Introduction

In their pioneering paper [1], Hardy and Littlewood proved the following theorem:

Theorem 1. *Let $1 < p < q < \infty$ and $u \in L^p(0,\infty)$ with $\frac{1}{q} = \frac{1}{p} + \lambda - 1$, then*

$$\|T_\lambda u\|_{L^q(0,\infty)} \leq C\|u\|_{L^p(0,\infty)}, \tag{1}$$

where C is a positive constant independent of u. Here T_λ is the one dimensional fractional integral operator on $(0,\infty)$ given by

$$T_\lambda u(x) = \int_0^\infty \frac{u(y)}{|x-y|^\lambda} dy, \quad 0 < \lambda < 1. \tag{2}$$

The multidimensional extention of (2) is

$$I_\lambda u(x) = \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^\lambda} dy, \quad 0 < \lambda < N. \tag{3}$$

Then, the corresponding generalisation of the Hardy-Littlewood inequality was proved by Sobolev in [2]:

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Theorem 2. *Let* $1 < p < q < \infty$, $u \in L^p(\mathbb{R}^N)$ *with* $\frac{1}{q} = \frac{1}{p} + \frac{\lambda}{N} - 1$. *Then*

$$\|I_\lambda u\|_{L^q(\mathbb{R}^N)} \leq C\|u\|_{L^p(\mathbb{R}^N)}, \tag{4}$$

where C *is a positive constant independent of* u.

Later, in [3] Stein and Weiss obtained the following two-weight Hardy-Littlewood-Sobolev inequality, which is also known as the Stein-Weiss inequality.

Theorem 3. *Let* $0 < \lambda < N$, $1 < p < \infty$, $\alpha < \frac{N(p-1)}{p}$, $\beta < \frac{N}{q}$, $\alpha + \beta \geq 0$ *and* $\frac{1}{q} = \frac{1}{p} + \frac{\lambda+\alpha+\beta}{N} - 1$. *If* $1 < p \leq q < \infty$, *then*

$$\||x|^{-\beta} I_\lambda u\|_{L^q(\mathbb{R}^N)} \leq C\||x|^{\alpha} u\|_{L^p(\mathbb{R}^N)}, \tag{5}$$

where C *is a positive constant independent of* u.

So, in the papers [4], [5] and [6], authors showed reversed Hardy-Littlewood-Sobolev inequality on the Euclidean space $\mathbb{R}^N$.

Theorem 4. *For any* $1 \leq N < \lambda$, $\frac{N}{\lambda} < p < 1$ *and* q *given by*

$$\frac{1}{q} = \frac{1}{p} - \frac{\lambda}{N}, \tag{6}$$

there exists a constant $C = C(n, \lambda, p) > 0$, *such that for all nonnegative* $u \in L^p(\mathbb{R}^N)$,

$$\|I_\lambda u\|_{L^q(\mathbb{R}^N)} \geq C\|u\|_{L^p(\mathbb{R}^N)}. \tag{7}$$

Nowadays, there is a number of studies related to this subject on $\mathbb{R}^N$. We refer the above excellent presentations [4], [5] and [6] as well as references therein for further discussions.

At the same time, there is another layer of intensive research over the years related to the Hardy-Littlewood-Sobolev inequalities in subelliptic settings. As expected, the subelliptic Hardy-Littlewood-Sobolev inequality was obtained on the most important example of the Heisenberg group by Folland and Stein in [7] (see, also [8]). In this case, we also note that the optimal constant for the inequality is given by Frank and Lieb in [9] (in the Euclidean case this was done earlier by Lieb in [10]). Futhermore, in this direction systematic studies of different functional inequalities on (general) homogeneous Lie groups were initiated by the paper [11]. Also, Hardy-Littlewood-Sobolev inequality in homogeneous Lie groups is proved in [12]. We refer to the open access book [13] for further discussions in this direction.

Let us consider Riesz operator in the following form:

$$I_\gamma u(x) = \int_{\mathbb{G}} \frac{u(y)}{|y^{-1}x|^\gamma} dy. \tag{8}$$

The main result of this paper is as follows:

- **Reversed Hardy-Littlewood-Sobolev inequality**: Let $\mathbb{G}$ be a homogeneous Lie group with $1 \leq Q < \alpha$, $\frac{Q}{\alpha} < p < 1$, such that
$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}. \tag{9}$$
Then we have
$$\|I_{Q-\alpha}u\|_{L^q(\mathbb{G})} \geq C\|u\|_{L^p(\mathbb{G})}, \tag{10}$$

where C is a positive constant independent of u.

2 Reversed Hardy-Littlewood-Sobolev inequality

In this section we prove the reversed Hardy-Littlewood-Sobolev inequality on homogeneous Lie groups. In order to do it, first we present reversed Young inequality on homogeneous Lie groups.

Let us recall that a Lie group (on $\mathbb{R}^N$) $\mathbb{G}$ with the dilation
$$D_\lambda(x) := (\lambda^{\nu_1}x_1, \ldots, \lambda^{\nu_N}x_N),\ \nu_1, \ldots, \nu_n > 0,\ D_\lambda : \mathbb{R}^N \to \mathbb{R}^N,$$
which is an automorphism of the group $\mathbb{G}$ for each $\lambda > 0$, is called a *homogeneous (Lie) group*. For simplicity, throughout this paper we use the notation λx for the dilation D_λ. The homogeneous dimension of the homogeneous group $\mathbb{G}$ is denoted by $Q := \nu_1 + \ldots + \nu_N$. Also, in this note we denote a homogeneous quasi-norm on $\mathbb{G}$ by $|x|$, which is a continuous non-negative function
$$\mathbb{G} \ni x \mapsto |x| \in [0, \infty), \tag{11}$$
with the properties

i) $|x| = |x^{-1}|$ for all $x \in \mathbb{G}$,

ii) $|\lambda x| = \lambda|x|$ for all $x \in \mathbb{G}$ and $\lambda > 0$,

iii) $|x| = 0$ iff $x = 0$.

Moreover, the following polarisation formula on the homogeneous Lie groups will be used in our proofs: there is a (unique) positive Borel measure σ on the unit quasi-sphere $\mathfrak{S} := \{x \in \mathbb{G} : |x| = 1\}$, so that for every $f \in L^1(\mathbb{G})$ we have
$$\int_{\mathbb{G}} f(x)dx = \int_0^\infty \int_{\mathfrak{S}} f(ry)r^{Q-1}d\sigma(y)dr. \tag{12}$$
The quasi-ball centred at $x \in \mathbb{G}$ with radius $R > 0$ can be defined by
$$B(x, R) := \{y \in \mathbb{G} : |x^{-1}y| < R\}. \tag{13}$$

Note that the standart Lebesque measure on $\mathbb{R}^N$ coincides with the Haar measure on the homogeneous Lie group $\mathbb{G}$. We refer to [14] for the original appearance of such groups, and to [13] for a recent comprehensive treatment.

Let us recall that for a measurable function f on $\mathbb{G}$ with $0 < p < \infty$, for weak L^p norm we define

$$\|f\|_{L^p_W} := \inf\{z > 0 : \ m\{|f| < t\} \le \frac{z^p}{t^p}\}.$$

In the case $p < 0$ we define weak L^p norm in the following form

$$\|f\|_{L^p_W} := \sup\{z > 0 : m\{|f| < t\} \le \frac{z}{t^p}\}.$$

Definition 1. For $q < 0 < p < 1$, we say operator L is of the weak-type (p, q), if there exists a constant $C(p, q) > 0$, such that for all $u \in L^p(\mathbb{G})$,

$$m\{x : |Lu(x)| < \zeta\} \le C(p,q) \left(\frac{\|u\|_{L^p(\mathbb{G})}}{\zeta} \right)^q, \ \forall \zeta > 0.$$

Proposition 1. (Proposition 2.5, [4]) *Let L be a linear operator which maps any nonnegative function to a nonnegative function. For a pair of numbers (p_1, q_1), (p_2, q_2) satisfying $q_i < 0 < p_i < 1$, $i = 1, 2$, $p_1 < p_2$ and $q_1 < q_2$, if L is of weak-types (p_1, q_1) and (p_2, q_2) for all nonnegative functions, then for any $\xi \in (0, 1)$, and*

$$\frac{1}{p} = \frac{1-\xi}{p_1} + \frac{\xi}{p_2}, \ \ \frac{1}{q} = \frac{1-\xi}{q_1} + \frac{\xi}{q_2},$$

L is reversed strong-type (p, q) for all nonnegative functions, that is,

$$\|Lu\|_{L^q(\mathbb{G})} \ge C\|u\|_{L^p(\mathbb{G})},$$

where $C = C(p_1, p_2, q_1, q_2, \gamma) > 0$.

Theorem 5. *Let $\mathbb{G}$ be a homogeneous Lie group. Let $0 < p < 1$ and $q, r < 0$ be such that*

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r},$$

and let f, g be nonnegative functions. Then we have

$$\|f * g\|_{L^q(\mathbb{G})} \ge \|f\|_{L^p(\mathbb{G})}\|g\|_{L^r(\mathbb{G})}. \tag{14}$$

Proof. By the definition with $\frac{1}{q} + \frac{1}{p'} + \frac{1}{r'} = 1$, $\frac{p}{q} + \frac{p}{r'} = 1$, $\frac{r}{q} + \frac{r}{p'} = 1$, and by using reversed Hölder's inequality, we calculate

$$f * g(x) = \int_{\mathbb{G}} f(x)g(y^{-1}x)dy = \int_{\mathbb{G}} f^{\frac{p}{r'}}(y) f^{\frac{p}{q}}(y) g^{\frac{r}{q}}(y^{-1}x) g^{\frac{r}{p'}}(y^{-1}x)dy$$

$$\geq \left(\int_{\mathbb{G}} f^p(y)dy\right)^{\frac{1}{r'}} \left(\int_{\mathbb{G}} g^r(y^{-1}x)dy)\right)^{\frac{1}{p'}} \left(\int_{\mathbb{G}} f^p(y)g^r(y^{-1}x)dy\right)^{\frac{1}{q}}$$

$$= \|f\|_{L^p(\mathbb{G})}^{\frac{p}{r'}} \|g\|_{L^r(\mathbb{G})}^{\frac{r}{p'}} \left(\int_{\mathbb{G}} f^p(y)g^r(y^{-1}x)dy\right)^{\frac{1}{q}}. \tag{15}$$

It implies

$$\int_{\mathbb{G}} (f * g(x))^q dx \leq \|f\|_{L^p(\mathbb{G})}^{\frac{pq}{r'}} \|g\|_{L^r(\mathbb{G})}^{\frac{qr}{p'}} \left(\int_{\mathbb{G}}\int_{\mathbb{G}} f^p(y)g^r(y^{-1}x)dydx\right)$$

$$= \|f\|_{L^p(\mathbb{G})}^{\frac{pq}{r'}} \|g\|_{L^r(\mathbb{G})}^{\frac{qr}{p'}} \|f\|_{L^p(\mathbb{G})}^{p} \left(\int_{\mathbb{G}} g^r(y)dy\right) = \|f\|_{L^p(\mathbb{G})}^{q} \|g\|_{L^r(\mathbb{G})}^{q}. \tag{16}$$

Then finally,

$$\|f * g\|_{L^q(\mathbb{G})} \geq \|f\|_{L^p(\mathbb{G})} \|g\|_{L^r(\mathbb{G})}. \tag{17}$$

It completes the proof. □

Now we state the reversed Hardy-Littlewood-Sobolev inequality on $\mathbb{G}$.

Theorem 6. *Let $\mathbb{G}$ be a homogeneous Lie group with $1 \leq Q < \alpha$, $\frac{Q}{\alpha} < p < 1$ and*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}. \tag{18}$$

Then,

$$\|I_{Q-\alpha} u\|_{L^q(\mathbb{G})} \geq C\|u\|_{L^p(\mathbb{G})}, \tag{19}$$

where C is a positive constant independent of u.

Proof. For the prove of this theorem we will use Marcinkiewicz interpolation theorem. We show first:

$$m\{x : |I_{Q-\alpha} u| \leq \zeta\} \leq C \left(\frac{\|u\|_{L^p(\mathbb{G})}}{\zeta}\right)^q, \ \ \zeta > 0. \tag{20}$$

Let us rewrite the Riesz operator in the following form:

$$I_{Q-\alpha} u(x) = K * u(x) = |x|^{\alpha-Q} * u(x) = K_1 * u(x) + K_2 * u(x),$$

where

$$K_1(x) := \begin{cases} |x|^{\alpha-Q}, \ \text{if } |x| \leq \theta, \\ 0, \ \text{if } |x| > \theta, \end{cases} \quad \text{and} \quad K_2(x) := \begin{cases} |x|^{\alpha-Q}, \ \text{if } |x| > \theta, \\ 0, \ \text{if } |x| \leq \theta. \end{cases} \tag{21}$$

Then, we have

$$m\{x : |K * u(x)| < 2\zeta\} \leq m\{x : |K_1 * u(x)| < \zeta\} + m\{x : |K_2 * u(x)| < \zeta\}, \tag{22}$$

where m is the Haar measure on $\mathbb{G}$. It is enough to prove inequality (20) with 2ζ instead of ζ in the left-hand side of the inequality. Without loss of generality we can assume $\|u\|_{L^p(\mathbb{G})} = 1$.

By taking $\beta_1 \in (\frac{Q}{Q-\alpha}, 0)$, we have

$$Q + (\alpha - Q)\beta_1 \geq Q + (\alpha - Q)\frac{Q}{Q-\alpha} = Q - Q = 0,$$

finally, $Q + (\alpha - Q)\beta_1 \geq 0$.

By using this and Theorem 5, we get

$$\|K_1 * u\|_{L^{r_1}(\mathbb{G})}$$

$$\geq \left(\int_{0<|x|\leq\theta} \frac{1}{|x|^{(Q-\alpha)\beta_1}} dx\right)^{\frac{1}{p'}} \left(\int_{0<|x|\leq\theta} \frac{1}{|x|^{(Q-\alpha)\beta_1}} dx\right)^{\frac{1}{r_1}} \|u\|_{L^p(\lambda)}$$

$$= C\left(\int_0^\theta r^{Q-1} r^{-\beta_1(Q-\alpha)} dr\right)^{\frac{1}{\beta_1}} = C\theta^{\frac{Q-\beta_1(Q-\alpha)}{\beta_1}}, \tag{23}$$

where $\frac{1}{p} + \frac{1}{\beta_1} = \frac{1}{r_1} + 1$, with $\beta_1 \in (\frac{Q}{Q-\alpha}, 0)$, $r_1 < 0$. Let $0 < \sigma < \infty$, $f \in L^\sigma(\lambda)$ and by using Chebychev's inequality with $\tau > 0$, we have

$$m\{x : |f(x)| > \tau\} \leq \frac{\int_{|f(x)|>\tau} |f(x)|^\sigma dx}{\tau^\sigma} \leq \frac{\|f\|^\sigma_{L^\sigma(\mathbb{G})}}{\tau^\sigma}, \tag{24}$$

then

$$m\{x : |f(x)|^{-1} < \frac{1}{\tau}\} \leq \frac{\|f\|^\sigma_{L^\sigma(\mathbb{G})}}{\tau^\sigma}, \tag{25}$$

and by changing $f(x) = \frac{1}{g(x)}$ and $\zeta = \frac{1}{\tau}$, we obtain

$$m\{x : |g(x)| < \zeta\} \leq \frac{\|g^{-1}\|^\sigma_{L^\sigma(\lambda)}}{\left(\frac{1}{\zeta}\right)^\sigma} = \frac{\int_{\mathbb{G}} g^{-\sigma}(x)dx}{\zeta^{-\sigma}}. \tag{26}$$

By taking $-\sigma = r$, we have

$$m\{x : |g(x)| < \zeta\} \leq \frac{\int_{\mathbb{G}} g^{-\sigma}(x)dx}{\zeta^{-\sigma}} = \frac{\|g\|^r_{L^r(\mathbb{G})}}{\zeta^r}. \tag{27}$$

Then with $r_1 < 0$, we have

$$m\{x : |K_1 * u(x)| \leq \zeta\} \leq C\frac{\|K_1 * u\|^{r_1}_{L^{r_1}(\lambda)}}{\zeta^{r_1}} \leq C\frac{\theta^{\frac{r_1(Q-\beta_1(Q-\alpha))}{\beta_1}}}{\zeta^{r_1}}. \tag{28}$$

Similarly, by using Theorem 5, we have

$$\|K_2 * u\|_{L^{r_2}(\mathbb{G})} \geq \left(\int_{\theta\leq|x|} \frac{1}{|x|^{\beta_2(Q-\alpha)}}dx\right)^{\frac{1}{p'}} \left(\int_{\mathbb{G}} \frac{1}{|x|^{\beta_2(Q-\alpha)}}dx\right)^{\frac{1}{r_2}} \|u\|_{L^p(\mathbb{G})}$$
$$\geq C\theta^{Q-\beta_2(Q-\alpha)}, \tag{29}$$

where $\frac{1}{p}+\frac{1}{\beta_2}=\frac{1}{r_2}+1$, with $\beta_2 \leq \frac{Q}{Q-\alpha}$, $r_2<0$. Then,

$$m\{x : K_2 * u(x) \leq \zeta\} \leq C\frac{\theta^{\frac{r_2(Q-\beta_2(Q-\alpha))}{\beta_2}}}{\zeta^{r_2}}. \tag{30}$$

By choosing $\theta=\zeta^{\frac{p}{\alpha p-Q}}$, $\frac{1}{p}+\frac{1}{\beta_i}=1+\frac{1}{r_i}$, $i=1,2$, and by the assumption we compute

$$\frac{r_i p}{p\alpha - Q}\left(\frac{Q}{\beta_i}+\alpha-Q\right)-r_i=\frac{r_i p}{p\alpha-Q}\left(\frac{Qp-r_iQ+\alpha p r_i}{pr_i}\right)-r_i$$
$$=\frac{Qp-r_id+\alpha pr_i}{p\alpha-Q}-r_i=\frac{Qp-r_iQ+\alpha pr_i-\alpha pr_i+Qr_i}{p\alpha-Q}=-\frac{pQ}{Q-\alpha p}=-q, \tag{31}$$

for $i=1,2$. By using this fact with $\theta=\zeta^{\frac{p}{\alpha p-Q}}$, we get

$$m\{x : |K * u(x)| > 2\zeta\} \leq m\{x : |K_1 * u(x)| > \zeta\} + m\{x : |K_2 * u(x)| > \zeta\}$$
$$\leq C\left(\frac{\theta^{\frac{r_1(Q-\beta_1(Q-\alpha))}{\beta_1}}}{\zeta^{r_1}}+\frac{\theta^{\frac{r_2(Q-\beta_2(Q-\alpha))}{\beta_2}}}{\zeta^{r_2}}\right)=\frac{C}{\zeta^q}. \tag{32}$$

Finally, we have

$$m\{x : I_{Q-\alpha}u \leq \zeta\} \leq C\left(\frac{\|u\|_{L^p(\mathbb{G})}}{\zeta}\right)^q, \ \ \zeta>0. \tag{33}$$

By using Definition 1 and Proposition 1 we have reversed Hardy-Littlewood-Sobolev inequality.

The proof of Theorem 6 is complete. □

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Касымов А., Сураган Д. БІРТЕКТІ ЛИ ТОПТАРЫНДАҒЫ ХАРДИ-ЛИТТЛВУД-СОБОЛЕВ КЕРІ ТЕҢСІЗДІГІ

Бұл қысқа мақалада, біз Харди-Литтлвуд-Соболев кері теңсіздігін біртекті Ли топтарында дәлелдедік. Бұл теңсіздіктің дәлелдеуі Янг кері теңсіздігі мен Марцинкевичтің кері интерполяциялық теоремасына негізделген.

Кілттік сөздер. Харди-Литтлвуд-Соболев теңсіздігі, Харди-Литтлвуд-Соболев кері теңсіздігі, бөлшектік интеграл, біртекті Ли тобы.

Касымов А., Сураган Д. ОБРАТНОЕ НЕРАВЕНСТВО ХАРДИ-ЛИТТЛВУД-СОБОЛЕВА НА ОДНОРОДНЫХ ГРУППАХ ЛИ

В этой короткой заметке, мы доказали обратное неравенство Харди-Литтлвуд-Соболева на однородных группах Ли. Доказательство этого неравентсва было основано на обратном неравенстве Янга и обратной интерполяционной теореме Марцинкевича.

Ключевые слова. Неравенство Харди-Литтлвуд-Соболева, обратное неравенство Харди-Литтлвуд-Соболева, дробный интеграл, однородное группа Ли.

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Solving two-phase spherical Stefan problem using heat polynomials

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Abstract. The inverse two-phase spherical Stefan problem for unknown boundary heat flux is solved by the method of the heat polynomials. Side by side with obtaining an exact solution, two methods for obtaining an approximate solution, collocation and variational methods, convenient for engineering applications are presented and compared. It is shown that both methods give very good approximation even for using only several points. However, the collocation method gives better result for the initial stage of heating, while the variational method is more preferable for the large values of the Fourier criterion. The approximation error estimate is obtained using the principle of maximum for the heat equation. The application of the obtained results for the calculation of the electrical arc heat flux at the contact opening is presented.

Keywords. Stefan problem, heat polynomials, heat flux, melting zone.

1 Introduction

The method of integral error functions and heat polynomials for solving heat equation in a domain with free boundary enables one to obtain the solution in the form handy for engineering application. The solution of the spherical Stefan problem with the boundary heat flux condition using this method is considered in [1]. It was shown that a given boundary function can be approximated by the linear combination of the system of the integral error functions $i^n erfc(x),\ n = 0, 1, 2, ...$, and the first five terms of this combination are sufficient to obtain the error less than 1%. It means according to the maximum principle for the heat equation that the error of approximation of the final solution has the same error. Then this

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approach was successfully applied for solving different Stefan type problems. One of the most important problems in the theory of phenomena in electrical contacts is determining the arc heat flux entering into electrodes. The experimental measuring the dynamics of this flux is very difficult, and sometimes the mathematical modeling only is capable to obtain required information [2]. The mathematical model describing the process of the interaction of the electrical arc with electrodes and the dynamics of their melting is based on the spherical Stefan problem, and if we want to define the arc heat flux, the inverse spherical Stefan problem should be considered [3].

2 Mathematical Model

The inverse Stefan problem consists in determining the arc heat flux $P(t)$ and the temperature distribution $\theta(r,t)$ in the molten contact hemisphere $r_0 < r < r + \alpha(t)$, if $\alpha(t)$ is given from the measurement. If the arc burning period is $0 \le t \le t_0$ and the final radius of the molten zone at $t = t_a$ is r_a, then the dynamics of the arc radius increasing at the melting can be approximated by the formula

$$\alpha(t) = r_0 + \alpha_0\sqrt{t} \quad \alpha_0 = (r_a - r_0)/\sqrt{t_0}. \tag{1}$$

The heat equation for the melting zone can be written in the form

$$\frac{\partial\theta}{\partial t} = a^2\left(\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial\theta}{\partial r}\right) \quad r_0 < r < \alpha(t), \quad 0 < t < t_a. \tag{2}$$

The initial and boundary conditions are

$$\theta\Big|_{t=0} = 0, \tag{3}$$

$$-\lambda\frac{\partial\theta}{\partial r}\Big|_{r=r_0} = P(t) \tag{4}$$

and on the interface of the phase transformation

$$\theta(\alpha(t), t) = \theta_m, \tag{5}$$

$$-\lambda\frac{\partial\theta}{\partial r}\Big|_{r=\alpha(t)} = L\gamma\frac{d\alpha}{dt}, \tag{6}$$

where θ_m is the melting temperature, α, L, γ are coefficients of the heat conductivity, latent heat of melting and density, respectively.

To simplify the calculation we can introduce the new dimensionless time $t_1 = t/t_a$, then the time interval of arcing changes to $0 < t_1 < 1$. Thus we can take $t_a = 1$ at once in (2).

This problem for the spherical heat equation can be reduced to the ordinary one-dimensional equation by the substitutions

$$\theta = \frac{u}{r}, \quad r - r_0 = x, \quad \beta(t) = \alpha(t) - r_0. \tag{7}$$

Then the problem transforms to the form

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \beta(t), \quad 0 < t < t_a, \tag{8}$$

$$\theta\big|_{t=0} = 0, \tag{9}$$

$$-\lambda \left[r_0 \frac{\partial u}{\partial x} - u \right] \Big|_{x=0} = r_0^2 P(t), \tag{10}$$

$$u(\beta(t), t) = U_m, \tag{11}$$

$$-\lambda \left[\beta(t) \frac{\partial u}{\partial x} - u \right] \Big|_{x=\beta(t)} = \beta^2(t) L \gamma \frac{d\beta}{dt}. \tag{12}$$

The solution of this problem can be represented in the form:

$$u(x,t) = \sum_{n=0}^{\infty} A_n v_{2n}(x,t) + \sum_{n=0}^{\infty} B_n v_{2n+1}(x,t), \tag{13}$$

where

$$v_{2n}(x,t) = \sum_{k=0}^{n} \frac{(2n)! a^{2k} x^{2n-2k}}{k!(2n-2k)!} t^k, \quad v_{2n+1}(x,t) = \sum_{k=0}^{n} \frac{(2n+1)! a^{2k} x^{2n-2k+1}}{k!(2n-2k+1)!} t^k, \tag{14}$$

are heat polynomials satisfying (8) at arbitrary coefficients A_n, B_n, which should be chosen to satisfy the boundary conditions. We represent the unknown heat flux in the form

$$P(t) = \sum_{k=0}^{l} P_k t^k. \tag{15}$$

From the conditions (9), (10) we have the following system of equations for A_n, B_n:

$$\sum_{n=0}^{m} A_n \sum_{k=0}^{n} \frac{(2n)! a^{2k} \alpha_0^{2n-2k}}{k!(2n-2k)!} t^n + \sum_{n=0}^{m} B_n \sum_{k=0}^{n} \frac{(2n+1)! a^{2k} \alpha_0^{2n-2k+1}}{k!(2n-2k+1)!} t^{n+\frac{1}{2}} = U_m, \tag{16}$$

$$\sum_{n=0}^{m} A_n \sum_{k=0}^{n} \frac{(2n)! a^{2k} \alpha_0^{2n-2k}}{k!(2n-2k)!} t^n + \sum_{n=0}^{m} B_n \sum_{k=0}^{n} \frac{(2n+1)! a^{2k} \alpha_0^{2n-2k+1}}{k!(2n-2k+1)!} t^{n+\frac{1}{2}}$$

$$= -\frac{1}{\lambda} L\gamma\alpha_0^3 \frac{\sqrt{t}}{2} + U_m. \quad (17)$$

To evaluate unknown coefficients we use two methods: variational and collocation.

3 Variational Method

Similarly, like in [4], we take $m = 5$, $U_m = 0$ and the basic points $t = t_i = \frac{2i}{10}$, $i = 0, 1, 2, 3, 4, 5$. To satisfy approximately the condition (16) we consider the functional:

$$J = \int_0^1 \left(\sum_{n=0}^{5} A_n \sum_{k=0}^{n} \frac{(2n)! a^{2k} \alpha_0^{2n-2k}}{k!(2n-2k)!} t^n + \sum_{n=0}^{5} B_n \sum_{k=0}^{n} \frac{(2n+1)! a^{2k} \alpha_0^{2n-2k+1}}{k!(2n-2k+1)!} t^{n+\frac{1}{2}} \right)^2 dt.$$

The minimum of this functional can be found from the equation

$$\frac{\partial J}{\partial A_m} = 2 \int_0^1 \left(\sum_{n=0}^{5} A_n v_{2n}(\beta(t), t) + \sum_{n=0}^{5} B_n v_{2n+1}(\beta(t), t) \right) v_{2m}(\beta(t), t) dt = 0, \ m = \overline{0, 5},$$

where

$$v_{2n}(\beta(t), t) = \sum_{k=0}^{n} \frac{(2n)! a^{2k} \alpha_0^{2n-2k}}{k!(2n-2k)!} t^n, \quad v_{2n+1}(\beta(t), t) = \sum_{k=0}^{n} \frac{(2n+1)! a^{2k} \alpha_0^{2n-2k+1}}{k!(2n-2k+1)!} t^{n+\frac{1}{2}},$$

$$\sum_{n=0}^{5} A_n C_{nm} = -D_m, \quad m = \overline{0, 5}, \quad (18)$$

$$C_{nm} = \int_0^1 v_{2n}(\beta(t), t) v_{2m}(\beta(t), t) dt, \quad D_m = \int_0^1 \sum_{n=0}^{5} B_n v_{2n+1}(\beta(t), t) v_{2m}(\beta(t), t) dt, \ m = \overline{0, 5}.$$

Solving the system (18) with respect to A_n from (18) and substituting the result into the expression (17) we get

$$J = \int_0^1 \left(\sum_{n=0}^{5} B_n w(n, t) - f(t) \right)^2 dt, \quad (19)$$

where

$$w(k, t) = \sum_{m=0}^{5} A_m \bar{v}_{2k}(m, \beta(t), t) + \bar{v}_{2k+1}(m, \beta(t), t),$$

$$\bar{v}_{2k}(m, \beta(t), t) = \sum_{n=0}^{k} \frac{(2k)! a^{2k} (2k-2m) \alpha_0^{2k-2m}}{m!(2k-2m)!} t^k,$$

$$\bar{v}_{2k+1}(m,\beta(t),t)=\sum_{n=0}^{k}\frac{(2k+1)!a^{2k}(2k-2m+1)\alpha_0^{2k-2m+1}}{m!(2k-2m+1)!}t^{k+\frac{1}{2}},$$

$$f(t)=\frac{1}{\lambda}L\gamma\alpha_0^3\frac{\sqrt{t}}{2}.$$

From the condition of maximum of (19) we have

$$\frac{\partial J}{\partial B_m}=2\int_0^1\left(\sum_{n=0}^{5}B_n w(n,t)-f(t)\right)w(m,t)dt=0,\ m=\overline{0,5}, \tag{20}$$

$$\sum_{k=0}^{5}E_{km}B_m=F_m,\quad m=\overline{0,5}, \tag{21}$$

where

$$E_{km}=\int_0^1 w(k,t)w(m,t)dt,\quad F_m=\int_0^1 f(t)w(m,t)dt,\quad m=0,1,...,k.$$

From the expression (21) we get the following results:

$$B_0=-0.784;\quad B_1=-0.062;\quad B_2=0.046\quad B_3=-9.712\times10^{-3};$$

$$B_4=6.788\times10^{-4};\quad B_5=-1.391\times10^{-5}.$$

Similarly, from the expression (16) we obtain:

$$A_0=0.058;\quad A_1=0.904;\quad A_2=-0.389;\quad A_3=0.071;$$

$$A_4=-4.716\times10^{-3};\quad A_5=9.516\times10^{-5}.$$

Now we should define the coefficients for the heat flux in the expression (15). The corresponding variational functional for the condition (10) is

$$J=\int_0^1\left(\sum_{m=0}^{5}P_m t^n+g(t)\right)^2 dt, \tag{22}$$

where

$$g(t)=-\frac{\lambda}{r_0^2}\left[\sum_{m=0}^{5}A_m\left(v_{2m}'(r_0,t)-v_{2m}(r_0,t)\right)+\sum_{m=0}^{5}B_m(v_{2n+1}'(r_0,t)-v_{2m+1}(r_0,t))\right],$$

$$v_{2m}'(r_0,t)=\sum_{k=0}^{m}\frac{(2m)!a^{2k}r_0^{2m-2k}}{k!(2m-2k)!}t^k,$$

$$v'_{2m+1}(r_0,t) = \sum_{k=0}^{n} \frac{(2m+1)!a^{2k}r_0^{2m-2k+1}}{k!(2m-2k+1)!}t^k,$$

$$v_{2m}(r_0,t) = \sum_{k=0}^{m} \frac{(2m)!a^{2k}r_0^{2m-2k}}{k!(2m-2k)!}t^k, \quad v_{2m+1}(r_0,t) = \sum_{k=0}^{m} \frac{(2m+1)!a^{2k}r_0^{2m-2k+1}}{k!(2m-2k+1)!}t^k.$$

The minimum of (22) gives the equation

$$\frac{\partial J}{\partial P_m} = 2\int_0^1 \left(\sum_{n=0}^{5} P_n t^n + g(t) \right) t^m dt = 0,$$

$$\sum_{n=0}^{5} P_n G_{nm} = -H_m, \quad m = \overline{0,5}, \tag{23}$$

where

$$G_{nm} = \int_0^1 t^{n+m} dt, \quad H_m = \int_0^1 g(t) t^m dt, \quad m = \overline{0,5}.$$

From the expression (23) we have the following results:

$$P_0 = -0.009; \quad P_1 = 0.085; \quad P_2 = -2.38; \quad P_3 = 6.572;$$

$$P_4 = -7.406; \quad P_5 = 2.872.$$

The results of testing for $a = 1, \alpha_0 = 1, r_0 = 1, L = 1, \gamma = 1, U_m = 1$ depict in Fig. 1 the approximated function

$$V(t) = -\frac{\lambda}{r_0^2} \left[\sum_{n=0}^{5} A_n \left(v'_{2n}(r_0,t) - v_{2n}(r_0,t) \right) + \sum_{n=0}^{5} B_n (v'_{2n+1}(r_0,t) - v_{2n+1}(r_0,t)) \right]$$

and the exact solution

$$P(t) = \sum_{n=0}^{5} P_n t^n, \tag{23}$$

which can be obtained by solving the direct Stefan problem [5], [6], [7], [8].

One can see the ideal coincidence of the exact and approximated solutions.

4 Collocation Method

Let us take for testing $m = 5$ the basic points $t = t_i = \frac{2i}{10}$, $i = 0,1,2,3,4,5$, $a = 1, \alpha_0 = 1, r_0 = 1, L = 1, \gamma = 1, U_m = 0$. Then we get the following values for A_n and B_n:

$$A_1 = 0.878; \; A_2 = -0.051; \; A_3 = -0.069;$$

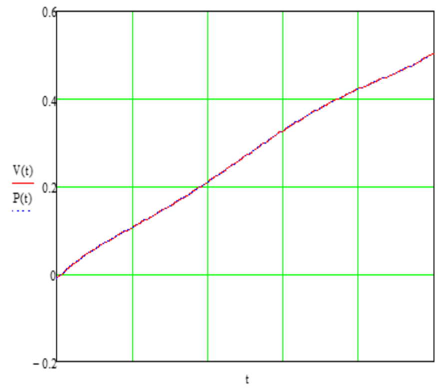

Figure 1 – Approximated and the exact heat fluxes

$$A_4 = 9.957 \times 10^{-3};\ A_5 = -3.073 \times 10^{-4},$$
$$B_1 = -1.448;\ B_2 = -0.514;\ B_3 = -0.05,$$
$$B_4 = -1.041 \times 10^{-3};\ B_5 = 1.383 \times 10^{-5}.$$

From the condition (10) we have the following results:

$$P_1 = 1.212;\ P_2 = -6.219;\ P_3 = 18.288;$$

$$P_4 = -21.378;\ P_5 = 8.597.$$

The Fig. 2 depicts the approximate function

$$V(t) = -\frac{\lambda}{r_0^2}\left[\sum_{n=1}^{5} A_n\left(v'_{2n}(r_0,t) - v_{2n}(r_0,t)\right) + \sum_{n=1}^{5} B_n(v'_{2n+1}(r_0,t) - v_{2n+1}(r_0,t))\right]$$

and the original function $P(t) = \sum_{n=0}^{5} P_n t^n$ taking for the corresponding direct Stefan problem.

The greatest error of approximation is in the neighborhood of zero and one. The error of approximation is approximately 2%.

5 Experimental Verification

Let us compare the results of approximation with the exact solution and experimental data presented in [9]. The contact material is $AgCdO$, the initial radius of the arc spot on the contact surface $r_0 = 10^{-4}m$, the current $I = 1.5kA$, the voltage $U = 42V$, the arc duration $t_a = 12\mu s$. Then we have the coefficients of the original function

$$P_1 = 1.755 \times 10^8;\ P_2 = -6.17 \times 10^7;\ P_3 = -8.254 \times 10^8;$$

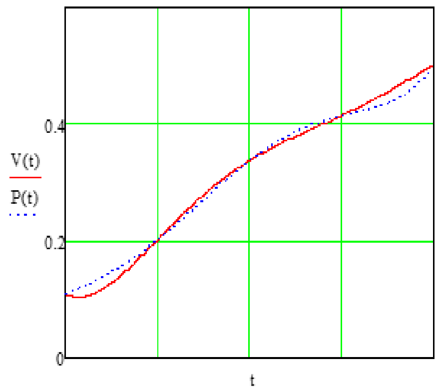

Figure 2 – Approximated $V(t)$ and the exact $P(t)$ heat fluxes

$$P_4 = -1.673 \times 10^9 \ P_5 = -9.292 \times 10^8.$$

Fig. 3 shows that the approximation and the original functions are identical everywhere without errors.

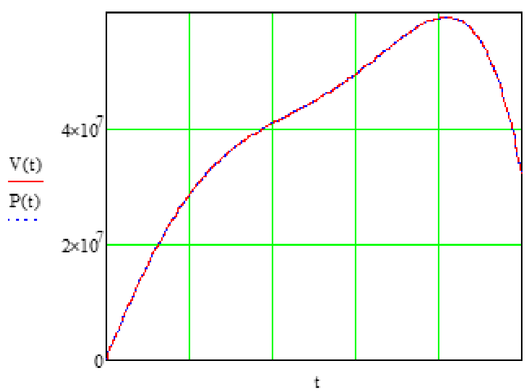

Figure 3 – The domain $\Omega_{x,t}$ in the case II

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Харин С.Н., Наурыз Т., Джаббарханов Х. ЕКІ ФАЗАЛЫҚ СФЕРАЛЫҚ СТЕФАН ЕСЕБІН ЖЫЛУЛЫҚ ПОЛИНОМДАРДЫ ПАЙДАЛАНА ОТЫРЫП ШЕШУ

Белгісіз шекаралық жылу ағыны үшін кері екі фазалық сфералық Стефан есебі жылулық полиномдар әдісімен шешіледі. Нақты шешіммен қоса, жуықтап шешудің инженерлік есептер үшін қолайлы болатын екі әдісі – вариациялық әдіс пен коллокациялық әдіс, ұсынылған және салыстырылған. Екі әдісте, бар болғаны тек бірнеше нүктелерді ғана пайдаланғанның өзінде өте жақсы жақындатуды көрсетеді. Дегенмен, коллокациялық әдіс жылудың бастапқы сатысында жақсы нәтиже берсе, вариациялық әдіс Фурье критерийінің үлкен мәндері үшін қолайлырақ болып отыр. Жылу өткізгіштік теңдеуі үшін аппроксимация қателігінің бағалауы максимум қағидатын пайдалану арқылы алынған. Алынған нәтижелердің контактіни ажырату кезіндегі электр доғасының жылу ағынын есептеуге қолданысы ұсынылды.

Кілттік сөздер. Стефан есебі, жылулық полиномдар, жылу ағыны, балқу аймағы.

Харин С.Н., Наурыз Т., Джаббарханов Х. РЕШЕНИЕ ДВУХФАЗНОЙ СФЕРИЧЕСКОЙ ЗАДАЧИ СТЕФАНА С ИСПОЛЬЗОВАНИЕМ ТЕПЛОВЫХ ПОЛИНОМОВ

Обратная двухфазная сферическая задача Стефана для неизвестного граничного теплового потока решается методом тепловых полиномов. Наряду с точным решением представлены и сопоставлены два метода приближенного решения - коллокационный и вариационный, удобные для инженерных задач. Показано, что оба метода дают очень хорошее приближение даже для использования только нескольких точек. Однако метод коллокации дает лучший результат для начальной стадии нагрева, тогда как вариационный метод более предпочтителен для больших значений критерия Фурье. Оценка погрешности аппроксимации получена с использованием принципа максимума для уравнения теплопроводности. Представлено применение полученных результатов для расчета теплового потока электрической дуги при размыкании контакта.

Ключевые слова. Задача Стефана, тепловые полиномы, тепловой поток, зона плавление.

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The existence of a solution to the special Cauchy problem for the system of nonlinear Fredholm integro-differential equations

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Abstract. Fredholm integro-differential equation with nonlinear differential part and linear integral term with degenerate kernel is considered on a finite interval. The interval is divided into N parts and the values of a solution to the nonlinear integro-differential equation at the left-end points of subintervals are introduced as additional parameters. The desired function is replaced by the sums of new unknown functions and additional parameters in the corresponding subintervals. The original integro-differential equation is reduced to the special Cauchy problem for the system of nonlinear integro-differential equations with parameters. The special Cauchy problem as the Cauchy problem for Fredholm integro-differential equations is not always solvable. Therefore, the questions of the existence of a solution to the special Cauchy problem at the fixed values of parameters are studied. To this end Arzela's theorem on compactness of a set of continuous functions on closed intervals is used. Conditions for the existence of a solution to the special Cauchy problem are established.

Keywords. Nonlinear Fredholm integro-differential equation, special Cauchy problem, parametrization's method, iterative method, compact set.

In [1]–[4] parametrization method is applied to study and solve the linear Fredholm integro-differential equations and boundary value problems for them. The interval is divided into N parts, values of desired function at the beginning points of subintervals are considered as additional parameters and the original integro-differential equation is reduced to a system of integro-differential equations with parameters, where unknown functions satisfy the initial conditions on the subintervals. At the fixed values of the parameters we get the special Cauchy problem for the system of linear integro-differential equations. The solutions

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to the special Cauchy problem are used in solving boundary value problems for the Fredholm integro-differential equations.

In the present paper, it is considered the nonlinear Fredholm integro-differential equation

$$\frac{dx}{dt} = f(t,x) + \sum_{k=1}^{m} \varphi_k(t) \int_0^T \psi_k(\tau) x(\tau) d\tau, \quad t \in [0,T], \quad x \in R^n, \tag{1}$$

where $(n \times n)$-matrices $\varphi_k(t)$, $\psi_k(\tau)$, $k = \overline{1,m}$, are continuous on $[0,T]$, $f : [0,T] \times R^n \to R^n$ is continuous; $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

Denote by $C\big([0,T], R^n\big)$ the space of continuous functions $x : [0,T] \to R^n$ with the norm $\|x\|_1 = \max\limits_{t\in[0,T]} \|x(t)\|$. A solution to Eq.(1) is a continuously differentiable on $(0,T)$ function $x(t) \in C\big([0,T], R^n\big)$, which satisfies the equation for any $t \in [0,T]$.

Let Δ_N be a partition of the interval $[0,T)$ into N parts: $[0,T) = \bigcup\limits_{r=1}^{N} [t_{r-1}, t_r)$, and $x_r(t)$ be the restriction of the function $x(t)$ to the r-th interval, i.e. $x_r(t) = x(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$.

We consider the value of functions $x_r(t)$ at the beginning points of the subintervals as additional parameters, and make the substitution $u_r(t) = x_r(t) - \lambda_r$, $r = \overline{1,N}$, on each r-th interval. Then system (1) is reduced to the special Cauchy problem for the system of nonlinear integro-differential equations with parameters

$$\frac{du_r}{dt} = f(t, u_r + \lambda_r) + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau)[u_j(\tau) + \lambda_j] d\tau,\ t \in [t_{r-1}, t_r), \tag{2}$$

$$u_r(t_{r-1}) = 0, \quad r = \overline{1,N}. \tag{3}$$

In [6], sufficient conditions for the existence of a unique solution to the special Cauchy problem for nonlinear Fredholm integro-differential equations are obtained. An algorithm for finding a solution to the special Cauchy problem for nonlinear integro-differential equations and a numerical implementation of the proposed algorithm are developed in [7]. Note that in these papers it is required that the lengths of subintervals be small.

The purpose of this paper is to establish conditions for the existence of a solution to the special Cauchy problem (2), (3) for any partition of the interval $[0,T]$.

Let $C\big([0,T], \Delta_N, R^{nN}\big)$ denote the space of function systems $u[t] = \big(u_1(t), u_2(t), \ldots, u_N(t)\big)$, where $u_r : [t_{r-1}, t_r) \to R^n$ is continuous and has the finite left-sided limit $\lim\limits_{t\to t_r-0} u_r(t)$ for any $r = \overline{1,N}$, with the norm $\big\|u[\cdot]\big\|_2 = \max\limits_{r=\overline{1,N}} \sup\limits_{t\in[t_{r-1},t_r)} \|u_r(t)\|$.

The solution to the special Cauchy problem (2), (3) at the fixed $\lambda = \lambda^* = \big(\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*\big) \in R^{nN}$ is a function system $u[t, \lambda^*] = \big(u_1(t,\lambda^*), u_2(t,\lambda^*), \ldots, u_N(t,\lambda^*)\big) \in$

$C\big([0,T],\Delta_N,R^{nN}\big)$, whose components $u_r(t,\lambda^*)$, $r=\overline{1,N}$, are continuously differentiable on their domains and satisfy the system of integro-differential equations (2) with $\lambda=\lambda^*$ and initial conditions (3).

We choose a vector $\lambda^{(0)}=\Big(\lambda_1^{(0)},\lambda_2^{(0)},\ldots,\lambda_N^{(0)}\Big)\in R^{nN}$ and define a piecewise constant vector function $x_0(t)$ on $[0,T]$ by the equalities $x_0(t)=\lambda_r^{(0)}$, $t\in[t_{r-1},t_r)$, $r=\overline{1,N}$, $x_0(T)=\lambda_N^{(0)}$.

Let $\rho_\lambda>0$, and

$$S\big(\lambda^{(0)},\rho_\lambda\big)=\big\{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_N)\in R^{nN}:\|\lambda_r-\lambda_r^{(0)}\|<\rho_\lambda,\ r=\overline{1,N}\big\},$$

$$G_0(\rho)=\big\{(t,x):t\in[0,T],\|x-x_0(t)\|<\rho\big\}.$$

To solve the boundary value problem we need the values of $\lim\limits_{t\to t_r-0}u_r(t)$, $r=\overline{1,N}$. So, we consider the special Cauchy problem for the system of nonlinear integro-differential equations with parameters on closed subintervals

$$\frac{dv_r}{dt}=f(t,v_r+\lambda_r)+\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)[v_j(\tau)+\lambda_j]d\tau,\ t\in[t_{r-1},t_r], \tag{4}$$

$$v_r(t_{r-1})=0,\quad r=\overline{1,N}. \tag{5}$$

Denote by $\widetilde{C}\big([0,T],\Delta_N,R^{nN}\big)$ the space of function systems $v[t]=\big(v_1(t),v_2(t),\ldots,v_N(t)\big)$, where $v_r:[t_{r-1},t_r]\to R^n$ is continuous for all $r=\overline{1,N}$, with the norm $\big\|v[\cdot]\big\|_3=\max\limits_{r=\overline{1,N}}\max\limits_{t\in[t_{r-1},t_r]}\|v_r(t)\|$.

It is obvious that if the function systems $u[t,\lambda]=\big(u_1(t,\lambda),u_2(t,\lambda),\ldots,u_N(t,\lambda)\big)$, and $v[t,\lambda]=\big(v_1(t,\lambda),v_2(t,\lambda),\ldots,v_N(t,\lambda)\big)$, are solutions to problems (2), (3) and (4), (5), respectively, then

$$u_r(t,\lambda)=v_r(t,\lambda),\quad t\in[t_{r-1},t_r),$$

$$\lim_{t\to t_r-0}u_r(t,\lambda)=v_r(t_r,\lambda),\quad r=\overline{1,N}.$$

For fixed parameter $\widehat{\lambda}\in S\big(\lambda^{(0)},\rho_\lambda\big)$, we get

$$\frac{dv_r}{dt}=f(t,v_r+\widehat{\lambda}_r)+\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)\big[v_j(\tau)+\widehat{\lambda}_j\big]d\tau,\ t\in[t_{r-1},t_r], \tag{6}$$

$$v_r(t_{r-1})=0,\quad r=\overline{1,N}. \tag{7}$$

Let $\rho_v>0$ and

$$S\big(0,\rho_v\big)=\big\{v[t]=(v_1(t),v_2(t),\ldots,v_N(t))\in\widetilde{C}\big([0,T],\Delta_N,R^{nN}\big):\|v[\cdot]\|_3<\rho_v\big\}.$$

We solve problem (6), (7) by the iterative method. We take $v^{(0)}[t] = (0, 0, \dots, 0)$ as an initial approximation for the solution to problem (6), (7) and successive approximations determined by the solutions to the special Cauchy problems for the system of linear integro-differential equations

$$\frac{dv_r}{dt} = F(t, v_r^{(\nu-1)}(t), \widehat{\lambda}) + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau) v_j(\tau) d\tau, t \in [t_{r-1}, t_r], \tag{8}$$

$$v_r(t_{r-1}) = 0, \quad r = \overline{1, N}, \tag{9}$$

where

$$F(t, v_r^{(\nu-1)}(t), \widehat{\lambda}) = f(t, v_r^{(\nu-1)}(t) + \widehat{\lambda}_r) + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau) d\tau \widehat{\lambda}_j,$$

$$t \in [t_{r-1}, t_r], \quad r = \overline{1, N}, \quad \nu = 1, 2, \dots . \tag{10}$$

By $\widehat{v}^{(\nu)}[t] = \big(\widehat{v}_1^{(\nu)}(t), \widehat{v}_2^{(\nu)}(t), \dots, \widehat{v}_N^{(\nu)}(t)\big)$ we denote the solution to the special Cauchy problem (8), (9).

Let $C\big([t_{r-1}, t_r], R^n\big)$ be the space of continuous functions $v_r : [t_{r-1}, t_r] \to R^n$ with the norm $\|v_r\| = \max\limits_{t \in [t_{r-1}, t_r]} \|v_r(t)\|$, $r = \overline{1, N}$.

For the fixed function system $v^{(\nu-1)}[t] = \big(v_1^{(\nu-1)}(t), v_2^{(\nu-1)}(t), \dots, v_N^{(\nu-1)}(t)\big) \in \widetilde{C}\big([0, T], \Delta_N, R^{nN}\big)$, $\nu = 1, 2, \dots$, problem (8), (9) turns into the special Cauchy problem for the system of linear integro-differential equations

$$\frac{dw_r}{dt} = \mathcal{F}_r(t, \widehat{\lambda}) + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} (\tau) w_j(\tau) \tau, \quad t \in [t_{r-1}, t_r], \tag{11}$$

$$w_r(t_{r-1}) = 0, \quad r = \overline{1, N}, \tag{12}$$

with $\mathcal{F}_r(t, \widehat{\lambda}) \in C\big([t_{r-1}, t_r], R^n\big)$.

We find the solution to the special Cauchy problem for the system of linear integro-differential equations (11), (12) by the method proposed in [3, p. 345-346].

Since the fundamental matrix of the differential part is the identity matrix of the dimension n, the special Cauchy problem (11), (12) is equivalent to the system of integral equations of the second kind

$$w_r(t) = \int_{t_{r-1}}^{t} \mathcal{F}_r(\tau, \widehat{\lambda}) d\tau + \int_{t_{r-1}}^{t} \sum_{k=1}^{m} \varphi_k(\tau) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(s) w_j(s) ds d\tau,$$

$$t \in [t_{r-1}, t_r], \quad r = \overline{1, N}. \tag{13}$$

Set

$$\xi_k = \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(s) w_j(s) ds, \quad k = \overline{1,m},$$

and rewrite system (13) in the next form

$$w_r(t) = \int_{t_{r-1}}^{t} \mathcal{F}_r(\tau, \widehat{\lambda}) d\tau + \int_{t_{r-1}}^{t} \sum_{k=1}^{m} \varphi_k(\tau) d\tau \xi_k, \ t \in [t_{r-1}, t_r], \ r = \overline{1,N}. \tag{14}$$

Multiplying both sides of (14) by $\psi_p(t)$, integrating on $[t_{r-1}, t_r]$ and summing up over r, we get the system of linear algebraic equations with respect to $\xi = (\xi_1, \xi_2, \ldots, \xi_m) \in R^{nm}$

$$\xi_p = \sum_{k=1}^{m} G_{p,k}(\Delta_N) \xi_k + g_p(\Delta_N, \mathcal{F}), \quad p = \overline{1,m}, \tag{15}$$

with $(n \times n)$-matrices

$$G_{p,k}(\Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \psi_p(\tau) \int_{t_{r-1}}^{\tau} \varphi_k(s) ds d\tau, \quad p, k = \overline{1,m},$$

and vectors of the dimension n

$$g_p(\Delta_N, \mathcal{F}) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \psi_p(\tau) \int_{t_{r-1}}^{\tau} \mathcal{F}_r(s, \widehat{\lambda}) ds d\tau, \quad p = \overline{1,m}. \tag{16}$$

Using the matrices $G_{p,k}\big(\Delta_N\big)$ and the vectors $g_p\big(\Delta_N, \mathcal{F}_r\big)$, we construct the $(nm \times nm)$-matrix $G(\Delta_N) = \big(G_{p,k}(\Delta_N)\big)$, $p, k = \overline{1,m}$, and the vector $g\big(\Delta_N, \mathcal{F}\big) = \big(g_1(\Delta_N, \mathcal{F}), g_2(\Delta_N, \mathcal{F}), \ldots, g_m(\Delta_N, \mathcal{F})\big)$. We can rewrite system (15) in the form

$$[I - G(\Delta_N)]\xi = g(\Delta_N, \mathcal{F}), \tag{17}$$

where I is the identity matrix of the dimension nm.

Assume that Δ_N is a regular partition [2]. Then the matrix $I - G(\Delta_N)$ is invertible and its inverse we write in the form $[I - G(\Delta_N)]^{-1} = \big(R_{k,p}(\Delta_N)\big)$, $k, p = \overline{1,m}$, where $R_{k,p}(\Delta_N)$ are square matrices of the dimension n. Now, a unique solution to Eq.(17) is determined by the equalities

$$\xi_k = \sum_{p=1}^{m} R_{k,p}(\Delta_N) g_p(\Delta_N, \mathcal{F}), \quad k = \overline{1,m}. \tag{18}$$

Let us introduce the following notation:

$$\overline{h} = \max_{r=\overline{1,N}} (t_r - t_{r-1}), \tag{19}$$

$$\overline{\varphi}_k = \max_{r=\overline{1,N}} \max_{t\in[t_{r-1},t_r]} \|\varphi_k(t)\|, \ \overline{\psi}_k = \max_{r=\overline{1,N}} \max_{t\in[t_{r-1},t_r]} \|\psi_k(t)\|, \ k=\overline{1,m}. \tag{20}$$

Substituting the right-hand side of (18) for ξ_k in (14), we get the functions

$$w_r(t,\widehat{\lambda}) = \int_{t_{r-1}}^{t} \sum_{k=1}^{m} \varphi_k(\tau) \sum_{p=1}^{m} R_{k,p}(\Delta_N) g_p(\Delta_N, \mathcal{F}) d\tau$$

$$+ \int_{t_{r-1}}^{t} \mathcal{F}_r(\tau,\widehat{\lambda}) d\tau, \quad t\in[t_{r-1},t_r], \quad r=\overline{1,N}.$$

The function system

$$w[t,\widehat{\lambda}] = \big(w_1(t,\widehat{\lambda}), w_2(t,\widehat{\lambda}), \ldots, w_N(t,\widehat{\lambda})\big)$$

is a unique solution to the special Cauchy problem for the system of linear integro-differential equations (11), (12) and the following estimate

$$\Big\|w[\cdot,\widehat{\lambda}]\Big\|_3 \le \chi \max_{r=\overline{1,N}} \max_{t\in[t_{r-1},t_r]} \Big\|\mathcal{F}_r(t,\widehat{\lambda})\Big\| \tag{21}$$

holds, where

$$\chi = \left[1 + \frac{N\overline{h}^2}{2} \max_{p=\overline{1,m}} \overline{\psi}_p \sum_{k=1}^{m}\sum_{p=1}^{m} \max_{t\in[t_{r-1},t_r]} \Big\|\varphi_k(t)\Big\|\Big\|R_{k,p}(\Delta_N)\Big\|\right]\overline{h}.$$

Theorem 1. *Let the matrix $I-G(\Delta_N)$ be invertible and the following conditions be fulfilled:*

1) $\|f(t,x)\| \le M_0$, $(t,x)\in G_0(\rho)$, M_0 is const;

2) $\displaystyle\sum_{j=1}^{N}\sum_{k=1}^{m} \Big\|\varphi_k(t)\Big\| \Big\|\int_{t_{j-1}}^{t_j} \psi_k(s)ds\Big\| \le M_1$, $t\in[0,T]$, M_1 is const;

3) $\chi \cdot \Big(M_0 + M_1\cdot\big(\rho_\lambda + \|\lambda^{(0)}\|\big)\Big) < \rho_v$;

4) $\rho_\lambda + \rho_v \le \rho$.

Then, for any $\widehat{\lambda} \in S(\lambda^{(0)},\rho_\lambda)$, the special Cauchy problem for the system of nonlinear integro-differential equations (6), (7) has a solution $v\big[t,\widehat{\lambda}\big] = \big(\widehat{v}_1^*(t), \widehat{v}_2^*(t), \ldots, \widehat{v}_N^*(t)\big) \in S(0,\rho_v)$.

Proof. By using the iterative method proposed above, we find the sequence of function systems $\big\{\widehat{v}^{(\nu)}[t]\big\}$, where

$$\widehat{v}^{(\nu)}[t] = \big(\widehat{v}_1^{(\nu)}(t), \widehat{v}_2^{(\nu)}(t), \ldots, \widehat{v}_N^{(\nu)}(t)\big).$$

It is easily seen that the functions $\widehat{v}_r^{(\nu)}(t)$ belong to $C\big([t_{r-1},t_r],R^n\big)$, $\nu=1,2,\dots$.

Using the elements of the function system $\widehat{v}^{(\nu)}[t]$ at the fixed value of r we compose the functional sequences

$$\big\{\widehat{v}_r^{(\nu)}(t)\big\},\quad t\in[t_{r-1},t_r],\quad r=\overline{1,N},\quad \nu=1,2,\dots\ .$$

Consider the set V_r of the functions $\widehat{v}_r^{(\nu)}(t)$.

Formula (16) implies the following estimates

$$\|g_p(\Delta_N,F)\|=\left\|\sum_{r=1}^{N}\int_{t_{r-1}}^{t_r}\psi_p(\tau)\int_{t_{r-1}}^{\tau}F(s,\widehat{v}_r^{(\nu-1)}(s),\widehat{\lambda})dsd\tau\right\|$$

$$\leq\sum_{r=1}^{N}\left\|\int_{t_{r-1}}^{t_r}\psi_p(\tau)\int_{t_{r-1}}^{\tau}F(s,\widehat{v}_r^{(\nu-1)}(s),\widehat{\lambda})dsd\tau\right\|$$

$$\leq\sum_{r=1}^{N}\int_{t_{r-1}}^{t_r}\left\|\psi_p(\tau)\int_{t_{r-1}}^{\tau}F(s,\widehat{v}_r^{(\nu-1)}(s),\widehat{\lambda})ds\right\|d\tau$$

$$\leq\sum_{r=1}^{N}\max_{\tau\in[t_{r-1},t_r]}\left\|F(\tau,\widehat{v}_r^{(\nu-1)}(\tau),\widehat{\lambda})\right\|\int_{t_{r-1}}^{t_r}\|\psi_p(\tau)\|(\tau-t_{r-1})d\tau$$

$$\leq\overline{\psi}_p\max_{r=\overline{1,N}}\max_{\tau\in[t_{r-1},t_r]}\left\|F(\tau,\widehat{v}_r^{(\nu-1)}(\tau),\widehat{\lambda})\right\|\sum_{r=1}^{N}\frac{(t_r-t_{r-1})^2}{2},\quad p=\overline{1,m}. \tag{22}$$

From (10), taking into account (20), we get

$$\left\|F(t,\widehat{v}_r^{(\nu-1)}(t),\widehat{\lambda})\right\|=\left\|f(t,\widehat{v}_r^{(\nu-1)}(t)+\widehat{\lambda}_r)+\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)d\tau\widehat{\lambda}_j\right\|$$

$$\leq\left\|f(t,\widehat{v}_r^{(\nu-1)}(t)+\widehat{\lambda}_r)\right\|+\left\|\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)d\tau\widehat{\lambda}_j\right\|$$

$$\leq\left\|f(t,\widehat{v}_r^{(\nu-1)}(t)+\widehat{\lambda}_r)\right\|+\left\|\sum_{k=1}^{m}\sum_{j=1}^{N}\varphi_k(t)\int_{t_{j-1}}^{t_j}\psi_k(\tau)d\tau\widehat{\lambda}_j\right\|$$

$$\leq\left\|f(t,\widehat{v}_r^{(\nu-1)}(t)+\widehat{\lambda}_r)\right\|+\sum_{j=1}^{N}\sum_{k=1}^{m}\left\|\varphi_k(t)\right\|\left\|\int_{t_{j-1}}^{t_j}\psi_k(\tau)d\tau\right\|\left\|\widehat{\lambda}_j\right\|$$

$$\leq \left\|f(t, \widehat{v}_r^{(\nu-1)}(t) + \widehat{\lambda}_r)\right\| + \sum_{j=1}^{N}\sum_{k=1}^{m}\left\|\varphi_k(t)\right\|\left\|\int_{t_{j-1}}^{t_j}\psi_k(\tau)d\tau\right\|\|\widehat{\lambda}\|,$$

$$\max_{r=\overline{1,N}}\max_{\tau\in[t_{r-1},t_r]}\left\|F(\tau, \widehat{v}_r^{(\nu-1)}(\tau), \widehat{\lambda})\right\| \leq \max_{r=\overline{1,N}}\max_{\tau\in[t_{r-1},t_r]}\left\{\left\|f(t, \widehat{v}_r^{(\nu-1)}(t) + \widehat{\lambda}_r)\right\|\right.$$

$$\left. + \sum_{j=1}^{N}\sum_{k=1}^{m}\left\|\varphi_k(t)\right\|\left\|\int_{t_{j-1}}^{t_j}\psi_k(\tau)d\tau\right\|\left(\|\widehat{\lambda} - \lambda^{(0)}\| + \|\lambda^{(0)}\|\right)\right\}$$

$$\leq \max_{r=\overline{1,N}}\max_{\tau\in[t_{r-1},t_r]}\left\{\left\|f(t, \widehat{v}_r^{(\nu-1)}(t) + \widehat{\lambda}_r)\right\|\right.$$

$$\left. + \sum_{j=1}^{N}\sum_{k=1}^{m}\left\|\varphi_k(t)\right\|\left\|\int_{t_{j-1}}^{t_j}\psi_k(\tau)d\tau\right\|\left(\rho_\lambda + \|\lambda^{(0)}\|\right)\right\}$$

$$\leq M_0 + M_1 \cdot \left(\rho_\lambda + \|\lambda^{(0)}\|\right). \tag{23}$$

By virtue of conditions of Theorem 1, estimates (22), (23) and notation (19), (20) we have

$$\left\|v_r^{(\nu)}(t, \widehat{\lambda})\right\| = \left\|\int_{t_{r-1}}^{t}\sum_{k=1}^{m}\varphi_k(\tau)\sum_{p=1}^{m}R_{k,p}(\Delta_N)g_p(\Delta_N, F)d\tau + \int_{t_{r-1}}^{t}F(\tau, \widehat{v}_r^{(\nu-1)}(\tau), \widehat{\lambda})d\tau\right\|$$

$$\leq \left\|\int_{t_{r-1}}^{t}\sum_{k=1}^{m}\varphi_k(\tau)\sum_{p=1}^{m}R_{k,p}(\Delta_N)g_p(\Delta_N, F)d\tau\right\| + \left\|\int_{t_{r-1}}^{t}F(\tau, \widehat{v}_r^{(\nu-1)}(\tau), \widehat{\lambda})d\tau\right\|$$

$$\leq \int_{t_{r-1}}^{t}\left\|\sum_{k=1}^{m}\varphi_k(\tau)\sum_{p=1}^{m}R_{k,p}(\Delta_N)g_p(\Delta_N, F)\right\|d\tau + \int_{t_{r-1}}^{t}\left\|F(\tau, \widehat{v}_r^{(\nu-1)}(\tau), \widehat{\lambda})\right\|d\tau$$

$$\leq \int_{t_{r-1}}^{t}\sum_{p=1}^{m}\sum_{k=1}^{m}\left\|\varphi_k(\tau)R_{k,p}(\Delta_N)\right\|\left\|g_p(\Delta_N, F)\right\|d\tau + \int_{t_{r-1}}^{t}\left\|F(\tau, \widehat{v}_r^{(\nu-1)}(\tau), \widehat{\lambda})\right\|d\tau$$

$$\leq \int_{t_{r-1}}^{t}\sum_{p=1}^{m}\sum_{k=1}^{m}\left\|\varphi_k(\tau)R_{k,p}(\Delta_N)\right\|d\tau \cdot \max_{p=\overline{1,m}}\left\|g_p(\Delta_N, F)\right\|$$

$$+ \int_{t_{r-1}}^{t}\left\|F(\tau, \widehat{v}_r^{(\nu-1)}(\tau), \widehat{\lambda})\right\|d\tau \leq \int_{t_{r-1}}^{t}\sum_{p=1}^{m}\sum_{k=1}^{m}\left\|\varphi_k(\tau)\right\|\left\|R_{k,p}(\Delta_N)\right\|d\tau$$

$$\times \max_{p=\overline{1,m}}\overline{\psi}_p\sum_{r=1}^{N}\frac{(t_r - t_{r-1})^2}{2}\max_{r=\overline{1,N}}\max_{\tau\in[t_{r-1},t_r]}\left\|F(\tau, \widehat{v}_r^{(\nu-1)}(\tau), \widehat{\lambda})\right\|$$

$$+\int_{t_{r-1}}^{t}\left\|F(\tau,\widehat{v}_r^{(\nu-1)}(\tau),\widehat{\lambda})\right\|d\tau\ \leq c_r,\quad t\in[t_{r-1},t_r],\quad r=\overline{1,N},\tag{24}$$

where

$$c_r=\left[1+\frac{N\overline{h}^2}{2}\max_{p=\overline{1,m}}\overline{\psi}_p\sum_{p=1}^{m}\sum_{k=1}^{m}\max_{t\in[t_{r-1},t_r]}\left\|\varphi_k(t)\right\|\left\|R_{k,p}(\Delta_N)\right\|\right]$$
$$\times\overline{h}\Big[M_0+M_1\cdot\big(\rho_\lambda+\|\lambda^{(0)}\|\big)\Big].$$

Since the inequality $\left\|\widehat{v}_r^{(\nu)}(t)\right\|\leq c_r$ holds for all $t\in[t_{r-1},t_r]$, the functions from the set V_r are uniformly bounded on $[t_{r-1},t_r]$, $r=\overline{1,N}$.

Now we take the points $t_r',t_r''\in[t_{r-1},t_r]$, $r=\overline{1,N}$. If $|t_r'-t_r''|<\delta_r$, $r=\overline{1,N}$, then by virtue of (21) and inequality (24) we have the inequality

$$\left\|\widehat{v}_r^{(\nu)}(t_r'')-\widehat{v}_r^{(\nu)}(t_r')\right\|=\Bigg\|\int_{t_r'}^{t_r''}F(\tau,\widehat{v}_r^{(\nu-1)}(\tau),\widehat{\lambda})d\tau$$
$$+\int_{t_r'}^{t_r''}\sum_{k=1}^{m}\varphi_k(\tau)\sum_{p=1}^{m}R_{k,p}(\Delta_N)g_p(\Delta_N,F)d\tau\Bigg\|\leq\varepsilon_r,$$

for all $\widehat{v}_r^{(\nu)}(t)$ on $[t_{r-1},t_r]$, where

$$\varepsilon_r=\max_{t\in[t_{r-1},t_r]}\left[1+\frac{N\overline{h}^2}{2}\max_{p=\overline{1,m}}\overline{\psi}_p\sum_{k=1}^{m}\sum_{p=1}^{m}\max_{t\in[t_{r-1},t_r]}\left\|\varphi_k(t)\right\|\left\|R_{k,p}(\Delta_N)\right\|\right]$$
$$\times\Big[M_0+M_1\cdot\big(\rho_\lambda+\|\lambda^{(0)}\|\big)\Big]|t_r''-t_r'|,\quad r=\overline{1,N},\quad \nu=1,2,\dots\ .$$

It follows that the functions from the set V_r, $r=\overline{1,N}$, are equicontinuous. By Arzela's theorem [5, p. 207], each set V_r, $r=\overline{1,N}$, is compact.

Since the set V_r is compact for each $r=\overline{1,N}$, we can select the subsequence $\widehat{v}_r^{(\nu_l)}(t)$, which is uniformly convergent to $\widehat{v}_r^{*}(t)$ as $l\to\infty$ on $[t_{r-1},t_r]$ for all $r=\overline{1,N}$.

We construct the function system

$$\widehat{v}^{*}\big[t\big]=\big(\widehat{v}_1^{*}(t),\widehat{v}_2^{*}(t),\dots,\widehat{v}_N^{*}(t)\big).$$

Now let us show that the function system $\widehat{v}^{*}\big[t\big]$ is a solution to the special Cauchy problem (6), (7).

Since the functions $\widehat{v}_r^{(\nu_l)}(t)$ are defined by using proposed iterative method, the following equality

$$\widehat{v}_r^{(\nu_l)}(t)=\int_{t_{r-1}}^{t}F(\tau,\widehat{v}_r^{(\nu_l-1)}(\tau),\widehat{\lambda})d\tau+\int_{t_{r-1}}^{t}\sum_{k=1}^{m}\varphi_k(\tau)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(s)\widehat{v}_j^{(\nu_l)}(s)dsd\tau,$$

$$t \in [t_{r-1}, t_r], \quad r = \overline{1, N}, \tag{25}$$

is true.

In (25), passing to the limit as $l \to \infty$, we get

$$\widehat{v}_r^*(t) = \int_{t_{r-1}}^{t} F(\tau, \widehat{v}_r^*(\tau), \widehat{\lambda}) d\tau + \int_{t_{r-1}}^{t} \sum_{k=1}^{m} \varphi_k(\tau) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(s) \widehat{v}_j^{*)}(s) ds d\tau,$$

$$t \in [t_{r-1}, t_r], \quad r = \overline{1, N}. \tag{26}$$

It is easily seen that $\widehat{v}_r^*(t_{r-1}) = 0$, $r = \overline{1, N}$. Differentiating both sides of (26), we obtain

$$\frac{d\widehat{v}_r^*(t)}{dt} = F(t, \widehat{v}_r^*(t), \widehat{\lambda}) + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(s) \widehat{v}_j^{*)}(s) ds,$$

$$t \in [t_{r-1}, t_r], \quad r = \overline{1, N}.$$

Thus, the function system $v[t, \widehat{\lambda}] = (\widehat{v}_1^*(t), \widehat{v}_2^*(t), \ldots, \widehat{v}_N^*(t))$ is a solution to the special Cauchy problem (6), (7). Theorem is proved.

Example. Consider the nonlinear Fredholm integro-differential equation

$$\frac{dx}{dt} = \varphi(t) \int_0^T \psi(\tau) x(\tau) d\tau + f(t, x), \quad t \in [0, T], \quad x \in R^2,$$

where $T = 2$, $\varphi(t) = \begin{pmatrix} \sqrt{t} & -t \\ 0 & \frac{t}{3} \end{pmatrix}$, $\psi(\tau) = \begin{pmatrix} 1 & \frac{\tau}{2} \\ \tau & 0 \end{pmatrix}$,

$$f(t, x) = \begin{pmatrix} \sqrt{t}\sin x_1 + \cos^3 x_2 + \frac{23t}{12} - \sqrt{t}\sin(t^2 - 2) - \frac{15\sqrt{t}}{8} - \cos^3(t + 1) \\ t\cos 3x_1 + \sqrt[3]{t}\sin x_2 + \frac{t}{36} - t\cos(3t^2 - 6) - \sqrt[3]{t}\sin(t + 1) + 1 \end{pmatrix}, \ t \in [0, 1],$$

$$f(t, x) = \begin{pmatrix} \sqrt{t}\sin x_1 + \cos^3 x_2 - \frac{t}{12} - \sqrt{t}\sin(2 - t) - \frac{15\sqrt{t}}{8} - \cos^3(t^2 + 1) \\ t\cos 3x_1 + \sqrt[3]{t}\sin x_2 + \frac{73t}{36} - t\cos(6 - 3t) - \sqrt[3]{t}\sin(t^2 + 1) \end{pmatrix}, \ t \in [1, 2].$$

Let us divide the interval $[0, T)$ into two equal parts and by Δ_2 denote the partition with the points $t_0 = 0$, $t_1 = 1$, $t_2 = 2$. Introducing parameters $\lambda_1 = x(0)$, $\lambda_2 = x(1)$ and making the substitutions

$$v_1(t) = x(t) - x(0), \quad t \in [0, 1], \ v_2(t) = x(t) - x(1), \ t \in [1, 2],$$

we obtain the special Cauchy problem

$$\frac{dv_r}{dt} = \varphi(t) \sum_{j=1}^{2} \int_{t_{j-1}}^{t_j} \psi(\tau)[v_j(\tau) + \lambda_j] d\tau + f(t, v_r + \lambda_r), \quad t \in [t_{r-1}, t_r], \tag{27}$$

$$v_r(t_{r-1}) = 0, \quad r = 1, 2. \tag{28}$$

Assume that $\rho_\lambda = 10$ and $\lambda^{(0)} = \big((1,0),(0,-1)\big)$.

By the equalities $x_0(t) = \lambda_1^{(0)}$, $t \in [0,1]$, $x_0(t) = \lambda_2^{(0)}$, $t \in [1,2]$, we define a piecewise constant function $x_0(t)$ on $[0,2]$. Then we set $G_0(\rho) = \{(t,x) : t \in [0,2], \|x - x_0(t)\| < \rho,\ \rho = 583\}$.

Since

$$\|f(t,x)\| \le 11.4, \quad (t,x) \in G_0(\rho),$$

$$\left\|\varphi(t) \sum_{j=1}^{2} \int_{t_{j-1}}^{t_j} \psi(s)ds\right\| \Big(\rho_\lambda + \|\lambda^{(0)}\|\Big) \le 90.7,$$

and $\chi \le 5.7$, conditions of Theorem 1 are satisfied.

Therefore, the special Cauchy problem (27), (28) has a solution $v[t,\widehat{\lambda}]$ belonging to $S(0,\rho_u)$, $\rho_u = 572$ for any $\widehat{\lambda} \in S(\lambda^{(0)},\rho_\lambda)$, $\rho_\lambda = 10$. If we take, for example, $\widehat{\lambda} = \big((-2,1),(1,2)\big) \in S(\lambda^{(0)},\rho_\lambda)$, then at the fixed value of parameters the special Cauchy problem (27), (28) has the solution $v[t,\widehat{\lambda}] = (\widehat{v}_1(t), \widehat{v}_2(t)) \in S(0,\rho_u)$ and $\widehat{v}_1(t) = \begin{pmatrix} t^2 \\ t \end{pmatrix}$, $\widehat{v}_2(t) = \begin{pmatrix} 1-t \\ t^2-1 \end{pmatrix}$.

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Мынбаева С.Т. СЫЗЫҚТЫ ЕМЕС ФРЕДГОЛЬМ ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІ ҮШІН АРНАЙЫ КОШИ ЕСЕБІНІҢ ШЕШІМІНІҢ БАР БОЛУЫ

Ақырлы аралықта дифференциалдық бөлігі сызықты емес және интегралдық бөлігі сызықты болатын ерекшеленген ядролы Фредгольм интегралдық-дифференциалдық теңдеуі қарастырылады. Аралық N бөлікке бөлінеді және сызықты емес интегралдық-дифференциалдық теңдеудің шешімінің ішкі аралықтардың сол жақ шеткі нүктелеріндегі мәндері қосымша параметрлер ретінде енгізіледі. Ізделінді функция сәйкес аралықтарда белгісіз функциялар мен қосымша параметрлердің қосындыларымен алмастырылады. Берілген интегралдық-дифференциалдық теңдеу сызықты емес интегралдық-дифференциалдық теңдеулер жүйесі үшін параметрлі арнайы Коши есебіне келтіріледі. Фредгольм интегралдық-дифференциалдық теңдеуі үшін Коши есебі сияқты, арнайы Коши есебі де барлық уақытта шешілімді бола бермейді. Сондықтан параметрлердің белгілі мәндерінде арнайы Коши есебінің шешімінің бар болуы мәселелері зерттеледі. Ол үшін кесіндіде үзіліссіз функциялар жиынының компактылығы туралы Арцела теоремасы қолданылады. Арнайы Коши есебінің шешімінің бар болуының шарттары алынған.

Кілттік сөздер. Сызықты емес Фредгольм интегралдық-дифференциалдық теңдеуі, арнайы Коши есебі, параметрлеу әдісі, итерациялық әдіс, компакт жиын.

Мынбаева С.Т. СУЩЕСТВОВАНИЕ РЕШЕНИЯ СПЕЦИАЛЬНОЙ ЗАДАЧИ КОШИ ДЛЯ СИСТЕМЫ НЕЛИНЕЙНЫХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ФРЕДГОЛЬМА

На конечном интервале рассматривается интегро-дифференциальное уравнение Фредгольма с нелинейной дифференциальной частью и линейной интегральной частью с вырожденным ядром. Интервал делится на N частей и значения решения нелинейного интегро-дифференциального уравнения в левых точках подинтервалов вводятся в качестве дополнительных параметров. Искомая функция заменяется на суммы новых неизвестных функций и дополнительных параметров в соответствующих подинтервалах. Исходное интегро-дифференциальное уравнение сводится к специальной задаче Коши для системы нелинейных интегро-дифференциальных уравнений с параметрами. Специальная задача Коши, как задача Коши для интегро-дифференциальных уравнений Фредгольма, не всегда разрешима. В связи с этим исследуются вопросы существования решения специальной задачи Коши при фиксированных значениях параметров. Для этого используется теорема Арцела о компактности множества непрерывных функций на отрезках. Установлены условия существования решения специальной задачи Коши.

Ключевые слова. Нелинейное интегро-дифференциальное уравнение Фредгольма, специальная задача Коши, метод параметризации, итерационный метод, компактное множество.

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Self-adjoint operators generated by integro-differential operators

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Abstract. This paper is devoted to description of self-adjoint extensions of an integro-differential operator. We find symmetric integro-differential operators of order 2α (with $\frac{1}{2} < \alpha < 1$). Indeed, it is an analogue of the fractional Sturm-Liouville operator in some sense. Moreover, an analogue of the Green's formula for fractional order differential equations is established with further applications in describing a class of self-adjoint operators. Finally, we discuss about global Fourier analysis and prove some results on spectral properties of fractional order self-adjoint operators associated with Caputo-Riemann-Liouville type derivatives.

Keywords. Integro-differential operator, Caputo derivative, Riemann-Liouville derivative, Self-adjoint problem, Fractional order differential equation, Fractional Sturm-Liouville operator, Extension theory.

1 Introduction

In the theory of differential equations the charming role plays describing and studying self-adjoint problems. One of the methods to describe them can be done by using the sufficiently developed theory of self-adjoint extension, for example, see monographs [1], [2]. The Green's formula is one of the main moments in the theory of extensions and contractions. In this paper the Green's formula is established for a differential equation of the fractional order. Moreover, we introduce the notion of a fractional differentiation of generalized functions.

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For clearity, we give a class of self-adjoint problems for a fractional analogue of the Sturm-Liouville operator. Due to the physical applications the spectral properties of the fractional operators are subject to intensive studies, especially, for the papers with applications [3]–[7].

One of the first investigations of the spectral properties of fractional differential equations is done by Dzhrbashyan [8]. After Dzhrbashyan's paper mathematicians began to pay attention to the properties of the special functions generated by the fractional differential the equations. For this, we refer the reader to the papers [9]–[14] and references therein. In general, fractional operators are not symmetric, and in all mentioned works non self–adjoint problems are considered (also, see [15]). In the weighted class of continuous functions one symmetric fractional order differential operator is described by Klimek and Agrawal [16]. In this work we continue researches started in [17], and we attempt to establish an analogue of the Green's formula for fractional order differential equations with further applications in describing a class of self–adjoint operators.

In this paper we deal with a fractional differential operator of the Caputo and Riemann-Liouville type. Moreover, we are aiming at describing a class of self-adjoint problems associated with this fractional order differential equation in the Hilbert space. Indeed, it is found a symmetric Caputo-Riemann-Liouville operator of order 2α (with $\frac{1}{2} < \alpha < 1$). In appreciate sense, it can be interpreted as an analogue of the classical Sturm-Liouville operator.

2 Main results

In what follows, we assume that $\frac{1}{2} < \alpha < 1$. Now, let us consider

$$Lu(x) := \mathcal{D}_1^{\alpha} \left[D_0^{\alpha} \left[u \right] \right] (x) \, , \, 0 < x < 1. \tag{1}$$

Here, our aim is to investigate spectral properties of operators generated by the fractional order differential equation (1) in $\mathrm{L}^2(0,1)$. To start, we define an operator in the Hölder classes. Consider the spectral problem

$$Lu(x) = \lambda u(x), \, 0 < x < 1, \tag{2}$$

in the space $H_0^{2\alpha+o}([0,1]) := \{\varphi \in H^{2\alpha+o}([0,1]) : \varphi(0) = 0, \ldots, \varphi^{(m)}(0) = 0\}$, where $m = [2\alpha+o]$, and $H^{2\alpha+o}([0,1])$ is the Hölder space with the parameter $2\alpha+o$. Here o is a sufficiently small positive number such that $o < 1-\alpha$. By other words, we deal with the following spaces:

$$H_0^{2\alpha+o}([0,1]) := \{\varphi \in H^{2\alpha+o}([0,1]) : \varphi(0) = 0, \varphi'(0) = 0\},$$
$$H_0^{\alpha+o}([0,1]) := \{\varphi \in H^{\alpha+o}([0,1]) : \varphi(0) = 0\},$$
$$H_0^{o}([0,1]) := \{\varphi \in H^{o}([0,1]) : \varphi(0) = 0\}.$$

From the book of Samko, Kilbas and Marichev [18, Chapter 1, Theorem 3.2] it follows that the integro–differential operator L is bounded from $H_0^{2\alpha+o}([0,1])$ to $H_0^{o}([0,1])$. Hence, the functionals

$$\xi_1^-(u) := I_0^{1-\alpha} \left[u \right] (0) \, , \;\; \xi_2^-(u) := I_0^{1-\alpha} \left[u \right] (1) \, ,$$

$$\xi_1^+(u) := D_0^\alpha [u](0) \text{ and } \xi_2^+(u) := D_0^\alpha [u](1),$$

are well-defined for all $H_0^{2\alpha+o}([0,1])$. Denote by L_0 an operator generated by the fractional differential expression (1) with "boundary" conditions

$$\xi_2^-(u) = 0 \text{ and } \xi_1^+(u) = 0. \tag{3}$$

Then due to the definitions and properties given by Appendix (see, [18, Chapter 1]) for

$$f \in \tilde{H}_0^o([0,1]) := \{v \in H_0^o([0,1]) : \int_0^1 v(s)s^{2\alpha}ds = 0 \text{ and } \int_0^1 v(s)s^{2\alpha-1}ds = 0\}$$

an inverse operator to L_0 has the form

$$L_0^{-1}f(x) = I_0^\alpha I_1^\alpha f(x) := \int_0^1 K(x,s)f(s)ds, \ \ 0 < x < 1,$$

as $L_0^{-1} : \tilde{H}_0^o \to H_0^{2\alpha+o}$, with the symmetric kernel $K(\cdot,\cdot)$ from $\mathrm{L}^2(0,1) \otimes \mathrm{L}^2(0,1)$. Since, $S := span\{x^k,\, k \in \mathbb{N}\} \subset H_0^o([0,1])$, and powers of the sets S and $\tilde{S} := \{v \in S : \int_0^1 v(s)s^{2\alpha}ds = 0 \text{ and } \int_0^1 v(s)s^{2\alpha-1}ds = 0\}$ are equal, then we conclude that a closure of the space $\tilde{H}_0^o([0,1])$ by the L^2–norm is $\mathrm{L}^2(0,1)$. Hence, L_0^{-1} has a continuous continuation to a compact operator in $\mathrm{L}^2(0,1)$. Compactness implies the fact that there exists non empty discrete spectrum with the eigenfunctions forming an orthogonal basis in the space $\mathrm{L}^2(0,1)$.

Denote by λ_k, $k \in \mathbb{N}$, eigenvalues of the spectral problem (2)–(3) in the ascending order and by u_k, $k \in \mathbb{N}$, corresponding eigenfunctions, i.e.

$$\mathcal{D}_1^\alpha [D_0^\alpha [u_k]](x) = \lambda_k u_k(x),\ 0 < x < 1,$$

$$\xi_2^-(u_k) = 0, \ \ \xi_1^+(u_k) = 0$$

for all $k \in \mathbb{N}$. Thus, the domain of the operator L_0

$$\mathrm{Dom}(L_0) := \{u \in H_0^{2\alpha+o}([0,1]) : \xi_2^-(u) = 0, \ \ \xi_1^+(u) = 0\}$$

is not empty.

Now, we introduce the space of test functions $C_{L_0}^\infty([0,1])$ (for more details, see [19,20]) as follows:

$$C_{L_0}^\infty([0,1]) := \bigcap_{k=1}^\infty \mathrm{Dom}(L_0^k),$$

where $\mathrm{Dom}(L_0^k)$ is a domain of L_0^k. Here L_0^k stands for the k times iterated L_0 with the domain

$$\mathrm{Dom}(L_0^k) := \{L_0^{k-j-1}u \in \mathrm{Dom}(L_0), j = 0,1,...,k-1\}$$

for $k \geq 2$. Since the linear combination of all eigenfunctions is in $C^{\infty}_{L_0}([0,1])$, then the space of test functions is not empty as a set. For further properties of the space $C^{\infty}_{L_0}([0,1])$ we refer to the papers [19], [20], where the properties of the test functions based on a basis are studied. The dual space to $C^{\infty}_{L_0}([0,1])$ we denote by $\mathcal{D}'_{L_0}(0,1)$ (the space of continuous functionals on $C^{\infty}_{L_0}([0,1])$).

Now, we are in a way to define a fractional derivation of generalized functions. To begin, note that for all $u, v \in C^{\infty}_{L_0}([0,1])$ we get

$$(\mathcal{D}^{\alpha}_{1}\left[D^{\alpha}_{0}u\right], v) = (u, \mathcal{D}^{\alpha}_{1}\left[D^{\alpha}_{0}v\right]). \tag{4}$$

Here, both sides exist in the classical sense.

Indeed, equality (4) follows by the direct computations of $(\mathcal{D}^{\alpha}_{1}\left[D^{\alpha}_{0}u\right], v)$. By the definition, we have

$$(\mathcal{D}^{\alpha}_{1}\left[D^{\alpha}_{0}u\right], v) = -\frac{1}{\Gamma(1-\alpha)}\int_0^1\int_x^1 (t-x)^{-\alpha}\frac{d}{dt}D^{\alpha}_{0}u(t)dtv(x)dx,$$

and by changing integration order, we obtain

$$\int_0^1\int_x^1 (t-x)^{-\alpha}\frac{d}{dt}D^{\alpha}_{0}u(t)dtv(x)dx$$

$$= \int_0^1 \frac{d}{dt}D^{\alpha}_{0}u(t)\int_0^t (t-x)^{-\alpha}v(x)dxdt. \tag{5}$$

Integrating by parts in the right-hand side of equation (5), we have

$$-\frac{1}{\Gamma(1-\alpha)}\int_0^1 \frac{d}{dt}D^{\alpha}_{0}u(t)\int_0^t (t-x)^{-\alpha}v(x)dxdt$$

$$= -D^{\alpha}_{0}u(t)I^{1-\alpha}_{0}v(t)\Big|_0^1 + (D^{\alpha}_{0}u, D^{\alpha}_{0}v)$$

$$= -D^{\alpha}_{0}u(t)I^{1-\alpha}_{0}v(t)\Big|_0^1 + I^{1-\alpha}_{0}u(t)D^{\alpha}_{0}v(t)\Big|_0^1 - \int_0^1 I^{1-\alpha}_{0}u(t)\frac{d}{dt}D^{\alpha}_{0}v(t)dt.$$

By applying the property to $\left(I^{1-\alpha}_{0}u, \frac{d}{dt}D^{\alpha}_{0}v\right)$, and due to the equivalent definitions of the Caputo derivation [18, Chapter 1], we obtain

$$\left(I^{1-\alpha}_{0}u, \frac{d}{dt}D^{\alpha}_{0}v\right) = -\left(u, \mathcal{D}^{\alpha}_{1}\left[D^{\alpha}_{0}\right]v\right).$$

As the result, one takes the Green's formula

$$(\mathcal{D}^{\alpha}_{1}\left[D^{\alpha}_{0}u\right], v) = (u, \mathcal{D}^{\alpha}_{1}\left[D^{\alpha}_{0}\right]v)$$

$$+\sum_{i=1}^{2}[\xi_i^-(u)\xi_i^+(v)-\xi_i^-(v)\xi_i^+(u)]. \tag{6}$$

Since $u, v \in C_{L_0}^\infty([0,1])$ the identity (6) implies (4).

Define an action of the operator L on a generalized function $u \in \mathcal{D}'_{L_0}(0,1)$. Put

$$(Lu, v) := (u, \mathcal{D}_1^\alpha \left[D_0^\alpha v\right]) \tag{7}$$

for all $v \in C_{L_0}^\infty([0,1])$. The term $(u, \mathcal{D}_1^\alpha \left[D_0^\alpha v\right])$ exists due to the fact that $v \in C_{L_0}^\infty([0,1])$ and also involves $\mathcal{D}_1^\alpha \left[D_0^\alpha v\right] \in C_{L_0}^\infty([0,1])$. Thus, the action of L introduced by the formula (7) is well defined on the space of the generalized functions $\mathcal{D}'_{L_0}(0,1)$.

Now, we consider the following expression

$$Lu(x) := \mathcal{D}_1^\alpha \left[D_0^\alpha \left[u\right]\right](x)\,,\; 0 < x < 1, \tag{8}$$

in the space $\mathrm{L}^2(0,1)$. To define correctly L in $\mathrm{L}^2(0,1)$, we introduce the space $\mathrm{W}_2^{2\alpha}(0,1)$ as a closure of $H_0^{2\alpha+o}([0,1])$ by the norm

$$\|u\|_{\mathrm{W}_2^{2\alpha}(0,1)} := \|u\|_{L_2(0,1)} + \|\mathcal{D}_1^\alpha D_0^\alpha u\|_{L_2(0,1)}.$$

Indeed, the space $\mathrm{W}_2^{2\alpha}(0,1)$ with the introduced norm is a Banach one. Moreover, it is the Hilbert space with the scalar product

$$(u, v)_{\mathrm{W}_2^{2\alpha}(0,1)} := (u, v) + (\mathcal{D}_1^\alpha D_0^\alpha u, \mathcal{D}_1^\alpha D_0^\alpha v).$$

We define L_m as an operator acting from $\mathrm{L}^2(0,1)$ to $\mathrm{L}^2(0,1)$ by formula (8) with the domain

$$\mathrm{Dom}(L_m) = \left\{u \in \mathrm{W}_2^{2\alpha}(0,1):\;\; \xi_1^-(u) = \xi_2^-(u) = \xi_1^+(u) = \xi_2^+(u) = 0\right\}.$$

Also, introduce an operator $L_M : \mathrm{L}^2(0,1) \to \mathrm{L}^2(0,1)$ generated by expression (8) with the domain $\mathrm{Dom}(L_M) := \left\{u \in \mathrm{W}_2^{2\alpha}(0,1)\right\}$.

Now, we are in a position to formulate the main result of the manuscript.

In what follows, we introduce a class of (2x4)-matrices. This class will be helpful to define boundary forms for $\mathcal{D}_1^\alpha \left[D_0^\alpha \left[u\right]\right]$.

Definition 1. *We say that the matrix*

$$\theta := \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\ \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \end{pmatrix}$$

is S–matrix, if it can be written in one of the following views:

$$\begin{pmatrix} 1 & 0 & r & c \\ 0 & 1 & -c & d \end{pmatrix}, \; \begin{pmatrix} d & 1 & 0 & r \\ c & 0 & 1 & d \end{pmatrix},$$

$$\begin{pmatrix} 1 & d & r & 0 \\ 0 & c & -d & 1 \end{pmatrix}, \quad \begin{pmatrix} r & c & 1 & 0 \\ -c & d & 0 & 1 \end{pmatrix},$$

where $r, c, d \in \mathbb{R}$. Here, the matrices

$$\begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\ \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \end{pmatrix}, \quad \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\ \gamma\theta_{21} & \gamma\theta_{22} & \gamma\theta_{23} & \gamma\theta_{24} \end{pmatrix} \ \textit{for} \ \ (\gamma \neq 0),$$

$$\begin{pmatrix} \theta_{11} \pm \theta_{21} & \theta_{12} \pm \theta_{22} & \theta_{13} \pm \theta_{23} & \theta_{14} \pm \theta_{24} \\ \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \end{pmatrix}$$

and

$$\begin{pmatrix} \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \\ \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \end{pmatrix}$$

are equivalent.

Theorem 1. *Let θ be an S-matrix. Then an operator L_θ generated by*

$$\mathcal{D}_1^\alpha D_0^\alpha u(x) = f(x), \ \ 0 < x < 1,$$

for $u \in \mathrm{W}_2^{2\alpha}(0,1)$ with "boundary" conditions

$$\begin{aligned} \theta_{11}\xi_1^-(u) + \theta_{12}\xi_2^-(u) + \theta_{13}\xi_1^+(u) + \theta_{14}\xi_2^+(u) = 0, \\ \theta_{21}\xi_1^-(u) + \theta_{22}\xi_2^-(u) + \theta_{23}\xi_1^+(u) + \theta_{24}\xi_2^+(u) = 0, \end{aligned}$$

is a self-adjoint extension of L_m in $\mathrm{W}_2^{2\alpha}(0,1)$.

Note that when $\alpha < 1/2$ the statement of Theorem 1, briefly speaking, is not true.

3 Proof of Theorem 1

3.1. Preliminaries

Below we formulate necessary results on the operators L_m and L_M.

Lemma 1. *Kernel of the operator L_M ($\mathrm{Ker}L_M$) consists of any linear combination of the functions $(x-\varepsilon)_*^\alpha$ and $(x-\varepsilon)_*^{\alpha-1}$ for arbitrary $\varepsilon \in [0,1]$.*

The proof of Lemma 1 follows from the statements of Properties A.2, A.3, A.4 and A.5.

Lemma 2. *The equation $L_m u = g$ has a solution $u \in \mathrm{Dom}(L_m)$ if and only if there exists a function $f \in \mathrm{L}^2(0,1)$ such that for arbitrary $v \in \mathrm{Ker}L_M$ we have $(f, v) = 0$, or*

$$\mathcal{R}(L_m) \oplus \mathrm{Ker}L_M = \mathrm{L}^2(0,1).$$

Proof. Let $f \in \mathcal{R}(L_m)$. Then there is a function $w \in \mathrm{L}^2(0,1)$ such that for any $v \in \mathrm{Ker}L_M$ we obtain

$$(f, v) = (L_m w, v) = (w, L_M v) = 0.$$

Now, fix a function $f \in \mathrm{L}^2(0,1)$ with the property $(f, v) = 0$ for all $v \in \mathrm{Ker}L_M$. Due to the definition of L_M there is a function $g \in \mathrm{Dom}(L_M)$ such that $L_M g = f$. It is easy to see that for arbitrary $v \in \mathrm{Ker}L_M$ we have

$$0 = (f, v) = (L_M g, v) = \sum_{i=1}^{2} [\xi_i^-(v)\xi_i^+(g) - \xi_i^-(g)\xi_i^+(v)]. \tag{9}$$

Finally, Lemma 1 implies that the kernel of the operator L_M consists of the infinite number of the linear independent functions. Thus, due to the arbitrariness of v from identity (9) we obtain

$$\xi_i^-(g) = \xi_i^+(g) = 0, \;\; i = 1, 2.$$

Hence, $f \in \mathcal{R}(L_m)$. This completes the proof of the lemma. □

Corollary 1. $\mathrm{Dom}(L_m)$ *is dense in* $\mathrm{L}^2(0,1)$.

Proof. Let $g \in \mathrm{L}_2(0,1)$ be orthogonal to the lineal $\mathrm{Dom}(L_m)$. Find a function v as an arbitrary solution of the equation $L_M v = g$. Then for any $u \in \mathrm{Dom}(L_m)$ we get

$$0 = (u, g) = (u, L_M v) = (L_m u, v).$$

Due to Lemma 2 we obtain $v \in \mathrm{Ker}L_M$. Hence, $g = L_M v = 0$. The corollary is proved. □

3.2. Proof of Theorem 1

By Definition [21] the operator L_m is hermit, since for any $u, v \in \mathrm{Dom}(L_m)$ we have

$$(L_m u, v) = (u, L_m v).$$

Moreover, due to Corollary 1 the operator L_m is symmetric. Thus, to show that L_θ is a self–adjoint operator it is enough to have

$$\mathrm{Dom}(L_\theta) = \mathrm{Dom}(L_\theta^*). \tag{10}$$

The last one can be proven by the direct calculations taking into account formula (6).

4 Global Fourier Analysis associated with Caputo–Riemann–Liouville type Fractional Order Operators

4.1. Spectral properties of L_θ

Theorem 2. *Let θ be as*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \theta_{22} & 0 & \theta_{24} \end{pmatrix}.$$

Then the following statements hold:

(i) *L_θ^{-1} is a compact operator in* $\mathrm{L}^2(0,1)$.

(ii) *The spectrum of L_θ is real and discrete, and the system of eigenfunctions forms a complete orthogonal basis of the space* $\mathrm{L}^2(0,1)$.

Proof. (i) Indeed, the inverse operator can be represented in the form

$$L_\theta^{-1} f(x) = -\frac{\theta_{22}}{\theta_{22}+\theta_{24}} \frac{x^\alpha}{\Gamma(\alpha)} I_1^{\alpha+1} f(0) + I_0^\alpha I_1^\alpha f(x).$$

For $\theta_{22} = 0$ we have

$$L_\theta^{-1} f(x) = I_0^\alpha I_1^\alpha f(x), \ \ 0 < x < 1.$$

Hence, it follows compactness of the operator L_θ^{-1} in $\mathrm{L}^2(0,1)$.

(ii) Compactness of L_θ^{-1} implies discreteness of the spectrum, and the system of eigenfunctions forms a complete orthogonal basis in $\mathrm{L}^2(0,1)$. From the self-adjoint property of L_θ one obtains real validity of all eigenvalues [21]. $\square$

Theorem 3. *Let θ be in one of the following forms:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

$$\begin{pmatrix} \rho & 1 & 0 & 0 \\ 0 & 0 & 1 & \rho \end{pmatrix}, \ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

Then for all $\rho \in \mathbb{R}$ the operator L_θ is positive in the space $\mathrm{L}^2(0,1)$.

Proof. To prove the theorem it is sufficient to show the inequality

$$(\mathcal{D}_1^\alpha \left[D_0^\alpha u\right], u) \geq 0.$$

Now, we calculate

$$(\mathcal{D}_1^\alpha \left[D_0^\alpha u\right], u) = -\frac{1}{\Gamma(1-\alpha)} \int_0^1 \int_x^1 (t-x)^{-\alpha} \frac{d}{dt} D_0^\alpha u(t) dt\, u(x) dx.$$

By changing integration order, we get

$$-\frac{1}{\Gamma(1-\alpha)}\int_0^1\int_x^1 (t-x)^{-\alpha}\frac{d}{dt}D_0^\alpha u(t)dt\,u(x)dx$$
$$=-\frac{1}{\Gamma(1-\alpha)}\int_0^1 \frac{d}{dt}D_0^\alpha u(t)\int_0^t (t-x)^{-\alpha}u(x)dxdt.$$

By integrating by parts in the right-hand side of the last integral, we obtain

$$-\frac{1}{\Gamma(1-\alpha)}\int_0^1 \frac{d}{dt}D_0^\alpha u(t)\int_0^t (t-x)^{-\alpha}u(x)dxdt$$
$$=-D_0^\alpha u(t)I_0^{1-\alpha}u(t)\Big|_0^1+\left(D_0^\alpha u, D_0^\alpha u\right).$$

As the result, we take the identity

$$D_0^\alpha u(t)I_0^{1-\alpha}u(t)\Big|_0^1=0,$$

which completes the proof. □

4.2. Schatten classes of L_θ^{-1}

The following assertion is proved by Delgado and Ruzhansky [26]: Let M be a closed manifold of the dimension n. Let $K\in H^\mu(M\times M)$ for some $\mu>0$. Then the integral operator T on $L^2(M)$, defined by

$$(Tf)(x)=\int\limits_M K(x,s)f(s)ds,$$

is in the Schatten classes $S_p(L^2(M))$ for $p>\frac{2n}{n+2\mu}$.

Now, we try to apply this result as follows:

Theorem 4. *Let θ be in one of the following forms:*

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{13}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix},\ \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{14}$$

Then the inverse operator L_θ^{-1} on $\mathrm{L}^2(0,1)$, defined by

$$L_\theta^{-1}f(x)=\int\limits_0^1 K(x,s)f(s)ds,$$

is in the Schatten classes $S_p\left(L^2(0,1)\right)$, for $p>\frac{2}{1+4\alpha}$.

Proof. Here, we give a full proof only for the case $\theta_{11} \neq 0, \theta_{24} \neq 0, \theta_{12} = \theta_{13} = \theta_{14} = \theta_{21} = \theta_{22} = \theta_{23} = 0$. Other three cases can be proved similarly.

From Theorem 2 it is known, that

$$L_\theta^{-1} f(x) = I_0^\alpha I_1^\alpha f(x), \ \ 0 < x < 1.$$

Then the operator L_θ^{-1} can be represented as

$$L_\theta^{-1} f(x) = I_0^\alpha I_1^\alpha f(x) = \int\limits_0^1 K(x,s) f(s) ds, \ \ 0 < x < 1,$$

where $K(x,s)$ has the form

$$K(x,s) = \frac{1}{\Gamma^2(\alpha)} \int_0^1 (x-\tau)_*^{\alpha-1} (s-\tau)_*^{\alpha-1} d\tau$$

$$= \frac{1}{\Gamma^2(\alpha)} \int_0^{\max\{x,s\}} (x-\tau)^{\alpha-1} (s-\tau)^{\alpha-1} d\tau.$$

Here

$$(z-\varepsilon)_*^\mu = \begin{cases} 0, \ z \leq \varepsilon, \\ (z-\varepsilon)^\mu, \ z > \varepsilon. \end{cases}$$

The fact that L_θ^{-1} is inverse to L_θ implies that $K(x,s)$ is the Green's function of L_θ. Hence $K(x,s)$ belongs to the class $W_2^{2\alpha}((0,1) \times (0,1))$. Consequently, by the Delgado-Ruzhansky's theorem 4, we obtain that the integral operator L_θ^{-1} is in the Schatten classes $S_p(L^2(0,1))$ for $p > \frac{2}{1+4\alpha}$. □

4.3. Global Analysis generated by L_θ

Here, we briefly discuss about the Global Analysis associated with the fractional order differential operator L_θ. Indeed, by using the Global Fourier Analysis commuted with L_θ developed in [19], [20], studied operators can be applied for solving problems of the sub-diffusion, super-diffusion, anomaly diffusion, etc (for instance, see, [23]–[25]). We note that the general case is developed in [26] with some applications given in [27] for the Landau Hamiltonian. More general setting is offered in the recent papers [28], [30]. One is worth to be mentioned that the theory of Pseudo-Differential Operators associated with fractional order differential equations can be started. Moreover, investigations of the spectral problems for fractional differential operators are helpful and important to enrich and develop the fractional calculus.

A. Fractional differentiation and its properties

In this Appendix, we define fractional integration and differentiation operators [18], [30], [30].

Definition A.1. *Let f be a function defined on the interval $[0,1]$. Assume that the following integrals exist*

$$I_0^{\alpha}[f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \;\; t \in (0,1],$$

and

$$I_1^{\alpha}[f](t) = \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} f(s)ds, \;\; t \in [0,1).$$

Then we call them the left and right Riemann-Liouville integral operators of the fractional order $\alpha > 0$, respectively.

Definition A.2. *Define left-side and right-side Riemann-Liouville differential operators of the fractional order α ($0 < \alpha < 1$) by*

$$D_0^{\alpha}[f](t) = \frac{d}{dt} I_0^{1-\alpha}[f](t)$$

and

$$D_1^{\alpha}[f](t) = -\frac{d}{dt} I_1^{1-\alpha}[f](t),$$

respectively.

Definition A.3. *For $0 < \alpha < 1$ we say that the actions*

$$\mathcal{D}_0^{\alpha}[f](t) = D_0^{\alpha}[f(t) - f(0)]$$

and

$$\mathcal{D}_1^{\alpha}[f](t) = D_1^{\alpha}[f(t) - f(1)],$$

are left and right differential operators of the fractional order α ($0 < \alpha < 1$) in the Caputo sense, respectively.

Note that in monographs [18], [30], [31] there are studied different types of fractional differentiations and their main properties. In what follows we formulate statements of necessary properties of integral and integro-differential operators of the Riemann-Liouville type and fractional Caputo operators.

Property A.1 [30, Pages 73, 76, 96]. *Let $0 < \alpha < 1$. Assume that*

$$f \in \mathrm{L}^1(0,1), \;\; I_1^{1-\alpha} f, \;\; I_0^{1-\alpha} f \in AC[0,1].$$

Then the following equalities are true:

$$I_0^\alpha I_0^\beta f(x) = I_0^{\alpha+\beta} f,$$

$$I_1^\alpha I_1^\beta f(x) = I_1^{\alpha+\beta} f,$$

for all $0 < \beta < 1$*;*

$$I_1^\alpha D_1^\alpha f(x) = f(x) - I_1^{1-\alpha} f(0) \frac{(1-x)^{\alpha-1}}{\Gamma(\alpha)},$$

$$I_0^\alpha D_0^\alpha f(x) = f(x) - I_0^{1-\alpha} f(0) \frac{x^{\alpha-1}}{\Gamma(\alpha)},$$

for $x \in (0,1)$*.*

Moreover, if $f \in AC[0,1]$*, then we have*

$$I_0^\alpha \mathcal{D}_0^\alpha f(x) = f(x) - f(0),$$

$$I_1^\alpha \mathcal{D}_1^\alpha f(x) = f(x) - f(1),$$

for all $x \in [0,1]$*.*

For any $\varepsilon \in (0,1)$ we denote

$$(x-\varepsilon)_* = \begin{cases} 0, \; x \le \varepsilon, \\ \\ x-\varepsilon, \; x > \varepsilon. \end{cases}$$

Property A.2 [18, Page 87]. *Let* $\alpha > 0$, $\beta > 0$, $C \equiv const$ *and*

$$f = C \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} (x-\varepsilon)_*^{\beta-1}.$$

Then we have

$$I_0^\alpha f(x) = C(x-\varepsilon)_*^{\alpha+\beta-1},$$

for $0 < x < 1$*.*

Property A.3. *Let* $0 < \alpha < 1$*. Then for all* $\varepsilon \in (0,1)$ *and any constant* C *the following function*

$$f(x) = C(x-\varepsilon)_*^{\alpha-1} = \begin{cases} 0, \; x \le \varepsilon, \\ \\ C(x-\varepsilon)^{\alpha-1} \; x > \varepsilon, \end{cases}$$

satisfies the equation

$$D_0^\alpha f(x) = 0, \;\; x \in (0,1).$$

Property A.4. *Let $0 < \alpha < 1$. Then for arbitrary $\varepsilon \in (0,1)$ and a constant C the function*

$$f(x) = C\theta(x-\varepsilon) = \begin{cases} 0, \, x \leq \varepsilon; \\ \\ C, \, x > \varepsilon, \end{cases}$$

satisfies

$$\mathcal{D}_1^\alpha f(x) = 0, \;\; 0 < x < 1,$$

where $\theta(x)$ is the Heaviside function.

Property A.5. *Let $0 < \alpha < 1$. Then*

$$f(x) = \frac{C}{\Gamma(\alpha)}(x-\varepsilon)_*^{\alpha-1} + \frac{1}{\Gamma(\alpha+1)}(x-\varepsilon)_*^{\alpha}, \;\; C = const, \;\; 0 < x < 1,$$

satisfies the equation

$$D_0^\alpha f(x) = \theta(x-\varepsilon), \;\; 0 < x < 1.$$

Property A.6 [31, Page 34]. *Let $u, v \in \mathrm{L}^2(0,1)$ and $0 < \alpha < 1$. Then we have the formula of integration by parts*

$$\left(I_1^\beta u, v\right) = \left(u, I_0^\beta v\right).$$

Here, by $(\cdot,\cdot)$ we denote the inner product of the Hilbert space $\mathrm{L}^2(0,1)$.

Let us formulate Theorem 3.2 of the book [18]:

Theorem A.1. *Assume that $\varphi \in H^\gamma([0,1])$, $\gamma \geq 0$. Then the fractional integral $I_0^\alpha \varphi$, $\alpha > 0$, has the form*

$$I_0^\alpha \varphi = \sum_{k=0}^{m} \frac{\varphi^{(k)}(0)}{\Gamma(\alpha+k+1)} x^{\alpha+k} + \psi(x),$$

where m is the greatest integer such that $m < \gamma$; and $\psi \in H^{\gamma+\alpha}([0,1])$, if $\gamma+\alpha$ is not integer, or if $\gamma, \alpha \in \mathbb{Z}$.

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Тоқмағамбетов Н., Төребек Б.Т. ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ОПЕРАТОРЛАРДАН ТУЫНДАҒАН ӨЗ-ӨЗІНЕ ТҮЙІНДЕС ОПЕРАТОРЛАР

Мақала интегралдық-дифференциалдық оператордың өз-өзіне түйіндес кеңейтулерін сипаттауға арналған. Реті 2α ($\frac{1}{2} < \alpha < 1$) болатын симметриялы интегралдық-дифференциалдық операторлар табылды. Бұл, шындығында белгілі мағынада бөлшек ретті Штурм-Лиувилль операторының аналогы болып табылады. Сондай-ақ, бөлшек ретті дифференциалдық теңдеу үшін Грин формуласының аналогы тағайындалып, әрі қарай оның өз-өзіне түйіндес операторларды сипаттауға қолданылуы келтірілген. Соңында глобалды Фурье талдауы талқыланған және Капуто-Риман-Лиувилль тектес туындылармен байланысқан өз-өзіне түйіндес бөлшек ретті операторлардың спектралдық қасиеттері туралы кейбір нәтижелер дәлелденген.

Кілттік сөздер. Интегралдық-дифференциалдық оператор, Капуто туындысы, Риман-Лиувилль туындысы, өз-өзіне түйіндес есеп, бөлшек ретті дифференциалдық теңдеу, бөлшек ретті Штурм-Лиувилль операторы, кеңейтулер теориясы.

Токмагамбетов Н., Торебек Б.Т. САМОСОПРЯЖЕННЫЕ ОПЕРАТОРЫ, ПОРОЖДЕННЫЕ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫМИ ОПЕРАТОРАМИ

Данная статья посвящена описанию самосопряженных расширений интегро-дифференциального оператора. Найдены симметричные интегро-дифференциальные операторы порядка 2α (где $\frac{1}{2} < \alpha < 1$). Действительно, в некотором смысле это аналог дробного оператора Штурма-Лиувилля. Кроме того, аналог формулы Грина для дифференциальных уравнений дробного порядка установлен с дальнейшими приложениями в описании класса самосопряженных операторов. Наконец, мы обсудим глобальный анализ Фурье и докажем некоторые результаты о спектральных свойствах самосопряженных операторов дробного порядка, связанных с производными типа Капуто-Римана-Лиувилля.

Ключевые слова. Интегро-дифференциальный оператор, производная Капуто, производная Римана-Лиувилля, самосопряженная задача, дифференциальное уравнение дробного порядка, дробный оператор Штурма-Лиувилля, теория расширений.

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An attempt to predict the arc to glow transition based on the experimental results

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Abstract. The paper presents an attempt to assess the transition of arc to glow discharge on the basis of a comparison of selected theoretical indicators and these provided by the experiments. The evaluation includes the dynamics change of the discharge volume during the opening of the contact. The tests were carried out for a low voltage DC circuit with a discharge energy not exceeding $10J$. Based on the results obtained, appropriate practical conclusions were formulated regarding the need for further consideration.

Keywords. Switching DC arc, low voltage and low power electric grid, arc-to-glow transition.

1 Introduction

The use of direct current in various areas shows increasing trend mainly due to the increasing use of renewable energy sources. However, this forces manufacturers and users to different approach to the application due to both the advantages and disadvantages of DC compared to AC. One of the problem is to ensure effective breaking of the DC circuit especially under inductive loads.

Tests carried out by authors in recent years have demonstrated the occurrence of the previously unknown effect of spontaneous transition of the DC switching arc into glow discharge [1]. This is a very positive effect because it significantly reduces contact erosion thereby increasing their switching life. However, there are significant problems with explaining the reasons of this effect due to the diversity, complexity and variability of physical phenomena and their mutual interaction each other. This applies both to the surface conditions of the contacts as well as the area of the contact space. The authors investigated this effect to

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the broadest limits possible, paying attention to the majority of factors affecting the discharge phenomena. This applies to both the type of contact material, voltage and switching current, type and pressure of protective gas inside the contact gap, contact opening speed, etc. The tests were performed using, among others, fast photo registration and optical fiber spectroscopy [2]. It allowed to draw specified conclusions, however, a small amount of data does not allow for the practical implementation of this process under operation conditions. No less, it can be concluded that for specified switching conditions this process is statistically predictable, unfortunately it is not repeatable for following switching.

At the same time, the authors made the attempt to theoretical analysis of this phenomenon via a mathematical description of this effect based on the experimental results. The mathematical model has been developed that describes the dynamics of the transition of a low-temperature electric arc plasma into a glow discharge. It is based on the system of differential equations for temperature and electromagnetic fields, the solution of which is found using the method of upper and lower functions. Based on Lyapunov's theory and the Hurwitz criteria, a system of inequalities is obtained for the parameters of the electric arc and the material of the electrodes, which makes it possible to obtain criteria for instability and bifurcation of voltage, current and temperature, which provide the required transition [3]. According to this the arc instability criteria can (for the given electric circuit R, L, C) be formulated as follows:

$$\frac{R}{L} + \frac{1}{CR_A} - \frac{1}{k_A} < 0, \tag{1}$$

$$\left(\frac{R}{L} + \frac{1}{CR_A} - \frac{1}{k_A}\right)\left(\frac{1}{LC} + \frac{R}{LCR_A} + \frac{1}{k_A CR_A} - \frac{R}{k_A L}\right) - \frac{1}{k_A LC}\left(\frac{R}{R_A} - 1\right) < 0, \tag{2}$$

$$\frac{R}{R_A} - 1 < 0. \tag{3}$$

Where arc resistance R_A:

$$R_A = \frac{U_A}{I_A} \tag{4}$$

and thermal (heat) constant

$$k_A = \frac{C_A V_{Arc} T_A}{P_A} \tag{5}$$

are particularly important for the relation between the arc time t_A and this of glowing t_g (V_{Arc} is arc volume, T_A is arc temperature, P_A is arc power, C_A is heat capacity of the arc). Unfortunately, the modeling of the arc transition into glowing is based on a large error due to complex phenomena and mutually dependent parameters. Therefore, despite the extensive experimental research, changes in transient cannot be determined precisely, mostly due to the lack of appropriate available high resolution measuring equipment. However, it should be noted that for the given switching conditions, the electrical circuit parameters R, L, C are already given. Selected physical quantities can be calculated from recorded waveforms of current and voltage drop. Therefore, it is possible to estimate the value of the resistance (R_A) of the discharge (arc and glow) as well as the power P_A provided to the electric discharge. Next, when using the literature data regarding the average arc T_A temperature and its heat capacity C_A, the thermal time constant k_A of the arc can be estimated. For analysis the value of the arc volume V_A as well as its variation with time under contact opening has to be taken into account. (It must be noted that in [3] only average arc volume was considered). These values were estimated by the authors on the basis of the measurement of the change in the length of the contact gap in time and its correlation with the results of measurements of photo-registration of the discharge process.

Thus, using the set of inequalities (1)–(3) one can estimate the values of both R_A and k_A for given switching conditions. Basing on (5) for the experimentally estimated values of the arc volume and its temperature one can obtain the threshold power value of the arc, i.e. the maximum value of the current at which interruption of the arc-to glow transition will take place. Note, that if the arc resistance R_A tends to infinity the k_A value is close to the circuit time constant ($T = L/R$).

The article attempts to define the limit parameters of the presented equations with reference to the measurement results obtained under testing. The test was carried out in a specially designed and made for this purpose the test stand with hermetic chamber for round, plain contacts (diameter $5\,mm$ and thickness of $1\,mm$) made of $CuCr$ composite material (made by means of the electron beam technology [4], for different Cr content) in air under normal pressure at room conditions (see Fig. 1). Voltage was fixed to be $110\,V$, current about $0.5\,A$ for the inductive time constant of the electric circuit equal to $40\,ms$ (discharge energy less than $10\,J$) and contact gap around $4\,mm$ [2]. The average contact opening speed was around $0.125\,m/s$.

2 Selection of the test results for analysis

The tests were performed for flat contacts (1) with the use of an insulating washer (2) that prevents the arc from moving beyond the contact area (Fig. 1).

Due to the different progress of the phenomenon for the same switching conditions, three different cases (waveforms) were selected for the analysis: only the arc discharge without transient to glowing (Fig. 2), only the glow discharge (Fig. 3) and the arc discharge with double transition into the glowing (Fig. 4).

Arc column radius r_{Arc} changes linearly from $1.5 \cdot 10^{-5}$ to $2 \cdot 10^{-3}\,m$.

Arc volume is:

$$V_{Arc} = V_{cilinder}(l, r_{Arc}) = \pi (r_{Arc})^2 l \tag{6}$$

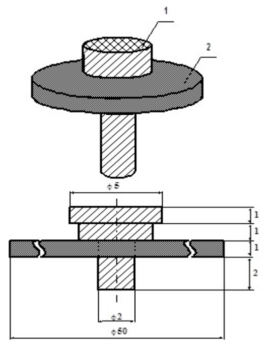

Figure 1 – Appearance of the contact sample for testing
(1 – contact member, 2 – textolite washer, dimensions in mm)

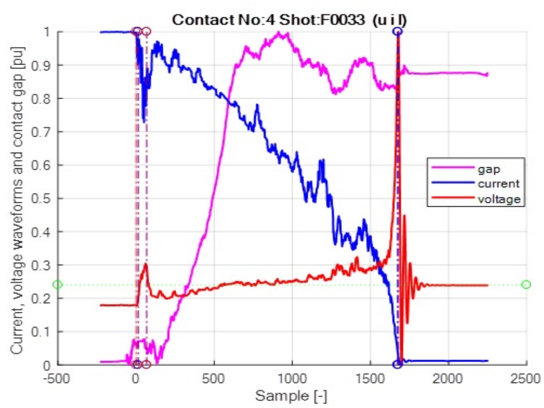

Figure 2 – Arc current waveform i, contact voltage u and contact gap l variation during interruption of DC inductive load ($110\,V$, $0.52\,A$, $L/R = 42\,ms$, $CuCr$ contacts materials)

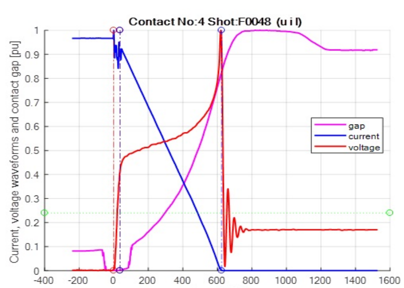

Figure 3 – Discharge current waveform i, contact voltage u and contact gap l
variation during interruption of DC inductive load
($110\,V$, $0.52\,A$, $L/R = 42\,ms$, $CuCr$ contacts materials), for only the glow appearance

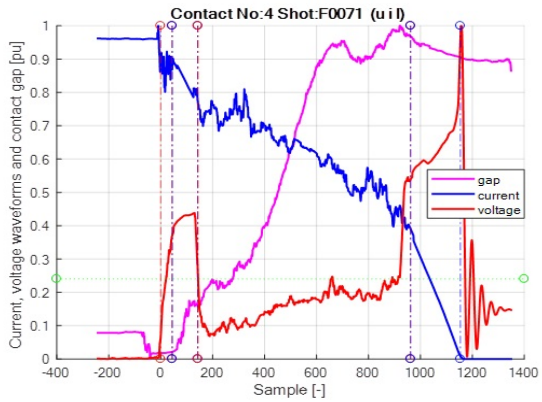

Figure 4 – Discharge current i, contact voltage waveform u and contact gap l variation during interruption of DC inductive load ($110\,V$, $0.52\,A$, $L/R = 42\,ms$, $CuCr$ contacts materials), for double transition of the switching arc into glowing

3 Estimation based on experiment

The assessment of the arc's transition into glow discharge was carried out for three presented above cases based on measured and estimated waveforms of the discharge column volume (arc) V_{Arc}, discharge resistance R_A, and discharge power P_A for two different radius values r_{Arcmin} and r_{Arcmax} of the arc column (discharge). All data for estimation are included in Table I.

Table 1 – Measured and theoretically estimated parameters for analysis

Parameters for analysis		
supplied voltage	U	$110\,V$
load current	I	$0.52\,A$
load resistance	R	$211.538\,Ohms$
load inductance	L	$8.931\,H$
circuit capacity	C	$5.86855 \cdot 10^{-8}\,F$
arc temperature	T_A	$6500\,K$
arc volume	V_{Arc}	$5.0 \cdot 10^{-19}; 1.5 \cdot 10^{-7}\,m^3$
arc heat capacity	C_A	$237.6\,J/m^3K$

For the case of the existence of only arcing (see Fig. 2), the trends of variation of the volume of the arc column, the resistance of the arc and the arc power with time under the discharge are shown in Fig. 5–9, respectively. The possibility of meeting the transition conditions (according to the equations (1)–(3)) for the arcing only as in Fig. 2 is illustrated by Fig. 10 and 11.

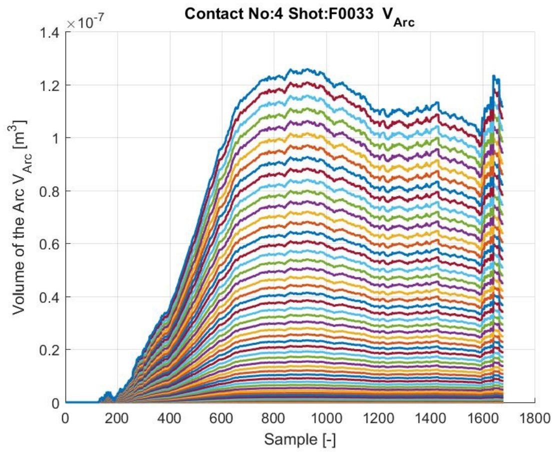

Figure 5 – Variation of the arc column V_{Arc} with time for different arc radius (discharge as in Fig. 2)

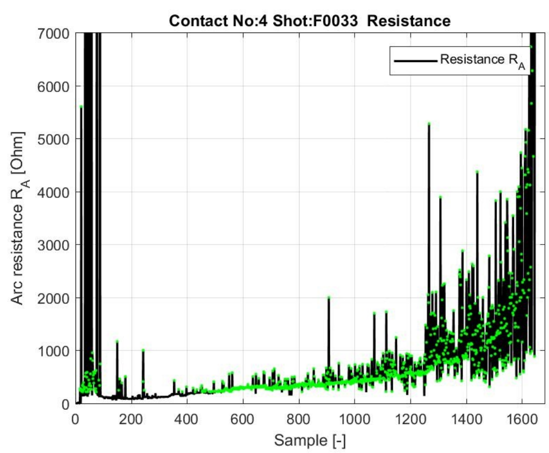

Figure 6 – Variation of the arc resistance R_A with time (discharge as in Fig. 2, dotted for $r_{Arcmin} = 1.5 \cdot 10^{-5}\, m$)

From the obtained waveforms it follows that for only arc appearance, both the resistance of the arc and the power supplied to the arc are quite stable and show change at the end of the discharge. However, the probability of meeting the logical conditions for the transition is quite possible, especially for a small radius of the arc column $r_{Arcmin} = 1.5 \cdot 10^{-5}\, m$ as indicated by dots in Fig. 6 and Fig. 8. It can be seen that with the increase in the radius value ($r_{Arcmax} = 2 \cdot 10^{-3}\, m$) the transition is theoretically possible only at the beginning of the course (as indicated by dots in Fig. 7 and Fig. 9). It should be also noted that the duration of the only arcing is the longest in comparison with the other runs.

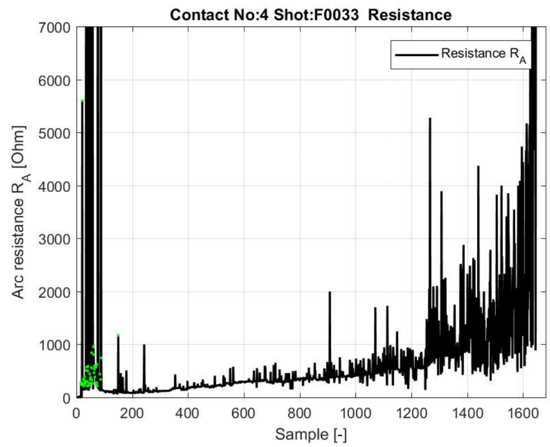

Figure 7 – Variation of the arc resistance R_A with time (discharge as in Fig .2, dotted for $r_{Arcmax} = 2 \cdot 10^{-3}\, m$)

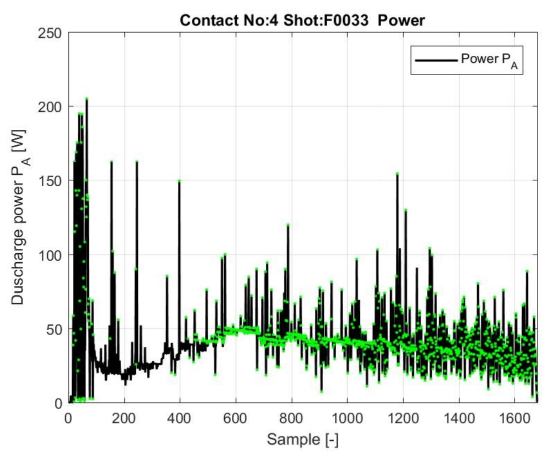

Figure 8 – Variation of the discharge power P_A with time (discharge as in Fig. 2, dotted for $r_{Arcmin} = 1.5 \cdot 10^{-5}\, m$)

For the case of the appearance of only glowing (see Fig. 3), the variation of the volume of the discharge column, the discharge resistance and the dissipated power with time under the discharge are shown in Fig. 12–16, respectively. Whereas, possibility of meeting the transition conditions (according to the equations (1)–(3)) for the glowing only as in Fig. 3 is illustrated by Fig. 17 and 18.

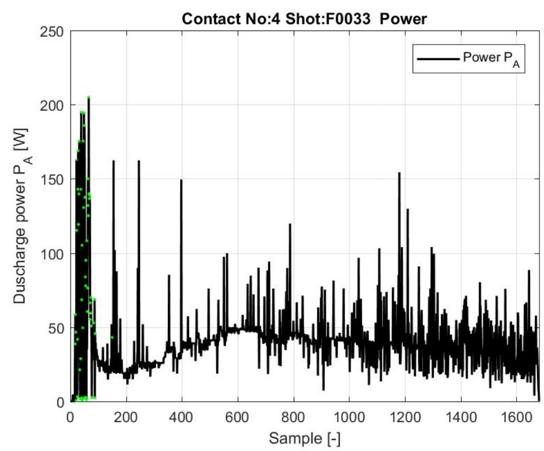

Figure 9 – Variation of the discharge power P_A with time
(discharge as in Fig. 2, dotted for $r_{Arcmax} = 2 \cdot 10^{-3}\, m$)

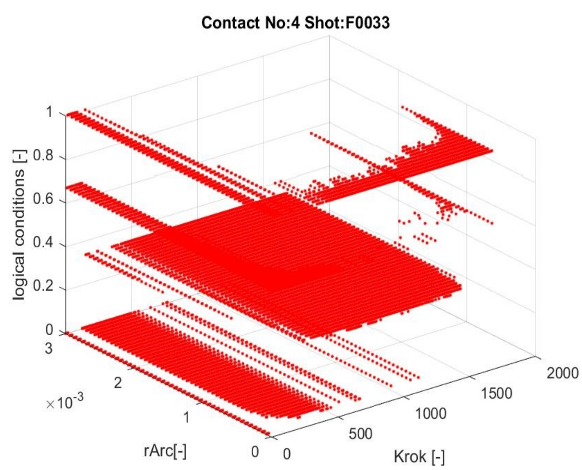

Figure 10 – Logical conditions $(3/3 = 1; 2/3 = 0.67; 1/3 = 0.33; 0/3 = 0)$
as a function of time and arc column radius r_{Arc} for discharge as in Fig. 2

During the tests, it was found that in the case of only glow discharge, the volume of the discharge column is smaller than in the case of only an arc discharge. However, this value increases with time, which is the result of the increase in the length of the contact gap under

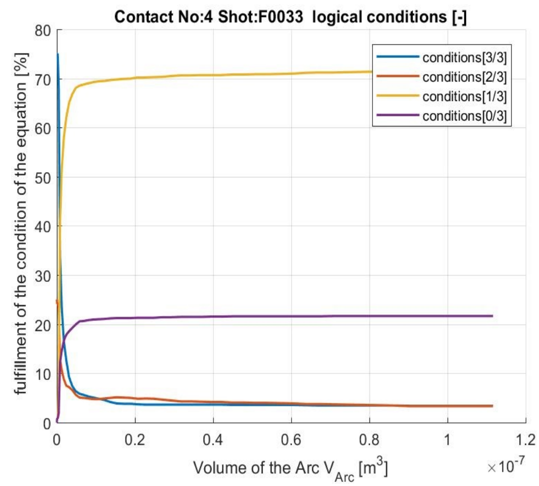

Figure 11 – Logical conditions with time and volume of the arc V_{Arc} for discharge as in Fig. 2.

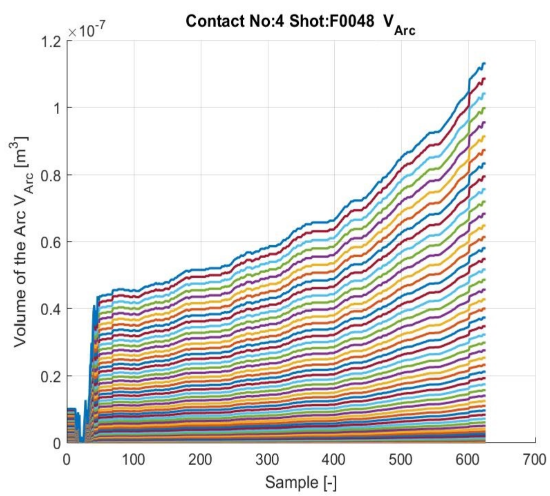

Figure 12 – Variation of the discharge column V_{Arc} with time for different discharge radius (discharge as in Fig. 3)

opening. The discharge resistance indicates much higher value and is exponentially increasing over time. The discharge power decreases practically linearly. The logical conditions for the occurrence of the transition effect are met, but practically for small values of both the radius and discharge volume. It should be emphasized here that the duration of the discharge is much shorter in this case. The glow discharge disappears before the contact opens.

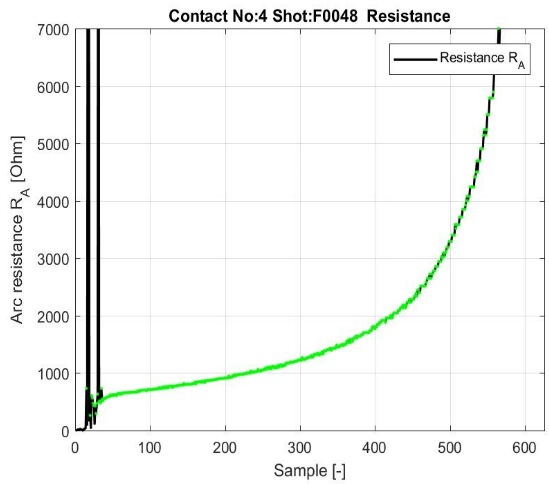

Figure 13 – Variation of the discharge resistance R_A
with time (discharge as in Fig. 3, dotted for $r_{Arcmin} = 1.5 \cdot 10^{-5}\, m$)

Figure 14 – Variation of the discharge resistance R_A
with time (discharge as in Fig.3, dotted for $r_{Arcmax} = 2 \cdot 10^{-3}\, m$)

For the case of the double transition of the arc into glowing (see Fig. 4), the trends of variation of the volume of the arc column, the resistance of the arc and the arc power with time under the discharge are shown in Fig. 19–23, respectively. Whereas, possibility of meeting the transition conditions (according to the equations (1)–(3)) for the double arc-glow transition, as in Fig. 4 is illustrated by Fig. 24 and 25.

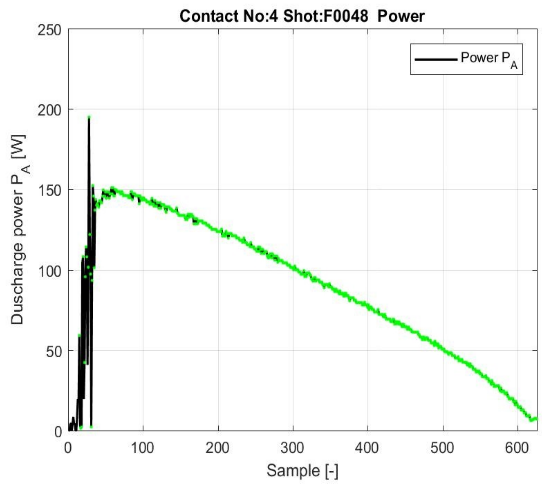

Figure 15 – Variation of the discharge power P_A
with time (discharge as inFig. 3, dotted for $r_{Arcmin} = 1.5 \cdot 10^{-5}\, m$)

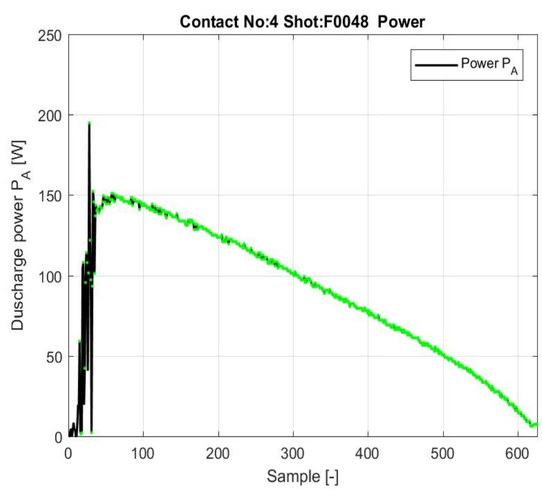

Figure 16 – Variation of the discharge power P_A
with time (discharge as inFig. 3, dotted for $r_{Arcmax} = 2 \cdot 10^{-3}\, m$)

The obtained measurements show that the volume of the discharge column is slightly smaller compared to the arcing. The duration of the contact opening is also shorter. However, the resistance at the transition points shows a significant increase. The course of power shows similarity and arcing and fluorescent discharge. The transition conditions are ensured for a small arc fault value. This possibility is also marked by dots on the power and resistance

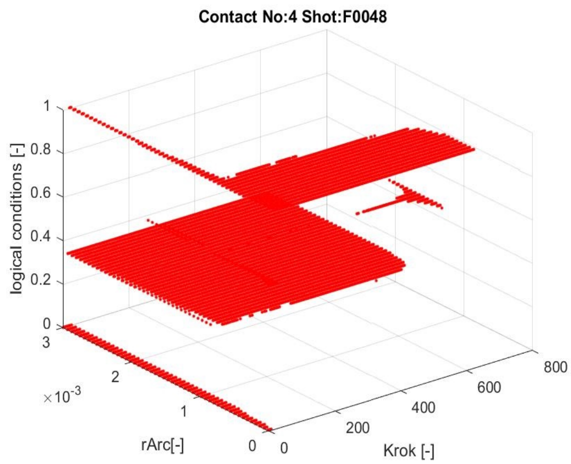

Figure 17 – Logical conditions $(3/3 = 1; 2/3 = 0.67; 1/3 = 0.33; 0/3 = 0)$ as a function of time and column radius r_{Arc} for discharge as in Fig. 3

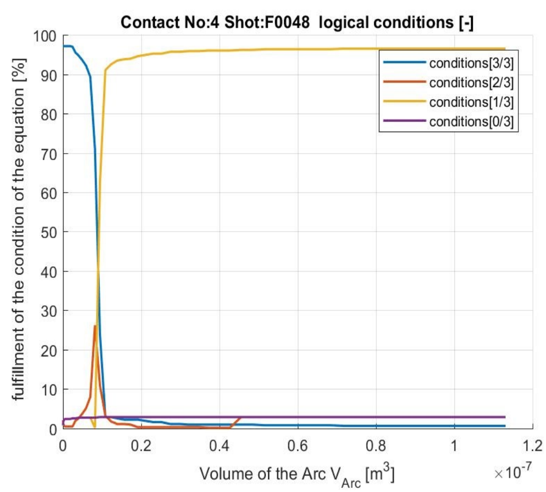

Figure 18 – Logical conditions with time and volume of the discharge V_{Arc} for discharge as in Fig. 3

waveforms as a function of time.

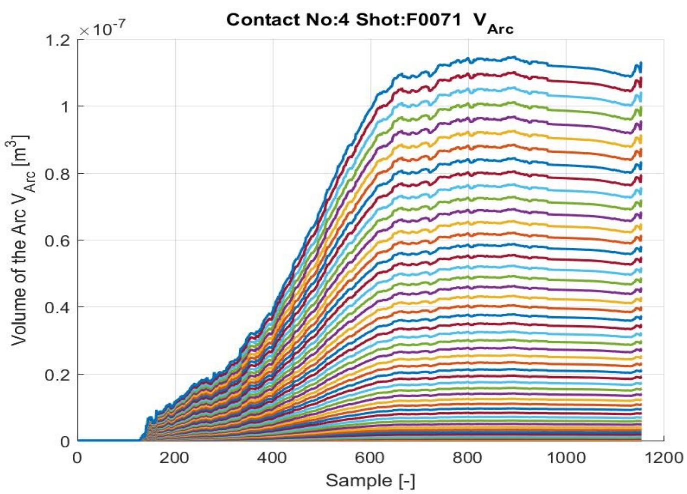

Figure 19 – Variation of the discharge column V_{Arc} with time for different discharge radius (discharge as in Fig. 4)

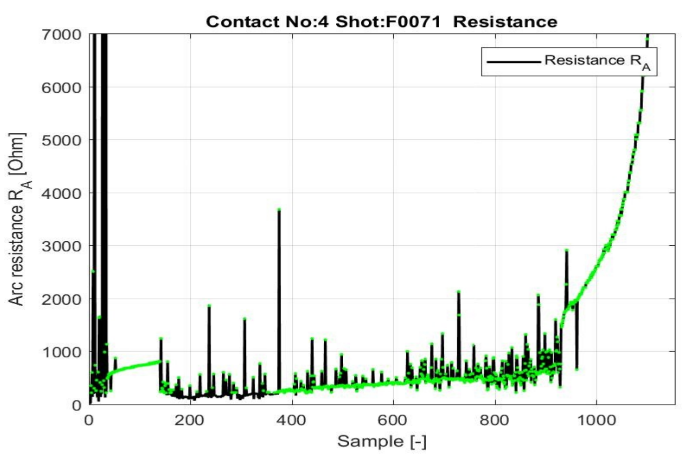

Figure 20 – Variation of the discharge resistance R_A with time (discharge as in Fig. 3, dotted for $r_{Arcmin} = 1.5 \cdot 10^{-5}\, m$)

4 Conclusions

The tests carried out showed that under certain conditions of interrupting the low-power inductive DC current, there is a spontaneous transition of the switching arc into glow discharge. Controlling this phenomenon is however, very difficult due to complexity and mutual interaction of physical phenomena both at the contact surfaces as well as inside the inter-contact space. Mathematical stability criteria derived are practically useful provided that

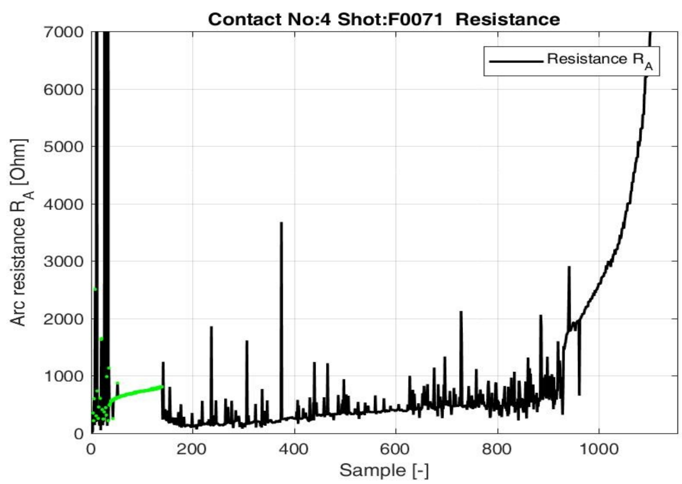

Figure 21 – Variation of the discharge resistance R_A with time (discharge as inFig. 3, dotted for $r_{Arcmax} = 2 \cdot 10^{-3}\, m$)

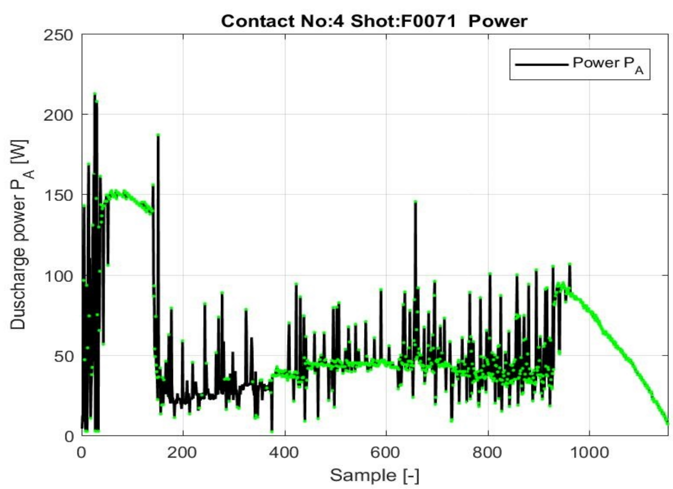

Figure 22 – Variation of the discharge power P_A with time (discharge as inFig. 3, dotted for $r_{Arcmin} = 1.5 \cdot 10^{-5}\, m$)

the values of nonlinear parameters in them are precisely determined. The research carried out by the authors showed that although there is no repeatability of the phenomenon under following breaking but it is theoretically possible to meet requirements of these conditions

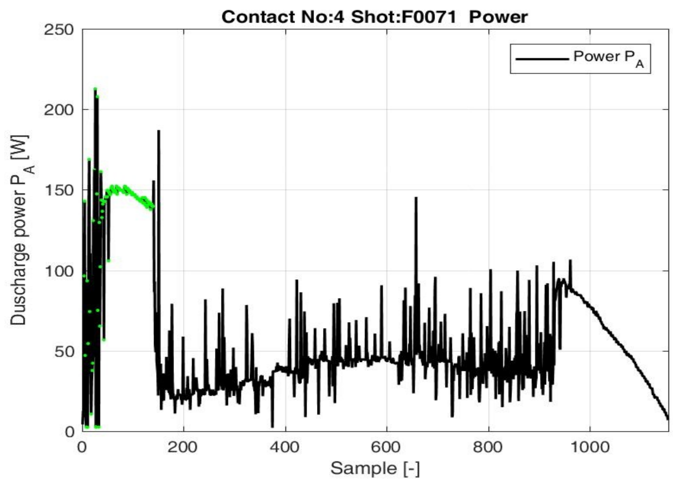

Figure 23 – Variation of the discharge power P_A with time (discharge as inFig. 3, dotted for $r_{Arcmax} = 2 \cdot 10^{-3}\, m$)

Figure 24 – Logical conditions $(3/3 = 1; 2/3 = 0.67; 1/3 = 0,33; 0/3 = 0)$ as a function of time and column radius r_{Arc} for discharge as in Fig. 3

(by, for example, decreasing discharge volume and its radius). However, it is necessary to analyze theoretically and to examine practically as accurately as possible the thermal process in transient states of breaking both at the surface of contacts as well as inside the contact gap volume. Unfortunately, it requires the use of suitable high-resolution measuring equipment.

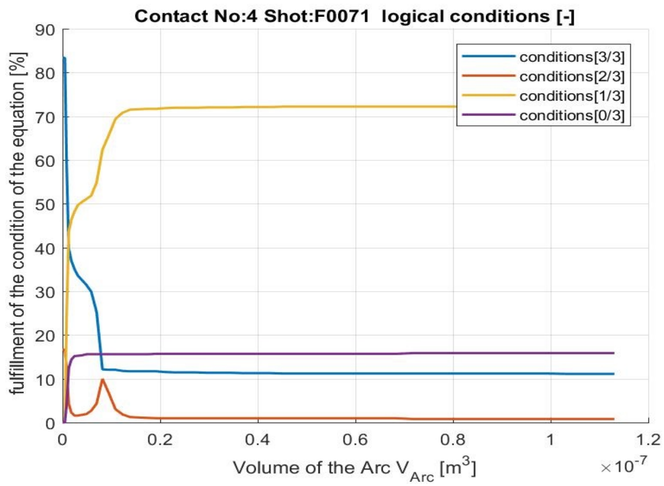

Figure 25 – Logical conditions with time and volume of the discharge V_{Arc} for discharge as in Fig. 3

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Вишневский Г.В., Харин С.Н., Меджинский Б. ДОҒАНЫҢ СОЛҒЫН РАЗРЯДҚА АУЫСУЫН ЭКСПЕРИМЕНТАЛДЫ НӘТИЖЕЛЕР НЕГІЗІНДЕ БОЛЖАУ ӘРЕКЕТІ

Мақалада доғаның солғын разрядқа ауысуын таңдап алынған теориялық көрсеткіштер мен эксперименттер деректерін салыстыру негізінде бағалауға әрекет жасалды. Бағалау контактіні ашқан кезде разряд көлемінің өзгеру динамикасын қамтиды. Сынақтар разряд қуаты 10 *Дж* аспайтын тұрақты тоқтың төмен вольтты тізбегі үшін жүргізілді. Алынған нәтижелер негізінде одан әрі қарай зерттеу қажеттілігіне қатысты сәйкес практикалық қорытындылар тұжырымдалды.

Кілттік сөздер. Тұрақты тоқтың доғасын ауыстырып қосу, төмен кернеулі және аз қуатты электр желісі, доғаның солғын разрядқа ауысуы.

Вишневский Г.В., Харин С.Н., Меджинский Б. ПОПЫТКА ПРЕДСКАЗАТЬ ПЕРЕХОД ДУГИ В ТЛЕЮЩИЙ РАЗРЯД НА ОСНОВЕ ЭКСПЕРИМЕНТАЛЬНЫХ РЕЗУЛЬТАТОВ

В статье предпринята попытка оценить возможность перехода дуги в тлеющий разряд на основе сравнения выбранных теоретических показателей и данных экспериментов. Оценка включает в себя динамику изменения объема разряда при открытии контакта. Испытания проводились для низковольтной цепи постоянного тока с энергией разряда не более 10 *Дж*. На основе полученных результатов были сформулированы соответствующие практические выводы относительно необходимости дальнейшего рассмотрения.

Ключевые слова. Переключение дуги постоянного тока, электрическая сеть низкого напряжения и малой мощности, переход дуги в тлеющий разряд.

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