ISSN 1682—0525

М А Т Е М А Т И Ч Е С К И Й Ж У Р Н А Л

Том 17 № 4 (66) 2017

Институт математики и математического моделирования
Алматы

ISSN 1682—0525

М А Т Е М А Т И К А Л Ы Қ   Ж У Р Н А Л

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МАТЕМАТИЧЕСКИЙ ЖУРНАЛ

Том 17, № 4 (66), 2017

Журнал выходит 4 раза в год

Издается с 2001 года

Адрес редакции:

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Журнал зарегистрирован в Комитете связи, информатизации и информации Министерства по инвестициям и развитию Республики Казахстан, Свидетельство № 15579-Ж от 25 сентября 2015 г.

СОДЕРЖАНИЕ

Том 17 **№ 4 (66)** **2017**

CONTENTS

Volume 17 **No. 4 (66)** **2017**

Математический журнал ISSN 1682–0525

2017. — Том 17, № 4. — С. 5–12.

МРНТИ 27.03.66

ON THE NUMBER OF COUNTABLE MODELS OF COMPLETE THEORIES WITH A PARTIAL ORDER

B. Baizhanov, F. Kobdikbayeva, T. Zambarnaya

Annotation. Given a formula φ defining a partial order on tuples of elements, we introduce a notion of a φ-chain, and prove that existence of an infinite discrete φ-chain in a small countable theory implies maximality of the number of countable models of this theory.

Keywords. Partial order, number of countable models.

1 Introduction

Question on the number of countable non-isomorphic models of theories is of a big importance in model theory. The two general steps in studying this question are studying the spectrum of theories, and describing properties which lead to non-isomorphism of structures.

In terms of counting spectrum countable theories are divided into natural classes. The Vaught conjecture was solved for $\aleph_1$-categorical (J.T. Baldwin, A.H. Lachlan [1]), ω-stable (S. Shelah, L. Harrington, M. Makkai [2]), superstable theories of a finite rank (S. Buechler [3]), o-minimal (L. Mayer [4]), quite o-minimal (S.V. Sudoplatov, B.Sh. Kulpeshov [5]), and weakly o-minimal theories (A. Alibek, B.S. Baizhanov [6]). But in general this problem is still unsolved.

A natural step towards the Vaught's conjecture constitutes in finding conditions, under which theories have continuum countable models. By modifying the approach to linearly ordered theories given in [7], we move to studying small countable theories with a definable partial order on tuples. Using the construction from [8] we prove a theorem on a sufficient condition of maximality of number of countable models of such theories.

2010 Mathematics Subject Classification: 03C64.

Funding: Комитет науки Министерства образования и науки РК, Грант № 5125/GF4.

2 Number of countable models

Let T be a countable complete theory, $\bar{c}$ be an element of some model of T.

We say that a formula $\varphi(\bar{x},\bar{y},\bar{c})$ $(ln(\bar{x}) = ln(\bar{y}))$ defines a *partial order* on T, if for any $\mathfrak{M} \models T$ with $\bar{c} \in M$

$\mathfrak{M} \models \forall\bar{x}\forall\bar{y}\big(\varphi(\bar{x},\bar{y},\bar{c}) \to \bar{x} \neq \bar{y}\big)$;

$\mathfrak{M} \models \forall\bar{x}\forall\bar{y}\neg\big(\varphi(\bar{x},\bar{y},\bar{c}) \wedge \varphi(\bar{y},\bar{x},\bar{c})\big)$;

$\mathfrak{M} \models \forall\bar{x}\forall\bar{y}\forall\bar{z}\Big(\big(\varphi(\bar{x},\bar{y},\bar{c}) \wedge \varphi(\bar{y},\bar{z},\bar{c})\big) \to \varphi(\bar{x},\bar{z},\bar{c})\Big)$.

Given a formula $\varphi(\bar{x},\bar{y},\bar{c})$ which defines a partial order on T, a *φ-chain* on T is a totally ordered by φ subset of a model $\mathfrak{M}$ of T with $\mathfrak{M} \models \exists\bar{x}\exists\bar{y}\,\varphi(\bar{x},\bar{y},\bar{c})$, and which is convex (by φ) in $\mathfrak{M}$.

Given a formula $\psi(\bar{x})$ (may be with parameters) a *convex-$\varphi(\bar{x},\bar{y},\bar{c})$-closure* [7] of ψ is the formula

$$\psi^c_{\varphi(\bar{x},\bar{y},\bar{c})}(\bar{x}) := \exists\bar{y}_1, \exists\bar{y}_2\Bigg(\psi(\bar{y}_1) \wedge \psi(\bar{y}_2)\wedge$$
$$\Big(\big(\varphi(\bar{y}_1,\bar{x},\bar{c}) \vee \bar{x} = \bar{y}_1\big) \wedge \big(\varphi(\bar{x},\bar{y}_1,\bar{c}) \vee \bar{x} = \bar{y}_2\big)\Big)\Bigg).$$

A *convex-$\varphi(\bar{x},\bar{y},\bar{c})$-closure* of a type $p(\bar{x})$, is the type

$$p^c_{\varphi(\bar{x},\bar{y},\bar{c})}(\bar{x}) := \{\psi^c_{\varphi(\bar{x},\bar{y},\bar{c})}(\bar{x}) \mid \psi(\bar{x}) \in p\}.$$

Theorem 1. *Let T be a small countable complete theory, $\mathfrak{M}$ be a countable model of T. If there exists $\bar{c} \in M$, and a formula $\varphi(\bar{x},\bar{y},\bar{c})$ with $\ln(\bar{x}) = \ln(\bar{y}) = l$, determining a partial order on T such that for any $n \in \mathbb{N}$ there is a finite discrete φ-chain of length at least n, then T has 2^ω countable non-isomorphic models.*

Proof. First note that by compactness there exists an infinite discrete φ-chain.

For convenience denote:

$\bar{x} <^* \bar{y} := \varphi(\bar{x},\bar{y},\bar{c})$;

$\bar{x} \leq^* \bar{y} := \bar{x} <^* \bar{y} \vee \bar{x} = \bar{y}$;

$s(\bar{x},\bar{y},\bar{c}) := \bar{x} <^* \bar{y} \wedge \neg\exists\bar{z}\big(\bar{x} <^* \bar{z} \wedge \bar{z} <^* \bar{y}\big)$;

$s^{(0)}(\bar{x},\bar{y}) := \bar{x} = \bar{y}$;

$s^{(n)}(\bar{x},\bar{y}) := \exists\bar{z}_1 ... \exists\bar{z}_n\big(\bar{z}_1 = \bar{x} \wedge \bar{z}_n = \bar{y} \wedge \bigwedge_{i=1}^{n-1} \bar{s}(z_i, \bar{z}_{i+1})\big)$;

$s^{(-n)}(\bar{x},\bar{y}) := \exists \bar{z}_1 ... \exists \bar{z}_n \big(\bar{z}_1 = \bar{x} \wedge \bar{z}_n = \bar{y} \wedge \bigwedge_{i=1}^{n-1} \bar{s}(z_{i+1}, \bar{z}_i)\big)$, where $n \in \mathbb{N} \backslash \{0\}$;
$\psi(\bar{x})^+ := \exists \bar{y}\big(\varphi(\bar{y}) \wedge \bar{y} <^* \bar{x}\big)$;
$\psi(\bar{x})^- := \exists \bar{y}\big(\varphi(\bar{y}) \wedge \bar{x} <^* \bar{y}\big)$.

For any natural number we can define a discrete chain of length greater or equal to that number: $\psi_n(\bar{x},\bar{y},\bar{c}) := \exists \bar{z}_1 ... \exists \bar{z}_n \Bigg(\bar{z}_1 = \bar{x} \wedge \bar{z}_n = \bar{y} \wedge \bigwedge_{i=1}^{n-1} \bar{z}_i <^* \bar{z}_{i+1} \wedge \forall \bar{z} \Big(\bar{x} \leq^* \bar{z} \wedge \bar{z} \leq^* \bar{y} \rightarrow \exists \bar{t}_1 \exists \bar{t}_2 \big(s(\bar{t}_1, \bar{z}) \wedge s(\bar{z}, \bar{t}_2)\big)\Big)\Bigg)$.

Let $p(\bar{x},\bar{y},\bar{c}) := \{\psi_n(\bar{x},\bar{y},\bar{c}) \mid n < \omega\}$. For a tuple $(\bar{a},\bar{b}) \models p$ denote $\gamma_n(\bar{x},\bar{a},\bar{b},\bar{c}) := \exists \bar{x}_1 ... \exists \bar{x}_n \exists \bar{y}_1 ... \exists \bar{y}_n \Big(\bar{x}_1 = \bar{a} \wedge \bar{y}_1 = \bar{b} \wedge \bigwedge_{i=1}^{n-1} \big(s(\bar{x}_i, \bar{x}_{i+1}) \wedge s(\bar{y}_{i+1}, \bar{y}_1)\big) \wedge \bar{x}_n <^* \bar{x} \wedge \bar{x} <^* \bar{y}_1 \Big)$ a formula saying that $\bar{x}$ is an element between $\bar{a}$ and $\bar{b}$, but is not an i-th $<^*$-successor (predecessor) of $\bar{a}$ ($\bar{b}$) for all $i \leq n$.

Let $q(\bar{x},\bar{a},\bar{b},\bar{c}) := \{\gamma_n(\bar{x},\bar{a},\bar{b},\bar{c}) \mid n < \omega\}$, it is a locally consistent, not necessary complete type over $\{\bar{a},\bar{b},\bar{c}\}$.

Let $\mathfrak{N}$ be an arbitrary countable saturated extension of $\mathfrak{M}(\bar{a},\bar{b},\bar{c})$, the prime model over $\{\bar{a},\bar{b},\bar{c}\}$.

For $\bar{\alpha},\bar{\beta} \in q(N)$ let $V_{q,\mathfrak{N}}(\bar{\alpha}) := \{\gamma \in q(N) \mid \exists n \in \mathbb{Z}\ \mathfrak{N} \models s^{(n)}(\bar{\alpha},\bar{\gamma})\}$ be all the elements from $q(N)$ that can be reached by use of finite s-steps from $\bar{\alpha}$. Also denote $\big(V_{q,\mathfrak{N}}(\bar{\alpha}), V_{q,\mathfrak{N}}(\bar{\beta})\big) := \{\bar{\gamma} \in q(N) \mid V_{q,\mathfrak{N}}(\bar{\alpha}) < \gamma < V_{q,\mathfrak{N}}(\bar{\beta})\}$.

Later on we will use the following notation $\tilde{a} := (\bar{a},\bar{b},\bar{c})$.

Lemma 1. *For all* $\bar{\gamma}_1, \bar{\gamma}_2 \in \big(V_{p,\mathfrak{N}}(\bar{a}), V_{p,\mathfrak{N}}(\bar{b})\big) = q(N)$,

$$tp^c_{\leq^*}(\bar{\gamma}_1 | \{\bar{a},\bar{b},\bar{c}\}) = tp^c_{\leq^*}(\bar{\gamma}_2 | \{\bar{a},\bar{b},\bar{c}\}).$$

Proof.

Suppose that the conclusion of the Lemma is not true, i.e. for some $\bar{\gamma}_1, \bar{\gamma}_2 \in \big(V_{p,\mathfrak{N}}(\bar{a}), V_{p,\mathfrak{N}}(\bar{b})\big)$, there exists an $\tilde{a}$-definable formula H such that $\bar{\gamma}_1 \in H(N,\tilde{a}) <^* \bar{\gamma}_2$. Replace H by $(H(N,\tilde{a})^+)^-$.

For $k, n_1, n_2 < \omega$ such that $n_1 + n_2 < k$ denote
$S_{k,n_1,n_2}(H)(\bar{x},\bar{y},\tilde{a}) := \Big(\big(\bar{x} <^* \bar{y} \wedge \neg s^k(\bar{x},\bar{y})\big) \rightarrow \exists \bar{z}_1, \exists \bar{z}_2 \big(\bar{x} < \bar{z}_1 <^* \bar{z}_2 <^* \bar{y} \wedge \neg s^{n_1}(\bar{x},\bar{z}_1) \wedge \neg s^{n_2}(\bar{z}_2,\bar{y}) \wedge H(\bar{z}_1,\bar{x},\bar{y},\bar{c}) \wedge \neg H(\bar{z}_2,\bar{x},\bar{y},\bar{c}) \wedge s(\bar{z}_1,\bar{z}_2,\tilde{a})\big)\Big)$.

From the Compactness Theorem it follows that:

Claim 1. *There exist two non-constant non-decreasing functions, $s_1, s_2 : \omega \to \omega$, such that $\exists m < \omega$, $\forall k > m$, $\forall \bar{\alpha}', \bar{\beta}' \in (\bar{a}, \bar{b})_{p(N)}$, for which the following is true:*

$$\mathcal{N} \models S_{k,s_1(k),s_2(k)}(H)(\bar{\alpha}', \bar{\beta}').$$

Denote $H_\emptyset(\bar{x}, \tilde{a}) := \neg H(\bar{x}, \tilde{a}) \wedge \exists \bar{y}\big(s(\bar{y}, \bar{x}) \wedge H(\bar{y}, \tilde{a})\big)$.

We have that $H_\emptyset(N, \tilde{a}) \cap q(N) \neq \emptyset$ and $H_\emptyset(N, \tilde{a}) \cap q(N) = \{\gamma_\emptyset\}$ for some $\gamma_\emptyset \in \big(V_{q,\mathfrak{N}}(\bar{a}), V_{q,\mathfrak{N}}(\bar{b})\big)$.

Then denote

$$G_0(\bar{x}, \tilde{a}) := \exists \bar{z}\big(H(\bar{x}, \bar{a}, \bar{z}, \bar{c}) \wedge H_\emptyset(\bar{z}, \tilde{a})\big);$$
$$G_1(\bar{x}, \tilde{a}) := \exists \bar{z}\big(H(\bar{x}, \bar{z}, \bar{b}, \bar{c}) \wedge H_\emptyset(\bar{z}, \tilde{a})\big).$$

So, we have $G_0(N, \tilde{a}) < V_{q,\mathfrak{N}}(\bar{\gamma}_\emptyset)$, $V_{p,\mathcal{N}}(\bar{a}) < G_0(N, \tilde{a})^+$ and $V_{q,\mathfrak{N}}(\bar{\gamma}_\emptyset) < G_1(N, \tilde{a})^+, G_1(N, \tilde{a}) < V_{p,\mathfrak{N}}(\bar{b})$.

Denote

$$H_0(\bar{x}) := \neg G_0(\bar{x}, \tilde{a}) \wedge \exists y(G_0(\bar{y}, \tilde{a}) \wedge s(\bar{y}, \bar{x}));$$
$$H_1(\bar{x}) := \neg G_1(\bar{x}, \tilde{a}) \wedge \exists y(G_1(\bar{y},) \wedge s(\bar{y}, \bar{x}));$$

$$\begin{aligned} G_{00}(\bar{x}, \tilde{a}) &:= \exists \bar{z}(H(\bar{x}, \bar{a}, \bar{z}, \bar{c}) \wedge H_0(\bar{z}, \tilde{a})); \\ G_{01}(\bar{x}, \tilde{a}) &:= \exists \bar{z}_1, \bar{z}_2(H(\bar{x}, \bar{z}_1, \bar{z}_2, \bar{c}) \wedge H_0(\bar{z}_1, \tilde{a}) \wedge H_\emptyset(\bar{z}_2, \tilde{a})); \\ G_{10}(\bar{x}, \tilde{a}) &:= \exists \bar{z}_1, \bar{z}_2(H(\bar{x}, \bar{z}_1, \bar{z}_2) \wedge H_\emptyset(\bar{z}_1, \tilde{a}) \wedge H_1(\bar{z}_2, \tilde{a})); \\ G_{11}(\bar{x}, \tilde{a}) &:= \exists \bar{z}(H(\bar{x}, \bar{z}, \bar{b}, \bar{c}) \wedge H_1(\bar{z}, \tilde{a})). \end{aligned}$$

Repeating this consideration ω times we obtain a countable number of $\tilde{a}$-definable formulas $H_\delta, \delta \in 2^{<\omega}$, such that for every $\tau \in 2^\omega$, $\tau(n) \in \{0, 1\}$ there is $q_\tau \in S_n(\{\tilde{a}\})$, which extends the following set of $\tilde{a}$-definable n-formulas:

$$\Gamma_\tau(x) := \{x < H_{\tau\restriction n}(N, \tilde{a}) \mid \tau(n+1) = 0\} \cup \{H_{\tau\restriction n}(x, \tilde{a}) \mid \tau(n+1) = 1\}.$$

This contradicts to our assumption that T is small.

□

As a corollary to the Lemma 1 we obtain the following lemma.

Lemma 2. *For every $\tilde{\alpha}_n :=< \bar{\alpha}_1, \ldots, \bar{\alpha}_n >$, $\bar{\alpha}_i \in (V_{p,\mathfrak{N}}(\bar{a}), V_{p,\mathfrak{N}}(\bar{b}))$, $1 \leq i \leq n$; such that $V_{q,\mathfrak{N}}(\bar{\alpha}_i) < V_{q,\mathfrak{N}}(\bar{\alpha}_{i+1})$ $(1 \leq i \leq (n-1))$, for every $\bar{\gamma} \in N$ such that $tp(\bar{\gamma}|\{\bar{a}\bar{b}, \bar{c}\} \cup \tilde{\alpha}_n)$ is isolated the following is true: $\forall \bar{\gamma}_1, \bar{\gamma}_2 \in (V_{p,\phi,}(\bar{\alpha}_i), V_{p,\phi}(\bar{\alpha}_{i+1}))$,*

$$tp^c(\gamma_1|A\cup\bar{\alpha}_n\cup\bar{\gamma}\cup\{\bar{\alpha},\bar{\beta}\})=tp^c(\gamma_2|A\cup\bar{\alpha}_n\cup\bar{\gamma}\cup\{\bar{\alpha},\bar{\beta}\}).$$

Let $\mathfrak{N}'$ be an $\aleph_1$-saturated extension of $\mathfrak{M}\cup\{\tilde{a}\}$.

Now for each sequence of zeros and ones, $\tau := \langle\tau(1),\tau(2),...,\tau(i),...\rangle_{i<\omega}$, we will use the construction from [8] to construct a countable model $\mathfrak{M}_\tau \prec \mathfrak{N}'$, such that for any $\tau_1\neq\tau_2$, $\mathfrak{M}_{\tau_1}\not\cong\mathfrak{M}_{\tau_2}$. Until the end of the proof fix such a sequence, τ.

Let $B'_\tau = \left\{\bar{e}^i_r \mid r\in\mathbb{Q}, i\in\mathbb{N}\right\}\cup\left\{\bar{f}^i_n \mid i\in\mathbb{N}, n\in\{0,1\},\text{ and }\tau(i)=0\right\}\cup\left\{\bar{f}^i_n \mid i\in\mathbb{N}, n\in\{0,1,2\},\text{ and }\tau(i)=1\right\}\subseteq q(N')$ be such that $V_{q,\mathfrak{N}'}(\bar{e}^i_{r_1})<V_{q,\mathfrak{N}'}(\bar{e}^i_{r_2})<V_{q,\mathfrak{N}'}(\bar{f}^i_{n_1})<V_{q,\mathfrak{N}'}(\bar{f}^i_{n_2})<V_{q,\mathfrak{N}'}(\bar{e}^{i+1}_r)$, where $i\in\mathbb{N}$, $r_1<r_2\in\mathbb{Q}$, $r\in Q$, $n_1<n_2\in\{0,1,2\}$. The union $B_\tau := B'\bigcup\limits_{\bar{b}\in B'}V_{q,\mathfrak{N}'}(\bar{b})$ is countable, so fix some enumeration $B_\tau=\{\bar{b}_i \mid i<\omega\}$. Also denote $\tilde{b}_n := \langle\bar{b}_1,\bar{b}_2,...,\bar{b}_n\rangle$, $n<\omega$. For the constructed model $\mathfrak{M}_\tau$ we will have $q(\mathfrak{M}_\tau)=B_\tau$.

Construction of $\mathfrak{M}_\tau$.

Step 1. Denote by Φ_1 the set of all $\tilde{a}$-definable 1-formulas, $\Phi_1 := \{\varphi^1_i(x,\tilde{a}) \mid i<\omega\}$. Choose the formula $\varphi^1_i(x,\tilde{a})\in\Phi_1$ with the smallest index i satisfying $\mathfrak{N}'\models\exists x\varphi^1_i(x,\tilde{a})$. Since T is small, there exists a principal over $\tilde{a}$ subformula $\varphi^1_{i,1}(x,\tilde{a})\subseteq\varphi^1_i(x,\tilde{a})$, which, in its turn, has a principal subformula over $\{\tilde{a},\bar{b}_1\}$. Repeating this procedure, we obtain a locally consistent infinite decreasing chain of principal over parameters formulas $\varphi^1_{i,j}(x,\tilde{a},\tilde{b}_j)$: $...\subseteq\varphi^1_{i,n+1}(N',\tilde{a},\tilde{b}_{n+1})\subseteq\varphi^1_{i,n}(N',\tilde{a},\tilde{b}_n)\subseteq...\subseteq\varphi^1_i(N',\tilde{a})$. Denote by d_1 realization of this chain, which exists since the model $\mathfrak{N}'$ is $\aleph_1$-saturated.

Step 2. Choose the formula $\varphi^1_i(x,\tilde{a})\in\Phi_1$ which was not considered before and having the smallest index i satisfying $\mathfrak{N}'\models\exists x\varphi^1_i(x,\tilde{a})$, find a realization d_2 by analogy with d_1.

Now take b_1 and consider the set of all $(\tilde{a}\cup\{\bar{d}_1\}\cup\{\bar{b}_1\})$-definable 1-formulas $\Phi_2 := \{\varphi^2_i(x,\tilde{a},d_1,\bar{b}_1)|i<\omega\}$. Choose the formula $\varphi^2_i(x,\tilde{a},d_1,\bar{b}_1)\in\Phi_2$ which was not considered previously and has the smallest index satisfying $\mathfrak{N}'\models\exists x\varphi^2_i(x,\tilde{a},\bar{b}_1,d_1)$, and find a realization d_3 (existing since $\mathfrak{N}'$ is $\aleph_1$-saturated) of the following infinite decreasing chain of principal formulas $\varphi^2_{i,j}(x,\tilde{a},d_1,\tilde{b}_j)$: $...\subseteq\varphi^2_{i,n+1}(x,\tilde{a},d_1,\tilde{b}_{n+1})\subseteq\varphi^2_{i,n}(x,\tilde{a},d_1,\tilde{b}_n)\subseteq...\subseteq\varphi^2_i(x,\tilde{a},d_1,\bar{b}_1)$.

Suppose by the end of the step k we constructed the following sets: for all m, $1\leq m\leq k$, the sets $D_m := \{d_1,d_2,...,d_{\frac{(m+1)m}{2}}\}$ (it is possible that $d_i=d_j$

for some i and j such that $1 \le i < j \le \frac{(m+1)m}{2}$), the set of all $\tilde{a}$-definable 1-formulas Φ_1, and for all m, $2 \le m \le k$, sets of all $(\{\tilde{a}\} \cup D_{m-1} \cup \tilde{b}_{m-1})$-definable 1-formulas, Φ_m.

Step $k+1$. For each m, $1 \le m \le k$, find a previously unused formula $\varphi^m_{i_m} \in \Phi_m$ of a minimal index, definable set of which in the model $\mathfrak{N}'$ is nonempty. And find realizations $d_{\frac{(k+1)k}{2}+m}$ of corresponding infinite decreasing chains of principal subformulas of the formulas $\varphi^m_{i_m}$.

Denote by Φ_{k+1} the set of all $(\{\tilde{a}\} \cup D_k \cup \tilde{b}_k)$-definable 1-formulas. And find $d_{\frac{(k+1)k}{2}+k+1}$ by analogy with the construction above. Let D_{k+1} stand for the set $\{d_1, d_2, ..., d_{\frac{(k+1)k}{2}+k+1}\}$.

Denote $M_\tau := \{\tilde{a}\} \cup B_\tau \cup \bigcup\limits_{i<\omega} D_i$.

Due to the condition of the theorem, for any realization $\bar{\delta} \in q(N')\backslash B$ and any tuple $\tilde{b}_n$ the type $tp(\delta/\tilde{a}, \tilde{b}_n)$ is non-isolated. Also, by the choice of $\bar{d}_i := \langle d_1, d_2, ..., d_i\rangle_{i<\omega}$, we have that $tp(\bar{d}_i/\tilde{a}, \tilde{b}_n, \bar{d}_{i-1})$ is isolated, and so does $tp(\bar{d}_i/\tilde{a}, \tilde{b}_n)$. Therefore, the Lemma 1 implies that the type $tp(\bar{\delta}/\tilde{a}, \tilde{b}_n, \bar{d}_i)$ is not isolated, and hence, that $\bar{\delta}$ is omitted in $\mathfrak{M}_\tau$.

The Tarski-Vaught criterion implies that the obtained model $\mathfrak{M}\tau$ is an elementary submodel of $\mathfrak{N}'$.

Now let us show that for any two different sequences of zeros and ones τ_1 and τ_2 the constructed models $\mathfrak{M}_{\tau_1}$ and $\mathfrak{M}_{\tau_2}$ are not isomorphic. Towards a contradiction suppose that $\mathfrak{M}_{\tau_1} \overset{\sigma}{\cong} \mathfrak{M}_{\tau_2}$ Take the smallest index i for which $\tau_1(i) \ne \tau_2(i)$. For simplicity suppose that $i = 1$ and that $0 = \tau_1(1) \ne \tau_2(1) = 1$. We will use the construction of B_τ and B'_τ which was mentioned above. Note that under an isomorphism the set of realizations of a type maps into the set of realizations of the same type. Also since σ is an isomorphism, for any $\bar{c}_1, \bar{c}_2 \in q(M_{\tau_1})$, $\mathfrak{M}_{\tau_1} \models \varphi(\bar{c}_1, \bar{c}_2)$ implies $\mathfrak{M}_{\tau_2} \models \varphi(\sigma(\bar{c}_1), \sigma(\bar{c}_2))$; $\bar{c}_1 \notin V_{q,\mathfrak{M}_{\tau_1}}(\bar{c}_2)$ implies $\sigma(\bar{c}_1) \notin V_{q,\mathfrak{M}_{\tau_1}}(\sigma(\bar{c}_2))$; and $\bar{c}_1 \in V_{q,\mathfrak{M}_{\tau_1}}(\bar{c}_2)$ implies $\sigma(\bar{c}_1) \in V_{q,\mathfrak{M}_{\tau_1}}(\sigma(\bar{c}_2))$. In other words, the $V_{q,\mathfrak{M}_{\tau_1}}$-neighborhoods are mapped to $V_{q,\mathfrak{M}_{\tau_2}}$-neighborhoods. Furthermore if there is no $\bar{c}_3 \in q(M_{\tau_1})$ with $V_{q,\mathfrak{M}_{\tau_1}}(\sigma(\bar{c}_1)) <^* \bar{c}_3 <^* V_{q,\mathfrak{M}_{\tau_1}}(\sigma(\bar{c}_2))$, that is the $V_{q,\mathfrak{M}_{\tau_1}}$ neighborhoods of $\bar{c}_1$ and $\bar{c}_2$ are discretely ordered in terms of $<^*$, then the neighborhoods of $\sigma(\bar{c}_1)$ and $\sigma(\bar{c}_2)$ should also be discretely ordered in terms of $<^*$. And the same holds for densely ordered neighborhoods. Therefore $\sigma(V_{q,\mathfrak{M}_{\tau_1}}(\bar{f}^1_1)) = (V_{q,\mathfrak{M}_{\tau_2}}(\bar{f}^1_1))$

$\sigma(V_{q,\mathfrak{M}_{\tau_1}}(\bar{f}_2^1)) = (V_{q,\mathfrak{M}_{\tau_2}}(\bar{f}_2^1))$ (since $\bar{f}_1^1$ and $\bar{f}_2^1$ are in the first two discretely ordered neighborhoods). And we have a contradiction since $\sigma^{-1}(V_{q,\mathfrak{M}_{\tau_2}}(\bar{f}_3^1))$ should be in the dense interval of neighborhoods.

Since the number of different infinite sequences of zeros and ones equals to 2^ω, $I(T \cup p(\tilde{a}), \omega) = 2^\omega$. Any model of the theory T generates maximum ω countable non-isomorphic models of $T \cup tp(\tilde{a})$, consequently, $I(T, \omega) = 2^\omega$.

As an immediate corollary we have the following:

Corollary 1. *Let T be a small countable theory having a few number of countable models. If there exists a formula $\varphi(\bar{x}, \bar{y}, \bar{c})$ determining a partial order with an infinite φ-chain, then this chain should be dense.*

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Байжанов Б., Көбдікбаева Ф., Замбарная Т. ҚАТАҢ РЕТПЕН ОРНАЛАСҚАН ТОЛЫҚ ТЕОРИЯЛАРДЫҢ САНАЛЫМДЫ МОДЕЛЬДЕРІНІҢ САНЫ ТУРАЛЫ

Кортеждердегі ішінара ретті анықтайтын φ формуласы берілсін. Біз φ-тізбегі ұғымын енгіземіз және шексіз дискреттік φ-тізбегі бар болатын шағын саналымды теорияның саналымды модельдерінің саны максималды екенін дәлелдейміз.

Кілттік сөздер. Ішінара рет, саналымды моделдердің саны.

Байжанов Б., Кобдикбаева Ф., Замбарная Т. О ЧИСЛЕ СЧЁТНЫХ МОДЕЛЕЙ ПОЛНЫХ ТЕОРИЙ СО СТРОГИМ ПОРЯДКОМ.

Пусть дана формула φ, определяющая частичный порядок на кортежах. Мы вводим понятие φ-цепи и доказываем, что число счётных моделей малой счётной теории, имеющей бесконечную дискретную φ-цепь, максимально.

Ключевые слова. Частичный порядок, число счётных моделей.

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Статья поступила в редакцию 25.12.2017

Математический журнал ISSN 1682–0525

2017. — Том 17, № 4. — С. 13–24.

МРНТИ 27.15.23:27.45.15

A NOTE ON THE INTRACTABILITY OF PARTITION, KNAPSACK, SUBSET SUM AND RELATED PROBLEMS

Vassilly Voinov

Annotation. Results of an empirical investigation of the time-complexity of both known and forgotten algorithms for solving partition, integer knapsack, subset sum, and related problems – important for both academic researchers and industry practitioners – are presented. The results obtained show that, contrary to prevailing opinion, the above-mentioned problems are solvable in polynomial time on a standard personal computer.

Keywords. Combinatorial optimization, Knapsack-like problems, P and NP classes.

1 Introduction

The following three classes of problems are usually considered in discrete combinatorial optimization: P, NP and NP-complete. The last class includes NP-hard problems that are at least as hard as the NP-complete ones. Karp [1] was probably the first one to list 21 NP-complete problems. Garey and Johnson [2] increased the list size of such problems to 320. Several problems from that list, such as partition, knapsack, subset sum and packing are probably the most interesting and important for academic researchers and industrial practitioners.

Two well-known basic techniques for solving those hard problems are usually used – branch and bounds (BB) and dynamic programming (DP). Both methods are search-based which means that they search for an optimal solution from an exponentially large number of probable candidates. Because of the principle on which they are based, both methods cannot be quick and cost-effective. In spite of this, during the last 30 years, researchers have tried to raise the performance of those algorithms as far as possible (e.g., see Andonov and Rajopadhye [3]; Andonov et al. [4]; Babayev et al. [5]; Chvátal [6]; Lodi

2010 Mathematics Subject Classification: 05A05, 05A15.

et al. [7]; Mansini and Speranza [8]; Martello et al. [9]; Martello and Toth [10]; Pisinger [11]; Poirriez et al. [12]; Seong et al. [13]; He et al. [14]). In spite of their efforts, it has to be noted that both BB and DP approaches usually cannot find all the optimal solutions of a particular problem.

2 Preliminaries

In this note, we analyze the time complexity of the enumerating (not searching) algorithm proposed by Voinov and Nikulin [15]. We start by considering the following linear Diophantine equation

$$a_1 s_1 + a_2 s_2 + \dots + a_l s_l = n, \tag{1}$$

where, without loss of generality, $a_1 \le a_2 \le \dots \le a_l$; $a_i > 0$; $s_i \ge 0$, $i = 1, 2, ..., l$; $n > 0$, all the variables involved being integers. The method of Voinov and Nikulin [15] is based on the generating function for the number of solutions of (1) proposed by Hardy and Littlewood [16]. Voinov and Nikulin [15] showed that their approach permits them not only to define the number of solutions of (1), but also to enumerate explicitly all the solutions, if they exist. It was shown that the number of solutions equals

$$R_n(l) = \sum_{s_l=0}^{[\frac{n}{a_l}]} \sum_{s_{l-1}=0}^{[\frac{n-s_l a_l}{a_{l-1}}]} \cdots \sum_{s_2=0}^{[\frac{n-s_l a_l-\dots-s_3 a_3}{a_2}]} 1, \tag{2}$$

if $s_1+s_2+\dots+s_l \le [n/a_1]$ and $(n-s_l a_l-\dots-s_2 a_2)/a_1$ is a nonnegative integer. Otherwise the value of $R_n(l)$ is zero. Mahmoudvand et al. [17] used (2) only for computing the number of solutions of equation (1). On the contrary, Voinov and Nikulin [15] showed that all the solutions $\{s_1, s_2, ..., s_l\}$ of equation (1) can themselves be written as $\{0^{[n/a_1]-s_1-\dots-s_l}, a_1^{s_1}, ..., a_l^{s_l}\}$, where $\{s_2, s_3, ..., s_l\}$ are the sets of summation indices in (2) and $s_1 = (n - s_l a_l - ... - s_2 a_2)/a_1$. This notation means that in a particular solution there will be $[n/a_1] - s_1 - \dots - s_l$ zeros, s_1 terms containing a_1, s_2 terms containing a_2, etcetera.

For example, consider the equation $2s_1 + 3s_2 + 5s_3 = 6$. Using (2), one easily gets the following 2 solutions of this equation: $\{3, 0, 0\}$ and $\{0, 2, 0\}$. The second solution, e.g., can be presented as: $\{0^1, 2^0, 3^2, 5^0\} = 0 + 3 + 3 = 6$.

3 Assessing the time complexity for partitioning

Consider first the classical partition problem when in (1) $a_i = i$, $i = 1, 2, ..., l$. To construct all the partitions of n Hankin [18] used the lexicographical algorithm. Estimates of the time needed to obtain a particular partition using the command *parts(n)* of the R-package "partitions" developed by Hankin (henceforth, software) are presented in Fig. 1.

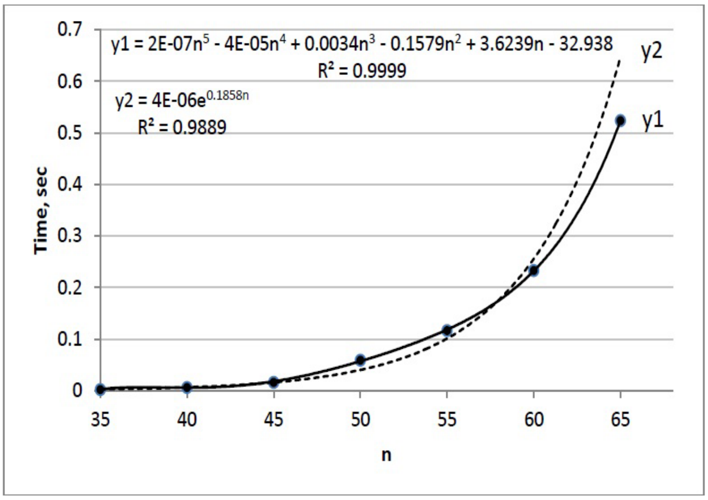

Figure 1 – Empirical (due to Hankin (2006)) dependence of partitions' computing time (solid line with black circles for assessed times) on the size of the problem. The dashed line corresponds to the exponential fit of data

Note that the R^2 for a polynomial fit is higher than that for an exponential one and that the time complexity of the algorithm is $O(n^5)$. Hankin published his result in 2006, but since then nobody has mentioned that his algorithm is polynomial in time, that is the time needed for a particular partition is bounded by a polynomial of n.

The application of the R-script developed in Voinov and Pya [19] for the algorithm of the same problem given by (2) produced the results shown in Fig. 2.

From Fig. 1 and Fig. 2, it follows that partition problem belongs to the complexity class P because it can be solved in polynomial time and has

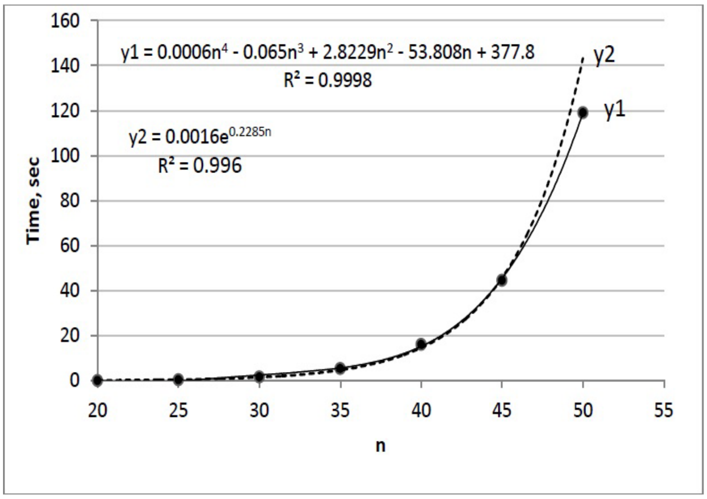

Figure 2 – Empirical dependence of partitions' computing time (solid line with black circles for assesses times) on the size of the problem. The dashed line corresponds to the exponential fit of data

complexity of $O(n^{\leq 5})$. Hence, the partition is not NP-complete problem. These results disprove the conclusion of Garey and Johnson ([2], p. 91).

4 ASSESSING THE TIME COMPLEXITY FOR KNAPSACK AND SUBSET SUM PROBLEMS

Since all problems of interest can be reduced to an equivalent problem involving the enumeration of all nonnegative integer solutions of the equation (1) with arbitrary positive integer coefficients a_i, $i = 1, 2, ..., l$, consider first the following equation: $2s_1 + 3s_2 + 5s_3 + 16s_4 = n$. A modified R-function (to be published elsewhere) of (Voinov and Pya [19]) was used for assessing the time complexity of this equation (see Fig. 3).

One sees that the time complexity is $O(n^3)$: hence, the problem belongs to complexity class P. Note that the time shown in Fig. 3 corresponds to the computing time for all solutions of the equation for a particular value of n (e.g., taking $n = 100$, we find that there are 499 solutions).

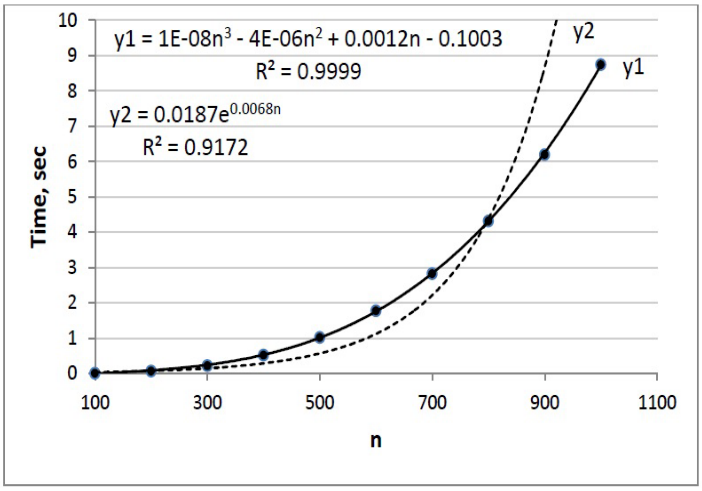

Figure 3 – Empirical dependence of computing time (solid line with black circles for assessed times) on the size of the problem. The dashed line corresponds to the exponential fit of data

Consider the following unbounded knapsack

$$\begin{aligned} &\text{maximize } C = 200s_1 + 201s_2 + 202s_3 + 203s_4 + 204s_5 + 205s_6 + 206s_7, \\ &\text{subject to } 100s_1 + 101s_2 + 102s_3 + 103s_4 + 104s_5 + 105s_6 + 106s_7 \le n, \quad (3) \\ &\qquad s_i \ge 0, \text{ integers}, \ i = 1, ..., 7, \end{aligned}$$

where $p = (p_1, p_2, ..., p_7) = (200, 201, ..., 206)$ are *profits*, $w = (w_1, w_2, ..., w_7) = (100, 101, ..., 106)$ are *weights*, and n is the *capacity* of the knapsack. Note that *weights* and *profits* are strongly correlated (Pearson's correlation coefficient is 1). According to Martello and Toth ([20], p. 91), "the optimal solution of such problems appears to be practically impossible". Introducing a slack variable (say s_8) into the constraining function the problem reduces to the enumeration of all nonnegative integer solutions of the linear Diophantine equation

$$100s_1 + 101s_2 + 102s_3 + 103s_4 + 104s_5 + 105s_6 + 106s_7 + s_8 = n.$$

Applying the approach based on the formula in (2) and the software, the empirical dependence of computing time on n for the problem in (3) is obtained, as shown in Fig. 4. Average percentage errors in the mean estimates of times used for constructing this figure do not exceed 1%.

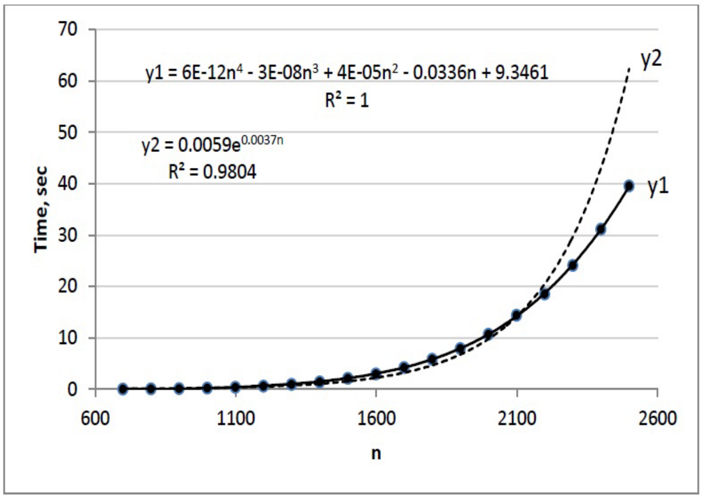

Figure 4 – Empirical dependence of computing time (solid line with black circles for assessed times) on the size of problem (3). The dashed line corresponds to the exponential fit of data

In Fig. 4, one can see that the optimal solution of the problem is not only possible but also easily obtained by the above approach. We also find that the algorithm has polynomial time complexity of $\mathrm{O}(n^4)$. Note that the computing time in Fig. 4 is the total time for all solutions corresponding to a particular value of n. For $n = 2299$, e.g., there are 1616 optimal solutions of (3). One may also conclude that the opinion of Martello and Toth ([20], p. 91) is disproved by this approach.

Out of the 1616 optimal solutions, consider two particular ones: $s_1 = (s_1, s_2, ..., s_7) = \{1, 0, 0, 0, 6, 15, 0\}$ and $s_2 = \{0, 0, 3, 7, 0, 0, 12\}$. One can easily check that both of them are optimal solutions of (3) if $n = 2299$ (note that $\sum s_i w_i{=}2299$). Let vector $A = (a_1, a_2, ..., a_7) = (1, 2, ..., 7)$ and $s(a_i) = p_i s_i, \quad i = 1, 2, ..., 7$. Consider a sub vector $A' \subseteq A = (1, 5, 6)$ corresponding to the solution s_1. For this vector $\sum\limits_{a \subseteq A'} s(a) = 4499$. At the same time for the solution s_2 $\sum\limits_{a \subseteq A-A'} s(a) = 4499$. The equality $\sum\limits_{a \subseteq A'} s(a) = \sum\limits_{a \subseteq A-A'} s(a)$ confirms the decision of Fig. 4 that the knapsack problem (3) is not NP-complete and, hence, is solvable in polynomial time. Its time complexity is O(n^4).

Consider below an instance of the subset sum problem (see Example 4.1 from Martello and Toth, [20], p. 111)

$$\text{maximize } C = 41s_1 + 34s_2 + 21s_3 + 20s_4 + 8s_5 + 7s_6 + 7s_7 + 4s_8 + 3s_9 + 3s_{10},$$

$$\text{subject to } 41s_1 + 34s_2 + 21s_3 + 20s_4 + 8s_5 + 7s_6 + 7s_7 + 4s_8 + 3s_9 + 3s_{10} \leq n, \quad (4)$$

$$s_i \geq 0, \text{ integers}, \ i = 1, 2, ..., 10,$$

Applying the approach defined by the formula in (2) and above-mentioned software the empirical dependence of computing time on n for problem (4) is obtained as in Fig. 5. Average percentage errors in the mean estimates of times used for constructing this figure do not exceed 1%.

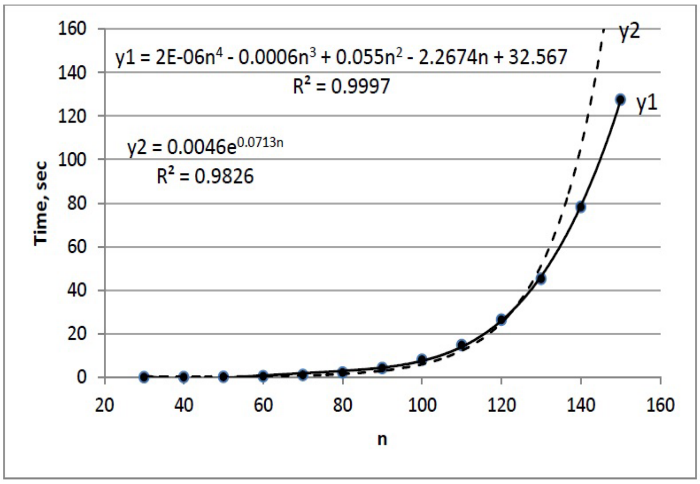

Figure 5 – Empirical dependence of computing time (solid line with black circles for assessed times) on the size of the problem (4). The dashed line corresponds to the exponential fit of data

In Fig. 5, one can see that all solutions of the problem are easily obtained by the above approach. We see that the algorithm implemented is polynomial in time with the complexity of O(n^4). Note that the computing time shown in Fig. 5, as in previous case, is the total time for all solutions corresponding to a particular value of n. For $n = 50$, e.g., there are 954 solutions of (4).

Out of those 954 solutions, consider two particular ones: $s_1 = (s_1, s_2, ..., s_{10}) = \{1,0,0,0,0,0,0,0,0,3\}$ and $s_2 = \{0,1,0,0,1,0,0,2,0,0\}$. One can easily check that both are optimal solutions of (4) if $n = 50$ (note that $\sum s_i w_i = 50$). Let vector $A = (1, 2, ..., 10)$ and $s(a_i) = p_i s_i, \quad i = 1, 2, ..., 10$. Consider a sub vector $A^{'} \subseteq A = (1, 10)$ corresponding to the solution s_1. For this vector $\sum\limits_{a \subseteq A'} s(a) = 50$. At the same time for the solution s_2 $\sum\limits_{a \subseteq A - A'} s(a) = 50$. The equality $\sum\limits_{a \subseteq A'} s(a) = \sum\limits_{a \subseteq A' - A'} s(a)$ theoretically confirms the conclusion of Fig. 5 that the subset sum problem in (4) is not NP-complete. Hence, it is solvable in polynomial time with the time complexity of $\mathrm{O}(n^4)$.

Consider one more "hard" problem which is a generalization of the 0-1 subset sum problem of Chvátal ([6], p. 1408) to the following unbounded one

$$\begin{gathered} \text{maximize } \; C = \sum_{j=1}^{10} a_j s_j, \\ \text{subject to } \; \sum_{j=10}^{10} a_j s_j \leq n, \\ s_i \geq 0, \text{ integers}, \; i = 1, 2, ..., 10, \end{gathered} \tag{5}$$

where $a_j = 2^{12} + 2^{1+j} + 1, \quad j = 1, 2, ..., 10$. Applying the approach and software used earlier the empirical dependence of the computing time on n for problem (5) is obtained, as shown in Fig. 6. Average percentage errors in the mean estimates of times used for constructing this figure do not exceed 1%.

One sees that the algorithm applied is polynomial in time with the complexity of $\mathrm{O}(n^4)$. Note that the computing time in Fig. 6 is the total time for all solutions corresponding to a particular value of n. For n = 44031, e.g., there are 23 optimal solutions of (5). By the date it was known (Hirschberg and Wong [21]) that knapsack problems are solvable in polynomial time only for two-dimensional case. Garey and Johnson ([2], p. 247) consider that problems of higher dimensionality can be solved by DP only in pseudo-polynomial time. Our empirical results disprove their contention by showing that the algorithm in (2) solves a knapsack problem of any dimensionality in polynomial time.

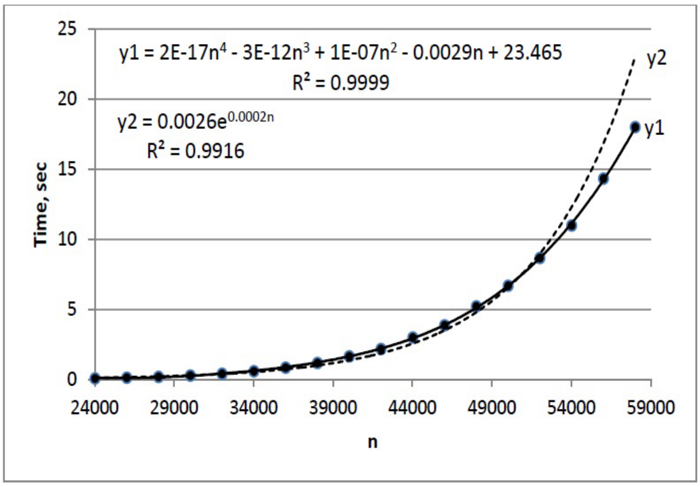

Figure 6 – Dependence of computing time (solid line with black circles for assessed times) on the size of problem (5). The dashed line corresponds to the exponential fit of data

In this regard, it is worth noting that if the results obtained above are true, then all other related problems such as 0-1 knapsack, bounded knapsack, multiple knapsack, and the two and three-dimensional bin packing problems, which can be reduced to the enumeration of all nonnegative integer solutions of a linear Diophantine equation are also not NP-complete, and therefore, can be solved in polynomial time.

5 Concluding remarks

The results of this note can be summarized as follows:

– Solution of partition, knapsack, subset sum and related problems is reduced to the enumeration of all nonnegative integer solutions of a linear Diophantine equation,

– The enumerative algorithm proposed by Voinov and Nikulin [15] is more effective, compared with the well-known searching algorithms, such as BB and DP,

– The algorithm of Voinov and Nikulin [15] enumerates all optimal solutions of partition, knapsack, subset sum, and related problems,

– The algorithm of Voinov and Nikulin [15] solves the above problems in polynomial time,

– Results obtained are in favor of the relation P = NP. If so, the theory of NP-completeness needs to be thoroughly reformulated.

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Воинов Василий О НЕРАЗРЕШИМОСТИ ПРОБЛЕМ РАЗБИЕНИЙ, УКЛАДКИ РАНЦА, СУММИРОВАНИЯ ПОДМНОЖЕСТВ И ДРУГИХ РОДСТВЕННЫХ ЗАДАЧ

Представлены результаты эмпирического анализа временной сложности известных и забытых алгоритмов для решения проблем разбиения натуральных чисел, укладки ранца, разбиений множеств и связанных с ними проблем. Полученные эмпирические результаты свидетельствуют, что все вышеупомянутые задачи разрешимы за полиномиальное время на стандартном персональном компьютере.

Ключевые слова. Комбинаторная оптимизация, проблемы укладки ранца, классы P и NP.

Воинов Василий БӨЛІКТЕУ, РАНЦ ТӨСЕУЛЕРІ, ІШКІ ЖИЫНТЫҚТАРДЫ ҚОСЫНДЫЛАУ МӘСЕЛЕЛЕРІНІҢ ЖӘНЕ БАСҚА ДА ҰҚСАС ЕСЕПТЕРДІҢ ШЕШІЛМЕЙТІНДІГІ ТУРАЛЫ

Натурал сандарды бөліктеу, ранц төсеулері, жиындарды бөліктеу мәселелерін және солармен байланысты есептерді шешуге арналған белгілі және ұмыт болған алгоритмдердің уақыттық күрделілігінің эмпирикалық талдауының нәтижелері ұсынылған. Алынған эмпирикалық нәтижелер жоұарыда айтылған есептердің барлығы стандарттық дербес компьютерде полиномдық уақыт аралығында шешілетіндігін айғақтайды.

Кілттік сөздер. Комбинаторлық тиімділеу, ранцты төсеу мәселелері, P және NP класстары.

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Received 04.11.2017

Математический журнал ISSN 1682–0525

2017. — Том 17, № 4. — С. 25–36.

МРНТИ 517.968.74

ЧИСЛЕННАЯ РЕАЛИЗАЦИЯ ОДНОГО АЛГОРИТМА НАХОЖДЕНИЯ РЕШЕНИЯ СПЕЦИАЛЬНОЙ ЗАДАЧИ КОШИ ДЛЯ НЕЛИНЕЙНЫХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ФРЕДГОЛЬМА

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Аннотация. Рассматривается специальная задача Коши для систем нелинейных интегро-дифференциальных уравнений, возникающая при применении метода параметризации к системе нелинейных интегро-дифференциальных уравнений Фредгольма. Разработан алгоритм нахождения численного решения рассматриваемой задачи.

Ключевые слова. Нелинейное интегро-дифференциальное уравнение, специальная задача Коши, численное решение, алгоритм.

1 Постановка задачи

Вопросы разрешимости и построения приближенных методов нахождения решения начальных и краевых задач для интегро-дифференицальных уравнений Фредгольма исследовались во многих работах [1]–[7].

В [8]–[11] предложен метод исследования и решения линейной двухточечной краевой задачи для интегро-дифференциальных уравнений, основанный на разбиении интервала и введении дополнительных параметров. Исходная задача сводится к эквивалентной многоточечной краевой задаче с параметрами для систем линейных интегро-дифференциальных уравнений. Предлагается алгоритм нахождения решения задачи с параметрами, где на каждом шаге решается специальная задача Коши для линейных интегро-дифференциальных уравнений при известных значениях параметров.

2010 Mathematics Subject Classification: 45J05.

В [12] получены достаточные условия существования единственного решения специальной задачи Коши для нелинейных интегро-дифференциальных уравнений Фредгольма и установлена оценка разности ее решений при соответствующих различным значениям параметрах.

Целью настоящей работы является разработка алгоритма нахождения численного решения специальной задачи Коши, возникающей при применении метода параметризации к системе нелинейных интегро-дифференциальных уравнений Фредгольма вида

$$\frac{dx}{dt} = f_0(t,x) + \int_0^T f_1(t,s,x(s))ds, \quad x \in R^n, \quad t \in [0,T], \tag{1}$$

где $f_0 : [0,T] \times R^n \to R^n$, $f_1 : [0,T] \times [0,T] \times R^n \to R^n$ непрерывны.

По шагу $h > 0 : \ h = T \ (N = 1, 2, \ldots)$ производится разбиение $[0,T) = \bigcup_{r=1}^{N} [(r-1)h, rh)$ и сужение функции $x(t)$ на r-ый интервал $[(r-1)h, rh)$ обозначается через $x_r(t)$, т.е. $x_r(t) = x(t)$ при $t \in [(r-1)h, rh)$. В качестве параметра λ_r будем рассматривать значение функций $x_r(t)$ в начальных точках подинтервалов и на каждом r-ом интервале произведем замену $u_r(t) = x_r(t) - \lambda_r$, $r = \overline{1,N}$. Тогда система (1) сведется к специальной задаче Коши

$$\frac{du_r}{dt} = f_0(t, u_r + \lambda_r) + \sum_{j=1}^{N} \int_{(j-1)h}^{jh} f_1(t,s,u_j(s)+\lambda_j)ds, \qquad t \in [(r-1)h, rh), \tag{2}$$

$$u_r[(r-1)h] = 0, \qquad r = \overline{1,N}. \tag{3}$$

Через $C([0,T], h, R^{nN})$ обозначим пространство систем функций $u[t] = (u_1(t), \, u_2(t), \ldots, u_N(t))$, где $u_r : [(r-1)h, rh) \to R^n$ непрерывна и имеет конечный предел $\lim\limits_{t \to rh-0} u_r(t)$, $r = \overline{1,N}$, с нормой $\|u[\cdot]\|_1 = \max\limits_{r=\overline{1,N}} \sup\limits_{t \in [(r-1)h, rh)} \|u_r(t)\|$.

Решением специальной задачи Коши (2), (3) является система функций $u[t] = (u_1(t), u_2(t), \ldots, u_N(t)) \in C([0,T], h, R^{nN})$, компоненты которой непрерывно дифференцируемы на своих интервалах определения и удовлетворяют системе интегро-дифференциальных уравнений (2) и начальным условиям (3).

По выбранному шагу $h > 0 : Nh = T$, $N \in \mathbb{N}$, и заданному вектору $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_N) \in R^{nN}$, равенствами: $x_0(t) = \widetilde{\lambda}_r$, $t \in [(r-1)h, rh)$, $r = \overline{1, N}$, $x_0(T) = \widetilde{\lambda}_N$ на $[0, T]$ определим кусочно-непрерывную вектор-функцию $x_0(t)$.

Взяв число $\rho > 0$, построим множества

$$G_0(\rho) = \{(t, x) : t \in [0, T], \|x - x_0(t)\| < \rho\},$$

$$G_1(\rho) = \{(t, s, x) : t \in [0, T], s \in [0, T], \|x - x_0(s)\| < \rho\},$$

$$S_h(0, \rho) = \{u[t] = (u_1(t), u_2(t), \ldots, u_N(t)) \in C([0, T], h, R^{nN}) : \|u[\cdot]\|_1 < \rho\}.$$

Условие *A*. Функции $f_0(t, x), f_1(t, s, x)$ соответственно в $G_0(\rho)$, $G_1(\rho)$ непрерывны, имеют непрерывные частные производные $\dfrac{\partial f_0(t, x)}{\partial x}$, $\dfrac{\partial f_1(t, s, x)}{\partial x}$ и выполняются неравенства

$$\left\| \frac{\partial f_0(t, x)}{\partial x} \right\| \le L_0, \quad (t, x) \in G_0(\rho), \qquad \left\| \frac{\partial f_1(t, s, x)}{\partial x} \right\| \le L_1, \quad (t, s, x) \in G_1(\rho).$$

где L_0, L_1 – положительные числа.

Теорема A [12, с. 44]. *Пусть выполняются условие A и неравенства*

$$a) \quad e^{L_0 h} L_1 T h < 1,$$

$$б) \quad \frac{e^{L_0 h}}{1 - e^{L_0 h} L_1 T h} \max_{r = \overline{1, N}} \sup_{t \in [(r-1)h, rh)} \Big\| \int\limits_{(r-1)h}^{t} f_0(\tau, \widetilde{\lambda}_r) d\tau +$$

$$+ \int\limits_{(r-1)h}^{t} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} f_1(\tau, s, \widetilde{\lambda}_j) ds d\tau \Big\| < \rho.$$

Тогда специальная задача Коши для систем интегро-дифференциальных уравнений (2), (3) при $\lambda = \widetilde{\lambda}$ в $S_h(0, \rho)$ имеет единственное решение $\widetilde{u}[t] = (\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_N(t))$.

Специальная задача Коши для систем интегро-дифференциальных уравнений (2), (3) при $\lambda = \widetilde{\lambda}$ эквивалентна системе интегральных уравнений

$$u_r(t) = \int\limits_{(r-1)h}^{t} f_0(\tau, \widetilde{\lambda}_r + u_r(\tau))d\tau +$$

$$+ \int\limits_{(r-1)h}^{t} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} f_1(\tau, s, \widetilde{\lambda}_j + u_j(s))dsd\tau, \ t \in [(r-1)h, rh), \ r = \overline{1,N}. \quad (4)$$

Решение специальной задачи Коши (2), (3) определяется по следующему алгоритму.

Шаг 0. Начальное приближение к решению (4) возьмем $u_r^{(0,0)}(t) = 0$. С помощью итерационного процесса построим функциональную последовательность

$$u_r^{(0,m+1)}(t) = \int\limits_{(r-1)h}^{t} f_0(\tau, \widetilde{\lambda}_r + u_r^{(0,m)}(\tau))d\tau +$$

$$+ \int\limits_{(r-1)h}^{t} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} f_1(\tau, s, \widetilde{\lambda}_j)dsd\tau, t \in [(r-1)h, rh), \ r = \overline{1,N}, \ m = 0, 1, 2, ..., \ldots$$

и, переходя к пределу при $m \to \infty$, найдем $u^{(0)}[t]$.

Шаг 1. Следующее приближение решения (4) – систему функций $u^{(1)}[t]$ – найдем по итерационному процессу: $u_r^{(1,0)}(t) = u_r^{(0)}(t)$,

$$u_r^{(1,m+1)}(t) = \int\limits_{(r-1)h}^{t} f_0(\tau, \widetilde{\lambda}_r + u_r^{(1,m)}(\tau))d\tau +$$

$$+ \int\limits_{(r-1)h}^{t} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} f_1(\tau, s, \widetilde{\lambda}_j + u_j^{(0)}(s))dsd\tau, \ t \in [(r-1)h, rh), \ r = \overline{1,N}. \quad (5)$$

Переходя в (5) к пределу при $m \to \infty$, найдем $u^{(1)}[t]$. И т.д.

Продолжая процесс, на $k+1$-ом шаге взяв за начальное приближение $u_r^{(k+1,0)}(t) = u_r^{(k)}(t)$, последущее приближение найдем по формуле:

$$u_r^{(k+1,m+1)}(t) = \int\limits_{(r-1)h}^{t} f_0(\tau, \widetilde{\lambda}_r + u_r^{(k+1,m)}(\tau))d\tau +$$

$$+ \int\limits_{(r-1)h}^{t} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} f_1(\tau, s, \widetilde{\lambda}_j + u_j^{(k)}(s))dsd\tau,$$

$$t \in [(r-1)h, rh), \quad r = \overline{1,N}, \quad k = 1,2,..., \quad m = 0,1,2,....$$

Сходимость предложенного алгоритма устанавливает следующее утверждение

Теорема 1. *Пусть выполняются условия Теоремы А. Тогда определяемая алгоритмом последовательность систем функций $u^{(k)}[t]$ при $k \to \infty$ сходится к $\widetilde{u}[t]$ – единственному решению задачи (2), (3) и справедливы оценки*

$$|\|u^{(k)}[\cdot] - \widetilde{u}[\cdot]\|_1 \leq \frac{(e^{L_0 h} L_1 T h)^k}{1 - e^{L_0 h} L_1 T h} \|u^{(1)}[\cdot] - u^{(0)}[\cdot]\|_1. \tag{6}$$

Доказательство. При предположениях теоремы в [12, с. 48] доказана справедливость оценки

$$\|u^{(k+1)}[\cdot] - u^{(k)}[\cdot]\|_1 \leq (e^{L_0 h} L_1 T h)\|u^{(k)}[\cdot] - u^{(k-1)}[\cdot]\|_1.$$

Отсюда получим, что

$$\|u^{(k+p)}[\cdot] - u^{(k)}[\cdot]\|_1 \leq \|u^{(k+p)}[\cdot] - u^{(k+p-1)}[\cdot]\|_1 + \|u^{(k+p-1)}[\cdot] - u^{(k+p-2)}[\cdot]\|_1 +$$

$$+ \ldots + \|u^{(k+2)}[\cdot] - u^{(k+1)}[\cdot]\|_1 + \|u^{(k+1)}[\cdot] - u^{(k)}[\cdot]\|_1 \leq \Big[(e^{L_0 h} L_1 T h)^{k+p-1} +$$

$$+ (e^{L_0 h} L_1 T h)^{k+p-2} + \ldots + (e^{L_0 h} L_1 T h)^{k+1} + (e^{L_0 h} L_1 T h)^{k}\Big]\|u^{(1)}[\cdot] - u^{(0)}[\cdot]\|_1.$$

Поэтому имеет место неравенство

$$\|u^{(k+p)}[\cdot] - u^{(k)}[\cdot]\|_1 \leq \frac{(e^{L_0 h} L_1 T h)^k}{1 - e^{L_0 h} L_1 T h} \|u^{(1)}[\cdot] - u^{(0)}[\cdot]\|_1. \tag{7}$$

Переходя к пределу при $p \to \infty$ в (7), установим оценку (6).

2 Алгоритм нахождения численного решения специальной задачи Коши для нелинейных интегро-дифференциальных уравнений

Основой предлагаемого алгоритма является задача Коши для обыкновенных дифференциальных уравнений на подинтервалах:

$$\frac{du_r}{dt} = f(t, u_r), \quad u_r[(r-1)h] = 0, \quad t \in [(r-1)h, rh), \quad r = \overline{1, N}.$$

Для решения этой задачи используется метод Рунге-Кутта четвертого порядка с шагом $h_1 = \frac{h}{N_1}$, $N_1 = 2M$, $M \in \mathbb{N}$:

$$u_r^{l+1} = u_r^l + \Delta u_r^l, \quad l = \overline{0, N_1 - 1},$$

$$u_r^l = u_r(\tau_l),\ \tau_l = (r-1)h + lh_1,\ \Delta u_r^l = \frac{1}{6}[K_1^l + 2K_2^l + 2K_3^l + K_4^l], \tag{8}$$

где K_1^l, K_2^l, K_3^l, K_4^l – n-векторы, определяемые по следующим формулам

$$K_1^l = h_1\Big(f_0(\tau_l, u_r^l)\Big),$$

$$K_2^l = h_1\Big(f_0(\tau_l + \frac{h_1}{2}, u_r^l + \frac{K_1^l}{2})\Big),$$

$$K_3^l = h_1\Big(f_0(\tau_l + \frac{h_1}{2}, u_r^l + \frac{K_2^l}{2})\Big),$$

$$K_4^l = h_1\Big(f_0(\tau_l + h_1, u_r^l + K_3^l)\Big),$$

и формула Симпсона [13, с. 165]

$$I_k = \sum_{j=1}^{N} \int_{(j-1)h}^{jh} f_1(t, s, u_j^{(k)}(s) + \widetilde{\lambda}_j)ds =$$

$$= \frac{h_1}{3} \sum_{j=1}^{N} \Big[f_1(t, s, \widetilde{\lambda}_j + u_j^{(k)}((j-1)h)) + f_1(t, s, \widetilde{\lambda}_j + u_j^{(k)}(jh)) +$$

$$+4\sum_{p=1}^{M} f_1(t,s,\widetilde{\lambda}_j+u_j^{(k)}((j-1)h+(2p-1)h_1))+$$

$$+2\sum_{p=1}^{M-1} f_1(t,s,\widetilde{\lambda}_j+u_j^{(k)}((j-1)h+2ph_1))\Big],\quad k=0,1,\ldots. \tag{9}$$

Решение специальной задачи Коши (2), (3) при $\lambda_r=\widetilde{\lambda}_r$, $r=\overline{1,N}$, найдем по следующему алгоритму.

Шаг 0. а) За начальное приближение к решению задачи (2), (3) возьмем функции $u_r^{(0)}(t)=0$, $r=\overline{1,N}$, и по формуле Симпсона (9) вычислим

$$I_0=\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh} f_1(t,s,\widetilde{\lambda}_j)ds,\quad t\in[(r-1)h,rh),\quad r=\overline{1,N}. \tag{10}$$

б) Подставляя значение (10) во второе слагаемое правой части (2), получим задачи Коши для обыкновенных дифференциальных уравнений на подинтервалах:

$$\frac{du_r^{(1)}}{dt}=f_0(t,\widetilde{\lambda}_r+u_r^{(1)})+I_0,\quad t\in[(r-1)h,rh), \tag{11}$$

$$u_r^{(1)}[(r-1)h]=0,\quad r=\overline{1,N}. \tag{12}$$

Решая задачи Коши (11), (12) методом Рунге-Кутта 4-го порядка, определим численные решения $u_r^{(1)}(t)$, $r=\overline{1,N}$.

Шаг 1. а) Используя значения систем функций $u_r^{(1)}(t)$ на $[(r-1)h,rh)$, $r=\overline{1,N}$, по формуле Симпсона (9) вычислим

$$I_1=\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh} f_1(t,s,u_j^{(1)}(s)+\lambda_j)ds,\quad t\in[(r-1)h,rh),\quad r=\overline{1,N}.$$

б) Решая задачи Коши для обыкновенных дифференциальных уравнений на подинтервалах:

$$\frac{du_r^{(2)}}{dt} = f_0(t, \widetilde{\lambda}_r + u_r^{(2)}(t)) + I_1, \quad t \in [(r-1)h, rh], \tag{13}$$

$$u_r^{(2)}[(r-1)h] = 0, \quad r = \overline{1, N}, \tag{14}$$

по формуле (8) находим численные значения функций $u_r^{(2)}(t)$, $r = \overline{1, N}$.

Повторяя процесс, на k-ом шаге алгоритма, используя формулу (9), вычислим I_k,и решая задачи Коши для обыкновенных дифференциальных уравнений

$$\frac{du_r^{(k+1)}}{dt} = f_0(t, \widetilde{\lambda}_r + u_r^{(k+1)}(t)) + I_k, \quad t \in [(r-1)h, rh),$$

$$u_r[(r-1)h] = 0, \quad r = \overline{1, N},$$

находим численные значения функций $u_r^{(k+1)}(t)$, $r = \overline{1, N}$.

Условия Теоремы 1 обеспечивают сходимость предложенного алгоритма нахождения численного решения специальной задачи Коши (2), (3).

Пример. На отрезке $[0, 1]$ рассмотрим нелинейное интегро-дифференциальное уравнение Фредгольма

$$\frac{dx}{dt} = x^2 + 0.2t^2 \int\limits_0^1 [x(s) + x^2(s)]ds + 1.4t^2 + f(t), \quad x \in R, \quad t \in [0, 1], \tag{15}$$

где $f(t) = 0.4t - 0.2 - \frac{896}{625}t^2 + \frac{2}{25}t^3 - \frac{1}{25}t^4$.

Применяя метод параметризации к уравнению (15) при $h = 0.125$, получим следующую специальную задачу Коши для нелинейного интегро-дифференциального уравнения с параметрами:

$$\frac{du_r}{dt} = (u_r + \lambda_r)^2 + 0.2t^2 \sum_{j=1}^{8} \int\limits_{(j-1)h}^{jh} [u_j(s) + \lambda_j +$$

$$+(u_j(s) + \lambda_j)^2]ds + 1.4t^2 + f(t), \quad t \in [(r-1)h, rh), \tag{16}$$

$$u_r[(r-1)h] = 0, \quad r = \overline{1, 8}. \tag{17}$$

Используя предложенный численный алгоритм, найдем решения специальной задачи Коши (16), (17) при $\lambda_r = \widetilde{\lambda}_r$, где $\widetilde{\lambda}_1 = 0$, $\widetilde{\lambda}_2 = -0.022$, $\widetilde{\lambda}_3 = -0.038$, $\widetilde{\lambda}_4 = -0.047$, $\widetilde{\lambda}_5 = -0.05$, $\widetilde{\lambda}_6 = -0.047$, $\widetilde{\lambda}_7 = -0.038$, $\widetilde{\lambda}_8 = -0.0022$. Решением специальной задачи Коши (16), (17) при этих значениях параметров является система функций $\widetilde{u}[t] = (\widetilde{u}_1(t), \widetilde{u}_2(t), \widetilde{u}_3(t), \widetilde{u}_4(t), \widetilde{u}_5(t), \widetilde{u}_6(t), \widetilde{u}_7(t), \widetilde{u}_8(t))$, где $\widetilde{u}_r(t) = 0.2t(t-1) - \widetilde{\lambda}_r$, $t \in [(r-1)h, rh)$, $r = \overline{1, N}$.

Проверим выполнение условия Теоремы А: $L_0 = 0.1$, $L_1 = 1$, $\rho = 1$,

$$\text{а)} \quad e^{L_0 h} L_1 T h = \frac{1}{8} e^{\frac{1}{80}} < 1,$$

$$\text{б)} \quad \frac{e^{L_0 h}}{1 - e^{L_0 h} L_1 T h} \|u[\cdot]\|_1 = 0.8887 < \rho.$$

Шаг 0. а) Выбрав $u_r^{(0)}(t) = 0$, $r = \overline{1, 8}$, по формуле Симпсона (9) вычислим

$$I_0 = \sum_{j=1}^{8} \int_{(j-1)h}^{jh} [\lambda_j + \lambda_j^2] ds = -0.0328125.$$

б) Решая задачу Коши для обыкновенного дифференциального уравнения на подинтервалах:

$$\frac{du_r^{(1)}}{dt} = (u_r^{(1)})^2 + 1.4t^2 + f(t) + 0.2t^2 \cdot I_0, \quad t \in [(r-1)h, rh),$$

$$u_r^{(1)}[(r-1)h] = 0, \quad r = \overline{1, 8},$$

методом Рунге-Кутта четвертого порядка найдем численные значения функций $u_r^{(1)}(t)$. Число разбиений на подинтервалах возьмем равным $N_1 = 2M$, $M = 2$, с одинаковым шагом $h_1 = 0.03125$.

Шаг 1. а) Используя значения систем функций $u_r^{(1)}(t)$ на $[(r-1)h, rh)$, $r = \overline{1, 8}$, по формуле Симпсона (9) вычислим

$$I_1 = \sum_{j=1}^{8} \int_{(j-1)h}^{jh} [u_j^{(1)}(s) + \lambda_j + (u_j^{(1)}(s) + \lambda_j)^2] ds = 0.032002985.$$

б) Подставляя значение I_1 в уравнение (16), снова получим задачу Коши для обыкновенного дифференциального уравнения:

$$\frac{du_r^{(2)}}{dt} = (u_r^{(2)})^2 + 1.4t^2 + f(t) + 0.2t^2 \cdot I_1, \quad t \in [(r-1)h, rh),$$

$$u_r^{(2)}[(r-1)h] = 0, \quad r = \overline{1,8}.$$

Решая задачу Коши для обыкновенного дифференциального уравнения на подинтервалах методом Рунге-Кутта четвертого порядка, находим численные решения функции $u_r^{(2)}(t)$. И т.д.

В следующей таблице приведены результаты вычислений 4-ой итерации нахождения численного решения задачи (16), (17)

Таблица 1 – Разность между точным и численным решениями задачи (16), (17) не превышает $\varepsilon = 0.3 \cdot 10^{-9}$

t	$x(t)$ (числ. реш.)	t	$x(t)$ (числ. реш.)	t	$x(t)$ (числ. реш.)
0.00000	0.000000000000	0.34375	-0.007617187338	0.68750	0.003906250124
0.03125	-0.006054687450	0.37500	-0.009374999783	0.71875	0.006445312688
0.06250	-0.011718749900	0.40625	-0.001367187444	0.75000	0.009375000253
0.09375	-0.016992187349	0.43750	-0.002343749888	0.78125	0.003320312567
0.12500	-0.021874999797	0.46875	-0.002929687332	0.81250	0.007031250135
0.15625	-0.004492187448	0.50000	-0.003124999775	0.84375	0.011132812704
0.18750	-0.008593749896	0.53125	0.000195312588	0.87500	0.015625000276
0.21875	-0.012304687343	0.56250	0.000781250117	0.90625	0.004882812574
0.25000	-0.015624999790	0.59375	0.001757812676	0.93750	0.010156250149
0.28125	-0.002929687446	0.62500	0.003125000236	0.96875	0.015820312728
0.31250	-0.005468749892	0.65625	0.001757812562	1.00000	0.021875000308

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Жұмабаев Д.С., Бакирова Э.А., Мынбаева С.Т. СЫЗЫҚТЫ ЕМЕС ФРЕДГОЛЬМ ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕРІ ҮШІН АРНАЙЫ КОШИ ЕСЕБІНІҢ ШЕШІМІН ТАБУҒА АРНАЛҒАН БІР АЛГОРИТМІНІҢ САНДЫҚ ЖҮЗЕГЕ АСЫРЫЛУЫ

Сызықты емес Фредгольм интегралдық-дифференциалдық теңдеулер жүйесіне параметрлеу әдісін қолдану кезінде туындайтын сызықты емес Фредгольм интегралдық-дифференциалдық теңдеулер жүйелері үшін арнайы Коши есебі қарастырылады. Қарастырылып отырған есептің сандық шешімін табудың алгоритмі жасалған.

Кілттік сөздер. Сызықтық емес интегралдық-дифференциалдық теңдеу, арнайы Коши есебі, сандық шешім, алгоритм.

Dzhumabaev D.S., Bakirova E.A., Mynbayeva S.T. NUMERICAL REALIZATION OF ONE ALGORITHM FOR FINDING A SOLUTION OF SPECIAL CAUCHY PROBLEM FOR FREDHOLM NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

Special Cauchy problem for nonlinear integro-differential equations, which arises when applying parametrization method to the system of Fredholm nonlinear integro-differential equations is considered. The algorithm for finding a numerical solution of the considering problem is elaborated.

Keywords. Integro-differential equations, special Cauchy problem, numerical method, algorithm.

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Статья поступила в редакцию 20.12.2017

Математический журнал ISSN 1682–0525

2017. — Том 17, № 4. — С. 37–49.

МРНТИ 27.29.19.

ОБ АСИМПТОТИКЕ ПО СПЕКТРАЛЬНОМУ ПАРАМЕТРУ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ НА ДЕРЕВЕ С УСЛОВИЯМИ КИРХГОФА В ЕГО ВНУТРЕННИХ ВЕРШИНАХ

Л.К. Жапсарбаева, Б.Е. Кангужин, Н. Кошкарбаев

Аннотация. В работе строятся асимптотически независимые по параметру решения дифференциального уравнения второго порядка на дереве с условиями Кирхгофа в его внутренних вершинах.

Ключевые слова. Ориентированный граф, условия Кирхгофа, граф типа дерево, асимптотически независимые по параметру решения, максимальный оператор.

1 Введение

Известно [1], что для дифференциального уравнения высшего порядка на отрезке

$$y^{(n)}(x) + \sum_{k=0}^{n-1} p_k(x) y^{(k)}(x) = \rho^n y(x),\ 0 < x < 1,$$

в секторе $0 < \arg\rho < \frac{\pi}{n}$ существует фундаментальная система решений $\{y_k(x,\rho), k = 1,\ldots,n\}$, которая удовлетворяет предельным соотношениям

$$\lim_{\rho\to\infty} y_k(x,\rho) e^{-\omega_k \rho x} = 1,$$

где $\omega_1,\ldots,\omega_n$ – различные корни n-ой степени из (-1). Смысл этого утверждения заключается в том, что решения $\{y_k(x,\rho), k = 1,\ldots,n\}$ составляют

2010 Mathematics Subject Classification: 34B45, 34B24, 34L20.
Funding: Комитет науки Министерства образования и науки РК, Грант № 0757/ГФ4.

не только линейно независимую систему на интервале $(0,1)$, но и представляют асимптотически линейно независимую систему. Существуют примеры, когда некоторая система решений линейно независима, но асимптотически не является таковой. К примеру, при $n = 2$, $p_1(x) = p_0(x) = 0$ решения $\{\cos\rho x, \sin\rho x\}$ составляют линейно независимую систему, но при $\rho \to \infty$ асимптотически $\cos\rho x = \frac{1}{2}e^{i\rho x}\,(1+\bar{o}(1))$, $\sin\rho x = \frac{1}{2i}e^{i\rho x}\,(1+o(1))$ между собой зависимы. При исследовании спектральных свойств дифференциальных уравнений высших порядков с регулярными по Бирхгофу граничными условиями эффективно используются асимптотически независимые системы решения.

Работа посвящена линейным дифференциальным операторам на графах типа дерево. Спектральная теория дифференциальных операторов на многообразиях типа сети изучалась в работах Ю.В. Покорного и его учеников [2]–[6]. Из последних публикаций, посвященных различным аспектам – теории обратных задач на графах, обратим внимание на работы [7]–[11]. Отметим недавние работы [12], [13], в которых изучены некоторые спектральные свойства дифференциальных операторов на простых графах типа звезда. В настоящей работе для дифференциальных уравнений второго порядка на графе типа дерево с условиями Кирхгофа в его внутренних вершинах строятся асимптотически независимые по параметру ρ решения.

2 Основные понятия

Пусть задан граф-дерево $\Im = \{\mathcal{V}, \mathcal{E}\}$, где $\mathcal{V}$ – множество вершин графа, $\mathcal{E}$ – множество его дуг. Ориентированный граф считается деревом, если в каждой вершине, кроме одной, только одна входящая дуга. Вершина, которая не имеет входящих дуг, называется корнем дерева и корню присваивается номер 0. Важным свойством дерева является то, что от корня до любой вершины существует единственный маршрут, соединяющий их [14]. Длина маршрута определяет высоту вершины графа. Вершины, у которых отсутствуют исходящие дуги, назовем граничными вершинами. Неграничные вершины будем называть внутренними вершинами. Сначала от 0 до p пронумеруем граничные вершины. Затем от $p+1$ до r присвоим номера внутренним вершинам по принципу: чем больше высота вершины, тем больше его номер. Через m_j обозначим количество дуг, исходящих из вершины j. Дугу, оканчивающуюся на вершине j, обозначим через e_j. Маршрут, исходящий из корня и оканчивающийся на вершине j, обозначим через s_j, а его длину – через $|s_j| - 1$. Всюду в дальнейшем считаем, что из корня исходит только одна дуга.

3 Определение максимального оператора на дереве

Рассмотрим пространство

$$L_2(\Im) \doteqdot \prod_{e\in\mathcal{E}} L_2(e)$$

с элементами

$$\vec{Y}(\vec{X}) \doteqdot [y_e(x_e), e \in \mathcal{E}]^T$$

(где $\vec{X} = (x_e, e \in \mathcal{E})$ и $\prod_{e\in\mathcal{E}}$ – декартово произведение подпространств) и с конечной нормой

$$||\vec{Y}||_{L_2(\Im)} = \sqrt{\sum_{e_j\in\mathcal{E}} \int_{e_j} |y_j(x_j)|^2 dx_j}.$$

Точно также стандартным образом вводится пространство

$$W_2^2(\Im) \doteqdot \prod_{e\in\mathcal{E}} W_2^2(e).$$

Введем множество функций $D(\Lambda_{max}) \subset W_2^2(\Im)$, элементы которых в каждой внутренней вершине $k = p+1, \dots, r$ удовлетворяют условиям Кирхгофа [15]:

$$y_k(1) = y_{s_{1k}}(0) = \ldots = y_{s_{m_k,k}}(0), \tag{1}$$

$$y'_k(1) = y'_{s_{1k}}(0) + \ldots + y'_{s_{m_k,k}}(0). \tag{2}$$

Здесь $s_{1k}, \dots, s_{m_k,k}$ – номера дуг, исходящих из вершины k. Оператор Λ_{max} с областью определения $D(\Lambda_{max})$ и задаваемый линейными дифференциальными выражениями

$$-y''_j(x_j) + q_j(x_j)y_j(x_j) = \rho^2 y_j(x_j),\ e_j \in \mathcal{E},\ 0 < x_j < 1, \tag{3}$$

$$j = 1, \dots, r,$$

назовем максимальным оператором.

При этом $\{q_j(x_j), x_j \in e_j \in \mathcal{E}, \, 0 < x_j < 1\}$ – набор вещественных непрерывных функций обычно называют потенциалами. Заметим, что общее количество условий Кирхгофа во внутренних вершинах равно $2r - p - 1$.

4 Основной результат

Основной результат сформулируем для случая, когда все $q_j(x_j) \equiv 0$, хотя его легко перефразировать для общего случая.

Пусть при $0 < x_{p+1} < 1$ решение $y_{p+1}(x_{p+1}) = e^{i\rho x_{p+1}}$ на дуге e_{p+1} известно. Какое воздействие оказывает $y_{p+1}(x_{p+1})$ на другие дуги, если выполняются соотношения (1), (2), (3)? Иначе говоря, надо найти решения $y_j(x_j)$ при $j \neq p+1$ на оставшихся дугах, если выполнены соотношения (1), (2), (3). Известно [16], что на дереве для любого $j \neq p+1$ существует единственный маршрут $s_j = \{0, p+1, \ldots, j\}$. Вершины, входящие в s_j, обозначим через $n_{1j} = 0, n_{2j} = p+1, \ldots, n_{|s_j|,j} = j$.

Теорема 1. *Пусть $j \neq p+1$. При $0 < x_j < 1$ справедливо представление*

$$y_j(x_j) = e^{i\rho(x_j + |s_j| - 2)} +$$

$$+ \sum_{t=1}^{|s_j|-2} (-1)^t \sum_{|s_j|-2 \geq j_t > j_{t-1} > \ldots > j_1 \geq 1} D_{j_1} \ldots D_{j_t} i e^{j_1 i\rho} \times$$

$$\times \cos\rho(j_2 - j_1) \ldots \cos\rho(j_t - j_{t-1}) \sin\rho(x_j - j_t + |s_j| - 2). \qquad (4)$$

Числа $D_1, \ldots, D_{|s_j|-2}$ удобно переобозначить через $D_1 = \theta_{n_{3j}}, \ldots, D_{|s_j|-2} = \theta_{|s_j|,j}$, причем числа $\theta_1, \ldots, \theta_p, \theta_{p+2}, \ldots, \theta_r$ удовлетворяют при $k = p+1, \ldots, r$ следующим $r - p$ соотношениям:

$$m_k - 1 = \theta_{s_{1k}} + \ldots + \theta_{s_{m_k,k}}, \qquad (5)$$

где $s_{1k}, \ldots, s_{m_k,k}$ – номера дуг, исходящих из вершины k.

Замечание 1. *На каждой дуге, кроме дуги e_{p+1} , определено одно число $\theta_j, j \neq p+1$, среди которых $p-1$ независимых.*

Замечание 2. *Из доказательства Теоремы 1 видно, что утверждение теоремы 1 легко сформулировать и доказать при $q_j(x_j) \in C(e_j)$ $\forall j$.*

5 Доказательство основной теоремы

Вначале приведем две вспомогательные леммы, затем докажем основную Теорему 1 методом математической индукции. Рассмотрим дерево, изображенное на Рис. 1. Количество вершин, исходящих из вершины j, пусть равно m. Граничные вершины дерева (Рис. 1) будем нумеровать от 0 до m, а соответствующие дуги – через $e_1, e_2, \ldots, e_m$. Считаем, что на дуге e_j задано решение $y(x) = e^{i\rho x}$. В вершине j выполняются условия (2), (3). Требуется найти решения $y_1(x_1), y_2(x_2), \ldots, y_m(x_m)$ на дугах $e_1, e_2, \ldots, e_m$ уравнения (1).

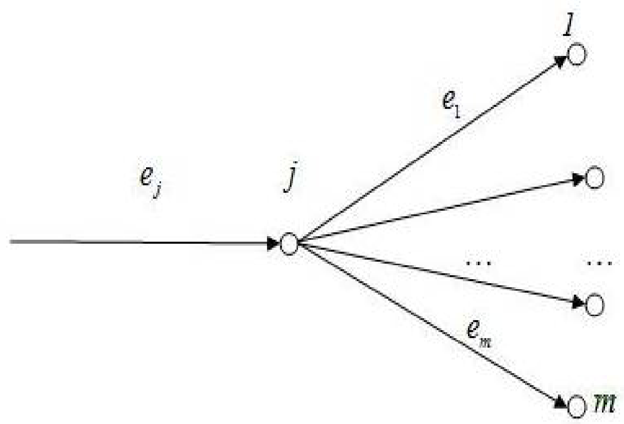

Рисунок 1 – Распространение решения по дереву

Лемма 1. *В случае дерева, изображенного на Рис. 1, решение имеет представление*

$$y_s(x_s) = e^{i\rho(x_s+1)} - A_s i e^{i\rho} \sin \rho x_s, \quad s = 1, \ldots, m, \ x_s \in e_s. \tag{6}$$

Причем постоянные $A_1, A_2, \ldots, A_m$ удовлетворяют соотношению

$$A_1 + A_2 + \ldots + A_m = m - 1. \tag{7}$$

Доказательство Леммы 1. Решения уравнения (1) ищем в виде (6) с некоторыми постоянными $A_1, A_2, \ldots, A_m$. Из вида решений (6) сразу же следует выполнение условий (2). Остается проверить выполнение условия (3). Понятно, что для выполнения условия (3) достаточно потребовать соотношение (7). Лемма 1 полностью доказана.

Следующая лемма – простое обобщение Леммы 1.

Лемма 2. *Пусть дано дерево, изображенное на Рис. 1, и на дуге* e_j *задано решение* $y_j(x) = e^{i\rho(x+1)} - A_j i e^{i\rho} \sin \rho x, \ \ x \in e_j$. *Тогда уравнение (1) на дугах* e_s *имеет решение*

$$y_s(x_s) = y_j(x_s + 1) - B_s y_j'(1) \frac{\sin \rho x_s}{\rho}, \ \ s = 1, \ldots, m.$$

Причем числа $B_1, B_2, \ldots, B_m$ *удовлетворяют соотношению*

$$B_1 + B_2 + \ldots + B_m = m - 1.$$

Последовательно применяя Леммы 1 и 2, можно вычислить распространение фиксированного решения $y(x) = e^{i\rho x}$ на дуге e_{p+1} вдоль произвольного маршрута $s_j = \{0, p+1, \ldots, j\}$.

Доказательство основной теоремы. Для любого $j \neq p+1$ выберем маршрут $s_j = \{0, p+1, \ldots, j\}$, длина которого равна $|s_j| - 1$. На этом маршруте s_j вершину, которая предшествует вершине j, обозначим через k. Итак, маршрут примет вид $s_j = \{0, p+1, \ldots, k, j\}$. Согласно индукционному предположению на дуге e_k решение имеет представление

$$y_k(x_k) = e^{i\rho(x_k + |s_j| - 3)} +$$
$$+ \sum_{t=1}^{|s_j|-3} (-1)^t \sum_{|s_j|-3 \geq j_t > j_{t-1} > \ldots > j_1 \geq 1} D_{j_1} \ldots D_{j_t} i e^{j_1 i \rho} \times$$
$$\times \cos \rho(j_2 - j_1) \ldots \cos \rho(j_t - j_{t-1}) \sin \rho(x_k - j_t + |s_j| - 3). \qquad (8)$$

Учитывая Лемму 2, решение $y_j(x_j)$ на дуге e_j примет вид

$$y_j(x_j) = y_k(x_k + 1) - D_{j_{t+1}} y_k'(1) \frac{\sin \rho x_j}{\rho}.$$

Подставляя в последнее соотношение равенство (8), имеем

$$y_j(x_j) = e^{i\rho(x_j + |s_j| - 2)} +$$

$$+\sum_{t=1}^{|s_j|-3}(-1)^t \sum_{|s_j|-3\geq j_t>j_{t-1}>\ldots>j_1\geq 1} D_{j_1}\ldots D_{j_t} i e^{j_1 i\rho}\times$$

$$\times\cos\rho(j_2-j_1)\ldots\cos\rho(j_t-j_{t-1})\sin\rho(x_j-j_t+|s_j|-2)-$$

$$-D_{j_{t+1}}\frac{\sin\rho x_j}{\rho}\Bigg[e^{i\rho(|s_j|-2)}+\sum_{t=1}^{|s_j|-3}(-1)^t \sum_{|s_j|-3\geq j_t>j_{t-1}>\ldots>j_1\geq 1} D_{j_1}\ldots D_{j_t} i e^{j_1 i\rho}\times$$

$$\times\cos\rho(j_2-j_1)\ldots\cos\rho(j_t-j_{t-1})\sin\rho(|s_j|-2-j_t)\Bigg].$$

Раскрывая скобки и приводя подобные члены, убеждаемся в справедливости формулы (4).

Соотношения (5) для чисел $\theta_1,\ldots,\theta_r$ проверяются точно также, как доказывались соотношения (7) для чисел $A_1,\ldots,A_m$.

Таким образом, Теорема 1 полностью доказана.

Замечание 3. *Если в решении $y(x)=e^{i\rho x}$ ρ заменить на $-\rho$, то можно получить еще одно асимптотически линейно независимое решение.*

6 Асимптотика решений дифференциального уравнения на дереве

В данном пункте рассмотрим общий случай, когда потенциалы $\{q_j(x_j), x_j\in e_j\}$ – набор непрерывных вещественных функций. В данном пункте сохраняются обозначения Пункта 4.

Теорема 2. *Пусть на дуге e_{p+1} решение $y_{p+1}(x_{p+1})=e^{i\rho x_{p+1}}\left[1+o\left(\frac{1}{\rho}\right)\right]$ при $\rho\to\infty$. Тогда при $Im\rho<0$ и $j\neq p+1$ на дуге e_j решения $y_j(x_j)$ задачи (1), (2), (3) имеют при $\rho\to\infty$ асимптотическое представление*

$$y_j(x_j)=e^{i\rho(x_j+|s_j|-2)}\left(1+\sum_{t=1}^{|s_j|-2}\frac{(-1)^t}{2^t}\times\right.$$

$$\left.\times\sum_{|s_j|-2\geq j_t>j_{t-1}>\ldots>j_1\geq 1} D_{j_1}\ldots D_{j_t}\right)\left[1+o\left(\frac{1}{\rho}\right)\right]. \quad (9)$$

Доказательство. Формула (9) получается из представления решения (4) вынесением за скобку множителя $e^{i\rho(x_j+|s_j|-2)}$ и объединением в скобках в $o\left(\frac{1}{\rho}\right)$ всех тех слагаемых, которые содержат $\frac{1}{\rho}$. Действительно,

$$y_j(x_j) = e^{i\rho(x_j+|s_j|-2)} + \sum_{t=1}^{|s_j|-2} (-1)^t \sum_{|s_j|-2\geq j_t>j_{t-1}>\ldots>j_1\geq 1} D_{j_1}\ldots D_{j_t} i e^{j_1 i\rho}\times$$

$$\times\cos\rho(j_2-j_1)\ldots\cos\rho(j_t-j_{t-1})\sin\rho(x_j-j_t+|s_j|-2) =$$

$$= e^{i\rho(x_j+|s_j|-2)} + \sum_{t=1}^{|s_j|-2} (-1)^t \sum_{|s_j|-2\geq j_t>j_{t-1}>\ldots>j_1\geq 1} D_{j_1}\ldots D_{j_t} i e^{j_1 i\rho}\times$$

$$\times\frac{e^{i\rho(j_2-j_1)}+e^{-i\rho(j_2-j_1)}}{2}\ldots\frac{e^{i\rho(j_t-j_{t-1})}+e^{-i\rho(j_t-j_{t-1})}}{2}\times$$

$$\times\frac{e^{i\rho(x_j-j_t+|s_j|-2)}-e^{-i\rho(x_j-j_t+|s_j|-2)}}{2i}.$$

Если $Im\rho<0$, то $iIm\rho>0$. Пусть $\rho=a+bi$, где $b<0$. Сравним значения $e^{i\rho(x_j+|s_j|-2)}$ и $e^{-i\rho(x_j+|s_j|-2)}$. Тогда $|e^{i\rho(x_j+|s_j|-2)}| = |e^{i(a+bi)(x_j+|s_j|-2)}| = e^{-b(x_j+|s_j|-2)}$ и $|e^{i\rho(x_j+|s_j|-2)}| = |e^{i(a+bi)(x_j+|s_j|-2)}| = e^{-b(x_j+|s_j|-2)}$. Отсюда следует, что $e^{-b(x_j+|s_j|-2)} > e^{b(x_j+|s_j|-2)}$ при $b<0$. Следовательно,

$$y_j(x_j) = e^{i\rho(x_j+|s_j|-2)} + \sum_{t=1}^{|s_j|-2}(-1)^t\times$$

$$\times\sum_{|s_j|-2\geq j_t>j_{t-1}>\ldots>j_1\geq 1} D_{j_1}\ldots D_{j_t} e^{(j_1+j_2-j_1+j_t-j_{t-1}+x_j-j_t+|s_j|-2)i\rho}\times$$

$$\times\frac{1+e^{-2i\rho(j_2-j_1)}}{2}\ldots\frac{1+e^{-2i\rho(j_t-j_{t-1})}}{2}\cdot\frac{1-e^{-2i\rho(x_j-j_t+|s_j|-2)}}{2} =$$

$$= e^{i\rho(x_j+|s_j|-2)} + \sum_{t=1}^{|s_j|-2}\frac{(-1)^t}{2^t}\sum_{|s_j|-2\geq j_t>j_{t-1}>\ldots>j_1\geq 1} D_{j_1}\ldots D_{j_t} e^{i\rho(x_j+|s_j|-2)}\times$$

$$\times(1+e^{-2i\rho(j_2-j_1)})\ldots(1+e^{-2i\rho(j_t-j_{t-1})})(1-e^{-2i\rho(x_j-j_t+|s_j|-2)}).$$

Таким образом, решение $y_j(x_j)$ запишется в виде

$$y_j(x_j) = e^{i\rho(x_j+|s_j|-2)}\Bigg[1 + \sum_{t=1}^{|s_j|-2} \frac{(-1)^t}{2^t} \sum_{|s_j|-2\geq j_t>j_{t-1}>\ldots>j_1\geq 1} D_{j_1}\ldots D_{j_t}\times$$

$$\times(1+e^{-2i\rho(j_2-j_1)})\ldots(1+e^{-2i\rho(j_t-j_{t-1})})(1-e^{-2i\rho(x_j-j_t+|s_j|-2)})\Bigg] =$$

$$= e^{i\rho(x_j+|s_j|-2)}\left(1 + \sum_{t=1}^{|s_j|-2} \frac{(-1)^t}{2^t} \sum_{|s_j|-2\geq j_t>j_{t-1}>\ldots>j_1\geq 1} D_{j_1}\ldots D_{j_t}\right)\times$$

$$\times\left[1+o\left(\frac{1}{\rho}\right)\right].$$

Теорема доказана.

Пример. Вычислим асимптотически независимые по параметру ρ решения дифференциальных уравнений

$$-y_j''(x_j) = \rho^2 y_j(x_j),\ e_j\in\mathcal{E},\ 0<x_j<1, \qquad (10)$$

$$j=1,2,3,$$

с распадающимися краевыми условиями

$$y_1'(1) - hy_1(1) = 0, \qquad (11)$$
$$y_2'(1) + Hy_2(1) = 0,$$
$$y_3'(0) + \gamma y_3(0) = 0,$$

где h, H, γ – фиксированные действительные числа.

Согласно формуле (6) уравнение (10) на дугах $e_1,\ e_2$ в случае дерева, изображенного на Рис. 2, имеет решение

$$y_1(x_1) = e^{i\rho(x_1+1)} - \theta_1 i e^{i\rho}\sin\rho x_1,\ x_1\in e_1,$$
$$y_2(x_2) = e^{i\rho(x_2+1)} - \theta_2 i e^{i\rho}\sin\rho x_2,\ x_2\in e_2,$$

где постоянные θ_1, θ_2 удовлетворяют соотношению

$$\theta_1+\theta_2 = 1.$$

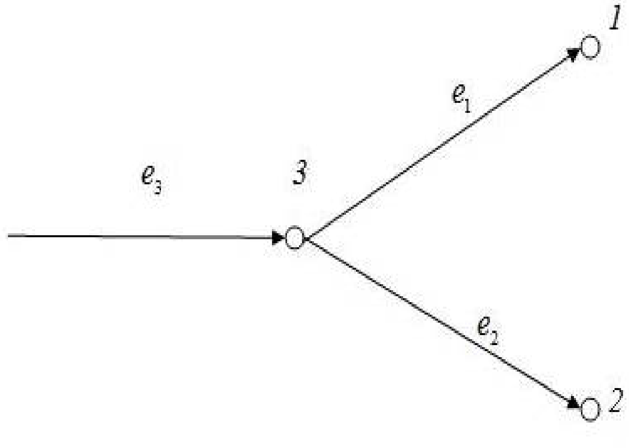

Рисунок 2 – Иллюстрация примера

Отсюда $\theta_1 = 1 - \theta_2$. Если переобозначить $\theta = \theta_1$, то $\theta_2 = 1 - \theta$. Построим общее решение на дугах e_1, e_2, e_3,

$$y_1(x_1) = A\left(e^{i\rho(x_1+1)} - \theta i e^{i\rho}\sin\rho x_1\right) + B\left(e^{-i\rho(x_1+1)} + \theta i e^{-i\rho}\sin\rho x_1\right), \quad (12)$$

$$x_1 \in e_1,$$

$$y_2(x_2) = A(e^{i\rho(x_2+1)} - (1-\theta)ie^{i\rho}\sin\rho x_2) + B(e^{-i\rho(x_2+1)} + (1-\theta)ie^{-i\rho}\sin\rho x_2),$$

$$x_2 \in e_2,$$

$$y_3(x_3) = Ae^{i\rho x_3} + Be^{-i\rho x_3}, \ x_3 \in e_3.$$

Для функций (12), вычисляя условия (11), получим систему уравнений относительно A, B, θ :

$$A\left(i\rho e^{2i\rho} - \theta i\rho e^{i\rho}\cos\rho\right) + B\left(-i\rho e^{-2i\rho} + \theta i\rho e^{-i\rho}\cos\rho\right) - \quad (13)$$

$$-hA\left(e^{2i\rho} - \theta i e^{i\rho}\sin\rho\right) - hB\left(e^{-2i\rho} + \theta i e^{-i\rho}\sin\rho\right) = 0,$$

$$A\left(i\rho e^{2i\rho} - (1-\theta)i\rho e^{i\rho}\cos\rho\right) + B\left(-i\rho e^{-2i\rho} + (1-\theta)i\rho e^{-i\rho}\cos\rho\right) +$$

$$+HA\left(e^{2i\rho} - (1-\theta)ie^{i\rho}\sin\rho\right) + HB\left(e^{-2i\rho} + (1-\theta)ie^{-i\rho}\sin\rho\right) = 0,$$

$$\gamma(A+B)+i\rho(A-B)=0.$$

Принимая $B=\tau A$, из третьего уравнения системы (13) находим

$$\tau=\frac{i\rho+\gamma}{i\rho-\gamma}.$$

Тогда получим систему линейных однородных уравнений относительно $A\theta$ и A, ее запишем в матричном виде

$$\begin{pmatrix} i(e^{i\rho}-\tau e^{-i\rho})(-\rho\cos\rho+h\sin\rho) & e^{2i\rho}(i\rho-h)-\tau e^{-2i\rho}(i\rho+h) \\ i(e^{i\rho}-\tau e^{-i\rho})(\rho\cos\rho+H\sin\rho) & e^{2i\rho}(i\rho+H)-\tau e^{-2i\rho}(i\rho-H)- \\ & -i(e^{i\rho}-\tau e^{-i\rho})(\rho\cos\rho+H\sin\rho) \end{pmatrix}\times$$

$$\times\begin{pmatrix} A\theta \\ A \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Обозначим матрицу системы через

$$Q=\begin{pmatrix} i(e^{i\rho}-\tau e^{-i\rho})(-\rho\cos\rho+h\sin\rho) & e^{2i\rho}(i\rho-h)-\tau e^{-2i\rho}(i\rho+h) \\ i(e^{i\rho}-\tau e^{-i\rho})(\rho\cos\rho+H\sin\rho) & e^{2i\rho}(i\rho+H)-\tau e^{-2i\rho}(i\rho-H)- \\ & -i(e^{i\rho}-\tau e^{-i\rho})(\rho\cos\rho+H\sin\rho) \end{pmatrix}.$$

Утверждение 3. *Собственные значения дифференциального оператора, определенного на графе, изображенном на Рис. 2, с граничными условиями (11), совпадают с нулями определителя* $detQ(\rho)$.

Стандартным образом доказывается существование бесконечного числа нулей определителя $detQ(\rho)$ [1].

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Жапсарбаева Л.Қ., Қангужин Б.Е., Қошқарбаев Н. АҒАШТА АНЫҚТАЛҒАН ЖӘНЕ ОНЫҢ ІШКІ ТӨБЕЛЕРІНДЕ КИРХГОФ ШАРТЫМЕН БЕРІЛГЕН ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ШЕШІМДЕРІНІҢ СПЕКТРАЛДЫҚ ПАРАМЕТР БОЙЫНША АСИМПТОТИКАСЫ ТУРАЛЫ

Жұмыста ағашта анықталған және оның ішкі төбелерінде Кирхгоф шартымен берілген екінші ретті дифференциалдық теңдеудің параметр бойынша асимптотикалық тәуелсіз шешімдері тұрғызылады.

Кілттік сөздер. Бағдарланған граф, Кирхгоф шарттары, ағаш тектес граф, параметр бойынша асимптотикалық тәуелсіз шешімдер, максимальды оператор.

Zhapsarbayeva L.K., Kanguzhin B.E., Koshkarbayev N. ON THE ASYMPTOTICS WITH RESPECT TO THE SPECTRAL PARAMETER FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS ON A TREE WITH KIRCHHOFF CONDITIONS AT ITS INTERNAL VERTICES

In this paper asymptotically independent solutions of second-order differential equation on a tree with Kirchhoff conditions at its internal vertices are constructed.

Keywords. Directed graph, maximal operator, graph of tree type, Kirchhoff conditions, asymptotic independent solutions with respect to parameter.

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Статья поступила в редакцию 12.07.2017

Математический журнал ISSN 1682–0525

2017. — Том 17, № 4. — С. 50–61.

МРНТИ 27.29.25, 27.33.19, 27.41.19

ON THE METHOD FOR SOLVING LINEAR BOUNDARY-VALUE PROBLEM FOR THE SYSTEM OF LOADED DIFFERENTIAL EQUATIONS WITH MULTIPOINT INTEGRAL CONDITION

Zh.M. Kadirbayeva

Annotation. A linear boundary value problem for the system of loaded differential equations with multipoint integral condition is considered. The method of parameterization is used for solving the considered problem. The linear boundary value problem for the system of loaded differential equations with multipoint integral condition by introducing additional parameters at the loading points is reduced to equivalent boundary value problem with parameters. The equivalent boundary value problem with parameters consists of the Cauchy problem for the system of ordinary differential equations with parameters, multipoint integral condition and continuity conditions. The solution of the Cauchy problem for the system of ordinary differential equations with parameters is constructed using the fundamental matrix of differential equation. The system of linear algebraic equations with respect to the parameters are composed by substituting the values of the corresponding points in the built solutions to the multipoint integral condition and the continuity condition. Numerical method for finding solution of the problem is suggested, which based on the solving the constructed system and Runge-Kutta method of the 4-th order for solving Cauchy problem on the subintervals.

Keywords. Loaded differential equation,integral condition, parameterization method, fundamental matrix.

In recent years the intensive research of the loaded differential equations connected with various applications of problems associated with the loaded equations is observed. The problem of long-term forecasting and regulation of level of ground waters and soil moisture [1]–[3], modeling of the processes of transfer of particles, some problems of optimal control belong to problems of application described by these equations [3], [4]. We will note that the loaded

2010 Mathematics Subject Classification: 34A99, 34B05.

differential equations describe processes with residual effects where the state of the process at some point or some time can affect the entire process [2]. Such equations can be used to describe processes in biology, ecology, and underground fluid dynamics.

A numerical method for solving systems of loaded linear nonautonomous ordinary differential equations with nonseparated multipoint and integral conditions is offered in works [4]–[6]. This method is based on the convolution of the integral conditions to obtain local conditions. This approach allows one to reduce solving the original problem to the solving a Cauchy problem for the system of ordinary differential equations and linear algebraic equations.

One of the constructive and effective methods for solving two-point boundary value problems for ordinary differential equations is parameterization method [7]. This method, except the proof of the unique solvability of the investigated problem, gives algorithm of constructing approximate solutions converging to its exact solution. In [8] parameterization method is developed for multi-point boundary value problems for the systems of ordinary differential equations, effective conditions of solvability are established and constructive algorithms of finding the solution are constructed.

Coefficient criterion of the unique solvability of linear two-point boundary value problem for the systems of loaded differential equations are found on the based of parametrization method and algorithms for finding solutions of this problem are constructed in works [9], [10].

In the present paper we consider a linear boundary-value problem for the system of loaded differential equations with a multipoint integral condition

$$\frac{dx}{dt} = A_0(t)x + \sum_{i=1}^{m} A_i(t)x(\theta_i) + f(t), \quad t \in (0,T), \quad x \in \mathbb{R}^n, \tag{1}$$

$$B_0x(0) + B_1x(\theta_1) + B_2x(T) + \int\limits_0^T M(t)x(t)dt = d, \qquad d \in \mathbb{R}^n, \tag{2}$$

where $(n \times n)$-matrices $A_j(t)$, $j = \overline{0,m}$, $M(t)$ and n-vector-function $f(t)$ are continuous on $[0,T]$, B_k, $k = \overline{0,2}$, are constant $(n\times n)$-matrices, $0 = \theta_0 < \theta_1 < \theta_2 < ... < \theta_m = T$.

The solution of the problem (1), (2) is a continuously differentiable vector function $x(t)$ on $[0,T]$ which satisfies the system of loaded differential equations (1) on $[0,T]$ and the multipoint integral condition (2).

In contrast to the previously investigated papers [9]–[12], in the present paper the right end of the interval $[0,T]$ is also the load point, i.e. the point $t=T$.

Interval $[0,T]$ is partitioned into the parts by the loading points:
$[0,T)=\bigcup\limits_{r=1}^{m}[\theta_{r-1},\theta_r)$.
Introduce the following spaces:

$C([0,T],\mathbb{R}^n)$ is the space of continuous functions $x:[0,T]\to\mathbb{R}^n$ with the norm $||x||_1=\max\limits_{t\in[0,T]}||x(t)||$;

$C([0,T],\theta,\mathbb{R}^{nm})$ is the space of function systems $x[t]=(x_1(t),x_2(t),...,x_m(t))$, where $x_r:[\theta_{r-1},\theta_r)\to\mathbb{R}^n$ are continuous and have finite left-sided limits $\lim\limits_{t\to\theta_r-0}x_r(t)$ for all $r=\overline{1,m}$ with the norm $||x||_2=\max\limits_{r=\overline{1,m}}\sup\limits_{t\in[\theta_{r-1},\theta_r)}||x_r(t)||$.

Let $x_r(t)$ be the restriction of function $x(t)$ to the r-th interval $[\theta_{r-1},\theta_r)$, i.e. $x_r(t)=x(t)$, for $t\in[\theta_{r-1},\theta_r)$, $r=\overline{1,m}$. Introducing the additional parameters $\lambda_r=x_r(\theta_{r-1})$, $r=\overline{1,m}$, $\lambda_{m+1}=x_m(\theta_m)$, and performing a replacement of the function $u_r(t)=x_r(t)-\lambda_r$ on each r-th interval, we obtain the boundary value problem with parameters

$$\frac{du_r}{dt}=A_0(t)[u_r(t)+\lambda_r]+\sum_{i=1}^{m}A_i(t)\lambda_{i+1}+f(t),\quad t\in[\theta_{r-1},\theta_r),\tag{3}$$

$$u_r(\theta_{r-1})=0,\quad r=\overline{1,m},\tag{4}$$

$$B_0\lambda_1+B_1\lambda_2+B_2\lambda_{m+1}+\sum_{k=1}^{m}\int\limits_{\theta_{k-1}}^{\theta_k}M(t)[u_k(t)+\lambda_k]dt=d,\tag{5}$$

$$\lambda_p+\lim_{t\to\theta_p-0}u_p(t)=\lambda_{p+1},\quad p=\overline{1,m}.\tag{6}$$

The solution of the problem (3)–(6) is a pair $(\lambda,u[t])$ with elements $\lambda=(\lambda_1,\lambda_2,...,\lambda_{m+1})\in\mathbb{R}^{n(m+1)}$, $u[t]=(u_1(t),u_2(t),...,u_m(t))\in$

$C([0,T],\theta,\mathbb{R}^{nm})$, where $u_r(t)$ are continuously differentiable on $[\theta_{r-1},\theta_r)$, $r=\overline{1,m}$, and satisfy the system of loaded differential equations (3) and the conditions (4)–(6) at $\lambda_r=\lambda_r^*$, $r=\overline{1,m+1}$.

The number of entered parameters will be one more than the number of unknown functions. This is due to the fact that in the given scheme of the parametrization method one more parameter is added at the point $t=T$ or $t=\theta_m$.

Problems (1), (2) and (3)–(6) are equivalent. If $x^*(t)$ is a solution of the problem (1), (2), then the pair $(\lambda^*,u^*[t])$ where $\lambda^*=(x^*(\theta_0),x^*(\theta_1),...,x^*(\theta_m))$, $u^*[t]=(x^*(t)-x^*(\theta_0)),x^*(t)-x^*(\theta_1),...,x^*(t)-x^*(\theta_{m-1})$ is a solution of the problem (3)–(6). Conversely, if the pair $(\widetilde{\lambda},\widetilde{u}[t])$, with elements $\widetilde{\lambda}=(\widetilde{\lambda}_1,\widetilde{\lambda}_2,...,\widetilde{\lambda}_{m+1})\in\mathbb{R}^{n(m+1)}$, $\widetilde{u}[t]=(\widetilde{u}_1(t),\widetilde{u}_2(t),...,\widetilde{u}_m(t))\in C([0,T],\theta,\mathbb{R}^{nm})$ is a solution to problem (3)–(6), then the function $\widetilde{x}(t)$ defined by the equalities $\widetilde{x}(t)=\widetilde{u}_r(t)+\widetilde{\lambda}_r$, $t\in[\theta_{r-1},\theta_r)$, $r=\overline{1,m}$, $\widetilde{x}(T)=\widetilde{\lambda}_{m+1}$, is a solution of the original boundary value problem (1), (2).

Using the fundamental matrix $\Phi(t)$ of differential equation $\dfrac{dx}{dt}=A(t)x$ on $[\theta_{r-1},\theta_r)$ we reduce the Cauchy problem for the system of ordinary differential equations with parameters (3), (4) to the equivalent system of integral equations

$$u_r(t)=\Phi(t)\int\limits_{\theta_{r-1}}^{t}\Phi^{-1}(\tau)\Big\{A_0(\tau)\lambda_r+\sum_{i=1}^{m}A_i(\tau)\lambda_{i+1}\Big\}d\tau+$$

$$+\Phi(t)\int\limits_{\theta_{r-1}}^{t}\Phi^{-1}(\tau)f(\tau)d\tau,\quad t\in[\theta_{r-1},\theta_r),\quad r=\overline{1,m}. \tag{7}$$

Solving (7), we find a representation of $u_r(t)$ in terms of $\lambda\in\mathbb{R}^{n(m+1)}$ and $f(t)$. Substituting them into (5) and (6) yields a system of algebraic equations for finding the unknown parameters λ_r, $r=\overline{1,m+1}$:

$$B_0\lambda_1+B_1\lambda_2+B_2\lambda_{m+1}+$$

$$+\sum_{k=1}^{m}\int\limits_{\theta_{k-1}}^{\theta_k} M(t)\Big\{\Phi(t)\int\limits_{\theta_{k-1}}^{t}\Phi^{-1}(\tau)\Big[A_0(\tau)\lambda_k+\sum_{i=1}^{m}A_i(\tau)\lambda_{i+1}\Big]d\tau+\lambda_k\Big\}dt=$$

$$=d-\sum_{k=1}^{m}\int\limits_{\theta_{k-1}}^{\theta_k} M(t)\Phi(t)\int\limits_{\theta_{k-1}}^{t}\Phi^{-1}(\tau)f(\tau)d\tau dt, \tag{8}$$

$$\lambda_p+\Phi(\theta_p)\int\limits_{\theta_{p-1}}^{\theta_p}\Phi^{-1}(\tau)\Big[A_0(\tau)\lambda_p+\sum_{i=1}^{m}A_i(\tau)\lambda_{i+1}\Big]d\tau-\lambda_{p+1}=$$

$$=-\Phi(\theta_p)\int\limits_{\theta_{p-1}}^{\theta_p}\Phi^{-1}(\tau)f(\tau)d\tau, \quad p=\overline{1,m}. \tag{9}$$

Denote the matrix corresponding to the left-hand side of the system (8), (9) by $Q_*(\theta)$, the system can be written as

$$Q_*(\theta)\lambda=-F_*(\theta), \quad \lambda\in\mathbb{R}^{n(m+1)}, \tag{10}$$

where

$$F_*(\theta)=\begin{pmatrix} -d+\sum\limits_{k=1}^{m}\int\limits_{\theta_{k-1}}^{\theta_k} M(t)\Phi(t)\int\limits_{\theta_{k-1}}^{t}\Phi^{-1}(\tau)f(\tau)d\tau dt \\ \Phi(\theta_1)\int\limits_{\theta_0}^{\theta_1}\Phi^{-1}(\tau)f(\tau)d\tau \\ \cdots\cdots\cdots \\ \Phi(\theta_m)\int\limits_{\theta_{m-1}}^{\theta_m}\Phi^{-1}(\tau)f(\tau)d\tau \end{pmatrix}.$$

It is not difficult to establish that the solvability of the boundary value problem (1), (2) is equivalent to the solvability of the system (10). The solution of the system (10) is a vector $\lambda^*=(\lambda_1^*,\lambda_2^*,...,\lambda_{m+1}^*)\in\mathbb{R}^{n(m+1)}$ consists of the values of the solutions of the original problem (1), (2) in the initial points of subintervals, i.e. $\lambda_r^*=x^*(\theta_{r-1})$, $r=\overline{1,m+1}$.

Further we consider the Cauchy problems for ordinary differential equations on subintervals

$$\frac{dz}{dt} = A_0(t)z + P(t), \quad z(\theta_{r-1}) = 0, \quad r = \overline{1,m}, \tag{11}$$

where $P(t)$ is either $(n \times n)$-matrix, or n-vector, both continuous on $[\theta_{r-1}, \theta_r]$, $r = \overline{1,m}$. Consequently, solution of the problem (11) is a square matrix or a vector of the dimension n. Denote by $a_r(P,t)$ the solution of the Cauchy problem (11). Obviously,

$$a_r(P,t) = \Phi(t) \int\limits_{\theta_{r-1}}^{t} \Phi^{-1}(\tau)P(\tau)d\tau, \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1,m}, \tag{12}$$

where $\Phi(t)$ is a fundamental matrix of differential equation (11) on the r-th interval.

We offer the following numerical implementation of the algorithm based on Runge-Kutta method of 4-th order and Simpson's method.

I. Suppose we have a partition $0 = \theta_0 < \theta_1 < \theta_2 < ... < \theta_{m-1} < \theta_m = T$. Divide each r-th interval $[\theta_{r-1}, \theta_r)$, $r = \overline{1,m}$, into N_r parts with the step $h_r = (\theta_r - \theta_{r-1})/N_r$. Assume on each interval $[\theta_{r-1}, \theta_r)$ the variable $\widehat{\theta}$ takes its discrete values: $\widehat{\theta} = \theta_{r-1}$, $\widehat{\theta} = \theta_{r-1} + h_r$, ..., $\widehat{\theta} = \theta_{r-1} + (N_r - 1)h_r$, $\widehat{\theta} = \theta_r$, and denote by $\{\theta_{r-1}, \theta_r\}$ the set of such points.

II. Solving Cauchy problem for the ordinary differential equations

$$\frac{dz}{dt} = A_0(t)z + A_i(t), \quad z(\theta_{r-1}) = 0, \quad i = \overline{0,m}, \quad r = \overline{1,m},$$

$$\frac{dz}{dt} = A_0(t)z + f(t), \quad z(\theta_{r-1}) = 0, \quad r = \overline{1,m},$$

by using again Runge-Kutta method of the 4-th order, we find the values of $(n \times n)$-matrix $a_r(A_i, \widehat{\theta})$, $i = \overline{0,m}$, and n-vector $a_r(f, \widehat{\theta})$ on $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1,m}$.

III. Applying Simpson's method on the set $\{\theta_{r-1}, \theta_r\}$, we evaluate the definite integrals

$$m_r^{h_r} = \int\limits_{\theta_{r-1}}^{\theta_r} M(\tau)d\tau, \quad m_r^{h_r}(A_i) = \int\limits_{\theta_{r-1}}^{\theta_r} M(\tau)a_r^{h_r}(A_i,\tau)d\tau, \ i = \overline{0,m},$$

$$m_r^{h_r}(f) = \int\limits_{\theta_{r-1}}^{\theta_r} M(\tau)a_r^{h_r}(f,\tau)d\tau, \;\; r=\overline{1,m}.$$

IV. Construct the system of linear algebraic equations with respect to parameters

$$Q_*^{\widetilde{h}}(\theta)\lambda = -F_*^{\widetilde{h}}(\theta), \quad \lambda \in \mathbb{R}^{n(m+1)}, \quad \widetilde{h} = (h_1, h_2, ..., h_{m+1}). \tag{13}$$

From it was the system of algebraic equations (13) we find $\lambda^{\widetilde{h}} \in \mathbb{R}^{n(m+1)}$.
As noted above, $\lambda^{\widetilde{h}} = (\lambda_1^{\widetilde{h}}, \lambda_2^{\widetilde{h}}, \ldots, \lambda_{m+1}^{\widetilde{h}}) \in R^{n(m+1)}$ components are values of the approximate solution of the problem (1), (2) in the starting points of subintervals: $x^{\widetilde{h}_r}(\theta_0) = \lambda_1^{\widetilde{h}}$, $x^{\widetilde{h}_r}(\theta_1) = \lambda_2^{\widetilde{h}}$,...,$x^{\widetilde{h}_r}(\theta_m) = \lambda_{m+1}^{\widetilde{h}}$.

V. To define the values of approximate solution at the remaining points of the set $\{\theta_{r-1}, \theta_r\}$, we solve Cauchy problems

$$\frac{dx}{dt} = A_0(t)x + \sum_{j=1}^{m} A_j(t)\lambda_{j+1}^{\widetilde{h}} + f(t), \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1,m}, \tag{14}$$

$$x(\theta_{r-1}) = \lambda_r^{\widetilde{h}}, \quad r = \overline{1,m}. \tag{15}$$

The numerical solving the problem (1), (2) will be determined by applying Runge-Kutta of the 4-th order method for numerical solving the Cauchy problem (14), (15).

To illustrate the proposed approach for the numerical solving linear boundary-value problem for the system of loaded differential equations with multipoint integral condition (1), (2) on the basis of parameterization method, let us consider the following example.

Example. Consider on $[0, T]$ the linear boundary-value problem for the system of loaded differential equations with a multipoint integral condition

$$\frac{dx}{dt} = A_0(t)x + \sum_{j=1}^{2} A_j(t)x(\theta_j) + f(t), \quad t \in (0,T), \quad x \in \mathbb{R}^2, \tag{16}$$

$$B_0x(0) + B_1x(\theta_1) + B_2x(T) + \int\limits_0^T M(t)x(t)dt = d, \qquad d \in \mathbb{R}^2. \tag{17}$$

Here $\theta_1 = \frac{1}{2}$, $\theta_2 = T = 1$, $M(t) = \begin{pmatrix} t & -1 \\ 0 & \frac{1}{2} \end{pmatrix}$, $d = \begin{pmatrix} 2e^{\frac{1}{2}} + \frac{56}{3} - e \\ -\frac{61}{12} + 7e^{\frac{1}{2}} \end{pmatrix}$,

$$A_0 = \begin{pmatrix} \sin t & 1 \\ 0 & t^2 \end{pmatrix}, \quad A_1(t) = \begin{pmatrix} t & 4 \\ 1 & e^t \end{pmatrix}, \quad A_2(t) = \begin{pmatrix} 0 & \cos t \\ 1 & t \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 1 & 2 \\ -5 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2 & -6 \\ 7 & 3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1 & 9 \\ 0 & -1 \end{pmatrix},$$

$$f(t) = \begin{pmatrix} e^t - e^t \sin t - \sin t - t^2 - 3t - te^{\frac{1}{2}} - 2\cos t \\ -t^4 - 2t^3 + t^2 - e^{\frac{1}{2}} - \frac{1}{4}e^t - e \end{pmatrix}.$$

In this example the matrix of differential part is variable and the construction of fundamental matrix breaks down. We use the numerical implementation of the algorithm of parametrization method. We provide the results of the numerical implementation of algorithm by partitioning the subintervals $[0, 0.5]$, $[0.5, 1]$ with steps $h_1 = h_2 = 0.05$.

Solving the system of equations (13), we obtain the numerical values of the parameters

$$\lambda_1^{\widetilde{h}} = \begin{pmatrix} 2.00000006 \\ -0.99999997 \end{pmatrix}, \quad \lambda_2^{\widetilde{h}} = \begin{pmatrix} 2.64872131 \\ 0.24999998 \end{pmatrix}, \quad \lambda_3^{\widetilde{h}} = \begin{pmatrix} 3.71828174 \\ 1.99999992 \end{pmatrix}.$$

We find the numerical solutions at the other points of the subintervals using Runge-Kutta method of the 4-th order to the following Cauchy problems

$$\frac{d\widetilde{x}_r}{dt} = A_0(t)\widetilde{x}_r + \sum_{j=1}^{2} A_j(t)\lambda_{j+1}^{\widetilde{h}} + f(t), \quad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1,2},$$

$$\widetilde{x}_r(\theta_{r-1}) = \lambda_r^{\widetilde{h}}, \quad r = \overline{1,2}.$$

Exact solution of the problem (16), (17) is $x^*(t) = \begin{pmatrix} 1 + e^t \\ t^2 + 2t - 1 \end{pmatrix}$.

The results of calculations of numerical and exact solutions at discrete points are presented in the tables

t	$\widetilde{x}_1(t)$	$x_1^*(t)$	$\widetilde{x}_2(t)$	$x_2^*(t)$
0	2.00000006	2	-0.99999997	-1
0.05	2.05127115	2.0512711	-0.89749998	-0.8975
0.1	2.10517097	2.10517092	-0.78999998	-0.79
0.15	2.16183429	2.16183424	-0.67749999	-0.6775
0.2	2.22140281	2.22140276	-0.55999999	-0.56
0.25	2.28402547	2.28402542	-0.4375	-0.4375
0.3	2.34985885	2.34985881	-0.31	-0.31
0.35	2.41906759	2.41906755	-0.17750001	-0.1775
0.4	2.49182474	2.4918247	-0.04000001	-0.04
0.45	2.56831222	2.56831219	0.10249998	0.1025
0.5	2.64872131	2.64872127	0.24999998	0.25
0.55	2.73325305	2.73325302	0.40249997	0.4025
0.6	2.82211883	2.8221188	0.55999997	0.56
0.65	2.91554085	2.91554083	0.72249996	0.7225
0.7	3.01375272	3.01375271	0.88999996	0.89
0.75	3.11700002	3.11700002	1.06249995	1.0625
0.8	3.22554092	3.22554093	1.23999995	1.24
0.85	3.33964682	3.33964685	1.42249994	1.4225
0.9	3.45960306	3.45960311	1.60999994	1.61
0.95	3.58570959	3.58570966	1.80249993	1.8025
1	3.71828174	3.71828183	1.99999992	2

For the difference of the corresponding values of the exact and constructed solutions of the problem in the example, the following estimate is true

$$\max_{j=\overline{0,20}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0,00000009.$$

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Қадырбаева Ж.М. КӨПНҮКТЕЛІ ИНТЕГРАЛДЫҚ ШАРТЫ БАР ЖҮКТЕЛГЕН ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕЛЕРІ ҮШІН СЫЗЫҚТЫ ШЕТТІК ЕСЕПТІҢ ШЕШІМІН ТАБУ ӘДІСІ ТУРАЛЫ

Көпнүктелі интегралдық шарты бар жүктелген дифференциалдық теңдеулер жүйесі үшін сызықты шеттік есеп қарастырылады. Қарастырылып отырған есепті шешу үшін параметрлеу әдісі қолданылады. Көпнүктелі интегралдық шарты бар жүктелген дифференциалдық теңдеулер жүйесі үшін сызықты шеттік есеп жүктелу нүктелерінде қосымша параметрлер енгізу арқылы пара-пар параметрлері бар шеттік есепке келтіріледі. Пара-пар параметрлері бар шеттік есеп жәй дифференциалдық теңдеулер жүйесі үшін параметрлері бар Коши есебінен, көпнүктелі интегралдық шартынан және үзіліссіздік шартынан тұрады. Параметрлері бар жәй дифференциалдық теңдеулер жүйесі үшін Коши есебінің шешімі дифференциалдық теңдеудің фундаменталдық матрицасының көмегімен тұрғызылады. Тұрғызылған шешімнің сәйкес нүктелердегі мәндерін көпнүктелі интегралдық шартқа және үзіліссіздік шартына қоя отырып, параметрлерге қатысты сызықтық алгебралық теңдеулер жүйесі құрылады. Қарастырылып отырған есепті шешудің құрылған жүйені және ішкі аралықтардағы Коши есептерін 4-ретті Рунге-Кутта әдісін қолданып шешуге негізделген сандық әдісі ұсынылады.

Кілттік сөздер. Жүктелген дифференциалдық теңдеу, интегралдық шарт, параметрлеу әдісі, фундаменталдық матрица.

Кадирбаева Ж.М. О МЕТОДЕ НАХОЖДЕНИЯ РЕШЕНИЯ ЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ СИСТЕМ НАГРУЖЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С МНОГОТОЧЕЧНЫМ ИНТЕГРАЛЬНЫМ УСЛОВИЕМ

Рассматривается линейная краевая задача для системы нагруженных дифференциальных уравнений с многоточечным интегральным условием. Для решения рассматриваемой задачи применяется метод параметризации. Разбиением интервала точками нагружения и введением дополнительных параметров линейная краевая задача для системы нагруженных дифференциальных уравнений с многоточечным интегральным условием сводится к эквивалентной краевой задаче с параметрами. Эквивалентная краевая задача с параметрами состоит из задачи Коши для системы обыкновенных дифференциальных уравнений с параметрами, многоточечного интегрального условия и условия склеивания. Решение задачи Коши для системы обыкновенных дифференциальных уравнений с параметрами строится с помощью фундаментальной матрицы дифференциального уравнения. Подставляя значения в соответствующих точках построенного решения в многоточечное интегральное условие и условие склеивания, составляется система линейных алгебраических уравнений относительно параметров. Предложен численный метод решения рассматриваемой задачи, основанный на решении построенной системы и методе Рунге-Кутта 4-го порядка точности для решения задач Коши на подинтервалах.

Ключевые слова. Нагруженное дифференциальное уравнение, интегральное условие, метод параметризации, фундаментальная матрица

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Received 22.12.2017

Математический журнал ISSN 1682–0525

2017. — Том 17, № 4. — С. 62–69.

МРНТИ 27.31.15

ЕДИНСТВЕННОСТЬ РЕШЕНИЯ ЗАДАЧИ ДИРИХЛЕ ДЛЯ МНОГОМЕРНЫХ ГИПЕРБОЛИЧЕСКИХ УРАВНЕНИЙ С ВЫРОЖДЕНИЕМ ТИПА И ПОРЯДКА

М.Н. Майкотов

Аннотация. Ранее автором была показана разрешимость многомерных гиперболических уравнений с вырождением типа и порядка в цилиндрической области. В данной работе показана единственность классического решения задачи Дирихле для многомерных гиперболических уравнений с вырождением типа и порядка.

Ключевые слова. Единственность, вырождение, цилиндрическая область, системы функций, краевые условия, коэффициенты ряда.

1 Введение

Известно, что для уравнений в частных производных гиперболического типа краевые задачи с данными на всей границе области служат примером некорректно поставленных задач [1], [2]. В [3], [4] показаны корректность задачи Дирихле в цилиндрической области для вырождающихся многомерных гиперболических уравнений. В [5] для многомерных гиперболических уравнений с вырождением типа и порядка в цилиндрической области показана разрешимость задачи Дирихле, а в данной работе доказывается единственность её решения.

2 Постановка задачи и результат

Пусть Ω_β – цилиндрическая область евклидова пространства E_{m+1} точек $(x_1, ..., x_m, t)$, ограниченная цилиндром $\Gamma = \{(x,t) : |x| = 1\}$, плоскостями $t = \beta > 0$ и $t = 0$, где $|x|$ – длина вектора $x = (x_1, ..., x_m)$. Части этих поверхностей, образующих границу ∂D_β области D_β, обозначим через Γ_β, S_β, S_0 соответственно.

2010 Mathematics Subject Classification: 35R12.

В области D_β рассмотрим взаимно-сопряженные уравнения

$$Lu \equiv g_1(t)\Delta_x u - g_2(t)u_{tt} + \sum_{i=1}^{m} a_i(x,t)u_{x_i} + b(x,t)u_t + c(x,t)u = 0, \qquad (1)$$

$$L^*\upsilon \equiv g_1(t)\Delta_x \upsilon - g_2(t)\frac{\partial^2 \upsilon}{\partial t^2} - \sum_{i=1}^{m} a_i \upsilon_{x_i} - \bar{b}\upsilon_t + d\upsilon = 0, \qquad (1^*)$$

где $g_j(t) > 0$ при $t > 0$ и обращаются в нуль при $t = 0, g_j(t) \in C([0,\beta]) \cap C^2((0,\beta)),\ j = 1,2,\ \Delta_x-$ оператор Лапласа по переменным $x_1, ..., x_m,\ m \geq 2$, а $d(x,t) = c - \sum\limits_{i=1}^{m} a_{ix_i} - b_t - g_2''(t),\ \bar{b}(x,t) = b(x,t) - 2g_2'(t)$.

Уравнение (1) гиперболично при $t > 0$, а вдоль плоскости $t = 0$ имеет место вырождение его типа и порядка.

В дальнейшем нам удобно перейти от декартовых координат $x_1, ..., x_m, t$ к сферическим $r, \theta_1, ..., \theta_{m-1}, t,\ r \geq 0,\ 0 \leq \theta_1 < 2\pi,\ 0 \leq \theta_i \leq \pi,\ i = 2, 3, ..., m-1$.

В качестве многомерной задачи Дирихле рассмотрим следующую задачу.

Задача 1. Найти решение уравнения (1) в области D_β из класса $C^1(\bar{D}_\beta) \cap C^2(D_\beta)$, удовлетворяющее краевым условиям

$$u\Big|_{S_\beta} = 0, u\Big|_{\Gamma_\beta} = 0, u\Big|_{S_0} = 0. \qquad (2)$$

Пусть $\left\{Y_{n,m}^k(\theta)\right\}$ – система линейно независимых сферических функций порядка $n,\ 1 \leq k \leq k_n,\ (m-2)!n!k_n = (n+m-3)!(2n+m-3),\ W_2^l(S)$, $l = 0, 1, ...,$ – пространства Соболева.

Через $\widetilde{a}_{in}^k(r,t),\ a_{in}^k(r,t),\ \widetilde{b}_n^k(r,t),\ \widetilde{c}_n^k(r,t),\ \widetilde{d}_n^k(r,t),\ \rho_n^k$ обозначим коэффициенты ряда

$$f(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} f_n^k(r,t)Y_{n,m}^k(\theta)$$

соответственно функций $a_i(r,\theta,t)\rho(\theta),\ a_i\frac{x_i}{r}\rho,\ \bar{b}(r,\theta,t)\rho, c(r,\theta,t)\rho, d(r,\theta,t)\rho$, $\rho(\theta),\ i = 1, ..., m$.

Пусть $\dfrac{a_i(r,\theta,t)}{g_2(t)}, \dfrac{b(r,\theta,t)}{g_2(t)}, \dfrac{c(r,\theta,t)}{g_2(t)} \in W_2^l(D_\beta) \subset C(\bar{D}_\beta), l \geq m+1, i = 1,...,m$. Тогда справедлива

Теорема. *Если выполняется условие*

$$\cos \mu_{s,n}\beta' \neq 0, \;\; s = 1,2,..., \tag{3}$$

то решение Задачи 1 тривиальное, где $\mu_{s,n}$ *– положительные нули функции Бесселя первого рода* $J_{n+\frac{(m-3)}{2}}(z)$, $\beta' = \int\limits_0^\beta \sqrt{\frac{g_1(\xi)}{g_2(\xi)}})d\xi$, $n = 0,1,....$

3. Доказательство Теоремы. Рассмотрим задачу (1),(2) и докажем, что ее решение нулевое. Для этого сначала построим решение задачи Дирихле для уравнения (1*) с данными

$$v\Big|_{S_\beta \cup \Gamma_\beta} = 0, v\Big|_{S_0} = \tau(r,\theta) = \bar{\tau}_n^k(r)Y_{n,m}^k(\theta), \quad k = \overline{1,k_n}, \quad n = 0,1,..., \tag{4}$$

где $\tau_n^k(r) \in V$, V – множество функций $\tau(r)$ из класса $C^1((0,1)) \cap C^2((0,1))$.

Множество V плотно всюду в $L_2((0,1))$ ([6]). Решение задачи (1*),(4) в сферических координатах будем искать в виде

$$v(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} \bar{v}_n^k(r,t)Y_{n,m}^k(\theta), \tag{5}$$

где $\bar{v}_n^k(r,t)$– функции, которые будут определены ниже.

В сферических координатах уравнение (1*) имеет вид

$$L^*v \equiv g_1(v_{rr} + \frac{m-1}{r}v_r - \frac{\delta v}{r^2}) - \frac{\partial^2}{\partial t^2}(g_2 v) -$$

$$-\sum_{i=1}^{m} a_i(r,\theta,t)v_{x_i} - b(r,\theta,t)v_t + d(r,\theta,t)v = 0, \tag{6}$$

$$\delta \equiv -\sum_{j=1}^{m-1} \frac{1}{g_j sin^{m-j-1}\theta_j}\frac{\partial}{\partial \theta_j}\left(sin^{m-j-1}\theta_j \frac{\partial}{\partial \theta_j}\right),$$

$$g_1 = 1, \; g_j = (sin\theta_1...sin\theta_{j-1})^2, \; j > 1.$$

Известно ([7]), что спектр оператора δ состоит из собственных чисел $\lambda_n = (n + m - 2), n = 0, 1...$, каждому из которых соответствует k_n ортонормированных собственных функций $Y^k_{n,m}(\theta)$.

Подставив (5) в (6), а затем умножив полученное выражение на $\rho(\theta) \neq 0$ и проинтегрировав по единичной сфере H из E_m, для $\bar{v}^k_n$ получим ([4])

$$g_1(t)\rho^1_0\bar{v}^1_{0rr} - g_2(t)\rho^1_0\bar{v}^1_{0tt} + \left(\frac{m-1}{r}g_1(t)\rho^1_0 - \sum_{i=1}^{m}\right)\bar{v}^1_{0r} - \tilde{b}^1_0\bar{v}^1_{0t} + \tilde{d}^1_0\bar{v}^1_0 +$$

$$+\sum_{n=1}^{\infty}\sum_{k=1}^{k_n}\left\{g_1(t)\rho^k_n\bar{v}^k_{nrr} - g_2(t)\rho^k_n\bar{v}^k_{ntt} + \left(\frac{m-1}{r}g_1(t)\rho^k_n - \sum_{i=1}^{m}a^k_{in}\right)\bar{v}^k_{nr} - \tilde{b}^k_n\bar{v}^k_{nt} +\right.$$

$$\left.+\left[\tilde{d}^k_n - \lambda_n\frac{\rho^k_n}{r^2}g_1(t) + \sum_{i=1}^{m}(-\tilde{a}^k_{in-1} + na^k_{in})\right]\bar{v}^k_n\right\} = 0. \qquad (7)$$

Теперь рассмотрим бесконечную систему дифференциальных уравнений

$$g_1(t)\rho^1_0\bar{v}^1_{0rr} - g_2(t)\rho^1_0\bar{v}^1_{0tt} + \frac{(m-1)}{r}g_1(t)\rho^1_0\bar{v}^1_{0r} = 0, \qquad (8)$$

$$g_1(t)\rho^k_1\bar{v}^k_{1rr} - g_2(t)\rho^k_1\bar{v}^k_{1tt} + \frac{(m-1)}{r}\rho^k_1\bar{v}^k_{1r} - \frac{\lambda_1}{r^2}g_1(t)\rho^k_1\bar{v}^k_1 =$$

$$= -\frac{1}{k_1}\left(-\sum_{i=1}^{m}a^1_{i0}\bar{v}^1_{0r} - \tilde{b}^1_0\bar{v}^1_{0t} + \tilde{d}^1_0\bar{v}^1_0\right),\ n = 1,\ k = \overline{1, k_1}, \qquad (9)$$

$$g_1(t)\rho^k_n\bar{v}^k_{nrr} - g_2(t)\rho^k_n\bar{v}^k_{ntt} + \frac{(m-1)}{r}g_1(t)\rho^k_n\bar{v}^k_{nr} - \frac{\lambda_n}{r^2}g_1(t)\rho^k_n\bar{v}^k_n =$$

$$= -\frac{1}{k_n}\sum_{k=1}^{k_{n-1}}\left\{-\sum_{i=1}^{m}a^k_{in-1}\bar{v}^k_{n-1r} - \tilde{b}^k_{n-1}\bar{v}^k_{n-1t} +\right.$$

$$\left.+\left[\tilde{d}^k_{n-1} + \sum_{i=1}^{m}(-\tilde{a}^k_{in-2} + (n-1)a^k_{in})\right]\bar{v}^k_{n-1}\right\},\ k = \overline{1, k_n},\ n = 2, 3\ldots. \qquad (10)$$

Суммируя уравнение (9) от 1 до k_1, а уравнение (10) – от 1 до k_n, затем сложив полученные выражения вместе с (8), приходим к уравнению (7).

Отсюда следует, что если $\{\bar{v}_n^k\}$, $k=\overline{1,k_n}$, $n=0,1,...,$ – решение системы (8)–(10), то оно является и решением уравнения (7).

Нетрудно заметить, что каждое уравнение системы (8)–(10) можно представить в виде

$$g(t)(\bar{v}_{nrr}^k+\frac{(m-1)}{r}\bar{v}_{nr}^k-\frac{\lambda_n}{r^2}\bar{v}_n^k)-\bar{v}_{ntt}^k=f_n^k(r,t), \qquad (11)$$

где $g(t)=\dfrac{g_1(t)}{g_2(t)}$, $f_n^k(r,t)$ определяются из предыдущих уравнений этой системы, при этом $f_0^1(r,t)\equiv 0$.

Далее, учитывая ортогональность ([6]) систем сферических функций $Y_{n,m}^k(\theta)$ из краевого условия (4), в силу (5) будем иметь

$$\bar{v}_n^k(r,\beta)=\bar{v}_n^k(1,t)=0, \bar{v}_n^k(r,0)=\bar{\tau}_n^k(r), k=\overline{1,\,k_n},\ n=0,1\ldots. \qquad (12)$$

В [4] показано, что задача (11), (12) однозначна разрешима, если выполняется условие (3).

Решив задачу (8), (12) (n=0), а затем (9), (12) (n=1) и т.д., найдем последовательно все $\bar{v}_n^k(r,t), k=\overline{1,\,k_n},\ n=0,1\ldots.$

Таким образом, решение задачи (1*), (4) в виде (5) построено.

Учитывая формулу $2J'_\nu(z)=J_{\nu-1}(z)-J_{\nu+1}(z)$ ([9]), оценки ([8], [9])

$$J_\nu(z)=\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)+0\left(\frac{1}{z^{3/2}}\right),\ \nu\geq 0,$$

$$|k_n|\leq c_1 n^{m-2},\ \left|\frac{\partial^q}{\partial\theta_j^q}Y_{n,m}^k(\theta)\right|\leq c_2 n^{\frac{m}{2}-1+q},$$

$$c_1,c_2=const, j=\overline{1,m-1},\ q=0,1,\ldots,$$

а также ограничения на коэффициенты уравнения (1*), как в [3], [4] можно показать, что полученное решение (5) принадлежит классу $C^1(\bar{D}_\beta)\cap C^2(D_\beta)$.

Для взаимно-сопряженных операторов $\bar{L}$, $\bar{L}^*$

$$\bar{L}\equiv g(t)\Delta_x-\frac{\partial^2}{\partial t^2}+\sum_{i=1}^m \bar{a}_i\frac{\partial}{\partial x_i}+\tilde{b}\frac{\partial}{\partial t}+\bar{c},$$

$$\bar{L^*} \equiv g(t)\Delta_x - \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{m} \bar{a}_i \frac{\partial}{\partial x_i} - \tilde{b}\frac{\partial}{\partial t} + \bar{d},$$

$$\bar{L} = g_2(t)L, \bar{L^*} = g_2(t)L^*, g(t) = g_1(t)/g_2(t),$$

$\dfrac{a_i(r,\theta,t)}{g_2(t)} = \bar{a}_i(r,\theta,t), \dfrac{\bar{b}(r,\theta,t)}{g_2(t)} = \tilde{b}(r,\theta,t), \dfrac{c(r,\theta,t)}{g_2(t)} = \bar{c}(r,\theta,t), \dfrac{d(r,\theta,t)}{g_2(t)} = \bar{d}(r,\theta,t)$, $i = \overline{1,m}$, имеет место формула Грина ([10])

$$\int\limits_{D_\beta} g_2(t)(\upsilon Lu - uL^*\upsilon)dD_\beta = \int\limits_{D_\beta} (\upsilon\bar{L}u - u\bar{L^*}\upsilon)dD_\beta =$$

$$= \int\limits_{D_\beta} \left(\left[\upsilon\frac{\partial u}{\partial N} - u\frac{\partial \upsilon}{\partial N}\right) + u\upsilon Q\right] ds, \tag{13}$$

где

$$\frac{\partial}{\partial N} = g_1 t \sum_{i=1}^{m} \cos\left(N^\perp, x_i\right)\frac{\partial}{\partial x_i} - \cos\left(N^\perp, t\right)\frac{\partial}{\partial t}, Q =$$

$$\sum_{i=1}^{m} \bar{a}_i \cos\left(N^\perp, x_i\right) + \tilde{b}\cos\left(N^\perp, t\right),$$

а $N^\perp$ – внутренняя нормаль к границе ∂D_β.

Из (13), принимая во внимание однородные граничные условия (2) и условия (12), получим

$$\int\limits_{S_0} \tau(r,\theta)u_t(r,\theta,0)ds = 0. \tag{14}$$

Поскольку линейная оболочка системы функций $\{\bar{\tau}_n^k(r)Y_{n,m}^k(\theta)\}$ плотна в $L_2(S_0)$([6]), то из (14) заключаем, что $u_t(r,\theta,0) = 0\,\forall(r,\theta) \in S_0$.

Следовательно, в силу единственности решения задачи Коши ([10]): $\bar{L}u = 0,\ u(x,0) = 0,\ u_t(x,0) = 0$ вытекает, что $u(x,t) = 0\ \forall(x,t) \in D_\beta$.

Так как $g_2(t)Lu = \bar{L}u = 0$ и $g_2(t) > 0$ при t>0, то будем иметь $Lu = 0$ и $u(x,t) \equiv 0$ в D_β.

Таким образом, Теорема доказана.

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Maikotov M.N. UNIQUENESS OF SOLUTION OF DIRICHLET PROBLEM FOR MULTIDIMENSIONAL HYPERBOLIC EQUATIONS WITH DEGENERACY OF TYPE AND ORDER

Earlier, the author showed solvability of multidimensional hyperbolic equations with degeneracy of type and order in a cylindrical domain. In this paper we show singularity of classical solution of Dirichlet problem for multidimensional hyperbolic equations with degeneration of type and order.

Keywords. Uniqueness, degeneracy, cylindrical domain, system of functions, boundary conditions, coefficients of the series.

Майкотов М.Н. ТЕГІ МЕН РЕТІ БОЙЫНША ӨЗГЕШЕЛЕНГЕН КӨП ӨЛШЕМДІ ГИПЕРБОЛАЛЫҚ ТЕҢДЕУЛЕР ҮШІН ДИРИХЛЕ ЕСЕБІНІҢ ЖАЛҒЫЗДЫҒЫ

Автордың бұдан бұрынғы жұмысында цилиндрлік облыстағы тегі мен реті бойынша өзгешеленген көп өлшемді гиперболалық теңдеулер үшін Дирихле есебінің шешімділігі көрсетілген болатын. Бұл жұмыста цилиндрлік облыстағы тегі мен реті бойынша өзгешеленген көп өлшемді гиперболалық теңдеулер үшін Дирихле есебінің классикалық шешімінің жалғыздығы дәлелденген.

Кілттік сөздер. Жалғыздық, өзгешелену, цилиндрлік аймақ, функциялардың жүйелері, шеттік шарттар, қатардың коэффициенттері.

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Статья поступила в редакцию 17.10.2017

Математический журнал ISSN 1682–0525

2017. — Том 17, № 4. — С. 70–98.

МРНТИ 27.03.19, 27.03.66

SIGNATURE REDUCTION PROCEDURES AND THE UNIVERSAL CONSTRUCTION AS TRANSFORMATION METHODS IN FIRST-ORDER COMBINATORIAL APPROACH

M.G. Peretyat'kin

Annotation. The work develops proposed earlier first-order combinatorial approach that represents a conceptual basis for studying isomorphism type of generalized Tarski-Lindenbaum algebras of predicate calculi of finite rich signatures under finitary and infinitary semantic layers. Technical conditions specifying status of the finite signature reduction procedures and the universal construction are given. Two kinds of semantic types are introduced, and operations on the types are defined within the first-order combinatorial approach.

Keywords. First-order logic, model-theoretic property, Tarski-Lindenbaum algebra, semantic type, signature reduction procedure, universal construction of finitely axiomatizable theories.

Results of the works [1] and [2] describe some special methods of constructing computable isomorphisms between the Tarski-Lindenbaum algebras of predicate calculi $PC(\sigma_1)$ and $PC(\sigma_2)$ of finite rich signatures σ_1 and σ_2. Finite-to-finite signature reduction procedure is involved in the main construction of the work [2]; it would be natural to call these transformations of theories as methods of *finitary first-order combinatorics*. On the other hand, an available version of the universal construction of finitely axiomatizable theories is involved in the proof of the main statement of the work [1]; thus, it would be natural to call these transformations of theories as methods of *infinitary first-order combinatorics*. It is important that the pointed out classes of transformations of theories preserve definite semantic layers of model-theoretic properties. Methods of finitary first-order combinatorics preserve a semantic

2010 Mathematics Subject Classification: 03B10.

Funding: The work was supported by grant (0665/GF4) of Ministry of Education and Science of the Republic of Kazakhstan (2015-2017).

layer called *finitary*, while methods of infinitary first-order combinatorics preserve a semantic layer called *infinitary*.

The works [3], [4], and [5] represent a conceptual framework of the first-order combinatorics. The main goal of this approach is to characterize the structure of the Tarski-Lindenbaum algebras of predicate calculi of finite rich signatures. In this paper, we present key technical specifications for the first-order combinatorial approach. The status of the finite signature reduction procedures and the universal construction of finitely axiomatizable theories in the framework of the first-order combinatorial approach is specified. The concept of a semantic type is introduced, and the most important properties of such types are considered in the context of the first-order finitary and infinitary combinatorial approach.

Preliminaries. We consider theories in *first-order predicate logic* with *equality* and use general concepts of model theory, algorithm theory, constructive models and Boolean algebras that can be found in [6], [7], and [8]. Generally, *incomplete* theories are considered. In this work, we consider just signatures admitting Godel's numberings of formulas. Such a signature is called *enumerable*. A finite signature is called *rich* if it contains at least one n-ary predicate or function symbol for $n \geqslant 2$, or two symbols of unary functions.

For a constant symbol c that is not in σ, statement "all σ-symbols are defined c-trivially"means the following set of formulas:

(a) $(\forall x_1 \dots x_n)\neg P(x_1, \dots, x_n),\ P^n \in \sigma,$
(b) $(\forall x_1 \dots x_m)\big(f(x_1, \dots, x_m) = x_1\big),\ f^m \in \sigma,$
(c) $a = c$, *for each constant symbol* $a \in \sigma$.

For a signature σ and a unary predicate $U^1 \notin \sigma$, statement "all σ-symbols are defined trivially outside $U(x)$" means the following set of formulas:

(a) $\neg U(x_i) \to \neg P(x_1, \dots, x_i, \dots, x_n),\ P^n \in \sigma,\ 1 \leqslant i \leqslant n,$
(b) $\neg U(x_j) \to f(x_1, \dots, x_j, \dots, x_m) = x_1,\ f^m \in \sigma,\ 1 \leqslant j \leqslant m.$

The following notations are used: $FL(\sigma)$ is the set of all formulas of signature σ, $FL_k(\sigma)$ is the set of all formulas of signature σ with free variables $x_0, \dots, x_{k-1}$, $SL(\sigma)$ is the set of all sentences of signature σ. By GR, we denote the Graph theory of signature $\sigma_{GR} = \{\Gamma^2\}$ defined by axioms $(\forall x)\neg\Gamma(x,x)$ and $(\forall x)(\forall y)[\Gamma(x,y) \leftrightarrow \Gamma(y,x)]$, while *GRE* denotes an extension of *GR* defined by extra axioms $(\exists x,y)\Gamma(x,y)$ and $(\exists x,y)\big[(x \neq y) \wedge \neg\Gamma(x,y)\big]$. For a theory, c.a. means *computably axiomatizable*, while f.a. means *finitely axiomatizable*.

Let σ be a signature and Σ a subset of $SL(\sigma)$. We denote by $[\Sigma]^\sigma$ a theory of signature σ generated by Σ as a set of its axioms. There is one more version of the definition. Let $\Sigma \subseteq SL(\sigma)$ be a set of sentences. We denote by $[\Sigma]^\star$ a theory of signature $\sigma' \subseteq \sigma$ generated by Σ as a set of its axioms, where σ' contains only those symbols from σ, which occur in formulas of Σ. By σ^∞, we denote a fixed (very large) enumerable signature containing countably many constant symbols, symbols of propositional variables, and predicate and function symbols of each arity $n \geqslant 1$. It is supposed that each considered signature σ is a part of the universal signature σ^∞. By $\mathfrak{S}$, we denote the set of all possible enumerable signatures $\sigma \subseteq \sigma^\infty$.

We use a fixed Godel's numbering Φ_k, $k \in \mathbb{N}$, for the set of sentences of a fixed signature σ, and Godel's numbering Φ_k^∞, $k \in \mathbb{N}$, for the set of sentences of the universal enumerable signature σ^∞. By using Post's numbering W_n, $n \in \mathbb{N}$, for the family of all computably enumerable sets, we organize an effective numbering for the class of all computably axiomatizable theories. Two versions of indices are possible. The first one presents c.e. indices of c.a. theories of an enumerable or finite signature σ. If a theory T of signature σ is defined by axioms $\{\Phi_i \mid i \in W_m\}$, the number m is called a *computably enumerable index* or simply *c.e. index* of T. The second version represents weak indices of theories of different enumerable signatures $\sigma \subseteq \sigma^\infty$. For a given $m \in \mathbb{N}$, we consider a set of axioms $\Sigma = \{\Phi_i^\infty \mid i \in W_m\}$ and construct theory $T = [\Sigma]^\star$. The number m is called a *weak computably enumerable index* or simply *weak c.e. index* of the theory T. As for finitely axiomatizable theories, any such theory F is defined by a finite system A of axioms and therefore, by a single formula Φ which is a conjunction of formulas in A. If a f.a. theory F of signature σ is defined by an axiom Φ_m, the number m is called Godel's number or simply *strong index* of F. For a given $m \in \mathbb{N}$, we consider an f.a. theory $F = [\Phi_m]^\star$. This number m is called a *universal Godel's number* or simply *universal strong index* of the theory F. By $T^\sigma\{n\}$, we denote a theory of signature σ with c.e. index n, while $T^\star\{n\}$ is a theory with weak c.e. index n. Furthermore, by $F^\sigma\{n\}$ we denote a f.a. theory of signature σ with Godel's number n, while $F^\star\{n\}$ is an f.a. theory with weak strong index n.

Initial definitions concerning main concepts of finitary first-order combinatorics, are given in [3] and [5]. We use a definition of the operation of a Cartesian extension of a theory and a Cartesian interpretation, as well as a scheme of layers relevant in the combinatorial approach, in the work [9]. By *ACL*, we denote the layer of model-theoretic properties preserved by all possible Cartesian interpretations between computably axiomatizable theories.

LEMMA 0.1. *Given a theory T of an enumerable signature σ and a sequence of $\exists \cap \forall$-formulas $\varkappa \in KC(\sigma)$. Special Cartesian-quotient interpretation $I_{T,\varkappa} : T \rightarrowtail T\langle\varkappa\rangle$ is effective, model-bijective, and isostone. In particular, the interpretation $I_{T,\varkappa}$ determines a computable isomorphism $\mu_{T,\varkappa} : \mathcal{L}(T) \to \mathcal{L}(T\langle\varkappa\rangle)$ between the Tarski-Lindenbaum algebras; moreover, it preserves semantic layer ACL.*

PROOF. By applying Definition 1.A(C) and Lemma 1.1 from [9]. □

REMARK 0.2. In this paper, we systematically use effective methods based on computable procedures for any object involved in our constructions. For this, we introduce either c.e. indices or Gödel numbers for all types of objects involved in the constructions. Moreover, the object yielded by a construction is always meant as defined by a single c.e. index coding a computable presentation of the object.

1 Finitary combinatorics and the finite signature reduction procedure

First, we formulate the main statement in a compact form:

THEOREM 1.1 [Finite-to-finite signature reduction statement: a compact form]. *Given two finite rich signatures σ_1 and σ_2. Effectively in their Gödel numbers, it is possible to construct a sentence ψ of signature σ_2 and a sequence of $\exists \cap \forall$-formulas $\varkappa = \langle \varphi_1^{m_1}, ..., \varphi_s^{m_s} \rangle$ of signature σ_2 together with an algebraic isomorphism $PC(\sigma_1) \approx_a PC(\sigma_2)[\psi]\langle\varkappa\rangle$.*

Now, we give an extended form of the same statement.

THEOREM 1.2 [Finite signature reduction procedure: a standard form]. *It is possible to determine in a regular way the operator of the following form*

$$\text{Redu} : \; T\mathfrak{S}^{\phi} \times \sigma FinRich \;\to\; T\mathfrak{S}^{\phi} \times ICartes^{\phi},$$

called the finite signature reduction procedure, where $\sigma FinRich$ is the set of all finite rich signatures $\sigma \subseteq \sigma^\infty$, $T\mathfrak{S}^\phi$ is the set of all theories of any finite signatures $\sigma \subseteq \sigma^\infty$, and $ICartes^\phi$ is the set of all Cartesian interpretations between theories of finite signatures. Moreover, all requirements listed below are satisfied.

Let T be a theory of a finite signature τ and σ be an arbitrary finite rich signature. Applying the mapping Redu *we obtain*

$$\mathrm{Redu}(T, \sigma) = (S, I),$$

where S is a theory of signature σ, while I is an interpretation of T in S, such that the following assertions are satisfied:

`Reference_Block` (1.1)

(a) *I is an $\exists \cap \forall$-presentable Cartesian interpretation of theories (thereby, the interpretation I defines a computable isomorphism $\mu : \mathcal{L}(T) \to \mathcal{L}(S)$ preserving model-theoretic properties of the semantic layer ACL),*

(b) *T is c.a. $\Leftrightarrow$ S is c.a.; in the case when T is a c.a. theory, c.e. indices of both S and I are found effectively in a pair of parameters consisting of a c.e. index of the input theory T and a Gödel number of the target finite rich signature σ,*

(c) *T is f.a. $\Leftrightarrow$ S is f.a.; in the case when T is a f.a. theory, both a Gödel number of S and a c.e. index of I are found effectively in a pair of parameters consisting of Gödel numbers of the input theory T and the target finite rich signature σ.*

`End_Ref`

Proofs of Theorem 1.1 and Theorem 1.2 are given in [9].

By specification, starting from a pair of input parameters (T, σ), the procedure Redu yields a theory S together with an interpretation I defining a computable isomorphism $\mu : \mathcal{L}(T) \to \mathcal{L}(S)$ between the Tarski-Lindenbaum algebras. A simpler record $S = \mathrm{Redu}(T, \sigma)$ is also possible, assuming that the interpretation I and isomorphism μ is omitted in the context.

Now, we pass to some generalization of the signature reduction statement. It represents a special method of reduction of finitely axiomatizable theories to the graph theory *GRE* controlled by a level parameter.

The following technical statement takes place:

Lemma 1.3 [Parameterized Signature Reduction Statement]. *Let* $\sigma_{GR} = \{\Gamma^2\}$ *be the signature of the graph theory* GRE. *There exist an effective sequence of sentences*

$$\theta_k, \ k \in \mathbb{N} \setminus \{0,1\},$$

of the signature σ_{GR} *and a procedure* RedLev *with two parameters*

$$\mathrm{RedLev}(e, T), \ e \in \mathbb{N}, \ T \text{ is a theory of a finite signature},$$

satisfying the following properties. Given an integer parameter $e \geqslant 2$ *(called the level parameter) and a finitely axiomatizable theory* F *of a finite signature* σ. *Effectively in* e *and* F, *a pair of objects*

$$(H, I) = \mathrm{RedLev}(e, F)$$

is constructed by the procedure RedLev *of the following form:*

H *is a finitely axiomatizable theory of signature* $\sigma_{GR} = \{\Gamma^2\}$,
I *is a Cartesian interpretation of* F *in* H.

Particularly, I *determines a computable isomorphism* $\mu : \mathcal{L}(F) \to \mathcal{L}(H)$ *preserving model-theoretic properties of Cartesian semantic layer* ACL *(the more, any smaller layer* $L \subseteq ACL$*).*

Moreover, the following assertions hold:

`Reference_Block` (1.2)

(a) $GRE \vdash \theta_{k+1} \to \theta_k$, *for all* $k \in \mathbb{N} \setminus \{0,1\}$,

(b) H *is an extension of the theory* $GRE_e = GRE \cup \{\theta_e\}$,

(c) $H \vdash \theta_e \wedge \neg\theta_{e+1}$,

(d) $GRE_\omega = GRE \cup \big\{\theta_2, \theta_3, .., \theta_k, ... \ ; \ k \in \mathbb{N} \setminus \{0,1\}\big\}$ *is a complete decidable theory without finite models.*

`End_Ref`

Proof. Use construction presented in main stage fP-to-Graph in the proof for signature reduction procedure in Theorem 1.2. It is possible to modify forms of coding configurations such that they become depending on the level parameter e, cf. Section 5 in [2], or Section 4.2 in [10]. □

2 Infinitary combinatorics and fixation of the status of the universal construction

By *MQL*, we denote the *infinitary* semantic layer consisting of model-theoretic properties that are preserved by any quasiexact interpretation between c.e. theories. The layer *MQL* is controlled by a standard version $\mathbb{F}\mathbb{U}$ of the universal construction of finitely axiomatizable theories, cf. [11].

Description of the construction $\mathbb{F}\mathbb{U}$ represents a sophisticated text that turns out to be difficult for reading and understanding. There are some weaker versions of the universal construction with a simplified or even omitted rigidity mechanism, thus, controlling smaller layers of model-theoretic properties. However, the difficulty of studying these constructions is practically the same as in the case of construction $\mathbb{F}\mathbb{U}$. There is even a weaker version $\mathbb{F}\mathbb{C}^{\circ}$ of the universal construction that is described in [12]. Construction $\mathbb{F}\mathbb{C}^{\circ}$ is obtained as a routine corollary of the canonical construction presented in [11, Ch. 3]. Therefore, $\mathbb{F}\mathbb{C}^{\circ}$ is said to be the *canonical-mini* construction, or *universal-under-canonical* construction. Canonical-mini construction has a standard formulation of the universal construction supporting a relatively small layer of model-theoretic properties. At the same time, the canonical-mini construction is significantly easier to understand than any normal version of the universal construction. Moreover, while studying canonical-mini construction no necessity to be familiar with the technically complicated definition of a quasiexact interpretation. On the other hand, a particular method of compact binary trees is required for the canonical-mini construction, while any normal version of the universal construction does without this method. Finally, we notice that the Hanf construction $\mathbb{H}$ (cf. either [13, Th. 1] or [11, Sec. 6.1]) can also be considered as a (weakest) release of the universal construction controlling an empty semantic layer of model-theoretic properties.

In this paper, we suppose that a fixed release of the universal construction is accepted, denoted by $\mathbb{U}$, that can control a sublayer

$$MQL \subseteq MQL \tag{2.1}$$

of the infinitary layer *MQL*. Moreover, we also suppose that the construction $\mathbb{U}$ is given by its *primitive form* $\hat{\mathbb{U}}$ without the effectiveness requirement in the passage from an input computably axiomatizable theory to the target finitely axiomatizable theory. In the paper, statements depending on the universal construction are marked with $[\mathbb{U}]$.

The pointed out primitive form $\hat{\mathbb{U}}$ of the construction $\mathbb{U}$ is presented by:

STATEMENT 2.1 [Generic universal construction: a primitive form]. *The following assertion holds for the sublayer* MQL *of the layer* MQL:

$$(\forall \text{ } c.a. \text{ } theory \text{ } T)(\exists \text{ } f.a. \text{ } theory \text{ } F)\big[T \equiv_{MQL} F\big], \tag{2.2}$$

where $T \equiv_{MQL} F$ *means that there is a computable isomorphism* $\mu : \mathcal{L}(T) \to \mathcal{L}(F)$ *between the Tarski-Lindenbaum algebras preserving all model-theoretic properties within the pointed out layer* MQL*.*

A more common *normal* formulation of the universal construction $\mathbb{U}$:

STATEMENT 2.2 [Generic universal construction: a normal form]. *Given an arbitrary computably axiomatizable theory* T *and a finite rich signature* σ*. Effectively in a weak c.e. index of* T *and a Gödel number of* σ*, one can construct a finitely axiomatizable theory* $F = \mathbb{U}(T, \sigma)$ *of signature* σ *together with a computable isomorphism* $\mu : \mathcal{L}(T) \to \mathcal{L}(F)$ *between the Tarski-Lindenbaum algebras preserving all model-theoretic properties within the layer* $MQL \subseteq MQL$*.*

The case $MQL = \varnothing$ in Statement 2.1, as well as in Statement 2.2, corresponds to the *Hanf construction* (called earlier as *Hanf's Localized Statement*), [13]. Obviously, the Hanf construction is a particular case of the universal construction corresponding to the case of an empty layer of the controlled model-theoretic properties.

LEMMA 2.3. *The following assertions hold*:

(a) *For a fixed semantic layer* $MQL \subseteq MQL$*, primitive form* (2.2) *of the universal construction is an immediate consequence of its normal form with this layer* MQL*.*

(b) *Having any version of the universal construction in a primitive form* (2.2) *that controls a semantic layer* $MQL \subseteq MQL$*, we can deduce a normal form of the universal construction presented in Statement* 2.2 *with this layer* MQL*, restoring by that the missing effectiveness requirement, as well as the possibility to have a given finite rich signature.*

PROOF. Part (a) is obvious, while proof of Part (b) is given in Section 4.

Earlier, we fixed a sublayer (2.1) of the infinitary semantic layer that is controlled by a primitive form (2.2) of the universal construction. Having this convention accepted, by virtue of Lemma 2.3, we will also have Statement 2.2 presenting a normal version of the universal construction with the layer (2.1).

3 Properties of effective numberings of the classes of theories

As a result, we have introduced c.e. indices and Gödel numbers for the following classes of theories:

`Reference_Block` (3.1)

- (a) $F^{\sigma}\{k\}$, $k \in \mathbb{N}$, *the set of all f.a. theories of a fixed finite signature* σ,
- (b) $F^{\star}\{k\}$, $k \in \mathbb{N}$, *the set of all f.a. theories of all possible finite signatures*,
- (c) $T^{\tau}\{k\}$, $k \in \mathbb{N}$, *the set of all c.a. theories of a fixed enumerable or finite signature* τ,
- (d) $T^{\star}\{k\}$, $k \in \mathbb{N}$, *the set of all c.a. theories of all possible enumerable signatures.*

`End_Ref`

Lemma 3.1. *The following assertions hold*:

(a) *collection* (3.1)(a) *represents a computable sequence of all possible, up to an algebraic isomorphism, finitely axiomatizable theories of a fixed finite signature* σ.

(b) *collection* (3.1)(b) *represents a computable sequence of all possible, up to an algebraic isomorphism, finitely axiomatizable theories of arbitrary finite signatures.*

Proof. Immediately. □

Lemma 3.2. *The following assertions hold*:

(a) *collection* (3.1)(c) *represents a computable sequence of all possible, up to an algebraic isomorphism, computably axiomatizable theories of a fixed enumerable or finite signature* τ.

(b) *collection* (3.1)(d) *represents a computable sequence of all possible, up to an algebraic isomorphism, computably axiomatizable theories of arbitrary enumerable signatures.*

Proof. Immediately. □

An effective reduction of Gödel numbers to other types of indices is possible.

Lemma 3.3. *There are general computable functions* $f_1(\sigma, x)$, $f_2(\tau, \sigma, x)$, *and* $f_3(x)$ *with* $\tau \in \sigma Enum$, $\sigma \in \sigma Fin$, *and* $x \in \mathbb{N}$, *such that we have for all* $n \in \mathbb{N}$:

(a) $F^{\sigma}{}_{\{n\}} \approx_a F^{\star}{}_{\{f_1(\sigma,n)\}}$,
(b) $F^{\sigma}{}_{\{n\}} \approx_a T^{\tau}{}_{\{f_2(\tau,\sigma,n)\}}$, *whenever* $\sigma \leqslant \tau$ (*see definition for* $\leqslant$ *in* [9]),
(c) $F^{\star}{}_{\{n\}} \approx_a T^{\star}{}_{\{f_3(n)\}}$;

moreover, c.e. indices of algebraic isomorphisms in (a), (b), and (c) are found effectively in τ, σ, *and* n.

Proof. By definition, finitely axiomatizable theory $F^{\sigma}{}_{\{n\}}$ is defined by an axiom $\varPhi_n$. Effectively in σ and n one can find an integer m such that $\varPhi^{\infty}_m$ represents the same theory as an axiom in another system of numbering of theories. Thereby, we have obtained instructions for the function $f_1(\sigma,n)$. Parts (b) and (c) are proved by similar schemes.

Lemma 3.3 is proved. □

In this subsection, we consider some computational properties for the numberings of the classes of c.a. and f.a. theories introduced in (3.1).

Lemma 3.4 [Effective cylinder property]. *Each of the numberings (a), (b), (c), and (d) in* (3.1) *is cylindric. More precisely, there are total computable functions* $f_k(n,i)$, $g_k(n,i,x)$, $k = 1,2,3,4$, *satisfying for all* n, i:

$$f_k(n,0) = n,\ \ f_k(n,i) < f_k(n,i+1),\ \ k = 1,2,3,4,$$

such that, the following properties are held for all $n, i \in \mathbb{N}$ *and all signatures* $\tau \in \sigma Enum$ *and* $\sigma \in \sigma FinRich$:

(a) $F^{\sigma}{}_{\{n\}} \approx_a F^{\sigma}{}_{\{f_1(n,i)\}}$; *moreover, the function* $\lambda x g_1(n,i,x)$ *represents this isomorphism*,

(b) $F^{\star}{}_{\{n\}} \approx_a F^{\star}{}_{\{f_2(n,i)\}}$; *moreover, the function* $\lambda x g_2(n,i,x)$ *represents this isomorphism*,

(c) $T^{\tau}{}_{\{n\}} \approx_a T^{\tau}{}_{\{f_3(n,i)\}}$; *moreover, the function* $\lambda x g_3(n,i,x)$ *represents this isomorphism*,

(d) $T^{\star}{}_{\{n\}} \approx_a T^{\star}{}_{\{f_4(n,i)\}}$; *moreover, the function* $\lambda x g_4(n,i,x)$ *represents this isomorphism*.

Proof. (a) Let σ be a finite rich signature and $\varPhi_i$, $i \in \mathbb{N}$, be a Gödel numbering of $SL(\sigma)$. By definition, formula $\varPhi_n$ is an axiom of theory $F^{\sigma}{}_{\{n\}}$. Consider the following formula

$$\varPhi^{(k)} = \varPhi_n \wedge (\exists x)\big[\underbrace{(x = x) \wedge (x = x) \wedge \ldots \wedge (x = x)}_{k\ times}\big]. \qquad (3.2)$$

Choose k such that Gödel number m of formula (3.2) satisfies $m > n$. Then, theory $F^{\sigma}{}_{\{m\}}$ defined by formula (3.2) as an axiom coincides with $F^{\sigma}{}_{\{n\}}$. Based on this construction, we obtain instructions for computing functions f_1 and g_1. Part (b) is proved by a similar method. Now, we pass to Part (c). By definition, $\{\Phi_i \mid i \in W_n\}$ is a set of axioms of theory $T^{\sigma}{}_{\{n\}}$. Based on the known properties of Post's numbering of c.e. sets, we can effectively find $m > n$ such that $W_m = W_n$, thus, theory $T^{\sigma}{}_{\{m\}}$ will coincide with $T^{\sigma}{}_{\{n\}}$. Based on this method, we can construct the functions required in (c). Part (d) can be proved by a similar method. □

In the following statement, for simplicity, we count that signatures τ and σ are parameters that are given by a c.e. index and, respectively, by a Gödel number.

Lemma 3.5 [Effective bijection property]. *There are computable functions p_i, f_i, $i = 1, 2, 3$, (whose arguments are seen below) satisfying the following properties for all $\tau \in \sigma EnumRich$, $\sigma \in \sigma FinRich$, and $n, x \in \mathbb{N}$:*

(a) $\lambda x p_1(\sigma, x) : \mathbb{N} \to \mathbb{N}$ *is a permutation of* $\mathbb{N}$ *for all* σ,

(a′) $T^{\star}{}_{\{n\}} \equiv_{MQL} F^{\sigma}{}_{\{p_1(\sigma,n)\}}$; *moreover,* $\lambda x f_1(\sigma, n, x)$ *represents a computable isomorphism* $\mu : \mathcal{L}(T^{\star}{}_{\{n\}}) \to \mathcal{L}(F^{\sigma}{}_{\{p_1(\sigma,n)\}})$ *of this similarity,*

(b) $\lambda x p_2(\tau, \sigma, x) : \mathbb{N} \to \mathbb{N}$ *is a permutation of* $\mathbb{N}$ *for all* τ, σ,

(b′) $T^{\tau}{}_{\{n\}} \equiv_{MQL} F^{\sigma}{}_{\{p_2(\tau,\sigma,n)\}}$; *moreover,* $\lambda x f_2(\tau, \sigma, n, x)$ *represents a computable isomorphism* $\mu : \mathcal{L}(T^{\tau}{}_{\{n\}}) \to \mathcal{L}(F^{\sigma}{}_{\{p_2(\tau,\sigma,n)\}})$ *of this similarity,*

(c) $\lambda x p_3(\sigma, x) : \mathbb{N} \to \mathbb{N}$ *is a permutation of* $\mathbb{N}$ *for all* σ,

(c′) $F^{\star}{}_{\{n\}} \equiv_{ACL} F^{\sigma}{}_{\{p_3(\sigma,n)\}}$; *moreover,* $\lambda x f_3(\sigma, n, x)$ *represents a computable isomorphism* $\mu : \mathcal{L}(F^{\star}{}_{\{n\}}) \to \mathcal{L}(F^{\sigma}{}_{\{p_3(\sigma,n)\}})$ *of this similarity.*

Proof. (a) A passage from $F^{\sigma}{}_{\{n\}}$ to $T^{\star}{}_{\{m\}}$ is provided by Lemma 3.3(a,c). A back passage is ensured by an accepted version of the universal construction controlling semantic layer (2.1). Lemma 3.4 provides cylindric properties for these numberings. By applying method of proof of the known Myhill Theorem in algorithm theory, [6, Sec. 7.4], we will construct the demanded permutation. The obtained permutation preserves semantic layer MQL because we have applied Statement 2.2 of the universal construction.

(b) A passage from $F^{\sigma}{}_{\{n\}}$ to $T^{\tau}{}_{\{m\}}$ is provided by Lemma 3.3(b); possible, finite signature reduction procedure should be used in addition. A back passage is ensured by an accepted version of the universal construction. Based on cylindric properties, by applying Myhill's method, we can construct the required permutation. The obtained permutation preserves semantic layer MQL because we have applied Statement 2.2 for the universal construction.

(c) A passage from $F^{\sigma}{}_{\{n\}}$ to $F^{\star}{}_{\{m\}}$ is provided by Lemma 3.3(a). A back passage is ensured by the finite-to-finite signature reduction procedure. Based on cylindric properties, by applying Myhill's method, we can construct the required permutation. Obtained passage preserves semantic layer ACL because we have applied Theorem 1.1 for the finite signature reduction procedure in the transformation. □

4 Effectiveness of the universal construction

Now, we pass to the Proof of Part (b) of Lemma 2.3.

First, we introduce an operation with a sequence of theories. We use sequence $T^{\star}{}_{\{n\}}$, $n \in \mathbb{N}$, including all, up to an algebraic isomorphism, c.a. theories, cf. (3.1)(d). Let $T^{\star}{}_{\{n\}}$ has signature σ_n. It is possible to assume that $\sigma_n \cap \sigma_k = \varnothing$ for all n, k such that $n \neq k$. Consider the following new signature

$$\sigma' = \{\mathcal{Z}_i^0 \mid i \in \mathbb{N}\} \cup \{U^1, c\} \cup \sigma_0 \cup \sigma_1 \cup \ldots \cup \sigma_k \cup \ldots,$$

where $\mathcal{Z}_i^0$, $i \in \mathbb{N}$, are symbols of nulary predicates (propositional variables). It is assumed that the symbols U, c, and $\mathcal{Z}_i$, $i \in \mathbb{N}$, do not belong to $\sigma_0 \cup \sigma_1 \cup \ldots \cup \sigma_k \cup \ldots$.

We are going to construct a theory $T^u_{c.a.}$ of signature σ' called a *simplest direct product of the sequence* $T^{\star}{}_{\{n\}}$, $n \in \mathbb{N}$, denoted

$$T^u_{c.a.} = \breve{\bigotimes}^{[EQ]}_{n \in \mathbb{N}} T^{\star}{}_{\{n\}}. \tag{4.1}$$

For comparison purposes, we use an alternative notation $T^{EQ}_{c.a.}$ for this theory. Theory $T^u_{c.a.}$ is defined by the following set of axioms (Ax-0):

1°. $U(x) \leftrightarrow (x \neq c)$,

2°. $(\exists x) U(x)$,

3°. $\mathcal{Z}_n \to \neg \mathcal{Z}_k$, $n, k \in \mathbb{N}$, $n \neq k$,

4°. $\mathcal{Z}_n \to$ (*all axioms of* $T^{\star}{}_{\{n\}}$ *are satisfied on* $U(x)$), $n \in \mathbb{N}$,

5°. $\mathcal{Z}_n \to$ (*outside* $U(x)$, σ_n*-symbols are defined trivially*), $n \in \mathbb{N}$,

6°. $\neg \mathcal{Z}_k \to$ (*all* σ_k*-symbols are defined* c*-trivially*), $k \in \mathbb{N}$.

Mention that, the term "defined c-trivially" is explained in Preliminaries.

Remark 4.0. Later (in a further article devoted the combinatorial approach) we are planning to describe a common version of the operation of a direct product of a sequence of theories. The version we present in (4.1) via (Ax-0) plays the role of the simplest realization of a natural idea to link a sequence of theories together. Mnemonic entry $\breve{\bigotimes}_{i\in\mathbb{N}}^{[EQ]}(...)$ for this operation can be applied to an arbitrary sequence of theories, although we are interested with just the case presented in (4.1).

Lemma 4.1. *The following assertions hold:*

(a) *theory* $T^u_{c.a.} = \breve{\bigotimes}_{n\in\mathbb{N}}^{[EQ]} T^{\star}{}_{\{n\}}$ *is computably axiomatizable*;

(b) *for any* $n \in \mathbb{N}$, *theory* $T^u_{c.a.} \cup \{\mathcal{Z}_n\}$ *is algebraically isomorphic to a singleton extension* $T^{\star}{}_{\{n\}}\langle c\rangle$ *of the theory* $T^{\star}{}_{\{n\}}$;

(c) *there is a computable isomorphism* $\mu_n : \mathcal{L}(T^{\star}{}_{\{n\}}) \to \mathcal{L}(T^u_{c.a.} \cup \{\mathcal{Z}_n\})$ *preserving model-theoretic properties within the semantic layer ASL.*

Proof. Part (a) is a consequence of the fact that the sequence $T^{\star}{}_{\{n\}}$, $n \in \mathbb{N}$, is computable. Part (b) is checked immediately. Statement of Part (c) is a consequence of Part (b). □

Now, we turn immediately to prove Part (b) of Lemma 2.3.

Applying primitive form (2.2) of the universal construction to the theory $T^u_{c.a.}$, we find a finitely axiomatizable theory $F^* = \hat{\mathbb{U}}(T^u_{c.a})$ together with a computable isomorphism

$$\mu^* : \mathcal{L}(T^u_{c.a.}) \to \mathcal{L}(F^*) \tag{4.2}$$

preserving model-theoretic properties of the accepted infinitary layer *MQL*. Denote $\hat{\mathcal{Z}}_i = \mu^*(\mathcal{Z}_i)$, $i \in \mathbb{N}$. The effectiveness requirement is obtained as an immediate consequence of the universality property for theory $T^u_{c.a.}$ stated in Parts (a)–(c) of Lemma 4.1. Namely, we have to perform the following chain of transformations:

$$T^{\star}{}_{\{n\}} \overset{\alpha}{\mapsto} T^u_{c.a.}+\{\mathcal{Z}_n\} \overset{\beta}{\mapsto} \underbrace{\hat{\mathbb{U}}(T^u_{c.a})+\{\hat{\mathcal{Z}}_n\}}_{S} \overset{\gamma}{\mapsto} \mathrm{Redu}(S,\sigma), \tag{4.3}$$

where $\mathrm{Redu}(S,\sigma)$ is an application of the finite-to-finite signature reduction procedure. By Lemma 4.1(c), passage α in chain (4.3) defines a computable isomorphism between the Tarski-Lindenbaum algebras preserving properties of semantic layer $ASL \supseteq MQL$, while passage γ, by Theorem 1.1, defines a computable isomorphism preserving semantic layer $ACL \supseteq MQL$. As for passage β, by construction, it is obtained by a restriction from isomorphism (4.2), thus, β also defines a computable isomorphism between corresponding Tarski-Lindenbaum algebras preserving the layer MQL. It is possible to check that all parts involved in the chain (4.3) are built effectively in n and σ yielding a finitely axiomatizable theory fitting to formulation of Statement 2.2. Thereby, the summary transformation (4.3) can play the role of a normal form of the universal construction preserving the semantic layer MQL.

Part (b) of Lemma 2.3 is proved. □

5 Introduction in semantic types

From the point of view of a semantic layer L, any computably axiomatizable theory T can be characterized via a 3-tuple $\not\mathcal{L}(T) = \big(\mathcal{L}(T), \gamma, \xi\big)$ where $\big(\mathcal{L}(T), \gamma\big)$ is the Tarski-Lindenbaum algebra of theory T with Godel's numbering γ, while ξ is a mapping from Stone space $St(\mathcal{L}(T))$ in the power-set $\mathcal{P}(L) = \{K | K \subseteq L\}$ which is defined as follows: for any theory T' from $St(\mathcal{L}(T))$ we put

$$\xi(T') = \{p \in L \mid T' \textit{ has the property } p\}.$$

Actually, so defined tuple $(\mathcal{L}(T), \gamma, \xi)$ represents a complete abstract description of theory T from the point of view of semantic layer L. This tuple $\not\mathcal{L}(T)$ is called the generalized Tarski-Lindenbaum algebra of theory T under semantic layer L.

Generalizing the situation, we introduce a special class of objects for presentation of isomorphism types of the generalized Tarski-Lindenbaum algebras under semantic layer L of model-theoretic properties. Namely, we consider an arbitrary 3-tuple of the form $\mathfrak{B} = (\mathcal{B}, \nu, \xi)$, where $(\mathcal{B}, \nu)$ is a computably enumerable Boolean algebra, while ξ is a mapping from Stone space $St(\mathcal{B})$ in the power-set $\mathcal{P}(L)$. So defined tuple $(\mathcal{B}, \nu, \xi)$ is called a semantic type under layer L, or simply a semantic type.

Let $\mathfrak{B}_1 = (\mathcal{B}_1, \nu_1, \xi_1)$ and $\mathfrak{B}_2 = (\mathcal{B}_2, \nu_2, \xi_2)$ be two semantic types under a layer L. The types $\mathfrak{B}_1$ and $\mathfrak{B}_1$ are said to be *isomorphic* or *equivalent*, written $\mathfrak{B}_1 \equiv_L \mathfrak{B}_2$, if there is a computable isomorphism $\mu : (\mathcal{B}_1, \nu_1) \to (\mathcal{B}_2, \nu_2)$ such that for any ultrafilter $\mathcal{F}_1 \in \mathrm{St}(\mathcal{B}_1)$ and corresponding ultrafilter $\mathcal{F}_2 \in \mathrm{St}(\mathcal{B}_2)$,

$\mathcal{F}_2 = \mu(\mathcal{F}_1)$, the following equality takes place:

$$\xi_1(\mathcal{F}_1) = \xi_2(\mathcal{F}_2). \tag{5.1}$$

Notice that, computability of the isomorphism μ requires that there are some general computable functions $f(x)$ and $g(x)$ for which the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{N} & \underset{g}{\overset{f}{\rightleftarrows}} & \mathbb{N} \\ {\scriptstyle \nu_1}\downarrow & & \downarrow{\scriptstyle \nu_2} \\ \mathcal{B}_1 & \xrightarrow{\ \mu\ } & \mathcal{B}_2 \end{array} \tag{5.2}$$

while an extra condition (5.1) is meant abstractly; that is, none supporting effective method is supposed to verify validity of the condition.

We say that theory T has semantic type $\mathfrak{B} = (\mathcal{B}, \nu, \xi)$ or that the semantic type $\mathfrak{B}$ is realized in theory T if $(\mathcal{L}(T), \gamma, \xi)$ and $(\mathcal{B}, \nu, \xi)$ are isomorphic according to the definition given above.

It is possible to see that the concept of a semantic type together with the equivalence relation for such objects are in exact correspondence with the relation of semantic similarity of theories under a semantic layer. Namely, the following statement takes place:

Lemma 5.1. *Let T and S be theories of enumerable signatures and L be a semantic layer. The following assertions are equivalent with each other*:

(a) *T and S are semantically similar under L*,

(b) $£(T) \equiv_L £(S)$.

Proof. Immediately, from definitions. □

The set of all possible abstract semantic types has the power of the continuum. This makes an obstacle for definition of the concept of an index for such objects. However, for isomorphisms of semantic types, indexes can be defined.

Let $\mathfrak{B}_1 = (\mathcal{B}_1, \nu_1, \xi_1)$ and $\mathfrak{B}_2 = (\mathcal{B}_2, \nu_2, \xi_2)$ be two abstract semantic types under a semantic layer L, and let μ be a computable isomorphism between $\mathfrak{B}_1$ and $\mathfrak{B}_2$ for which diagram (5.2) is commutative with the function $f(x) = \lambda x \varphi_n(x)$ which should be total, where φ_n is nth function in Kleene's numbering of all partially computable functions (a suitable computable function $g(x)$ for the back passage can then be found in a standard way). If so, the number n is called a *c.e. index* or simple *index* of the isomorphism μ.

A number of operations similar to those on c.e. Boolean algebras are defined on the class of semantic types. Operations of factorization, restriction on an element, the direct product of two types, and the direct product of a countable computable sequence of types are defined. In the last operation, we use the concept of computability of a sequence of types in some narrow sense. Namely, a sequence of semantic types $\mathfrak{B}_n = (\mathcal{B}_n, \nu_n, \xi_n)$, $n \in \mathbb{N}$, is said to be basically computable, if the corresponding sequence of c.e. Boolean algebras $\mathfrak{B}_n = (\mathcal{B}_n, \nu_n)$, $n \in \mathbb{N}$, is computable (computability does not concern the function ξ).

We pass to the operations in detail.

We start with the operation of restriction on an element. Consider a semantic type $\mathfrak{B} = (\mathcal{B}, \nu, \xi)$ under a semantic layer L, and let $a = \nu[n_0]$ be an element in $\mathcal{B}$. Denote by $(\mathcal{B}, \nu, \xi)[a]$ a semantic type $(\mathcal{B}', \nu', \xi')$, where $(\mathcal{B}', \nu') = (\mathcal{B}[a], \nu[a])$, while ξ' is the restriction of ξ up to Stone space of the restricted algebra defined as $\mathrm{St}(\mathcal{B}[a]) = \{\mathfrak{F} \in \mathrm{St}(\mathcal{B}) \mid a \in \mathfrak{F}\}$. So defined function ξ' is denoted by $\xi[a]$. Thus, the operation as a whole has the following form

$$(\mathcal{B}, \nu, \xi)[a] = (\mathcal{B}[a], \nu[a], \xi[a]).$$

One can check that, if the source type $\mathfrak{B}$ corresponds to a theory T, its restriction $\mathfrak{B}[a]$ corresponds to a finitely axiomatizable extension of the theory T.

Turn to the *quotient operation*. Consider a semantic type $\mathfrak{B} = (\mathcal{B}, \nu, \xi)$, and let $\mathcal{F}$ be a filter of the Boolean algebra $\mathcal{B}$. Denote by ${}^{(\mathcal{B},\nu,\xi)}/_{\mathcal{F}}$ a semantic type of the form $(\mathcal{B}', \nu', \xi')$ where $(\mathcal{B}', \nu')$ is the quotient of $(\mathcal{B}, \nu)$ modulo $\mathcal{F}$, while the assignment function ξ' is defined by the following rule

$$\xi'(\mathcal{F}') = \xi(\mathcal{F}'), \quad \textit{for } \mathcal{F}' \in \mathrm{St}({}^{\mathcal{B}}/_{\mathcal{F}}) = \{\mathcal{F}' \in \mathrm{St}(\mathcal{B}) \mid \mathcal{F} \subseteq \mathcal{F}'\}.$$

So defined function ξ' is denoted by ${}^{\xi}/_{\mathcal{F}}$. This is simply a restriction of ξ on the subspace of $\mathrm{St}(\mathcal{B})$ defined by the filter $\mathcal{F}$. Then, the quotient operation as a whole has the following form

$${}^{(\mathcal{B},\nu,\xi)}/_{\mathcal{F}} = ({}^{\mathcal{B}}/_{\mathcal{F}}, {}^{\nu}/_{\mathcal{F}}, {}^{\xi}/_{\mathcal{F}}).$$

One can check that, if the source semantic type $\mathfrak{B}$ corresponds to a theory T, then the quotient type ${}^{\mathfrak{B}}/_{\mathcal{F}}$ corresponds to an extension of T defined by this filter $\mathcal{F}$ considered as an extra set of axioms.

Some natural relation between the two operations:

Lemma 5.2. *Let $\mathfrak{B} = (\mathcal{B}, \nu, \xi)$ be a semantic type under a semantic layer L and a be an element of $\mathcal{B}$. Suppose that $\mathcal{F}$ is a principal filter of $\mathcal{B}$ generated by a; i.e., $\mathcal{F} = \{c \in \mathcal{B} \mid a \subseteq c\}$. Then, we have ${}^{\mathfrak{B}}/_{\mathcal{F}} \equiv_L \mathfrak{B}[a]$.*

Proof. Immediately, from elementary properties of Boolean algebras. □

Now, we define the operation of p direct product of two semantic types.

Let $\mathfrak{B}_1 = (\mathcal{B}_1, \nu_1, \xi_1)$ and $\mathfrak{B}_2 = (\mathcal{B}_2, \nu_2, \xi_2)$ be two semantic types under a semantic layer L. Define some new semantic type

$$\mathfrak{B} = (\mathcal{B}, \nu, \xi) = (\mathcal{B}_1, \nu_1, \xi_1) \otimes (\mathcal{B}_2, \nu_2, \xi_2) = \mathfrak{B}_1 \otimes \mathfrak{B}_2$$

under the same semantic layer L as follows. We put $(\mathcal{B}, \nu) = (\mathcal{B}_1, \nu_1) \otimes (\mathcal{B}_2, \nu_2)$, while the assignment function ξ is determined by the rule

$$\xi(\mathcal{F}) = \begin{cases} \xi_1(\mathcal{F}), & \text{if } \mathcal{F} \in \mathrm{St}(\mathcal{B}_1), \\ \xi_2(\mathcal{F}), & \text{if } \mathcal{F} \in \mathrm{St}(\mathcal{B}_2). \end{cases}$$

So defined operation $\mathfrak{B}_1 \otimes \mathfrak{B}_2$ is called the *direct product* of the semantic types $\mathfrak{B}_1$ and $\mathfrak{B}_2$, while the function ξ is called the *direct product* of functions ξ_1 and ξ_2, using for this an entry $\xi = \xi_1 \otimes \xi_2$. The idea of the assignment function in the operation $\otimes$ is based on the following algebraic relation

$$\mathrm{St}(\mathcal{B}_1 \otimes \mathcal{B}_2) = \mathrm{St}(\mathcal{B}_1) \cup \mathrm{St}(\mathcal{B}_2)$$

Thus, the two given assignment functions ξ_1 and ξ_2 defined on the disjoint parts $\mathrm{St}(\mathcal{B}_1)$ and $\mathrm{St}(\mathcal{B}_2)$ of $\mathrm{St}(\mathcal{B})$ are simply assembled in one function $\xi = \xi_1 \cup \xi_2$.

We pass to a more complicated operation of the direct product of a sequence of semantic types. Let $\mathfrak{B}_n = (\mathcal{B}_n, \nu_n, \xi_n)$, $n \in \mathbb{N}$ be a basically computable sequence of semantic types under a semantic layer L, while P a complete theory of an enumerable signature that is used as an additional parameter in the operation.

Let us define a new semantic type

$$(\mathcal{B}, \nu, \xi) = \bigotimes_{n\in\mathbb{N}}^{[P]} \mathfrak{B}_n = \bigotimes_{n\in\mathbb{N}}^{[P]} \big(\mathcal{B}_n, \nu_n, \xi_n\big) \tag{5.3}$$

as follows. We put $(\mathcal{B}, \nu) = \bigotimes_{n\in\mathbb{N}} (\mathcal{B}_n, \nu_n)$; moreover, the appointment operation ξ is defined by the following rule

$$\xi(\mathfrak{F}) = \begin{cases} \xi_n(\mathfrak{F}), & \textit{if } \mathfrak{F} \in \mathrm{St}(\mathcal{B}_n),\ n \in \mathbb{N}, \\ \mathrm{prop}(P) \upharpoonright L, & \textit{if } \mathfrak{F} = \hat{\mathfrak{F}},\ \hat{\mathfrak{F}} = \textit{Filter}\{-\mathbf{1}_i \mid i \in \mathbb{N}\}, \end{cases}$$

where prop(P) is set of model-theoretic properties connected with the complete theory P. An idea behind the appointment function in operation $\bigotimes$ for a sequence of semantic types is based on the following algebraic relation for Boolean algebras

$$\mathrm{St}(\bigotimes_{i\in\mathbb{N}}\mathcal{B}_i)=\bigcup_{i\in\mathbb{N}}\mathrm{St}(\mathcal{B}_i)\cup\{\hat{\mathfrak{F}}\},$$

where $\hat{\mathfrak{F}}$ is a new ultrafilter appearing in the operation of the direct product of a sequence of algebras. The appointment functions ξ_i provide natural definitions for the appointment function ξ on ultrafilters in the members of the sequence. A special value $\xi(\hat{\mathfrak{F}})$ is required because any of functions ξ_i, $i\in\mathbb{N}$ does not define value for ξ on the filter $\hat{\mathfrak{F}}$. An alternative entry $\bigotimes_{i\in\mathbb{N}}^{[K]}\mathfrak{B}_i$ with a parameter K, $K\subset L$, is possible that directly represents model-theoretic properties appointed to the special ultrafilter $\hat{\mathfrak{F}}$.

In addition to the operations over types we have introduced, a method of splitting and pasting of the sequences of semantic types is often applied. Given a basically computable sequence of semantic types

$$\mathfrak{B}_i,\ i\in\mathbb{N}. \tag{5.4}$$

Fix a computable function $d(x)$ satisfying $0=d(0)<d(1)<d(2)<\ldots<d(k)<\ldots$, and construct a new sequence $\mathfrak{B}'_t$, $t\in\mathbb{N}$, of types obtained by the operation of finite pasting of adjacent members in the initial sequence as follows:

$$\mathfrak{B}'_t=\bigotimes_{i=d(t)}^{d(t+1)-1}\mathfrak{B}_i,\ t\in\mathbb{N}. \tag{5.5}$$

We say that sequence (5.5) is built from the sequence (5.4) by a *finite gluing procedure*. Similarly, we say that the sequence (5.4) is obtained by a *finite splitting procedure* from the sequence (5.5).

The following interdependencies between the operations exist.

Lemma 5.3. *Let $L\subseteq ASL$ be a semantic layer and P be an arbitrary complete theory of an enumerable signature. Then, for any semantic types under the semantic layer L the following relations are satisfied*:

(a) $(\mathcal{B}_1,\nu_1,\xi_1)\otimes(\mathcal{B}_2,\nu_2,\xi_2)\equiv_L(\mathcal{B}_2,\nu_2,\xi_2)\otimes(\mathcal{B}_1,\nu_1,\xi_1)$,

(b) $\Big((\mathcal{B}_1,\nu_1,\xi_1)\otimes(\mathcal{B}_2,\nu_2,\xi_2)\Big)\otimes(\mathcal{B}_3,\nu_3,\xi_3)\equiv_L$
$(\mathcal{B}_1,\nu_1,\xi_1)\otimes\Big((\mathcal{B}_2,\nu_2,\xi_2)\otimes(\mathcal{B}_3,\nu_3,\xi_3)\Big)$,

(c) $(\mathcal{B},\nu,\xi)\equiv_L(\mathcal{B},\nu,\xi)[a]\otimes(\mathcal{B},\nu,\xi)[-a]$, *for any* $a\in\mathcal{B}$,

(d) $\bigotimes^{[P]}_{n\in\mathbb{N}}(\mathcal{B}_n,\nu_n,\xi_n) \equiv_L (\mathcal{B}_0,\nu_0,\xi_0)\otimes\bigotimes^{[P]}_{n\in\mathbb{N}\setminus\{0\}}(\mathcal{B}_{n+1},\nu_{n+1},\xi_{n+1})$,

(e) $\bigotimes^{[P]}_{n\in\mathbb{N}}(\mathcal{B}_n,\nu_n,\xi_n) \equiv_L \bigotimes^{[P]}_{n\in\mathbb{N}}(\mathcal{B}_{f(n)},\nu_{f(n)},\xi_{f(n)})$,
for any computable sequence of semantic types and any computable permutation f of the set $\mathbb{N}$,

(f) $\bigotimes^{[P]}_{n\in\mathbb{N}}(\mathcal{B}_n,\nu_n,\xi_n) \equiv_L \bigotimes^{[P]}_{n\in\mathbb{N}}(\mathcal{B}'_n,\nu'_n,\xi'_n)$,
whenever the latter sequence is obtained by a finite computable splitting procedure from the former one.

Proof. Immediately, based on definitions of the operations. □

6 Two classes of semantic types for first-order combinatorics

It is not difficult to show that there are semantic types that are not realized in any first-order theory. The term *abstract semantic type* is often used when we want to emphasize that its assigning function is not limited to the condition of realizability in a theory. In first-order combinatorial approach we are developing, the classes of types realized in computably axiomatizable and finitely axiomatizable theories are of particular importance. The restriction to consider only the types realized in theories does not give any advantages in comparison with the study of the theories themselves. On the contrary, a comprehensive study of a wider class of abstract semantic types will be more fruitful, and some of the results thus obtained can be applied to theories.

Introduce the following notations for classes of semantic types:

$\mathcal{A}(L)$, *is the set of all abstract semantic types under the layer L,*
$\mathcal{E}(L)$, *is the set of all computably axiomatizable types under L,*
$\mathcal{F}(L)$, *is the set of all finitely axiomatizable types under L.*

We use simpler notations $\mathcal{A}$, $\mathcal{E}$, and $\mathcal{F}$, considering that semantic layer L is defined in the context. Obviously, inclusions $\mathcal{F}\subseteq\mathcal{E}\subseteq\mathcal{A}$ take place.

Lemma 6.1 [$\mathbb{U}$]. *Any $\mathcal{E}$-type under layer $MQL\subseteq MQL$, cf.* (2.1), *is an $\mathcal{F}$-type under MQL.*

Proof. Immediately, from Statement 2.1. □

Although we cannot define indices for the set of all abstract semantic types, nevertheless, it is possible to define indices for semantic types in the classes $\mathcal{F}$ and $\mathcal{E}$.

Given a semantic layer $L \subseteq AL$. If a semantic type $\mathfrak{B} \in \mathcal{E}(L)$ is presented in computably axiomatizable theory $T^{\star}{}_{\{n\}}$ with an index n, the number n is said to be an *$\mathcal{E}$-index* of this type $\mathfrak{B}$, symbolically $\mathfrak{B} = \mathcal{E}^{*}{}_{\{n\}}$. Similarly, if a type $\mathfrak{B} \in \mathcal{F}(L)$ is presented in finitely axiomatizable theory $\mathcal{F}^{\star}{}_{\{n\}}$ having Gödel number n, the number n is said to be an *$\mathcal{F}$-index* of this type $\mathfrak{B}$, symbolically $\mathfrak{B} = \mathcal{F}^{*}{}_{\{n\}}$. We often use common term *index* instead of either $\mathcal{F}$-index or $\mathcal{E}$-index whenever kind of the type is clear from context.

We introduce definitions for universal semantic types.

Definition 6.A. Given a semantic layer L of model theoretic properties (the value *ACL* is meant for L by default, when it is not defined evidently in the statement). Semantic $\mathcal{F}$-type $\mathfrak{B} = (\mathcal{B}, \nu, \xi)$ is called *weakly $\mathcal{F}$-universal* under L if any $\mathcal{F}$-type $\mathfrak{B}'$ under L can be presented in the form $\mathfrak{B}[a]$ for some $a \in |\mathfrak{B}|$. Furthermore, the $\mathcal{F}$-type $\mathfrak{B} = (\mathcal{B}, \nu, \xi)$ is called *$\mathcal{F}$-universal* under L if it is weakly $\mathcal{F}$-universal under L; moreover, a transition from an index of $\mathfrak{B}'$ to the element a is performed effectively, i.e., there are computable functions $g(n)$ and $h(n,t)$ satisfying to the following properties:

$$\mathcal{F}^{\star}{}_{\{n\}} \equiv_L \mathfrak{B}[\nu(g(n))],\ \textit{for all } n \in \mathbb{N};\ \textit{moreover, the function } (\lambda t)h(n,t) \textit{ represent this computable isomorphism.} \tag{6.1}$$

Definition 6.B. Given a semantic layer M of model theoretic properties (the value *MQL* is meant for M by default, when it is not defined evidently in the statement). Semantic $\mathcal{E}$-type $\mathfrak{B} = (\mathcal{B}, \nu, \xi)$ is called *weakly $\mathcal{F}$-universal* under M if any $\mathcal{F}$-type $\mathfrak{B}'$ under M can be presented in the form $\mathfrak{B}[a]$ for some $a \in |\mathfrak{B}|$. Furthermore, the $\mathcal{E}$-type $\mathfrak{B} = (\mathcal{B}, \nu, \xi)$ is called *$\mathcal{F}$-universal* under M if it is weakly $\mathcal{E}$-universal under M; moreover, a transition from an index of $\mathfrak{B}'$ to the element a is performed effectively, i.e., there are computable functions $g(n)$ and $h(n,t)$ satisfying to the following properties:

$$\mathcal{E}^{\star}{}_{\{n\}} \equiv_M \mathfrak{B}[\nu(g(n))],\ \textit{for all } n \in \mathbb{N};\ \textit{moreover, the function } (\lambda t)h(n,t) \textit{ represent this computable isomorphism.} \tag{6.2}$$

We mention some elementary properties of the introduced concepts.

Lemma 6.2. *The following statements are satisfied*:

(a) *for any finite rich signature σ, semantic $\mathcal{F}$-type of predicate calculus $\mathfrak{L}(PC(\sigma))$ is $\mathcal{F}$-universal under semantic layer ACL (the more, under any its sublayer $L \subseteq ACL$)*,

(b) *semantic $\mathcal{F}$-type $\text{₤}(GRE)$ of theory GRE is $\mathcal{F}$-universal under semantic layer ACL (the more, under any its sublayer $L \subseteq ACL$),*

(c) *semantic $\mathcal{E}$-type $\mathscr{E}^u_{c.a.} = \text{₤}(T^u_{c.a.})$ is $\mathcal{E}$-universal under semantic layer MSL (the more, under any its sublayer $K \subseteq MSL$).*

Proof. (a) Immediately, from Theorem 1.2; (b) immediately, from Theorem 1.3; (c) immediately, from Statement 2.1. □

Lemma 6.3. *The following statements are satisfied*:

(a) *any weak $\mathcal{F}$-universal semantic $\mathcal{F}$-type $\mathfrak{B}$ under a semantic layer $L \subseteq ACL$ is $\mathcal{F}$-universal under L.*

(b) *any weak $\mathcal{E}$-universal semantic $\mathcal{E}$-type $\mathfrak{B}$ under a semantic layer $M \subseteq ASL$ is $\mathcal{E}$-universal under M.*

Proof. (a) Suppose that $\mathfrak{B}$ is a weak $\mathcal{F}$-universal semantic $\mathcal{F}$-type. By definition, there is an element a in $\mathfrak{B}$ such that $\mathfrak{B}[a] \equiv_L$ *GRE*. By applying Statement 1.3, we immediately obtain the wished effectiveness property; thereby, $\mathfrak{B}$ is indeed an $\mathcal{F}$-universal semantic type. (b) Suppose that $\mathfrak{B}$ is a weak $\mathcal{E}$-universal semantic $\mathcal{E}$-type. By definition, there is an element a in $\mathfrak{B}$ such that $\mathfrak{B}[a] \equiv_L T^u_{c.a.}$. By applying Lemma 4.1, we obtain the wished effectiveness property; thereby, $\mathfrak{B}$ is indeed an $\mathcal{E}$-universal semantic type. □

7 Representative computable sequences of semantic types

We introduce a few kinds of rich computable sequences of semantic types. They are intended to be used in assembling universal semantic types.

A sequence $\mathfrak{B}_n$, $n \in \mathbb{N}$, of $\mathcal{F}$-types under a semantic layer L is called *computable* if there is a general computable function $f(x)$ such that $\mathfrak{B}_n \equiv_L \mathcal{F}^\star\{f(n)\}$ for all $n \in \mathbb{N}$. A sequence $\mathfrak{B}_n$, $n \in \mathbb{N}$, of $\mathcal{E}$-types under a layer L is called *computable*, if there is a general computable function $f(x)$ such that $\mathfrak{B}_n \equiv_L \mathcal{E}^\star\{f(n)\}$ for all $n \in \mathbb{N}$.

Two computable sequences $\mathfrak{B}_n$, $n \in \mathbb{N}$, and $\mathfrak{B}'_n$, $n \in \mathbb{N}$, of semantic types under a semantic layer L are said to be *equivalent*, if there are two computable functions $p(x)$ and $f(n,x)$, such that $p(x)$ is a permutation of the set $\mathbb{N}$, and for all $n \in \mathbb{N}$ we have $\mathfrak{B}_n \equiv_L \mathfrak{B}'_{p(n)}$; moreover, the function $\lambda t\, f(n,t)$ represents this computable isomorphism.

Introduce a few classes of special computable sequences of semantic types.

Definition 7.A. A sequence of $\mathcal{F}$-types $\mathfrak{B}_n = (\mathcal{B}_n, \nu_n, \xi_n)$, $n \in \mathbb{N}$, is called *$\mathcal{F}$-rich* under a semantic layer L, if it is computable and, effectively in an index, any $\mathcal{F}$-type $\mathfrak{B}$ under L can be presented as $\mathfrak{B}_i[a]$ for some $a \in |\mathfrak{B}_i|$ with $i \geqslant k$ for arbitrarily large k. A sequence of $\mathcal{E}$-types $\mathfrak{B}_n = (\mathcal{B}_n, \nu_n, \xi_n)$, $n \in \mathbb{N}$, is called *$\mathcal{E}$-rich* under a semantic layer M, if it is computable and, effectively in an index, any $\mathcal{E}$-type $\mathfrak{B}$ under M can be presented as $\mathfrak{B}_i[a]$ for some $a \in |\mathfrak{B}_i|$ with $i \geqslant k$ for arbitrarily large k.

Definition 7.B. A sequence $\mathfrak{B}_n = (\mathcal{B}_n, \nu_n, \xi_n)$, $n \in \mathbb{N}$, of $\mathcal{F}$-types under a semantic layer L is called *uniformly $\mathcal{F}$-universal* under L, if it is computable and the universality condition is satisfied uniformly effective. $\mathcal{E}$-types under a semantic layer M is called *uniformly $\mathcal{E}$-universal* under the semantic layer M, if it is computable and the universality condition is satisfied uniformly effective.

Definition 7.C. A computable sequence of $\mathcal{F}$-types $\mathfrak{B}_n$, $n \in \mathbb{N}$, is called *$\mathcal{F}$-representative* under a semantic layer L if the following conditions are satisfied: (a) the sequence has effective cylindric properties; (b) the sequence presents effectively all possible $\mathcal{F}$-types under L. A computable sequence of $\mathcal{E}$-types $\mathfrak{B}_n$, $n \in \mathbb{N}$, is called *$\mathcal{E}$-representative* under a semantic layer K if the following conditions are satisfied: (a) the sequence has effective cylindric properties; (b) the sequence presents effectively all possible $\mathcal{E}$-types under K.

Establish computability of the most important sets of semantic types.

Theorem 7.1. *The following assertions hold*:

(a) *the set of all semantic types under an arbitrary semantic layer L, which are types of finitely axiomatizable theories of a fixed finite signature σ, is computable.*

(b) *the set of all semantic types under an arbitrary semantic layer L, which are types of finitely axiomatizable theories of arbitrary finite signatures, is computable,*

(c) *the set of all semantic types under an arbitrary semantic layer K, which are types of computably axiomatizable theories of arbitrary enumerable signatures, is computable.*

Proof. For (a) and (b), consider the following sequences of semantic types:

$$\begin{aligned} &\texttt{(a)}\quad \mathcal{F}^{\sigma}{}_{\{n\}} = \text{\textlira}(F^{\sigma}{}_{\{n\}}),\ n \in \mathbb{N}. \\ &\texttt{(b)}\quad \mathcal{F}^{\star}{}_{\{n\}} = \text{\textlira}(F^{\star}{}_{\{n\}}),\ n \in \mathbb{N}, \end{aligned} \tag{7.1}$$

By applying Lemma 3.1, we obtain exactly what is required.

(c) Consider the following sequences of semantic types:

$$\mathcal{E}^{\star}{}_{\{n\}} = \text{\textit{€}}(T^{\star}{}_{\{n\}}),\ n \in \mathbb{N}. \qquad (7.2)$$

By applying Lemma 3.2(b), we obtain exactly what is required. □

Theorem 7.2. *The following assertions hold*:

(a) *for any semantic layer $L \subseteq ACL$, there is a computable $\mathcal{F}$-representative sequence of $\mathcal{F}$-types under L; sequence* (7.1)(*a*) *provided that σ is a finite rich signature, as well as sequence* (7.1)(*b*), *satisfies this demand,*

(b) *for any semantic layer $K \subseteq ASL$, there is a computable $\mathcal{E}$-representative sequence of $\mathcal{E}$-types under K; the sequence* (7.2) *satisfies this demand,*

(c) *any two computable $\mathcal{F}$-representative sequences of $\mathcal{F}$-types $\mathfrak{B}_i$, $i \in \mathbb{N}$, and $\mathfrak{B}'$, $i \in \mathbb{N}$, under an arbitrary layer L are equivalent with each other,*

(d) *any two computable $\mathcal{E}$-representative sequences of $\mathcal{E}$-types $\mathfrak{B}_i$, $i \in \mathbb{N}$, and $\mathfrak{B}'$, $i \in \mathbb{N}$, under an arbitrary layer K are equivalent with each other,*

(e) *any computable $\mathcal{E}$-representatative sequence $\mathfrak{B}_i$, $i \in \mathbb{N}$, of $\mathcal{F}$-types under a layer L is a $\mathcal{F}$-representatative sequence of $\mathcal{F}$-types under the layer $K = MQL \cap L$.*

Proof. Parts (a) and (b) are proved based on Lemma 3.4 and Lemma 3.5. Parts (c) and (d) are proved by a standard method similar to the proof used in known Myhill's Theorem, cf. [6].

(e) By regular applying of Statement 2.2 presenting an effective version of the universal construction that controls semantic layer (2.1). □

Exercise 7.3. Let $\mathfrak{B}_i$, $i \in \mathbb{N}$, be a computable representative sequence of semantic $\mathcal{F}$-types under $L \subseteq ACL$. Show that there is a computable permutation p of $\mathbb{N}$ such that we have uniformly effective

$$\mathfrak{B}_i \equiv_L \mathcal{F}_{\{p(i)\}},\ \textit{for all } i \in \mathbb{N}. \qquad (7.3)$$

Conversely, availability of relation (7.3) with a computable permutation $p(x)$ ensures that $\mathfrak{B}_i$, $i \in \mathbb{N}$, is a computable representative sequence of $\mathcal{F}$-types under L. Prove the same statement for sequences of $\mathcal{E}$-types under any layer $K \subseteq ASL$.

Hint. Use Theorem 7.2 together with definitions of indices for $\mathcal{F}$-types and $\mathcal{E}$-types, cf. Section 6.

The following statement represents an effective version of Lemma 6.1.

THEOREM 7.4 [$\mathbb{U}$]. *Given semantic layers L and K such that $L \subseteq ACL$ and $K \subseteq MSL$. Let $\mathcal{F}_i$, $i \in \mathbb{N}$, be a computable $\mathcal{F}$-representative sequence of $\mathcal{F}$-types under L and $\mathcal{E}_i$, $i \in \mathbb{N}$, a computable $\mathcal{E}$-representative sequence of $\mathcal{E}$-types under the layer K. There is a computable permutation $p : \mathbb{N} \to \mathbb{N}$ together with a computable function $h(n, x)$ such that $\mathcal{F}_i \equiv_{L\cap K\cap MQL} \mathcal{E}_{p(i)}$ for all $n \in \mathbb{N}$; moreover, the function $\lambda x h(n, x)$ represents an isomorphism for the pointed out similarity relation.*

PROOF. Immediately, from Theorem 7.2(d,e). □

8 TWO KINDS OF DENSE THEORIES FOR FIRST-ORDER COMBINATORICS

We introduce two key technical definitions.

DEFINITION 8.A. Given a semantic layer E of model-theoretic properties (the value ACL is meant for E by default, when it is not defined evidently in the statement). A theory R of an enumerable signature σ, is called *inf*-dense under the layer E if there is a computably enumerable set $\Sigma \subseteq SL(\sigma)$, called a *framework* for R, for which the following conditions are satisfied:

(a) the theory R is complete and decidable,

(b) for any $\Phi \in SL(\sigma)$ satisfying $R \vdash \Phi$, there is a sentence $\Psi \in SL(\sigma)$ and a computable isomorphism μ having the following properties: $R \vdash \neg\Psi$, $\Sigma \vdash \Psi \to \Phi$, and $₤([\Sigma \cup \{\Psi\}]^\sigma) \equiv_E ₤(T^u_{c.a.})$ by means of μ; moreover, a Godel's number of Ψ and a c.e. index of isomorphism μ are found effectively from a Godel's number of the sentence Φ.

DEFINITION 8.B. Given a semantic layer D of model-theoretic properties (the value ASL is meant for D by default, when it is not defined evidently in the statement). A theory P of a finite signature σ, is called f-dense under the layer D if the following conditions are satisfied:

(a) the theory P is complete and decidable,

(b) for any $\Phi \in SL(\sigma)$ satisfying $P \vdash \Phi$, there is a sentence $\Psi \in SL(\sigma)$ and a computable isomorphism μ having the following properties: $P \vdash \neg\Psi$, $\vdash \Psi \to \Phi$, and $₤([\Psi]^\sigma) \equiv_D ₤(GRE)$ by means of μ; moreover, a Godel's number of sentence Ψ and c.e. index of isomorphism μ are found effectively from a Godel's number of the sentence Φ.

The concepts of an f-dense and *inf*-dense theory we have introduced are used in the subsequent as a source providing a collection of model theoretic properties assigned for special ultrafilters in the operation of direct product of a sequence of semantic types.

Exercise 8.1. Given a rich computable sequence of semantic $\mathcal{F}$-types together with a complete theory P of a finite signature. Show that theory P is f-dense whenever semantic type $(\mathcal{B}, \nu, \xi) = \bigotimes_{n\in\mathbb{N}}^{[P]} \mathfrak{B}_n = \bigotimes_{n\in\mathbb{N}}^{[P]} (\mathcal{B}_n, \nu_n, \xi_n)$ is finitely axiomatizable.

Hint. Use Definition 8.A together with description of the operation of direct product of a computable sequence of semantic types. □

Exercise 8.2. Given a rich computable sequence of semantic $\mathcal{E}$-types together with a complete theory R of an enumerable signature. Show that theory R is *inf*-dense whenever semantic type $(\mathcal{B}, \nu, \xi) = \bigotimes_{n\in\mathbb{N}}^{[P]} \mathfrak{B}_n = \bigotimes_{n\in\mathbb{N}}^{[P]} (\mathcal{B}_n, \nu_n, \xi_n)$ is computably axiomatizable.

Hint. Use Definition 8.B together with description of the operation of direct product of a computable sequence of semantic types. □

Lemma 8.3 [$\mathbb{U}$]. *Let a theory P be f-dense under a semantic layer L. Then, P is an inf-dense theory under any semantic layer $K \subseteq MQL \cap L$.*

Proof. This fact is a routine consequence of the two definitions. One can check that the definition of an *f*-dense theory is obtained by way of strengthen of some internal parts in the definition of an *inf*-dense theory. A finite signature is considered instead of an enumerable signature. A finite (or even empty) set of sentences is considered instead of a computable frame Σ. Apart from that, we can apply the universal construction that provides the existence of an equivalence $\pounds(T^u_{c.a.}) \equiv_{MQL} GRE\,[\Theta]$ which is a similarity relation under the semantic layer *MQL*, where *GRE* $[\Theta]$ is an appropriate finitely axiomatizable extension of the graph theory *GRE*. As a result, we obtain that the theory P must be *inf*-dense under any semantic layer $K \subseteq MQL \cap L$. □

9 Main transformations between the types and sequences

First, we consider a passage from types to sequences of types.

Lemma 9.1. *If $\mathfrak{B}$ is a week $\mathcal{F}$-universal semantic $\mathcal{F}$-type under a layer L, then the sequence $\mathfrak{B}[\nu(n)]$, $n \in \mathbb{N}$, is an $\mathcal{F}$-rich sequence of $\mathcal{F}$-types under L. The same statement is valid for the case of $\mathcal{E}$-types.*

Proof. Immediately. □

In the following statement, various schemes of transformations between universal types and different classes of rich computable sequences of semantic types are collected.

THEOREM 9.2. *Transitions between different types of sequences of types and universal types are available in accordance with the scheme in Fig. 1.*

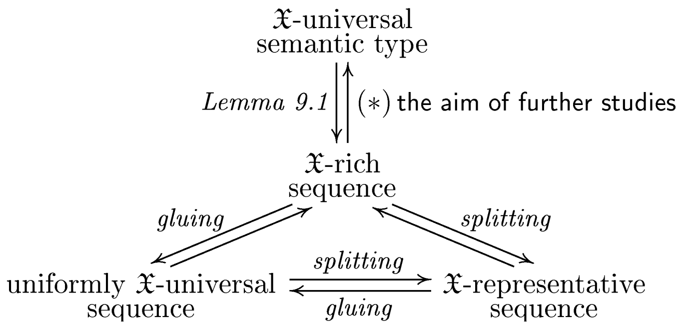

Figure 1 – Passages between sequences of $\mathfrak{X}$-types, $\mathfrak{X} \in \{\mathcal{F}, \mathcal{E}\}$

A SKETCH OF PROOF. By using standard methods of algorithm theory, [6]. For instance, any uniformly $\mathfrak{X}$-universal sequence is $\mathfrak{X}$-rich that, while having an $\mathfrak{X}$-rich sequence, by finite gluing operation, we can transform it into a uniformly $\mathfrak{X}$-universal sequence etc. Furthermore, having an $\mathfrak{X}$-universal semantic type $\mathfrak{B} = (\mathcal{B}, \nu, \xi)$, we can construct sequence of types $\mathfrak{B}[\nu(n)]$, $n \in \mathbb{N}$, that is obviously an $\mathfrak{X}$-rich sequence. As for the back passage marked with $(*)$, it is both an important and an interesting problem how to construct a universal type from a rich sequence of types.

CONCLUSION

It can be checked that if the semantic type of the product (5.3) is finitely axiomatizable and the corresponding sequence of types is rich (the more uniformly universal or representative), then the special ultrafilter of this product determines the f-dense theory; under the same conditions, if the type is computably axiomatizable, then the special ultrafilter of this product corresponds to the *inf*-dense theory. The inverse relationship also holds. Namely, if the sequence is rich and the theory P in the product (5.3) is f-dense or *inf*-dense, then this product defines a finitely axiomatizable or, respectively, computably axiomatizable type. The proof of these and a number of other statements requires complex techniques and a large number of technical concepts.

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Перетятькин М.Г. ПРОЦЕДУРА РЕДУКЦИИ СИГНАТУР И УНИВЕРСАЛЬНАЯ КОНСТРУКЦИЯ В КАЧЕСТВЕ МЕТОДОВ ПРЕОБРАЗОВАНИЯ В КОМБИНАТОРНОМ ПОДХОДЕ ПЕРВОГО ПОРЯДКА

Работа развивает предложенный ранее комбинаторный подход первого порядка, представляющий концептуальную основу для исследования структуры обобщённых алгебр Тарского-Линденбаума исчислений предикатов конечных богатых сигнатур над финитарным и инфинитарным семантическими слоями. В настоящей работе приведены ключевые технические спецификации, определяющие статус процедур редукции сигнатур и универсальной конструкции конечно аксиоматизируемых теорий в рамках комбинаторного подхода первого порядка.

Ключевые слова. Логика первого порядка, теоретико-модельное свойство, алгебра Тарского-Линденбаума, семантический тип, процедура редукции сигнатур, универсальная конструкция конечно аксиоматизируемых теорий.

Перетятькин М.Г. СИГНАТУРАЛАРДЫ РЕДУКЦИЯЛАУ РӘСІМІ МЕН ӘМБЕБАП ҚҰРЫЛЫМ БІРІНШІ РЕТТІ КОМБИНАТОРЛЫҚ ТӘСІЛДЕГІ ТҮРЛЕНДІРУ ӘДІСТЕРІ РЕТІНДЕ

Жұмыс бұрын ұсынылған финитарлы және инфинитарлы семантикалық қабаттар үстінде ақырлы бай сигнатуралар предикаттарының есептеулерінің жалпыланған Тарский-Линденбаум алгебраларының құрылымын зерттеуге арналған тұжырымдамалық негізді кейіптейтін бірінші ретті комбинаторлық тәсілді дамытады. Ақырлы аксиомдалатын теориялардың бірінші ретті комбинаторлық тәсіл шеңберіндегі сигнатураларды редукциялау рәсімдерінің және әмбебап құрылымның мәртебесін анықтайтын негізгі техникалық спецификациялары келтірілген.

Кілттік сөздер. Бірінші ретті логика, теориялық-моделдік қасиет, Тарский-Линденбаум алгебрасы, семантикалық тип, сигнатураларды редукциялау рәсімі, ақырлы аксиомдалатын теориялардың әмбебап құрылымы.

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Received 17.10.2017

Математический журнал ISSN 1682–0525

2017. — Том 17, № 4. — С. 99–104.

МАТЕМАТИЧЕСКАЯ ЖИЗНЬ

АЛЕКСАНДР АЛИПКАНОВИЧ ЖЕНСЫКБАЕВ

(к 70-летию со дня рождения)

21 августа нынешнего года исполнилось бы 70 лет выдающемуся математику и крупному организатору науки Республики Казахстан, действительному члену НАН РК Александру Алипкановичу Женсыкбаеву.

Александр Алипканович Женсыкбаев родился 21 августа 1947 года в г. Бургас (Болгария) в семье военнослужащего. В связи с частыми переездами во время учебы он сменил семь школ в Азербайджане, России, Украине. Тем не менее, среднюю школу Александр Алипканович окончил с золотой медалью.

В 1965 г. А.А. Женсыкбаев поступил в Днепропетровский государственный университет, который с отличием закончил в 1970 г. В том же году он поступил в аспирантуру ДГУ. Его научным руководителем по аспирантуре был выдающийся советский математик Н.П. Корнейчук. Досрочно завершив работу над диссертацией (по специальности 01.01.01 – теория функций и функциональный анализ), Александр Алипканович успешно защитил ее в 1973 г. Основное содержание диссертации составили точные оценки приближения классов непрерывных и дифференцируемых функций интерполяционными сплайнами минимального дефекта.

С 1973 г. А.А. Женсыкбаев – ассистент кафедры математического анализа ДГУ. В 1974 г. А.А. Женсыкбаев по приглашению академика АН

КазССР О.А. Жаутыкова и тогдашнего ректора Казахского государственного университета им. С.М. Кирова академика АН Каз ССР У.А. Джолдасбекова переезжает в Алма-Ату и становится сначала старшим преподавателем, затем (через короткое время) доцентом кафедры математического анализа Университета.

В апреле 1980 г. Александр Алипканович защищает докторскую диссертацию на тему "Экстремальные свойства моносплайнов и наилучшие квадратурные формулы" (специальность 01.01.01. – математический анализ) в Математическом институте им. В.А. Стеклова АН СССР. В диссертации решена широко известная задача Колмогорова-Никольского о наилучшей квадратурной формуле для классов Соболева. Методы, разработанные в диссертации для ее решения, нашли также важные применения при решении ряда других экстремальных задач теории интерполяции функций, теории поперечников, теории квадратур и получили дальнейшее развитие в работах многих математиков.

С 1981 г. по 2000 г. А.А. Женсыкбаев заведовал кафедрой математического анализа КазНУ им. аль-Фараби (до 1992г. – КазГУ им. С.М. Кирова). В 1982 г. ему присвоено ученое звание профессора.

С 1994 г. по 1997 г. Александр Алипканович работал заместителем по науке начальника Алматинского Высшего технического училища, где, в частности, организовал адъюнктуру для подготовки специалистов высшей квалификации.

С 1998 г. по 2000 г. Александр Алипканович – председатель ВАК РК.

С 2000 г. по 2006 г. А.А. Женсыкбаев – директор Института математики НАН РК.

С 1983 г. по 1992 г. А.А. Женсыкбаев был членом специализированного совета по защите докторских диссертаций при Институте Математики СО АН СССР (г. Новосибирск). Он также был экспертом INTAS (Brussels, 1994 г., Алматы 1996 г.) и научным руководителем международной программы INTAS, объединявшей ученых Германии, Испании, Казахстана, России, Франции.

Александр Алипканович был одним из основателей и первым главным редактором "Математического журнала", издаваемого Институтом математики (ныне Институтом математики и математического моделирования), членом редколлегии международного журнала "East Journal on

Approximation", издававшегося в Болгарии (с момента его основания), членом Американского математического общества.

А.А. Женсыкбаев – признанный в мире специалист в области теории функций и приближений, основатель школы по теории приближений и сплайнов в Казахстане. Под его руководством защищено 10 кандидатских диссертаций.

Александр Алипханович внес принципиальный вклад в теорию приближений и теорию оптимального восстановления операторов.

В научном творчестве Александра Алипкановича можно выделить (несколько условно) три периода.

Первый период (1970-1973 г.г.) посвящен, как уже было отмечено выше, экстремальным задачам приближения классов гладких функций интерполяционными сплайнами получил свое логическое завершение в его кандидатской диссертации, хотя Александр Алипканович и позднее обращался к этой проблематике. В частности, в 1979 г. он получил тонкий отрицательный результат: оператор сплайн-интерполяции не является экстремальным в смысле линейного поперечника классов гладких периодических функций одной переменной (Женсыкбаев А.А. Сплайн-интерполяция и наилучшее приближение тригонометрическими многочленами // Матем. заметки. – 1979. – Т. 26, № 3. – С. 355-366).

Второй период (1974-1990 г.г.) – экстремальные задачи теории квадратур. Кульминация этого периода – уже отмеченная докторская диссертация Александра Алипкановича. Но и позднее Александр Алипканович получает ряд выдающихся результатов в этом направлении: в частности, им решены известные задачи о нулях моносплайнов произвольной кратности и о наилучших гауссовых квадратурных формулах для слабых чебышевских систем. Итоги этого этапа были подведены Александром Алипкановичем в его монографии "Сплайны в теории восстановления", которая была написана по заказу Всесоюзного издательства "Наука" (г. Москва), прошла полное рецензирование и корректуру, была включена в темплан издательства на 1992 г., однако, так и не была опубликована в связи с распадом Советского Союза. Позднее Александр Алипканович опубликовал ее в переработанном и дополненном виде в издательстве КазГосИНТИ (Алматы) в 2001 г.

В третий период (1991-2009 г.г.) научные интересы Александра Алипкановича, в основном, переключаются на многомерную теорию приближений и родственные задачи оптимального восстановления операторов. Здесь им также был получен ряд важных результатов: в частности, были разработаны новые оптимальные методы восстановления операторов на классах функций многих переменных, новые многомерные аппараты интерполяции и сглаживания – информационно-ядерные сплайны. Результаты этого периода частично отражены в упоминавшейся монографии 2001 г. и достаточно полно освещены в монографии "Проблемы восстановления операторов" (Москва-Ижевск: РХД. – 2003 г.).

А.А. Женсыкбаев имеет свыше 90 научных публикаций, а также 3 монографии и 2 учебных пособия, изданные в Казахстане и России. Основные научные статьи Александра Алипкановича опубликованы в ведущих советских (позднее российских) и международных математических журналах: "Доклады АН СССР", "Доклады РАН", "Успехи математических наук", "Известия АН СССР, серия математическая", "Analysis Mathematica", "Journal of Approximation Theory", "East Journal on Approximations" и др. Он выступал с научными докладами на математических конгрессах (Варшава, Цюрих) и многих международных конференциях (Казахстан, Россия, США, Франция, Испания, Югославия, Польша, Венгрия, Болгария, Индия и др.).

Большое место в жизни А.А. Женсыкбаева занимала преподавательская деятельность: с 1974 по 2000 г.г. Александр Алипканович читал лекции в КазНУ им. Аль-Фараби (все основные курсы анализа, множество специальных курсов), во второй половине 90-х – начале 2000-х – в КазНПУ им. Абая (по совместительству математический анализ, спецкурсы), с 2000 по 2009 г.г. он читал лекции в КИМЭП (по совместительству). А.А. был прекрасным лектором, умел увлечь студентов, щедро делился своими идеями и замыслами с учениками и сотрудниками. Кроме того, Александр Алипканович в разные годы по приглашению читал лекции в университетах и научных центрах США, Франции, Испании, Польши, Пакистана.

Научные достижения, педагогическая и научно-организационная деятельность А.А. Женсыкбаева получили высокую оценку. Александр Алипканович был избран члены-корреспондентом НАН РК в 1995 г., а в 2003 г. стал ее действительным членом. В 1999 г. он был избран действительным членом МАН ВШ. Александр Алипканович – лауреат Премии ВЛКСМ в области науки и техники (СССР, 1978 г.), кавалер ордена "Знак Почета" (СССР, 1984 г.), лауреат международной премии им. Хорезми (ИР Иран, 1999 г.). В 2000 г. Александр Алипканович был избран почетным членом Американской Ассоциации научных советников. В 2001 г. его наградили юбилейной медалью, посвященной 10-летию независимости Республики Казахстан.

А.А. Женсыкбаев ушел из жизни 4 сентября 2009 года.

Светлая память об Александре Алипкановиче Женсыкбаеве будет всегда с нами – его друзьями, коллегами, учениками.

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Правила "Математического журнала" для авторов статей

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В "Математическом журнале" публикуются оригинальные статьи по основным разделам современной математики: теория функций, функциональный анализ, обыкновенные дифференциальные уравнения, уравнения с частными производными, алгебра, математическая логика, теория чисел, геометрия, топология, теория вероятностей и математическая статистика, вычислительная математика, математическая физика, математическое моделирование.

Журнал выпускается ежеквартально, четыре номера составляют том.

Статья должна быть написана на высоком научном уровне, содержать новые, четко сформулированные математические результаты и их доказательства. Во введении необходимо привести имеющиеся результаты по теме представленной работы, дать краткое содержание статьи и отразить актуальность, новизну полученных автором результатов.

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МАТЕМАТИЧЕСКИЙ ЖУРНАЛ

Том 17, №4 (66), 2017

Собственник "Математического журнала":
Институт математики и математического моделирования

Журнал подписан в печать
и выставлен на сайте http://www.math.kz
Института математики и математического моделирования МОН РК
28.12.2017 г.

Тираж 300 экз. Объем 107 стр.
Формат 70×100 1/16. Бумага офсетная № 1

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