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DISCRETE ORDER ON A DEFINABLE SET AND THE NUMBER OF MODELS

In this paper there is considered a condition, which is similar to a 2-ary relation of following. We prove that a small theory, which has the formula called the quasi-successor has the maximum number of countable models.
Keywords: *ordered structure, number of countable models.*

Introduction

This paper is devoted on finding of a condition of maximality of the number of countable non-isomorphic models of complete theories with definable linear orders. The number of countable models of theories with an $\emptyset$-definable linear order had been studied in the works [1]–[6] and others.

The main theorem

Further $\mathcal{N}$ will be a countable saturated model of a small theory T. We shall consider ordered theories and we shall suppose that $<$ is an $\emptyset$−definable relation of a linear order.

The formulas of the first order will be often written by the relations of definable sets.

Keywords: *ordered structure, number of countable models*
2010 Mathematics Subject Classification: 03C15, 03C64.

For example,

$$\begin{aligned} x < \phi(N) &\equiv \forall y(\phi(y) \to x < y), \\ x \in (\beta_1, \beta_2) &\equiv \beta_1 < x < \beta_2, \\ \phi(N) \cap \theta(N) \neq \emptyset &\equiv N \models \exists x(\phi(x) \wedge \theta(x)), \\ \phi(N) < \theta(N)^+ &\equiv N \models \forall t(\forall y(\theta(y) \to y < t) \to \forall x(\phi(x) \to x < t)). \end{aligned}$$

For any $A \subset N$ (not necessary definable) we denote

$$A^+ := \{\gamma \in N | \forall a \in A : N \models a < \gamma\};$$
$$A^- := \{\gamma \in N | \forall a \in A : N \models \gamma < a\}.$$

Definition 1. *Let $A \subset B$. The set A is said to be convex in B, if*

$$\forall x, y \in A(x < y), \forall z \in B(x < z < y \to z \in B).$$

If A is convex in N, we say that A is convex.

Definition 2. *For a formula $\phi(x, \bar{\alpha})$ a convex closure of ϕ is a formula ϕ^c, such that*

$$\phi^c(x, \bar{\alpha}) := \exists y_1, \exists y_2(\phi(y_1, \bar{\alpha}) \wedge \phi(y_2, \bar{\alpha}) \wedge (y_1 \leq x \leq y_2)).$$

Definition 3. *Let $p \in S_1(A)$ be an 1-type. A convex closure of p is a type p^c, such that*

$$p^c = \{\phi^c(x, \bar{\alpha}) | \phi(x, \bar{\alpha} \in p)\}.$$

Definition 4. *Let A and $B \subset N$, and $\phi(x, y)$ be an A-definable 2-formula. We say that $\phi(x, y)$ is B-stable, if $\forall \alpha \in B$, $\exists \gamma_1$, $\gamma_2 \in B$ $(\gamma_1 < \alpha < \gamma_2)$, such that*

$$\gamma_1 < \phi(\alpha, N) < \gamma_2, \text{ and } \phi(\alpha, N) \cap B \neq \emptyset.$$

If $B = \Theta(N)$ and Θ is an A-definable 1-formula or $B = p(N)$ and $p \in S_1(A)$ is an one-type then we we say that $\phi(x, y)$ is Θ-stable or p-stable.

Definition 5. *We say that a B-stable 2-formula $\phi(x, y)$ is convex to the right on B, if*

$$\forall \alpha \in B, \forall \beta(\beta \in \phi(\alpha, N) \to \alpha \leq \beta \land \forall \gamma \in B(\alpha < \gamma < \beta \to \gamma \in \phi(\alpha, N)).$$

If for $\Theta \in F_1(A), p \in S_1(A)$, we have $B = \Theta(N)$ or $B = p(N)$, then we say that the 2-formula is convex to the right on $\Theta(x)$, or on $p(x)$.

Definition 6. *We say that a B-stable 2-formula $\phi(x, y)$ is convex to the left on B, if*
$\forall \alpha \in B, \forall \beta(\beta \in \phi(\alpha, N) \to \beta \leq \alpha) \land \forall \gamma \in B(\beta < \gamma < \alpha \to \gamma \in \phi(\alpha, N)).$

If for $\Theta \in F_1(A), p \in S_1(A)$, we have $B = \Theta(N)$ or $B = p(N)$, then we say that the 2-formula is convex to the left on $\Theta(x)$, or on $p(x)$.

The definitions 4, 5 and 6 generalize the notions for weakly o-minimal theories, defined in [7] and [8], and introduced in [9]. The other generalization of p-stability was represented in [10]. In this work instead of the notion "p-stable" the notion "p-preserving" is used.

Definition 7. *We say that a convex to the right 2-formula $\phi(x, y)$ increases on B, if $\forall \alpha$, $\beta \in B$,*

$$(\alpha < \beta \to \phi(\beta, N)^+ \subseteq \phi(\alpha, N)^+).$$

We are interested in the case, when $\beta \in \phi(\alpha, N)$.

Definition 8. *We say that a convex to the left 2-formula $\phi(x, y)$ decreases on B, if $\forall \alpha$, $\beta \in B$,*

$$\big(\alpha < \beta \to \phi(\alpha, N)^- \subseteq \phi(\beta, N)^-\big).$$

Definition 9. *We say that an A-definable increasing (decreasing) on B 2-formula $\phi(x, y)$ is a quasi-successor on B, if $\forall \alpha \in B$, $\exists \beta \in \phi(\alpha, N) \cap B$, such that*

$$B \bigcap \big(\phi(\beta, N) \setminus \phi(\alpha, N)\big) \neq \emptyset.$$

If $\phi(x, y)$ is a quasi-successor denote

$$\phi^0(x, y) := \{x = y\};$$
$$\phi^n(x, y) := \exists y_1, \ldots, \exists y_{n-1} \ \big(\phi(x, y_1) \land \phi(y_1, y_2) \land \cdots \land \phi(y_{n-1}, y)\big);$$
$$\phi^{-n}(x, y) := \exists x_1, \ldots, \exists x_{n-1}\big(\phi(x_1, x) \land \phi(x_2, x_1) \land \cdots \land \phi(y, x_{n-1}) \land y \leq x \land \textstyle\bigwedge_{i=1}^{n-1} x_i \leq x\big).$$

Let $\phi(x, y)$ be a quasi-successor on B. Denote for $\alpha \in B$

$$V_{B,\phi}(\alpha) := \{\gamma \in B | \exists n \in \mathbb{Z}, \gamma \in \phi^n(\alpha, N) \cap B\}.$$

Theorem 1. *Let A be a finite subset of N, $p \in S_1(A)$, and $\phi(x, y)$ be an A-definable quasi-successor on $p(x)$. Then T has 2^ω countable non-isomorphic models.*

Proof. Without loss of generality we assume that $\phi(x, y)$ is convex to the right on $p(x)$.

Let $q(x, y) := \{x < y\} \cup p(x) \cup p(y) \cup \{y \notin \phi^n(x, N) | n < \omega\} \cup \{R(y, x) | R(x, y)$ is an A-definable convex to the right on p 2-formula such that $\forall n < \omega, \forall \alpha \in p(N), \phi^n(\alpha, N) \cap p(N) \subset R(\alpha, N)\} \cup \{L(x, y) | L(x, y)$ is an A-definable convex to the left on p 2-formula, such that $\forall n < \omega$, $\exists \alpha_1, \alpha_2 \in p(N), \alpha_1 < \phi^n(\alpha_2, N), \alpha_1 \in L(\alpha_2, N)\}$,

The consistence of $q(x, y)$ is verified directly.

Let the tuple $< \alpha, \beta >$ realizes $q(x, y)$. Then we fix this tuple until the end of the proof of the Theorem 1. Denote

$$\left(V_{p,\phi}(\alpha), V_{p,\phi}(\beta)\right)_{p(N)} := \{\gamma \in p(N) | V_{p,\phi}(\alpha) < \gamma < V_{p,\phi}(\beta)\}.$$

Lemma 1. $\forall \gamma_1, \gamma_2 \in \left(V_{p,\phi,}(\alpha), V_{p,\phi}(\beta)\right)_{p(N)}$,

$$tp^c(\gamma_1 | A \cup \{\alpha, \beta\}) = tp^c(\gamma_2 | A \cup \{\alpha, \beta\}).$$

Proof of Lemma 1. Suppose that the conclusion of Lemma 1 is not true, i.e. $\exists \gamma_1, \gamma_2 \in (V_{p,\phi}(\alpha), V_{p,\phi}(\beta))_{p(N)}$ and there exists an $(A \cup \{\alpha, \beta\})$-definable formula, such that $\gamma_1 \in H(N, \alpha, \beta) < \gamma_2$. We can suppose that $H(N, \alpha, \beta)$ is convex. If it is not we can take an $(A \cup \{\alpha, \beta\})$-definable formula, which determines the set $(H(N\alpha, \beta)^+)^-$.

By the theorem of compactness we can suppose that

$(*)$ there is an A-definable formula $\Theta(x) \in p$ such that for every two elements $\alpha', \beta' \in \Theta(N), \alpha' < \beta'$ if $V_{\Theta,\phi}(\alpha') < V_{\Theta,\phi}(\beta')$, then $\exists \gamma_1, \gamma_2 \in$ $(V_{\Theta,\phi}(\alpha'), V_{\Theta,\phi}(\beta'))_{\Theta(N)}$, such that $\gamma_1 \in H(N, \alpha', \beta') < \gamma_2$ and $\gamma_2 \in \phi(\gamma_1, N)$.

For $k, n_1, n_2 < \omega$ such that $n_1 + n_2 < k$ we denote

$S_{k,n_1,n_2}(H)(x, y) := (x < y \wedge y \notin \phi^k(x, N)) \rightarrow$
$\exists z_1, \exists z_2\ (x < z_1 < z_2 < y \wedge z_1 \notin \phi^{n_1}(x, N) \wedge y \notin \phi^{n_2}(z_2, N) \wedge$
$z_1 \in H(N, x, y) \wedge H(N, x, y) < z_2 \wedge z_2 \in \phi(z_1, N))$.

Claim 1. *There exist two non-constant non-decreasing functions, $s_1, s_2 : \omega \to \omega$, such that $\exists m < \omega$, $\forall k > m$, $\forall \alpha'$, $\beta' \in (\alpha, \beta)_{p(N)}$, for which the following is true:*

$$\mathcal{N} \models S_{k,s_1(k),s_2(k)}(H)(\alpha', \beta').$$

Proof of Claim 1. In the opposite case, by the theorem of compactness, we obtain a contradiction with the definition of $H(x, \alpha, \beta)$. □

Continuation of the proof of Lemma 1. We denote $H_\emptyset(x, \alpha, \beta) := \neg H(x, \alpha, \beta) \wedge \exists y(\phi(y, x) \wedge H(y, \alpha, \beta))$. It follows from the Claim 1 that $H_\emptyset(N, \alpha, \beta) \cap p(N) \neq \emptyset$ and $H_\emptyset(N, \alpha, \beta) \cap p(N) \subset V_{p,\phi}(\gamma_\emptyset)$ for some $\gamma_\emptyset \in (V_{p,\phi}(\alpha), V_{p,\phi}(\beta))_{p(N)}$.

Then we denote

$$G_0(x, \alpha, \beta) := \exists z(H(x, \alpha, z) \wedge H_\emptyset(z, \alpha, \beta));$$
$$G_1(x, \alpha, \beta) := \exists z(H(x, z, \beta) \wedge H_\emptyset(z, \alpha, \beta)).$$

So, by $(*)$ we have $G_0(N, \alpha, \beta) < V_{p,\phi}(\gamma_\emptyset)$, $V_{p,\phi}(\alpha) < G_0(N, \alpha, \beta)^+$ and $V_{p,\phi}(\gamma_\emptyset) < G_1(N, \alpha, \beta)^+, G_1(N, \alpha, \beta) < V_{p,\phi}(\beta)$.

Let

$$H_0(x) := \neg G_0(x, \alpha, \beta) \wedge \exists y(G_0(y, \alpha, \beta) \wedge \phi(y, x));$$
$$H_1(x) := \neg G_1(x, \alpha, \beta) \wedge \exists y(G_1(y, \alpha, \beta) \wedge \phi(y, x)).$$

Then by the Claim 1 we have

$H_0(N, \alpha, \beta) \cap p(N) \neq \emptyset$ and $H_0(N, \alpha, \beta) \cap p(N) \subset V_{p,\phi}(\gamma_0)$ for some $\gamma_0 \in (V_{p,\phi}(\alpha), V_{p,\phi}(\gamma_0))_{p(N)}$.

$H_1(N, \alpha, \beta) \cap p(N) \neq \emptyset$ and $H_1(N, \alpha, \beta) \cap p(N) \subset V_{p,\phi}(\gamma_1)$ for some $\gamma_1 \in (V_{p,\phi}(\gamma_\emptyset), V_{p,\phi}(\beta))_{p(N)}$.

Thus, we have $(**)$ $\alpha < H_0(N) < H_\emptyset(N) < H_1(N) < \beta$,

$V_{p,\phi}(\alpha) < V_{p,\phi}(\gamma_0) < V_{p,\phi}(\gamma_\emptyset) < V_{p,\phi}(\gamma_1) < V_{p,\phi}(\beta)$,

$H_0(N) \subset V_{p,\phi}(\gamma_0), H_\emptyset(N) \subset V_{p,\phi}(\gamma_\emptyset), H_1(N) \subset V_{p,\phi}(\gamma_1)$.

Then we designate

$$\begin{aligned}
G_{00}(x, \alpha, \beta) &:= \exists z(H(x, \alpha, z) \wedge H_0(z, \alpha, \beta));\\
G_{01}(x, \alpha, \beta) &:= \exists z_1, z_2(H(x, z_1, z_2) \wedge H_0(z_1, \alpha, \beta) \wedge H_\emptyset(z_2, \alpha, \beta));\\
G_{10}(x, \alpha, \beta) &:= \exists z_1, z_2(H(x, z_1, z_2) \wedge H_\emptyset(z_1, \alpha, \beta) \wedge H_1(z_2 \alpha, \beta));\\
G_{11}(x, \alpha, \beta) &:= \exists z(H(x, z, \beta) \wedge H_1(z, \alpha, \beta)).
\end{aligned}$$

So, by $(**)$ we have $G_{00}(N,\alpha,\beta) < V_{p,\phi}(\gamma_0)$, $V_{p,\phi}(\alpha) < G_{00}(N,\alpha,\beta)^+$ and
$V_{p,\phi}(\gamma_0) < G_{01}(N,\alpha,\beta)^+, G_{01}(N,\alpha,\beta) < V_{p,\phi}(\gamma_\emptyset)$,
$G_{10}(N,\alpha,\beta) < V_{p,\phi}(\gamma_1)$, $V_{p,\phi}(\gamma_\emptyset) < G_{10}(N,\alpha,\beta)^+$ and
$V_{p,\phi}(\gamma_\emptyset) < G_{11}(N,\alpha,\beta)^+, G_{11}(N,\alpha,\beta) < V_{p,\phi}(\beta)$.

Then by the Lemma 1 we have

$H_{00}(N,\alpha,\beta) \cap p(N) \neq \emptyset$ and $H_{00}(N,\alpha,\beta) \cap p(N) \subset V_{p,\phi}(\gamma_{00})$ for some $\gamma_{00} \in (V_{p,\phi}(\alpha), V_{p,\phi}(\gamma_0)) - p(N)$.

$H_{01}(N,\alpha,\beta) \cap p(N) \neq \emptyset$ and $H_{01}(N,\alpha,\beta) \cap p(N) \subset V_{p,\phi}(\gamma_{01})$ for some $\gamma_{01} \in (V_{p,\phi}(\gamma_0), V_{p,\phi}(\gamma_\emptyset))_{p(N)}$

$H_{10}(N,\alpha,\beta) \cap p(N) \neq \emptyset$ and $H_{10}(N,\alpha,\beta) \cap p(N) \subset V_{p,\phi}(\gamma_{10})$ for some $\gamma_{10} \in (V_{p,\phi}(\gamma_\emptyset), V_{p,\phi}(\gamma_1))_{p(N)}$ $H_{11}(N,\alpha,\beta) \cap p(N) \neq \emptyset$ and $H_{11}(N,\alpha,\beta) \cap p(N) \subset V_{p,\phi}(\gamma_{11})$ for some $\gamma_{11} \in (V_{p,\phi}(\gamma_1), V_{p,\phi}(\beta))_{p(N)}$.

Repeating this consideration ω times we obtain a countable number of A-definable formulas $H_\delta, \delta \in 2^{<\omega}$ such that for every $\tau \in 2^\omega$, $\tau(n) \in \{0,1\}$ there is $p_\tau \in S_1(A)$, one-type over A which extends the following set of A-definable 1-formulas:

$$\Gamma_\tau(x) := \{x < H_{\tau(1),\ldots,\tau(n)}(N,\bar{\alpha},\beta) | \tau(n+1) = 0\} \cup \{H_{\tau(1),\ldots,\tau(n)}(x,\alpha,\beta) | \tau(n+1) = 1\}.$$

This contradicts to our assumption that T is small. □

As a corollary of the proof of the Lemma 1 we obtain the following lemma.

Lemma 2. *For every* $\bar{\alpha}_n := <\alpha_1,\ldots,\alpha_n>$, $\alpha_i \in (V_{p,\phi}(\alpha), V_{p,\phi}(\beta))_{p(N)}$, $1 \le i \le n$; *such that* $V_{p,\phi}(\alpha_i) < V_{p,\phi}(\alpha_{i+1})$ $(1 \le i \le (n-1))$, *for every* $\bar{\gamma} \in N$ *such that* $tp(\bar{\gamma}|A \cup \bar{\alpha}_n \cup \{\alpha,\beta\})$ *is isolated the following is true:*

$\forall \gamma_1, \gamma_2 \in (V_{p,\phi,}(\alpha_i), V_{p,\phi}(\alpha_{i+1}))$ *we have*

$$tp^c(\gamma_1|A \cup \bar{\alpha}_n \cup \bar{\gamma} \cup \{\alpha,\beta\}) = tp^c(\gamma_2|A \cup \bar{\alpha}_n \cup \bar{\gamma} \cup \{\alpha,\beta\}).$$

It follows from the Lemma 2 that any element $\gamma \in (V_{p,\phi,}(\alpha_i), V_{p,\phi}(\alpha_{i+1}))$ has a non-isolated one-type over $A \cup \bar{\alpha}_n \cup \bar{\gamma} \cup \{\alpha,\beta\}$ because it is irrational.

Continuation of the proof of Theorem 1. Let $2^{<\omega}$ be a set of all finite tuples of elements from $\{0,1\}$. Then for every $\eta \in 2^{<\omega}$, $\eta := <\eta(1), \eta(2), \ldots, \eta(n)>$ we denote $l(\eta) := n$. Let $\eta \neq \pi \in 2^{<\omega}$, then we say that η is less than π $(\eta < \pi)$ if either $\eta \subset \pi \wedge \pi(l(\eta)+1) = 1$, or $\exists i \le min\{l(\eta), l(\pi)\}$, $\forall j < i, \eta(j) = \pi(j) \wedge \eta(i) = 0 \wedge \pi(i) = 1$.

Let $< \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots >_{n<\omega}$ be an ω-consequence of elements from $p(N)$, such that

$$V_{p,\phi}(\alpha) < V_{p,\phi}(\alpha_i) < V_{p,\phi}(\alpha_{i+1}) < V_{p,\phi}(\beta),\ (1 \leq i \leq \omega).$$

Then for every $\tau \in 2^\omega$ we shall construct, by using of the Lemma 2, the countable model $\mathcal{M}_\tau \prec \mathcal{N}$ such that for every $n < \omega, \alpha_n \in M_\tau$,

$\tau(n) = 0 \iff$ in M_τ there is no element from $(V_{p,\phi}(\alpha_{2n}), V_{p,\phi}(\alpha_{2n+1}))_{p(N)}$, and

$\tau(n) = 1 \iff$ for any $\eta \in 2^{<\omega}$, there exists an element $\alpha_{n,\eta} \in (V_{p,\phi}(\alpha_{2n}), V_{p,\phi}(\alpha_{2n+1}))_{p(N)} \cap M_\tau$, such that for every two $\eta \neq \pi \in 2^{<\omega}$ if $\eta < \pi$, then $V_{p,\phi}(\alpha_{n,\eta}) < V_{p,\phi}(\alpha_{n,\pi})$.

Construction of $\mathcal{M}_\tau$.

Let $\tau \in 2^\omega$. We shall construct M_τ as a union of increasing chain of finite sets $M_\tau = \bigcup_{m<\omega} B_m$, $B_{m-1} \subset A_m \subset B_m$, such that $|B_m|, |A_m| < \omega$; $|B_m \setminus A_m| = m^2$; $tp(B_m \setminus A_m | A_m)$ is isolated and for every $i \leq m$ we have some fixed enumeration of $F_1(B_i)$, where $F_1(B_i)$ is the set of all B_i-definable 1-formulas.

Step 0. We denote $B_0 := A$. Fix some enumeration of $F_1(B_0)$.

Step $m+1$. By the Lemma 2, and by using the approach in the choice of γ_η from the proof of the Lemma 1 we can determinate

$$A_{m+1} := B_m \cup \{\alpha_{i,\eta} | \eta \in 2^{<\omega},\ l(\eta) \leq m+1,\ \tau(i) = 1\}.$$

For every $k < m+1$ we denote $B_{m+1,k} := A_{m+1} \cup \{\beta_{m+1,k'} | k' < k\}$.

We define $\beta_{m+1,k}$. Let $\Theta_{k,j}(x)$ be a 1-formula from $F_1(B_k)$, such that $\Theta_{m+1,k}(N) \cap B_{m+1,k} = \emptyset$ and j is minimal with this property. Then take $G(x)$ – an arbitrary atom from $F_1(B_{m+1,k})$ (i.e. for every $K(x) \in F_1(B_{m+1,k})$ if $G(N) \cap K(N) \neq \emptyset$ then $G(N) \subseteq K(N)$), such that $G(N) \subseteq \Theta_{k,j}(N)$, and arbitrary element from $\beta_{m+1,k} \in G(N)$. The existence of $G(x)$ follows from our assumption that T is small, and because $B_{m+1,k}$ is finite.

Then put $B_{m+1} := \bigcup_{k<m+1} B_{m+1,k}$ and fix some enumeration of $F_1(B_{m+1})$.

Let us to verify that M_τ is model. Consider an arbitrary M_τ-definable 1-formula $\Psi(x, \bar{\gamma}), \bar{\gamma} \in M_\tau$, such that $N \models \exists x \Psi(x, \bar{\gamma})$. Then there exists $k < \omega$ such that $\bar{\gamma} \cap (B_k \setminus B_{k-1}) \neq \emptyset$. Thus, for some $m < \omega, k < m$ we have $N \models \Psi(\beta_{m,k}, \bar{\gamma}), \beta \in M_\tau$. This means [11] that $\mathcal{M}_\tau \prec \mathcal{N}$.

It is clear that if $\tau \neq \tau' \in 2^\omega$, then in language $L^* := L \cup A \cup \{\alpha, \beta\}$, $\mathcal{M}^*_\tau \not\cong_{L^*} \mathcal{M}^*_{\tau'}$. So, because any countable model of T can generate maximum countable number of non-isomorphic models in language L^*, T has 2^ω countable non-isomorphic models. $\square$

References

1 Rubin M. Theories of linear order // Israel J. Math. — 1974. — V. 17. — P. 392–443.

2 Shelah S. End extensions and numbers of countable models // The Journal of Symbolic Logic. — 1978. V. 43. — P. 550–562.

3 Peretiat'kin M. On theories with three countable models // Algebra i Logika. — 1980. — V. 19, №2. — P. 224–236.

4 Omarov B. Non-essential expansions of complete theories // Algebra i Logika. — 1983. — V. 22, №5. — P. 542–550.

5 Mayer L. Vaught's conjecture for o-minimal theories // The Journal of Symbolic Logic. — 1988. — V. 53. — P. 146–159.

6 Alibek A. and Baizhanov B.S. Examples of countable models of weakly o-minimal theory // International Journal of Mathematics and Physics. — 2012. — V. 3, №2. — P. 1–8.

7 Baizhanov B.S. Classification of one-types in weakly o-minimal theories and its corollaries // Preprint. — 1996. — 33 p.

8 Baizhanov B.S. Expansion of a model of a weakly o-minimal theory by a family of unary predicates // The Journal of Symbolic Logic. — 2001. — V. 66. — P. 1382–1414.

9 Baizhanov B.S. and Hodges W. Countable ordered models with small theories // Preprint. — 2001. — 22 p.

10 Baizhanov B.S., Sudoplatov S.V. and Verbovskiy V.V. Conditions for non-symmetric relations of semi-isolation // Siberian electronic mathematical reports. — 2012. — V. 9. — P. 161–184.

11 Hodges W. A Shorter Model Theory. — Cambridge: Cambridge University Press, 1997. — 772 p.

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Алибек А.А., Байжанов Б.С., Замбарная Т.С. ФОРМУЛАЛЫ ЖИЫН-ДАҒЫ ДИСКРЕТТІ РЕТ ЖӘНЕ МОДЕЛЬДЕР САНЫ

Мақалада екі орынды ілесу қатынасына ұқсас шарт қарастырылады және де квази-ілесу деп аталатын формулаға ие кіші реттелген теорияда максималды саналымды модельдер саны бар екені дәлелденеді.

Алибек А.А., Байжанов Б.С., Замбарная Т.С. ДИСКРЕТНЫЙ ПОРЯДОК НА ФОРМУЛЬНОМ МНОЖЕСТВЕ И ЧИСЛО МОДЕЛЕЙ

В статье рассматривается условие, похожее на 2-х местное отношение следования и доказывается, что малая упорядоченная теория, обладающая формулой, называемой квази-следователем, имеет максимальное число счётных моделей.

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ESTIMATES OF THE SOLUTION OF THE TWO–PHASE SINGULARLY PERTURBED PROBLEM FOR THE PARABOLIC EQUATIONS. II

Two-phase boundary value problem for the parabolic equations with two small parameters at the principle terms in the boundary condition is studied in the Hölder space. There is derived an estimate of the solution of the problem with the constant independent on the small parameters, an estimate of the time derivative of the solution at the small parameter in the boundary condition is obtained.
Keywords: *parabolic equation, small parameters in the boundary condition, estimates of the solution, Hölder space.*

1 Statement of the problem. Main results

Let $D_1 := \mathbb{R}^n_- = \{x : x' \in \mathbb{R}^{n-1},\ x_n < 0\}$, $D_2 := \mathbb{R}^n_+ = \{x : x' \in \mathbb{R}^{n-1},\ x_n > 0\}$, $n \geq 2$, $R := \{x : x' \in \mathbb{R}^{n-1}, x_n = 0\}$, $D_{pT} := D_p \times (0,T)$, $p = 1,2$, $R_T := R \times [0,T]$, $x = (x', x_n)$, $x' = (x_1, \dots, x_{n-1})$.

Consider the problem with the unknown functions $v_1(x,t)$ and $v_2(x,t)$

$$\partial_t v_1 - a\,\Delta v_1 = 0 \text{ in } D_{1T}, \tag{1.1}$$

Keywords: *parabolic equation, small parameters in the boundary condition, estimates of the solution, Hölder space*
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$$\partial_t v_2 - \sum_{i,j=1}^{n} a_{i\,j}\,\partial^2_{x_i x_j} v_2 = 0 \text{ in } D_{2T}, \tag{1.2}$$

$$v_p|_{t=0} = 0 \text{ in } D_p,\ p = 1, 2, \tag{1.3}$$

$$\big(v_1 - v_2\big)|_{x_n=0} = 0 \text{ on } R_T, \tag{1.4}$$

$$\big(\varepsilon\,\partial_t v_1 + \kappa d\nabla^T v_1 - h\nabla^T v_2\big)\big|_{x_n=0} = \varphi(x', t) \text{ on } R_T, \tag{1.5}$$

where all coefficients are constant, $a > 0$, $d = (d', d_n)$, $d' = (d_1, \ldots, d_{n-1})$, $h = (h', h_n)$, $h' = (h_1, \ldots, h_{n-1})$, $\nabla^T = \text{colon}(\partial_{x_1}, \ldots, \partial_{x_n})$ — column-vector, $d\,h^T = d_1h_1 + \ldots + d_nh_n$ - scalar product, $\Delta = \partial^2_{x_1} + \ldots + \partial^2_{x_n}$, $\partial_t = \partial/\partial t$, $\partial_{x_i} = \partial/\partial x_i$, $\kappa > 0$ and $\varepsilon > 0$ – small parameters,

$$a_{i\,j} = a_{j\,i},\ i,\ j = 1, \ldots, n,\ \sum_{i,j=1}^{n} a_{i\,j}\,\xi_i\xi_j \geq a_0\xi^2,\ \xi \in \mathbb{R}^n,\ a_0 = \text{const} > 0.$$

This paper is a continuation of a previous one [1], where there was constructed the solution of the problem (1.1) – (1.5) in the explicit form, the estimates of Green functions were derived. Now we shall obtain the estimates of the solution with the constant independent on the small parameters κ and ε (Theorem 1.1), the estimate of the time derivative $\varepsilon\,\partial_t v_1|_{x_n=0}$ in the boundary condition (Theorem 1.2) in the Hölder space.

This model problem is on the basis of the establishment of the unique solvability of the linear and free boundary problems for the parabolic equation with two small parameters in the boundary condition. The Theorems 1.1 and 1.2 proved in this paper permit also to justify the convergence of the solution of the perturbed problem to the solution of the unperturbed one, to obtain the existence, uniqueness and estimates of the unperturbed problem without loss of the smoothness of given functions.

We shall study the problem (1.1) – (1.5) in the Hölder space $C_x^{2+l,\,1+l/2}{}_t(\overline{\Omega}_T)$, l — positive non-integer, of the functions $u(x,t)$ with the norm [2]:

$$|u|^{(2+l)}_{\Omega_T} = \sum_{2m_0+|m|\leq 2+[l]} |\partial_t^{m_0}\partial_x^m u|_{\Omega_T} + \sum_{2m_0+|m|=2+[l]} [\partial_t^{m_0}\partial_x^m u]^{(\alpha)}_{\Omega_T} +$$

$$+ \sum_{2m_0+|m|=1+[l]} [\partial_t^{m_0}\partial_x^m u]^{(\frac{1+\alpha}{2})}_{t,\Omega_T},\ \alpha = l - [l] \in (0,1), \tag{1.6}$$

where $\Omega_T := \Omega \times (0,T)$, Ω is a domain in $\mathbb{R}^n$, $n \geq 2$, $m = (m_1, \ldots, m_n)$, m_i, $i = 1, \ldots, n$, — non-negative integers, $|m| = m_1 + \ldots + m_n$,

$$|v|_{\Omega_T} = \max_{(x,t)\in\overline{\Omega}} |v|, \quad [v]^{(\alpha)}_{\Omega_T} = [v]^{(\alpha)}_{x,\Omega_T} + [v]^{(\alpha/2)}_{t,\Omega_T},$$

$$[v]^{(\alpha)}_{x,\Omega_T} = \max_{(x,t),(z,t)\in\overline{\Omega}_T} \frac{\big|v(x,t) - v(z,t)\big|}{|x-z|^\alpha}, \quad [v]^{(\alpha)}_{t,\Omega_T} = \max_{(x,t),(x,t_1)\in\overline{\Omega}_T} \frac{\big|v(x,t) - v(x,t_1)\big|}{|t-t_1|^\alpha}.$$

By $\overset{\circ}{C}{}^{2+l,\,1+l/2}_{x\ \ t}(\overline{\Omega}_T)$ we designate the subset of the functions $u(x,t) \in C^{2+l,\,1+l/2}_{x\ \ t}(\overline{\Omega}_T)$, such that $\partial_t^k u\big|_{t=0} = 0$, $k = 0, \ldots, 1 + [l/2]$.

The following lemma is valid.

Lemma 1.1. [3] *In* $\overset{\circ}{C}{}^{l,\,l/2}_{x\ t}(\overline{\Omega}_T)$, l *— positive non-integer, the norm* $|u|^{(2+l)}_{\Omega_T}$ *defined by formula* (1.6) *is equivalent to the norm*

$$||u||^{(l)}_{\Omega_T} = \sup_{(x,t)\in\Omega_T} t^{-l/2}|u(x,t)| +$$

$$+ \sum_{2m_0+|m|=2+[l]} [\partial_t^{m_0}\partial_x^m u]^{(\alpha)}_{\Omega_T} + \sum_{2m_0+|m|=1+[l]} [\partial_t^{m_0}\partial_x^m u]^{(\frac{1+\alpha}{2})}_{t,\Omega_T}, \ \alpha = l - [l] \quad (1.7)$$

(for $[l] = 0$ *the last sum is omitted).*

We formulate the main results of a present work.

Theorem 1.1. *Let* $d_n > 0$, $h_n > 0$, $\kappa \in [0, \kappa_0]$, $\varepsilon \in (0, \varepsilon_0]$.

For every function $\varphi(x',t) \in \overset{\circ}{C}{}^{1+l,\frac{1+l}{2}}_{x'\ \ t}(R_T)$, l *— positive non-integer, the problem (1.1) – (1.5) has a unique solution* $v_p(x,t) \in \overset{\circ}{C}{}^{2+l,1+l/2}_{x\ \ \ t}(\overline{D}_{pT})$, $p = 1, 2$, $\varepsilon\,\partial_t v_1|_{x_n=0} \in \overset{\circ}{C}{}^{1+l,\frac{1+l}{2}}_{x'\ \ t}(R_T)$ *and it satisfies an estimate*

$$\sum_{p=1}^{2} |v_p|^{(2+l)}_{D_{pT}} + |\varepsilon\partial_t v_1|^{(1+l)}_{R_T} \leq C_1 |\varphi|^{(1+l)}_{R_T}, \quad (1.8)$$

where a constant C_1 *does not depend on* κ *and* ε.

Theorem 1.2. *Let* $d_n > 0$, $h_n > 0$, $\kappa \in [0, \kappa_0]$, $\varepsilon \in (0, \varepsilon_0]$, $l = k + \alpha$, $k = 0, 1, \ldots$, $\alpha \in (0,1)$.

For every function $\varphi(x', t) \in \overset{\circ}{C}{}_{x'}^{1+k+\alpha, \frac{1+k+\alpha}{2}}{}_{t}(R_T)$ *the time derivative* $\varepsilon\partial_t v_1(x,t)|_{x_n=0}$ *in the condition* (1.5) *of the problem* (1.1) – (1.5) *satisfies an estimate*

$$|\varepsilon\partial_t v_1|_{C_{x'}^{1+k+\beta, \frac{1+k+\beta}{2}}{}_{t}(R_T)} \leq C_2\varepsilon^{\alpha/4}|\varphi|_{C_{x'}^{1+k+\alpha, \frac{1+k+\alpha}{2}}{}_{t}(R_T)}, \ \beta \in (0, \alpha/2), \qquad (1.9)$$

where a constant C_2 *is independent on* κ *and* ε*.*

In [1] under the conditions $d_n > 0, \ h_n > 0$, there was constructed the solution of the problem (1.1) – (1.5) in the explicit form

$$v_p(x,t) = \frac{1}{\varepsilon}\int_0^t d\tau \int_{\mathbb{R}^{n-1}} \varphi(y', \tau) G_p(x' - y', x_n, t - \tau)\, dy', \qquad (1.10)$$

where

$$G_p(x,t) = \int_0^t K_p(x, \sigma, t - \sigma) d\sigma,$$

$$K_1(x,\sigma,t) = -4a\int_0^t d\tau_1 \int_{\mathbb{R}^{n-1}} \partial_{x_n}\Gamma_1(x - \eta - \frac{\kappa}{\varepsilon}d\sigma, t - \tau_1)\times$$

$$\times\frac{1}{\sqrt{|A_n|}}\sum_{k=1}^{n} a_{kn}\partial_{\eta_k}\Gamma_2(\eta + \frac{h\sigma}{\varepsilon}, \tau_1)\big|_{\eta_n=0}\, d\eta' \equiv$$

$$\equiv \int_0^t d\,\tau_1 \int_{\mathbb{R}^{n-1}} \frac{-x_n + \eta_n + \kappa d_n\sigma/\varepsilon}{(2\sqrt{a\pi(t-\tau_1)})^n(t-\tau_1)} e^{-\frac{(x-\eta-\kappa d\,\sigma/\varepsilon)^2}{4a(t-\tau_1)}}\times$$

$$\times\frac{h_n\sigma/\varepsilon}{(2\sqrt{\pi\tau_1})^n\tau_1}\,\frac{1}{\sqrt{|A_n|}}e^{-\frac{\sum_{i,j=1}^n a^{ij}(\eta_i + h_i\sigma/\varepsilon)(\eta_j+h_j\sigma/\varepsilon)}{4\tau_1}}\big|_{\eta_n=0}\, d\,\eta', \ x_n < 0, \qquad (1.11)$$

$$K_2(x,\sigma,t) = -4a\int_0^t d\tau_1 \int_{\mathbb{R}^{n-1}} \partial_{\eta_n}\Gamma_1(\eta - \frac{\kappa}{\varepsilon}d\sigma, \tau_1)\times$$

$$\times\frac{1}{\sqrt{|A_n|}}\sum_{k=1}^{n} a_{kn}\partial_{x_k}\Gamma_2(x - \eta + \frac{h\sigma}{\varepsilon}, t - \tau_1)\big|_{\eta_n=0}\, d\eta' \equiv$$

$$\equiv \int_0^t d\,\tau_1 \int_{\mathbb{R}^{n-1}} \frac{\kappa d_n\sigma/\varepsilon}{(2\sqrt{a\pi\tau_1})^n\tau_1} e^{-\frac{(\eta-\kappa d\sigma/\varepsilon)^2}{4a\tau_1}} \frac{x_n - \eta_n + h_n\sigma/\varepsilon}{(2\sqrt{\pi(t-\tau_1)})^n(t-\tau_1)}\times$$

$$\times \frac{1}{\sqrt{|A_n|}} e^{-\frac{\sum_{i,j=1}^n a^{ij}(x_i-\eta_i+h_i\sigma/\varepsilon)(x_j-\eta_j+h_j\sigma/\varepsilon)}{4(t-\tau_1)}}\Big|_{\eta_n=0} d\eta',\ x_n>0, \qquad (1.12)$$

where

$$\Gamma_1(x,t)=\frac{1}{(2\sqrt{a\pi t})^n}e^{-\frac{x^2}{4at}},\quad \Gamma_2(x,t)=\frac{1}{(2\sqrt{\pi t})^n}e^{-\frac{\sum_{i,j=1}^n a^{ij}x_ix_j}{4t}} \qquad (1.13)$$

are the fundamental solutions of the equations (1.1), (1.2) respectively, $|A_n|>0$ – determinant of a matrix $A_n=\{a_{ij}\}_{i,j=1}^n$, a^{ij}, $i,j=1,\dots,n$, are the elements of the inverse matrix A_n^{-1}.

In [1] there was obtained also the following estimates of the kernels (1.11), (1.12) for $\kappa\in[0,\,\kappa_0]$, $\varepsilon\in(0,\,\varepsilon_0]$:

$$|\partial_t^k\partial_x^m K_p(x,\sigma,t)|\le C_3\frac{1}{t^{\frac{n+2k+|m|+1}{2}}}\,e^{-\frac{q_1^2x^2}{t}-\frac{q_2^2\sigma^2}{\varepsilon^2 t}}, \qquad (1.14)$$

where

$$q_1^2=\frac{c_0^2h_n^2}{2(h^2+\kappa_0^2d^2+2\kappa_0|d'h'|)},\quad q_2^2=\frac{c_0^2h_n^2}{2},$$

constant C_3 does not depend on ε and κ, constant c_0^2 is from the estimates

$$|\partial_t^k\partial_x^m\Gamma_p(x,t)|\le C_4\frac{1}{t^{\frac{n+2k+|m|}{2}}}\,e^{-c_0^2\frac{x^2}{t}},\ p=1,2,\ c_0=\text{const}>0.$$

2 Proof of Theorem 1.1.

In [4, 5] there was considered the problem with the unknown functions $u_1(x,t)$ and $u_2(x,t)$ satisfying zero initial data

$$\begin{gathered}\partial_t u_1-a_p\,\Delta u_p=0 \text{ in } D_{pT},\ p=1,2,\\ (u_1-u_2)|_{x_n=0}=0,\ (\varepsilon\,\partial_t u_1+b\nabla^T u_1-c\nabla^T u_2)|_{x_n=0}=\varphi(x',t) \text{ on } R_T,\end{gathered} \qquad (2.1)$$

where $a_p>0$, $p=1,2$, $\varepsilon>0$ is a small parameter.

For this problem the following theorem is valid [5].

Theorem 2.1. *Let* $b_n>0$, $c_n>0$, $0<\varepsilon\le\varepsilon_0$.

For every function $\varphi(x',t) \in \overset{\circ}{C}{}_{x'}^{1+l,\frac{1+l}{2}}{}_{t}(R_T)$, l *— positive non-integer, the problem (2.1) has a unique solution* $u_p(x,t) \in \overset{\circ}{C}{}_{x}^{2+l,1+l/2}{}_{t}(\overline{D}_{pT})$, $p=1,2$, $\varepsilon\,\partial_t u_1(x,t) \in \overset{\circ}{C}{}_{x'}^{1+l,\frac{1+l}{2}}{}_{t}(R_T)$, *and it satisfies the estimate*

$$\sum_{p=1}^{2} |u_p|_{D_{pT}}^{(2+l)} + |\varepsilon\partial_t u_1|_{R_T}^{(1+l)} \le C_1 |\varphi|_{R_T}^{(1+l)}, \tag{2.2}$$

where a constant C_1 *does not depend on* ε.

The estimate (1.14) of the Green functions K_p of the problem (1.1) – (1.5) does not depend on κ, the Green functions of the problem (2.1) satisfy the same inequality [4, 5]. The theorem 2.1 for $l=\alpha\in(0,1)$ was proved in [4] by direct evaluations of the norms $|u_p|_{D_{pT}}^{(2+\alpha)}$ only with the help of the estimates of the Green functions, the estimate of the norm $|\varepsilon\partial_t u_1|_{R_T}^{(1+l)}$ follows from the second boundary condition. Further, in [5] there was proved Theorem 2.1 for any positive non–integer l with the help of the estimetes derived in [4]. Thus, Theorem 1.1 is valid and the constant C_1 in the estimate (1.8) does not depend on κ and ε.

3 Proof of Theorem 1.2

First, we prove the following lemma.

Lemma 3.1. *Let the conditions of the Theorem 1.2 be fulfilled.*

Then the derivatives of the function $\varepsilon\partial_t v_1(x,t)|_{x_n=0}$ *from the condition* (1.5) *of the problem* (1.1) – (1.5) *may be represented in the form*

$$\varepsilon\partial_t^{m_0}\partial_{x'}^{m'}\partial_t v_1|_{x_n=0} = -W_1^{(s)}(x',t) + W_2^{(s)}(x',t) - W_3^{(s)}(x',t), \tag{3.1}$$

$$W_1^{(s)}(x',t) = \frac{\kappa}{\varepsilon}\int_0^t d\tau \int_{\mathbb{R}^{n-1}} dy' \int_0^\tau \big(\Phi_s(y',\tau-\sigma) - \Phi_s(y',\tau)\big)\times \\ \times d\nabla_x^T K_1(x-y',\sigma,t-\tau)\,d\sigma\big|_{x_n=0}, \tag{3.2}$$

$$W_2^{(s)}(x',t) = \frac{1}{\varepsilon}\int_0^t d\tau \int_{\mathbb{R}^{n-1}} dy' \int_0^\tau \big(\Phi_s(y',\tau-\sigma) - \Phi_s(y',\tau)\big)\times \\ \times h\nabla_x^T K_2(x-y',\sigma,t-\tau)\,d\sigma\big|_{x_n=0}, \tag{3.3}$$

$$W_3^{(s)}(x',t)=\int_0^t d\tau\int_{\mathbb{R}^{n-1}}\Phi_s(y',\tau)K_1(x-y',\tau,t-\tau)\,dy'\big|_{x_n=0}\equiv$$
$$\equiv\int_0^t d\tau\int_{\mathbb{R}^{n-1}}\Phi(y',t-\tau)K_1(x-y',t-\tau,\tau)\,dy'\big|_{x_n=0}, \qquad (3.4)$$

$\Phi_s(x',t)=\partial_t^{m_0}\partial_{x'}^{m'}\varphi(x',t),\quad 2m_0+|m'|=s,\ s=0,1,2,\dots,k,1+k,\quad k=0,1,\dots,\ m'=(m_1,\dots,m_{n-1})$.

Proof of Lemma 3.1. We differentiate the solution (see (1.10))

$$v_p(x,t)=1/\varepsilon\int_0^t d\tau\int_{\mathbb{R}^{n-1}}\varphi(y',t-\tau)G_p(x-y',t-\tau)\,dy',$$

$$G_p(x-y',t-\tau)=\int_0^{t-\tau}K_p(x-y',\sigma,t-\tau-\sigma)d\sigma,$$

of the problem (1.1) – (1.5) and after the substitution $\tau_1=\tau+\sigma$ in the integral with respect to τ we obtain

$$\partial_t^{m_0}\partial_{x'}^{m'}v_p(x,t)=1/\varepsilon\int_0^t d\tau\int_{\mathbb{R}^{n-1}}\partial_\tau^{m_0}\partial_{y'}^{m'}\varphi(y',\tau)G_p(x-y',t-\tau)\,dy'=$$

$$=1/\varepsilon\int_0^t d\tau\int_{\mathbb{R}^{n-1}}dy'\int_0^\tau\Phi_s(y',\tau-\sigma)K_p(x-y',\sigma,t-\tau)\,d\sigma,\ p=1,2, \qquad (3.5)$$

$\Phi_s(x',t)=\partial_t^{m_0}\partial_{x'}^{m'}\varphi(x',t),\ 2m_0+|m'|=s.$

From the condition (1.5) we find

$$\varepsilon\,\partial_t^{m_0}\partial_{x'}^{m'}\partial_t v_1\big|_{x_n=0}=\Phi_s(x',t)-\left(\kappa d\nabla^T\partial_t^{m_0}\partial_{x'}^{m'}v_1-h\nabla^T\partial_t^{m_0}\partial_{x'}^{m'}v_2\right)\Big|_{x_n=0} \qquad (3.6)$$

and applying formula (3.5) we represent the difference of the derivatives in (3.6) in the form

$$\left(\kappa d\nabla^T\partial_t^{m_0}\partial_{x'}^{m'}v_1-h\nabla^T\partial_t^{m_0}\partial_{x'}^{m'}v_2\right)\big|_{x_n=0}=:$$

$$=:I(x,t)\big|_{x_n=0}+W_1^{(s)}(x',t)-W_2^{(s)}(x',t)+W_3^{(s)}(x',t), \qquad (3.7)$$

here the functions $W_1^{(s)}(x',t),\ W_2^{(s)}(x',t)$ are determined by the formulas (3.2), (3.3),

$$W_3^{(s)}(x',t) = \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \Phi_s(y',\tau)dy' \times \\ \times \int_\tau^\infty \Big(\frac{\kappa}{\varepsilon} d\nabla_x^T K_1(x-y',\sigma,t-\tau) - \frac{1}{\varepsilon} h\nabla_x^T K_2(x-y',\sigma,t-\tau)\Big)\, d\sigma\big|_{x_n=0}, \tag{3.8}$$

$$I(x,t) = \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \Phi_s(y',\tau)dy' \times \\ \times \int_0^\infty \big(\frac{\kappa}{\varepsilon} d\nabla_x^T K_1(x-y',\sigma,t-\tau) - \frac{1}{\varepsilon} h\nabla_x^T K_2(x-y',\sigma,t-\tau)\big)\, d\sigma. \tag{3.9}$$

We show that

$$I(x,t)\big|_{x_n=0} = \Phi_s(x',t), \tag{3.10}$$

then from (3.6), (3.7) the required formula (3.1) will follow.

Consider an integral (3.9). Let $\Phi_s(x',t) = 1$, then

$$I(x,t) = \int_0^t d\tau \int_{\mathbb{R}^{n-1}} dy' \int_0^\infty \frac{\kappa}{\varepsilon} d\nabla_x^T K_1(x-y',\sigma,t-\tau)\, d\sigma -$$

$$- \int_0^t d\tau \int_{\mathbb{R}^{n-1}} dy' \int_0^\infty \frac{1}{\varepsilon} h\nabla_x^T K_2(x-y',\sigma,t-\tau)\, d\sigma. \tag{3.11}$$

As in [1] we integrate (3.11) with respect to $y',\ \eta', \tau_1$ and τ applying the formulas

$$\int_0^t \frac{a\,b}{4\sqrt{\pi}(t-\tau_1)^{3/2}\tau_1^{3/2}} e^{-\frac{a^2}{4(t-\tau_1)} - \frac{b^2}{4\tau_1}} d\tau_1 = \frac{(a+b)}{2t^{3/2}} e^{-\frac{(a+b)^2}{4t}},\ a>0,\ b>0,$$

$$\int_0^t \frac{a}{2\sqrt{\pi}(t-\tau)^{3/2}} e^{-\frac{a^2}{4(t-\tau)}} d\tau = \frac{2}{\sqrt{\pi}} \int_{\frac{a}{2\sqrt{t}}}^\infty e^{-\zeta^2} d\zeta := \operatorname{erfc}\frac{a}{2\sqrt{t}},\ a>0,$$

$$\int_{\mathbb{R}^{n-1}} e^{-\frac{\sum_{i,j=1}^n a^{ij}\xi_i\xi_j}{4t}} d\xi' = \frac{(2\sqrt{\pi t})^{n-1}}{\sqrt{|A_{n-1}^{-1}|}} e^{-\frac{|A_n^{-1}|}{|A_{n-1}^{-1}|}\frac{\xi_n^2}{4t}},$$

where A_{n-1}^{-1} is a matrix of $(n-1)$–order obtained from a matrix A_n^{-1}, in which the n–th line and n-th row are crossed out, $|A_n^{-1}| = \det A_n^{-1} = 1/|A_n|$, then we obtain

$$I(x,t)\big|_{x_n=0} = \frac{\kappa}{\varepsilon} d\nabla_x^T \int_0^\infty \operatorname{erfs}\frac{\frac{-x_n+\kappa d_n\sigma_1/\varepsilon}{\sqrt{a}} + \frac{h_n\sigma_2/\varepsilon}{\sqrt{|A_n||A_{n-1}^{-1}|}}}{2\sqrt{t}}\Big|_{\sigma_1=\sigma_2=\sigma} d\sigma\big|_{x_n=0} -$$

$$-\frac{1}{\varepsilon} h\nabla_x^T \int_0^\infty \operatorname{erfs}\frac{\frac{\kappa d_n\sigma_1/\varepsilon}{\sqrt{a}} + \frac{x_n+h_n\sigma_2/\varepsilon}{\sqrt{|A_n||A_{n-1}^{-1}|}}}{2\sqrt{t}}\Big|_{\sigma_1=\sigma_2=\sigma} d\sigma\big|_{x_n=0} =$$

$$= -\int_0^\infty \partial_{\sigma_1} \operatorname{erfs}\frac{\frac{-x_n+\kappa d_n\sigma_1/\varepsilon}{\sqrt{a}} + \frac{h_n\sigma_2/\varepsilon}{\sqrt{|A_n||A_{n-1}^{-1}|}}}{2\sqrt{t}}\Big|_{\sigma_1=\sigma_2=\sigma} d\sigma\big|_{x_n=0} -$$

$$-\int_0^\infty \partial_{\sigma_2} \operatorname{erfs}\frac{\frac{\kappa d_n\sigma_1/\varepsilon}{\sqrt{a}} + \frac{x_n+h_n\sigma_2/\varepsilon}{\sqrt{|A_n||A_{n-1}^{-1}|}}}{2\sqrt{t}}\Big|_{\sigma_1=\sigma_2=\sigma} d\sigma\big|_{x_n=0} =$$

$$= -\int_0^\infty \frac{d}{d\sigma} \operatorname{erfs}\frac{\frac{\kappa d_n\sigma/\varepsilon}{\sqrt{a}} + \frac{h_n\sigma/\varepsilon}{\sqrt{|A_n||A_{n-1}^{-1}|}}}{2\sqrt{t}} d\sigma = \operatorname{erfc} 0 = 1 \qquad (3.12)$$

here $d_n > 0, h_n > 0$, erfs ζ satisfies an estimate erfc $\zeta \le \sqrt{2}\, e^{-\zeta^2/2},\ \zeta \ge 0$.

Thus, for $\Phi_s(x',t) = 1$ due to (3.6), (3.7), (3.12) the formula (3.1) is proved.

Let $\Phi_s(x',t)$ be the function from the Hölder space $\overset{\circ}{C}{}_{x'\ \ t}^{\alpha,\alpha/2}(R_T),\ \alpha \in (0,1)$. We represent the potential $I(x,t)$ determined by (3.9) as follows:

$$I(x,t) = J_1(x,t) + \Phi_s(x',t)\int_0^t d\tau \int_{\mathbb{R}^{n-1}} dy' \times$$

$$\times \int_0^\infty \Big(\frac{\kappa}{\varepsilon} d\nabla_x^T K_1(x-y',\sigma,t-\tau) - \frac{1}{\varepsilon} h\nabla_x^T K_2(x-y',\sigma,t-\tau)\Big) d\sigma, \qquad (3.13)$$

$$J_1(x,t) = \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \big(\Phi_s(y',\tau) - \Phi_s(x',t)\big)\, dy' \int_0^\infty \big(\frac{\kappa}{\varepsilon} d\nabla_x^T K_1 - \frac{1}{\varepsilon} h\nabla_x^T K_2\big)\, d\sigma, \qquad (3.14)$$

where as it was proved for $\Phi_s = 1$

$$\int_0^t d\tau \int_{\mathbb{R}^{n-1}} dy' \int_0^\infty \big(\frac{\kappa}{\varepsilon} d\nabla_x^T K_1 - \frac{1}{\varepsilon} h \nabla_x^T K_2\big)\, d\sigma\big|_{x_n=0} = 1, \tag{3.15}$$

that is the second term in the right – hand side of (3.13) goes to $\Phi_s(x', t)$ as $x_n \to 0$.

We show that $J_1(x,t)|_{x_n=0} = 0$. We can see that

$$\begin{aligned} \int_0^\infty \frac{\kappa}{\varepsilon}\, d\nabla_x^T K_1(x-y',\sigma, t-\tau)\, d\sigma &= -\int_0^\infty \partial_{\sigma_1} K_1\, d\sigma, \\ \int_0^\infty \frac{1}{\varepsilon}\, h\nabla_x^T K_2(x-y',\sigma, t-\tau)\, d\sigma &= \int_0^\infty \partial_{\sigma_2} K_2\, d\sigma, \end{aligned} \tag{3.16}$$

here

$$\begin{aligned} K_1(\cdot) = \int_0^{t-\tau} d\,\tau_1 \int_{\mathbb{R}^{n-1}} \frac{-x_n + \eta_n + \kappa d_n \sigma_1/\varepsilon}{(2\sqrt{a\pi(t-\tau-\tau_1)})^n (t-\tau-\tau_1)} e^{-\frac{(x-y'-\eta-\kappa d\,\sigma_1/\varepsilon)^2}{4a(t-\tau-\tau_1)}} \times \\ \times \frac{h_n\sigma_2/\varepsilon}{(2\sqrt{\pi\tau_1})^n\tau_1} \frac{1}{\sqrt{|A_n|}} e^{-\frac{\sum_{i,j=1}^n a^{ij}(\eta_i + h_i\sigma_2/\varepsilon)(\eta_j + h_j\sigma_2/\varepsilon)}{4\tau_1}} \Big|_{\substack{\eta_n=0,\\ \sigma_1=\sigma_2=\sigma}} d\,\eta' \equiv \\ \equiv \int_0^{t-\tau} d\tau_1 \int_{\mathbb{R}^{n-1}} \frac{-x_n + \kappa d_n \sigma_1/\varepsilon_1}{(2\sqrt{a\pi\tau_1})^n\tau_1} e^{-\frac{(\eta-\kappa d\,\sigma_1/\varepsilon)^2}{4a\tau_1}} \times \\ \times \frac{1}{\sqrt{|A_n|}} \frac{h_n\sigma_2/\varepsilon}{(2\sqrt{\pi(t-\tau-\tau_1)})^n (t-\tau-\tau_1)} \times \\ \times \Big(e^{-\frac{\sum_{i,j=1}^n a^{ij}(x_i - y_i - \eta_i + h_i\sigma_2/\varepsilon)(x_j - y_j - \eta_j + h_j\sigma_2/\varepsilon)}{4(t-\tau-\tau_1)}} \Big|_{\substack{x_n=0,\,\eta_n=0,\\ \sigma_1=\sigma_2=\sigma}} \Big) d\,\eta' \end{aligned} \tag{3.17}$$

(in the last exponent we added variable x_n for the sake of convenience to do not divide the sum $\Sigma_{i,j=1}^n = \Sigma_{i,j=1}^{n-1} + 2\Sigma_{i=1}^{n-1}\Sigma_{j=n} + \Sigma_{i,j=n}$),

$$\begin{aligned} K_2(\cdot) = \int_0^{t-\tau} d\,\tau_1 \int_{\mathbb{R}^{n-1}} \frac{\kappa d_n \sigma_1/\varepsilon}{(2\sqrt{a\pi\tau_1})^n\tau_1} e^{-\frac{(\eta-\kappa d\sigma_1/\varepsilon)^2}{4a\tau_1}} \frac{x_n - \eta_n + h_n\sigma_2/\varepsilon}{(2\sqrt{\pi(t-\tau-\tau_1)})^n (t-\tau-\tau_1)} \times \\ \times \frac{1}{\sqrt{|A_n|}} e^{-\frac{\sum_{i,j=1}^n a^{ij}(x_i - y_i - \eta_i + h_i\sigma_2/\varepsilon)(x_j - y_j - \eta_j + h_j\sigma/\varepsilon)}{4(t-\tau-\tau_1)}} \Big|_{\substack{\eta_n=0,\, y_n=0,\\ \sigma_1=\sigma_2=\sigma}} d\,\eta'. \end{aligned} \tag{3.18}$$

Comparing the formulas (3.17) and (3.18) we can see that

$$\big(K_1 - K_2\big)\big|_{x_n=0} = 0, \tag{3.19}$$

so due to (3.19), (3.16) we obtain formally

$$\int_0^\infty \Big(\frac{\kappa}{\varepsilon}\, d\nabla_x^T K_1 - \frac{1}{\varepsilon} h \nabla_x^T K_2\Big)\big|_{x_n=0}\, d\sigma = \int_0^\infty \big(-\partial_{\sigma_1} K_1 - \partial_{\sigma_2} K_2\big)\big|_{\substack{\sigma_1=\sigma_2=\sigma\\ x_n=0}}\, d\sigma =$$

$$= -\int_0^\infty \big(\partial_{\sigma_1} K_1 + \partial_{\sigma_2} K_1\big)\big|_{\substack{\sigma_1=\sigma_2=\sigma,\\ x_n=0}}\, d\sigma = -\int_0^\infty \partial_\sigma K_1\big|_{x_n=0}\, d\sigma = 0. \tag{3.20}$$

We shall show that an integral $J_1(x,t)$ (see (3.14)) converges uniformly with respect to $(x,t) \in D_{1T}$. Applying the inequalities (1.14) for K_1,

$$|\Phi_s(y',\tau) - \Phi_s(x',t)| \le \big[\Phi_s\big]_{R_T}^{(\alpha)} \Big(|x'-y'|^\alpha + (t-\tau)^{\alpha/2}\Big)$$

and

$$|\xi|^\gamma e^{-\xi^2} \le C_\gamma e^{-\xi^2/2},\ \gamma \ge 0, \tag{3.21}$$

then integrating with respect to y' and σ we derive

$$|J_1| \le C_1 \big[\Phi_s\big]_{R_T}^{(\alpha)} \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \frac{|x'-y'|^\alpha + (t-\tau)^{\alpha/2}}{(t-\tau)^{\frac{n+2}{2}}} dy' \int_0^\infty e^{-\frac{q_1^2(x'-y')^2}{t-\tau} - \frac{q_2^2\sigma^2}{\varepsilon^2(t-\tau)}}\, d\sigma \le$$

$$\le C_2 \varepsilon (1+\varepsilon^{\alpha/2}) \big[\Phi_s\big]_{R_T}^{(\alpha)} \int_0^t \frac{d\tau}{(t-\tau)^{1-\alpha/2}} \le C_3 \varepsilon \big[\Phi_s\big]_{R_T}^{(\alpha)} T^{\alpha/2}.$$

Thus, in the integral $J_1(x,t)$ we can make use of the identity (3.20) and for the integrals $J_1,\ I$ determined by (3.13), (3.14), (3.15) we have got

$$J_1(x,t) \to 0,\ \ I(x,t) \to \Phi_s(x',t) \text{ as } x_n \to 0$$

and from the formulas (3.7), (3.6) we obtain formula (3.1): $\varepsilon \partial_t^{m_0} \partial_{x'}^{m'} \partial_t v_1|_{x_n=0} = -W_1^{(s)}(x',t) + W_2^{(s)}(x',t) - W_3^{(s)}(x',t)$.

We transform the function $W_3^{(s)}(x',t)$ defined by (3.8). As it is seen from the formulas (3.20), (3.19)

$$\begin{aligned}\Big(\frac{\kappa}{\varepsilon} d\nabla_x^T K_1(x-y',\sigma,t-\tau) - \frac{1}{\varepsilon} h \nabla_x^T K_2(x-y',\sigma,t-\tau)\Big)\big|_{x_n=0} =\\ = -\big(\partial_{\sigma_1} K_1 + \partial_{\sigma_2} K_1\big)\Big|_{\substack{\sigma_1=\sigma_2=\sigma,\\ x_n=0}} = -\partial_\sigma K_1\Big|_{x_n=0},\end{aligned} \tag{3.22}$$

moreover, function $W_3^{(s)}(x',t)$ converges uniformly in R_T thanks to the density Φ_s satisfying an estimate $|\Phi_s(y',\tau)| \le \tau^{\alpha/2} \le \sigma^{\alpha/2}$, $\sigma \in (\tau,\infty)$, so we can write

$$W_3^{(s)}(x',t) = -\int_0^t d\tau \int_{\mathbb{R}^{n-1}} \Phi_s(y',\tau)dy' \int_\tau^\infty \partial_\sigma K_1(x-y',\sigma,t-\tau)\,d\sigma\big|_{x_n=0}$$

and after integrating with respect to σ we shall have formula (3.4).

We point out that we could not apply formula (3.20) in (3.11) (in the case $\Phi_s = 1$) because an integral (3.11) is a potential of double layer and it has a jump as $x_n \to 0$. □

Proof of Theorem 2.2. Due to Lemma 1.1 we shall estimate the norm

$$\begin{gathered} ||\varepsilon\partial_t v_1||_{C_{x'}^{1+k+\beta,\frac{1+k+\beta}{2}}{}_t(R_T)} := \sup_{(x',t)\in R_T} t^{-(1+k+\beta)/2}|\varepsilon\partial_t v_1| + \\ + \sum_{2m_0+|m'|=1+k} [\varepsilon\partial_t^{m_0}\partial_{x'}^{m'}\partial_t v_1]^{(\beta)}_{x',R_T} + \sum_{2m_0+|m'|=1+k} [\varepsilon\partial_t^{m_0}\partial_{x'}^{m'}\partial_t v_1]^{(\beta/2)}_{t,R_T} + \\ + \sum_{2m_0+|m'|=k} [\varepsilon\partial_t^{m_0}\partial_{x'}^{m'}\partial_t v_1]^{(\frac{1+\beta}{2})}_{t,R_T}, \end{gathered} \tag{3.23}$$

where by Lemma 3.1 $\varepsilon\partial_t^{m_0}\partial_{x'}^{m'}\partial_t v_1|_{x_n=0} = -W_1^{(s)}(x',t) + W_2^{(s)}(x',t) - W_3^{(s)}(x',t)$, potentials $W_\mu^{(s)}$ are determined by the formulas (3.2) – (3.4), $s = 2m_0 + |m'|$.

The density in the potentials $W_\mu^{(s)}$ is $\Phi_s(x',t) = \partial_t^{m_0}\partial_{x'}^{m'}\varphi(x',t)$, $s = 2m_0 + |m'|$, $\Phi_0 \equiv \varphi(x',t) \in \overset{\circ}{C}{}_{x'}^{1+k+\alpha,\frac{1+k+\alpha}{2}}{}_t(R_T)$, $k = 0,1,\dots$, and $\Phi_{1+k}(x',t) \in \overset{\circ}{C}{}_{x'}^{\alpha,\alpha/2}{}_t(R_T)$, $\Phi_k(x',t) \in \overset{\circ}{C}{}_{x'}^{1+\alpha,\frac{1+\alpha}{2}}{}_t(R_T)$.

For the function $\varphi(x',t)$ we have the following estimates:

$$\begin{gathered} |\varphi(y',\tau-\sigma) - \varphi(y',\tau)| \le M_0\,\sigma^{\frac{1+\alpha}{2}},\ k=0, \\ |\varphi(y',\tau-\sigma) - \varphi(y',\tau)| \le M_k\,\sigma\tau^{\frac{k-1+\alpha}{2}},\ k=1,2,\dots, \\ |\varphi(y',\tau)| \le M_k\,\tau^{\frac{1+k+\alpha}{2}},\ M_k = \big[\partial_t^{[\frac{1+k}{2}]}\varphi\big]^{(\frac{1+k+\alpha}{2}-[\frac{1+k}{2}])}_{t,R_T},\ k=0,1,\dots. \end{gathered} \tag{3.24}$$

The functions Φ_{1+k}, Φ_k satisfy the inequalities

$$|\Phi_{k+j}(y',\tau)| \le N_{k+j}\,\tau^{\frac{1+\alpha-j}{2}},\ |\Phi_{k+j}(y',\tau-\sigma) - \Phi_{k+j}(y',\tau)| \le N_{k+j}\,\sigma^{\frac{1+\alpha-j}{2}},$$

$$N_{k+j}=\left[\partial_t^{m_0}\partial_{x'}^{m'}\varphi\right]_{t,R_T}^{(\frac{1+\alpha-j}{2})},\ 2m_0+|m'|=k+j,\ j=0,1. \tag{3.25}$$

Comparing the functions $W_1^{(s)}$ and $W_2^{(s)}$ determined by the formulas (3.2), (3.3) we can see that they are similar. So for definiteness, we shall estimate the functions $W_1^{(s)}$ both with $W_3^{(s)}$.

First, due to (3.23) we estimate modulo of $\partial_t v_1|_{x_n=0}$, that is $|W_1^{(0)}|$ and $|W_3^{(0)}|$ with $\Phi_0=\varphi$.

Consider the potential $W_1^{(0)}(x',t)$. Applying an estimate (1.14) for K_1, (3.24) for φ, using an inequalities $\sigma^{1/2}\ \le\tau^{1/2}$ and

$$\sigma^{\alpha/2}e^{-\frac{q_2^2\sigma^2}{\varepsilon^2(t-\tau)}}\le C_{\alpha/2}\varepsilon^{\alpha/2}\tau^{\alpha/4}e^{-\frac{q_2^2\sigma^2}{2\varepsilon^2(t-\tau)}}, \tag{3.26}$$

which follows from (3.21) and integrating with respect to y' and σ we shall have for $k=0$

$$|W_1^{(0)}(x',t)|\le C_4M_0\frac{\kappa}{\varepsilon}\int_0^t d\tau\int_0^\tau\frac{\sigma^{\frac{1+\alpha}{2}}}{(t-\tau)^{3/2}}e^{-\frac{q_2^2\sigma^2}{\varepsilon^2(t-\tau)}}\,d\sigma\le$$

$$\le C_5M_0\frac{\kappa_0}{\varepsilon}\varepsilon^{\alpha/2}\int_0^t\frac{\tau^{1/2}}{(t-\tau)^{3/2-\alpha/4}}\,d\tau\int_0^\tau e^{-\frac{q_2^2\sigma^2}{2\varepsilon^2(t-\tau)}}\,d\sigma\le$$

$$\le C_6M_0\kappa_0\varepsilon^{\alpha/2}\int_0^t\frac{\tau^{1/2}}{(t-\tau)^{1-\alpha/4}}\,d\tau\le C_7M_0\kappa_0\varepsilon^{\alpha/2}t^{\frac{1+\alpha/2}{2}},$$

$$|W_1^{(0)}(x',t)|\le C_7M_0\kappa_0\varepsilon^{\alpha/2}t^{\frac{1+\beta}{2}}T^{\frac{\alpha-2\beta}{4}},\ \beta\in(0,\alpha/2),\ k=0. \tag{3.27}$$

For $k=1,2,\ldots$, taking into account (3.24), the estimates $\sigma^{1-\alpha/2}\le\tau^{1-\alpha/2}$ and (3.26) we obtain

$$|W_1^{(0)}(x't)|\le C_8M_k\frac{\kappa_0}{\varepsilon}\varepsilon^{\alpha/2}\int_0^t\frac{\tau^{\frac{k-1+\alpha}{2}+1-\alpha/2}}{(t-\tau)^{3/2}}\,d\tau\int_0^\tau\sigma^{\alpha/2}e^{-\frac{q_2^2\sigma^2}{\varepsilon^2(t-\tau)}}\,d\sigma\le$$

$$\le C_9M_k\kappa_0\varepsilon^{\alpha/2}\int_0^t\frac{\tau^{\frac{k+1}{2}}}{(t-\tau)^{1-\alpha/4}}\,d\tau=C_{10}M_k\kappa_0\varepsilon^{\alpha/2}t^{\frac{k+1+\alpha/2}{2}},$$

$$|W_1^{(0)}(x',t)^{(0)}|\le C_{10}M_k\kappa_0\varepsilon^{\alpha/2}t^{\frac{k+1+\beta}{2}}T^{\frac{\alpha-2\beta}{4}},\ \beta\in(0,\alpha/2),\ k=1,2,\ldots. \tag{3.28}$$

The function $W_2^{(0)}$ satisfies the same estimate as $W_1^{(0)}$ (without κ_0).

Consider potential $W_3^{(0)}(x',t)$ defined by formula (3.4). With the help of the estimates (1.14) for K_1, (3.24) for φ and integrating with respect to y' we find

$$|W_3^{(0)}(x',t)| \le C_{11} M_k \int_0^t \frac{\tau^{\frac{1+k+\alpha}{2}}}{t-\tau} e^{-\frac{q_2^2 \tau^2}{\varepsilon^2 (t-\tau)}}\, d\tau.$$

Applying the inequality (3.21) and integrating with respect to τ we obtain

$$|W_3^{(0)}| \le C_{12} M_k \varepsilon^{\alpha/2} \int_0^t \frac{\tau^{\frac{1+k}{2}}}{(t-\tau)^{1-\alpha/4}}\, d\tau \le C_{13} M_k \varepsilon^{\alpha/2} t^{\frac{1+k+\beta}{2}} t^{\frac{\alpha-2\beta}{4}}, \qquad (3.29)$$

$\beta \in (0, \alpha/2)$.

Gathering the estimates (3.27) – (3.29) we have got

$$|\varepsilon \partial_t v_1| \le C_{14} t^{\frac{1+k+\beta}{2}} [\varphi]_{t,R_T}^{(\frac{1+k+\alpha}{2})} \varepsilon^{\alpha/2} (1+\kappa_0), \; \beta \in (0, \alpha/2). \qquad (3.30)$$

Now we estimate the Hölder constants in (3.23) with respect to t. We compose the differences letting $t_1 < t$

$$\triangle_{p,j} := W_p^{(k+j)}(x',t) - W_p^{(k+j)}(x',t_1), \; p = 1,3, \; j = 0,1,$$

with density $\Phi_{k+j} \in \overset{\circ}{C}{}_{x'}^{j+\alpha, \frac{j+\alpha}{2}}{}_{t}(R_T)$, $j = 0,1$.

We represent the potential $W_1^{(k+j)}$ in the form

$$W_1^{(k+j)}(x',t) = \frac{\kappa}{\varepsilon} \int_0^t d\tau \int_{\mathbb{R}^{n-1}} dy' \int_0^{t-\tau} \big(\Phi_{k+j}(y', t-\tau-\sigma) - \Phi_{k+j}(y', t-\tau)\big) \times$$
$$\times d\nabla_x^T K_1(x-y', \sigma, \tau)\, d\sigma\big|_{x_n=0},$$

then

$$\triangle_{1,j} = \frac{\kappa}{\varepsilon} \int_{t_1}^t d\tau \int_{\mathbb{R}^{n-1}} dy' \int_0^{t-\tau} \big(\Phi_{k+j}(y', t-\tau-\sigma) - \Phi_{k+j}(y', t-\tau)\big) \times$$
$$\times d\nabla_x^T K_1(x-y', \sigma, \tau)\, d\sigma\big|_{x_n=0} +$$
$$+ \frac{\kappa}{\varepsilon} \int_0^{t_1} d\tau \int_{\mathbb{R}^{n-1}} dy' \int_{t_1-\tau}^{t-\tau} \big(\Phi_{k+j}(y', t-\tau-\sigma) - \Phi_{k+j}(y', t-\tau)\big) d\nabla_x^T K_1(\cdot)\, d\sigma\big|_{x_n=0} +$$

$$+\frac{\kappa}{\varepsilon}\int_{t_1}^{t}d\tau\int_{\mathbb{R}^{n-1}}dy'\int_0^{t_1-\tau}\tilde{\triangle}_j(y',t,t_1,\tau,\sigma)d\nabla_x^T K_1(\cdot)\,d\sigma\big|_{x_n=0}, \qquad (3.31)$$

where

$$\widetilde{\triangle}_{1,j}=\Phi_{k+j}(y',t-\tau-\sigma)-\Phi_{k+j}(y',t-\tau)-\Phi_{k+j}(y',t_1-\tau-\sigma)+\Phi_{k+j}(y',t_1-\tau),$$

$$\triangle_{2,j}=W_3^{(k+j)}(x',t)-W_3^{(k+j)}(x',t_1)=$$

$$=\int_{t_1}^{t}d\tau\int_{\mathbb{R}^{n-1}}\Phi_{k+j}(y',t-\tau)K_1(x-y',t-\tau,\tau)\,|_{x_n=0}dy'+$$

$$+\int_0^{t_1}d\tau\int_{\mathbb{R}^{n-1}}\widetilde{\triangle}_{2,j}(y',t,t_1,\tau)K_1(x-y',t-\tau,\tau)|_{x_n=0}dy'+$$

$$+\int_{t_1}^{t}d\tau_1\int_0^{t_1}d\tau\int_{\mathbb{R}^{n-1}}\Phi_{k+j}(y',t_1-\tau)\partial_{\tau_1}K_1(x-y',\tau_1-\tau,\tau)|_{x_n=0}\,dy', \quad (3.32)$$

where $\widetilde{\triangle}_{2,j}=\Phi_{k+j}(y',t-\tau)-\Phi_{k+j}(y',t_1-\tau)$.

We evaluate $|\widetilde{\triangle}_{1,j}|=|\widetilde{\triangle}_{1,j}|^{\theta}|\widetilde{\triangle}_{1,j}|^{1-\theta}$. Letting $\theta=\frac{1+\alpha/2-j}{1+\alpha-j},\ j=0,1$, and applying an inequality (3.25) for Φ_{k+j} we have got

$$|\widetilde{\triangle}_{1,j}|=$$

$$=|\big(\Phi_{k+j}(y',t-\tau-\sigma)-\Phi_{k+j}(y',t_1-\tau-\sigma)\big)+\big(\Phi_{k+j}(y',t_1-\tau)-\Phi_{k+j}(y',t-\tau)\big)|^{\theta}\times$$

$$|\Phi_{k+j}(y',t-\tau-\sigma)-\Phi_{k+j}(y',t-\tau)\big)+\big(\Phi_{k+j}(y',t_1-\tau)-\Phi_{k+j}(y',t_1-\tau-\sigma)\big)|^{1-\theta}\le$$

$$\le C_{15}\big(N_{k+j}(t-t_1)^{\frac{1+\alpha-j}{2}}\big)^{\theta}\big(N_{k+j}\sigma^{\frac{1+\alpha-j}{2}}\big)^{1-\theta}=C_{15}N_{k+j}(t-t_1)^{\frac{1+\alpha/2-j}{2}}\sigma^{\alpha/4}, \qquad (3.33)$$

here $t_1<t,\ \sigma\ge 0$.

Consider the difference (3.31). Applying an estimate (1.14) for K_1, (3.25) for Φ_j, and (3.33) and integrating with respect to y' we shall have

$$|\triangle_{1,j}|\le C_{16}N_{k+j}\frac{\kappa}{\varepsilon}\Big(\int_{t_1}^{t}d\tau\int_0^{t-\tau}\frac{\sigma^{\frac{1+\alpha-j}{2}}}{\tau^{3/2}}e^{-\frac{q_2^2\sigma^2}{\varepsilon^2\tau}}\,d\sigma+$$

$$+\int_0^{t_1}d\tau\int_{t_1-\tau}^{t-\tau}\frac{\sigma^{\frac{1+\alpha/2-j}{2}}\sigma^{1+\alpha/4}}{\tau^{3/2}\sigma}e^{-\frac{\gamma_2^2\sigma^2}{\varepsilon^2\tau}}\,d\sigma+(t-t_1)^{\frac{1+\alpha/2-j}{2}}\int_0^{t_1}d\tau\int_0^{t_1-\tau}\frac{\sigma^{\alpha/4}}{\tau^{3/2}}e^{-\frac{q_2^2\sigma^2}{\varepsilon^2\tau}}\,d\sigma\Big).$$

We make use the inequality $\sigma^{\frac{1+\alpha/2-j}{2}} \le (t-\tau)^{\frac{1+\alpha/2-j}{2}} \le (t-t_1)^{\frac{1+\alpha/2-j}{2}}$ in the first integral and (3.26) in all ones and integrate with respect to σ,

$$|\triangle_{1,j}| \le C_{17} N_{k+j} \kappa_0 \varepsilon^{\alpha/4} \Big((t-t_1)^{\frac{1+\alpha/2-j}{2}} \int_0^t \frac{d\tau}{\tau^{1-\alpha/4}} +$$

$$+\varepsilon \int_0^{t_1} \frac{d\tau}{\tau^{1-\alpha/8}} \int_{t_1-\tau}^{t-\tau} \sigma^{-1+\frac{j+\alpha/2}{2}}\, d\sigma + (t-t_1)^{\frac{1+\alpha/2-j}{2}} \int_0^{t_1} \frac{d\tau}{\tau^{1-\alpha/8}} \Big),$$

$$|\triangle_{1,j}| = |W_1^{(k+j)}(x',t) - W_1^{(k+j)}(x',t_1)| \le$$

$$\le C_{18} N_{k+j} \kappa_0 \varepsilon^{\alpha/4} (t-t_1)^{\frac{1+\alpha/2-j}{2}} (t^{\alpha/8} + t^{\alpha/4}) \le$$

$$\le C_{19} N_{k+j} \kappa_0 \varepsilon^{\alpha/4} (t-t_1)^{\frac{1+\beta-j}{2}} t^{\frac{\alpha-2\beta}{2}} (t^{\alpha/8} + t^{\alpha/4})$$

and

$$\big[W_1^{(k+j)}\big]_{t,R_T}^{(\frac{1+\beta-j}{2})} = \max_{(x,t),(x,t_1)\in R_T} |\triangle_{1,j}| |t-t_1|^{-\frac{1+\beta-j}{2}} \le$$

$$\le C_{20} \kappa_0 \varepsilon^{\alpha/4} \big[\partial_t^{m_0} \partial_{x'}^{m'} \varphi\big]_{t,R_T}^{(\frac{1+\alpha-j}{2})}, \qquad (3.34)$$

where $2m_0 + |m'| = k + j$, $\beta \in (0,\alpha/2)$, $j = 0,1$, $\triangle_{1,j} := W_1^{(k+j)}(x',t) - W_1^{(k+j)}(x',t_1)$.

Potential $W_2^{(k+j)}$ satisfies the same estimate as $W_1^{(k+j)}$ (without κ_0).

Consider the difference $\triangle_{2,j}$ determined by (3.32), where $\widetilde{\triangle}_{2,j} = \Phi_{k+j}(y',t-\tau) - \Phi_{k+j}(y',t_1-\tau)$ satisfies an estimate

$$|\widetilde{\triangle}_{2,j}| \equiv |\widetilde{\triangle}_{2,j}|^{\theta} |\widetilde{\triangle}_{2,j}|^{1-\theta} \le C_{21} N_{k+j} (t-t_1)^{\frac{1+\alpha/2-j}{2}} (t-\tau)^{\alpha/4}, \qquad (3.35)$$

$\theta = \frac{j+\alpha/2-j}{1+\alpha-j}$, $t_1 < t$, $j = 0,1$.

Now as above we apply the estimates (3.25) for Φ_{k+j}, (1.14) for the kernel K_1 , (3.35) and integrating with respect to y' we shall have

$$|\triangle_{2,j}| \le C_{22} N_{k+j} \Big(\int_{t_1}^{t} \frac{(t-\tau)^{\frac{1+\alpha-j}{2}}}{\tau} e^{-\frac{q_2^2 (t-\tau)^2}{\varepsilon^2 \tau}}\, d\tau +$$

$$+(t-t_1)^{\frac{1+\alpha/2-j}{2}} \int_0^{t_1} \frac{(t-\tau)^{\alpha/4}}{\tau} e^{-\frac{q_2^2 (t-\tau)^2}{\varepsilon^2 \tau}} d\tau + \int_{t_1}^{t} d\tau_1 \int_0^{t_1} \frac{(t_1-\tau)^{\frac{1+\alpha-j}{2}}}{\tau^{3/2}} e^{-\frac{q_2^2 (\tau_1-\tau)^2}{\varepsilon^2 \tau}} d\tau \Big).$$

In the first and last integrals we make use of the estimates

$$(t-\tau)^{\frac{1+\alpha-j}{2}} \le (t-t_1)^{\frac{1+\alpha/2-j}{2}}(t-\tau)^{\alpha/4},\ \tau\in(t_1,t),$$

$$(t_1-\tau)^{\frac{1+\alpha-j}{2}} \le (\tau_1-\tau)^{\frac{1+\alpha-j}{2}} \le \frac{(\tau_1-\tau)^{1+\alpha/4}}{(\tau_1-\tau)^{1-\frac{1+\alpha/2-j}{2}}} \le \frac{(\tau_1-\tau)^{1+\alpha/4}}{(\tau_1-t_1)^{1-\frac{1+\alpha/2-j}{2}}},$$

$\tau_1\in(t_1,t)$, $\tau\in(0,t_1)$ respectively, and in all integrals an inequality (3.26), then we obtain

$$|\triangle_{2,j}| \le C_{23}N_{k+j}\Big(2(t-t_1)^{\frac{1+\alpha/2-j}{2}}\int_{t_1}^{t}\frac{(t-\tau)^{\alpha/4}}{\tau}e^{-\frac{q_2^2(t-\tau)^2}{\varepsilon^2\tau}}\,d\tau+$$

$$+\int_{t_1}^{t}\frac{d\tau_1}{(\tau_1-t_1)^{1-\frac{1+\alpha/2-j}{2}}}\int_0^{t_1}\frac{(\tau_1-\tau)^{1+\alpha/2}}{\tau^{3/2}}\,e^{-\frac{q_2^2(\tau_1-\tau)^2}{\varepsilon^2\tau}}\,d\tau\Big) \le$$

$$\le C_{24}N_{k+j}(\varepsilon^{\alpha/2}+\varepsilon^{\alpha/4})(t-t_1)^{\frac{1+\alpha/2-j}{2}}\Big(\int_0^t\frac{d\tau}{\tau^{1-\alpha/8}}+\varepsilon\int_0^{t_1}\frac{d\tau}{\tau^{1-\alpha/4}}\Big) \le$$

$$\le C_{25}N_{k+j}\varepsilon^{\alpha/4}(t-t_1)^{\frac{1+\alpha/2-j}{2}}t^{\alpha/8}(1+\varepsilon\, t^{\alpha/8}) \le C_{26}\varepsilon^{\alpha/4}N_{k+j}(t-t_1)^{\frac{1+\beta-j}{2}}t^{\frac{\alpha/2-\beta}{2}},$$

$$\big[W_3^{(k+j)}\big]_{t,R_T}^{(\frac{1+\beta-j}{2})} = \max_{(x',t),(x',t_1)\in R_T}|\triangle_{2,j}||t-t_1|^{-\frac{1+\beta-j}{2}} \le$$

$$\le C_{27}\varepsilon^{\alpha/4}\big[\partial_t^{m_0}\partial_{x'}^{m'}\varphi\big]_{t,R_T}^{(\frac{1+\alpha-j}{2})}, \tag{3.36}$$

where $\beta\in 0,\alpha/2)$, $2m_0+|m'|=k+j$, $j=0,1$, $\triangle_{2,j}:=W_3^{(k+j)}(x',t)-W_3^{(k+j)}(x',t_1)$.

Now we evaluate the Hölder constants of the potentials $W_p^{(1+k)}$, $p=1,3$ with respect to the spatial variables x' with the density $\Phi_{1+k}(x',t)\in \overset{\circ}{C}{}^{\alpha,\alpha/2}_{x'\ \ t}(R_T)$. For this we compose the differences with $r=|x'-z'|$

$$\triangle_3 := W_1^{(1+k)}(x',t)-W_1^{(1+k)}(z',t) =$$

$$=-\frac{\kappa}{\varepsilon}\int_0^t d\tau\int_{|y'-z'|\le 2r}dy'\int_0^{\tau}\big(\Phi_{1+k}(y',\tau-\sigma)-\Phi_{1+k}(y',\tau)\big)\big(d'\nabla_{y'}'^{T}-d_n\partial_{y_n}\big)\times$$

$$\times\Big(K_1(x'-y',y_n,\sigma,t-\tau)-K_1(z'-y',y_n,\sigma,t-\tau)\Big)\Big|_{y_n=0}d\sigma+ \tag{3.37}$$

$$+\frac{\kappa}{\varepsilon}\int_0^t d\tau\int_{|y'-z'|>2r}dy'\int_0^\tau\big(\Phi_{1+k}(y',\tau-\sigma)-\Phi_{1+k}(y',\tau)\big)\int_0^1\sum_{\nu=1}^{n-1}(x_\nu-z_\nu)\times$$
$$\times\partial_{y_\nu}\big(d'\nabla_{y'}'^T-d_n\partial_{y_n}\big)K_1(z'-y'+\lambda(x'-z'),y_n,\sigma,t-\tau)\big|_{y_n=0}\,d\lambda\,d\sigma,$$

here we have used an identity for convenience

$$d\nabla_x^T K_1(x'-y',x_n,\sigma,t-\tau)|_{x_n=0}=-(d'\nabla_{y'}'^T-d_n\partial_{y_n})K_1(x'-y',y_n,\sigma,t-\tau)|_{y_n=0},$$

$$\triangle_4:=W_3^{(1+k)}(x',t)-W_3^{(1+k)}(z',t)=$$
$$=\int_0^t d\tau\int_{|y'-z'|\le 2r}\Phi_{1+k}(y',\tau)\Big(K_1(x'-y',y_n,\sigma,t-\tau)-K_1(z'-y',y_n,\sigma,t-\tau)\Big)\Big|_{y_n=0}dy'-$$
$$-\int_0^t d\tau\int_{|y'-z'|\le 2r}\Phi_{1+k}(y',\tau)\int_0^1\sum_{\nu=1}^{n-1}(x_\nu-z_\nu)\times \tag{3.38}$$
$$\times\partial_{y_\nu}K_1(z'-y'+\lambda(x'-z'),y_n,\sigma,t-\tau)\big|_{y_n=0}\big|_{y_n=0}d\lambda dy',$$

in both differences we have applied the formula

$$K_1(x'-y',x_n,\cdot)\big|_{x_n=0}-K_1(z'-y',z_n,\cdot)\big|_{z_n=0}=$$
$$=-\int_0^1\sum_{\nu=1}^{n-1}(x_\nu-z_\nu)\partial_{y_\nu}K_1(z'-y'+\lambda(x'-z'),y_n,\cdot)\big|_{y_n=0}\,d\lambda$$

and rename x_n by y_n for the convenience.

Consider $\triangle_3$ defined by (3.37). We apply the inequalities (1.14) to K_1, (3.25) for Φ_{1+k}, pass to the spherical coordinates letting $\rho=|x'-y'|$, $\rho=|z'-y'|$ and $\rho=|z'-y'+\lambda(x'-z')|$ in the first, second and third integrals respectively, then we obtain

$$|\triangle_3|\le C_{28}N_{1+k}\frac{\kappa}{\varepsilon}\Big(\int_0^t d\tau\Big(\int_0^{3r}+\int_0^{2r}\Big)\rho^{n-2}d\rho\int_0^\tau\frac{\sigma^{\alpha/2}}{(t-\tau)^{\frac{n+2}{2}}}e^{-\frac{q_1^2\rho^2}{t-\tau}-\frac{q_2^2\sigma^2}{\varepsilon^2(t-\tau)}}\,d\sigma$$
$$+r\int_0^t d\tau\int_r^\infty\rho^{n-2}\,d\rho\int_0^\tau\frac{\sigma^{\alpha/2}}{(t-\tau)^{\frac{n+3}{2}}}e^{-\frac{q_1^2\rho^2}{t-\tau}-\frac{q_2^2\sigma^2}{\varepsilon^2(t-\tau)}}d\sigma\Big).$$

We make use of the estimates

$$\frac{\rho^{n-\mu-\beta}}{(t-\tau)^{\frac{n-\mu-\beta}{2}}}e^{-\frac{q_1^2\rho^2}{t-\tau}}\leq C_{29},\ \beta\in(0,\alpha/2),\ \mu=1,0, \tag{3.39}$$

in the two first integrals with $\mu=1$ and in the last one with $\mu=0$ and also (3.26) and integrate with respect to σ, then we obtain

$$|\triangle_3|\leq C_{30}N_{1+k}\kappa_0\varepsilon^{\alpha/2}\int_0^t\frac{d\tau}{(t-\tau)^{1-\frac{\alpha-2\beta}{4}}2}\Big(\Big(\int_0^{3r}+\int_0^{2r}\Big)\rho^{-1+\beta}\,d\rho+$$

$$+r\int_r^{\infty}\rho^{-2+\beta}\,d\rho\Big)\leq C_{31}N_{1+k}\kappa_0\varepsilon^{\alpha/2}t^{\frac{\alpha-2\beta}{2}}r^{\beta},\ r=|x'-z'|,\ \beta\in(0,\alpha/2),$$

and

$$\big[W_1^{(1+k)}\big]^{(\beta)}_{x',R_T}=\max_{(x',t),(z',t)\in R_T}\frac{|\triangle_3|}{|x'-z'|^{\beta}}\leq C_{32}\kappa_0\varepsilon^{\alpha/2}\big[\partial_t^{m_0}\partial_{x'}^{m'}\varphi\big]^{(\alpha/2)}_{t,R_T}, \tag{3.40}$$

where $2m_0+|m'|=1+k$, $\triangle_3:=W_1^{(1+k)}(x',t)-W_1^{(1+k)}(z',t)$.

The difference $W_2^{(1+k)}(x',t)-W_2^{(1+k)}(z',t)$ is estimated as $\triangle_3$ and subjected to the inequality (3.40) (without κ_0).

We evaluate the difference $\triangle_4$ (see (3.38)) with the help of the estimates (1.14), (3.25), (3.21), (3.39)

$$|\triangle_4|\leq C_{33}N_{1+k}\Big(\int_0^t\frac{\tau^{\alpha/2}}{(t-\tau)^{\frac{n+1}{2}}}d\tau\Big(\int_0^{3r}+\int_0^{2r}\Big)\rho^{n-2}e^{-\frac{q_1^2\rho^2}{t-\tau}-\frac{q_2^2\tau^2}{\varepsilon^2(t-\tau)}}\,d\rho+$$

$$+r\int_0^t\frac{\tau^{\alpha/2}}{(t-\tau)^{\frac{n+2}{2}}}d\tau\int_r^{\infty}\rho^{n-2}e^{-\frac{q_1^2\rho^2}{t-\tau}-\frac{q_2^2\tau^2}{\varepsilon^2(t-\tau)}}\,d\rho\Big)\leq$$

$$\leq C_{34}N_{1+k}\varepsilon^{\alpha/2}t^{\frac{\alpha-2\beta}{2}}r^{\beta},\ \beta\in(0,\alpha/2),\ r=|x'-z'|,$$

$$\big[W_3^{(1+k)}\big]^{(\beta)}_{x',R_T}=\max_{(x',t),(z',t)\in R_T}\frac{|\triangle_4|}{|x'-z'|^{\beta}}\leq C_{35}\kappa_0\varepsilon^{\alpha/2}\big[\partial_t^{m_0}\partial_{x'}^{m'}\varphi\big]^{(\alpha/2)}_{t,R_T}, \tag{3.41}$$

here $2m_0+|m'|=1+k,\ \beta\in(0,\alpha/2),\ \triangle_4:=W_3^{(1+k)}(x',t)-W_3^{(1+k)}(z',t)$.

Remembering that $\varepsilon\partial_t^{m_0}\partial_{x'}^{m'}\partial_t v_1|_{x_n=0} = -W_1^{(s)}(x',t) + W_2^{(s)}(x',t) + W_3^{(s)}(x',t)$, $s = 2m_0 + |m'|$, in the norm (3.23) of the derivative $\varepsilon\partial_t v_1|_{x_n=0}$ we apply obtained estimates (3.30), (3.34), (3.36), (3.40), (3.41) of the modulo $|\varepsilon\partial_t v_1|$ and the Hölder constants

$$\left[W_1^{(k+j)}\right]_{t,R_T}^{(\frac{1+\beta-j}{2})}, \left[W_3^{(k+j)}\right]_{t,R_T}^{(\frac{1+\beta-j}{2})}, \ \left[W_1^{(1+k)}\right]_{x',R_T}^{(\beta)}, \ \left[W_3^{(1+k)}\right]_{x',R_T}^{(\beta)}$$

respectively, then we shall have

$$||\varepsilon\partial_t v_1||_{C_{x'}^{1+k+\beta}, \frac{1+k+\beta}{2}_t (R_T)} \le C_{36}(\varepsilon^{\alpha/4} + \varepsilon^{\alpha/2})\,(1+\kappa_0)\times$$

$$\times\Big(\sum_{2m_0+|m'|=1+k}\left[\partial_t^{m_0}\partial_{x'}^{m'}\varphi\right]_{t,R_T}^{(\alpha/2)} + \sum_{2m_0+|m'|=k}\left[\partial_t^{m_0}\partial_{x'}^{m'}\varphi\right]_{t,R_T}^{(\frac{1+\alpha}{2})}\Big),$$

and

$$|\varepsilon\partial_t u|_{C_{x'}^{1+k+\beta}, \frac{1+k+\beta}{2}_t (R_T)} \le C_{37}\varepsilon^{\alpha/4}(1+\kappa_0)|\varphi|_{R_T}^{(1+k+\alpha)}, \ \beta \in (0, \alpha/2),$$

where the constant C_{37} is independent on ε and κ. This is an estimate (1.9)

Theorem 1.2 is proved completely. □

References

1 Bizhanova G.I. Estimates of the solution of the two-phase singularly perturbed problem for the parabolic equations. I // Matem. zhurnal. — Almaty. — 2013. — №2. — P. 31 – 49.

2 Ladyženskaja O.A., Solonnikov V.A., Ural'çeva, N.N. Linear and quasilinear equations of parabolic type. — M.: Nauka, 1967. — 736 p.

3 Solonnikov, V.A. On an estimate of the maximum of a derivative modulus for a solution of a uniform parabolic initial-boundary value problem // Preprint № P-2-77, LOMI. — 1977. — P.20.

4 Bizhanova G.I. Uniform estimates of the solution to the linear two-phase Stefan problem with a small parameter // Matem. zhurnal. — Almaty. — 2005. — №1. — P. 20 – 29.

5 Bizhanova G.I. On the solutions of the linear free boundary problems of Stefan type with a small parameter. I // Matem. zhurnal. — Almaty. — 2012. — №1. — P. 24 – 37.

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Бижанова Г.И. ПАРАБОЛАЛЫҚ ТЕҢДЕУЛЕР ҮШІН ЕКІФАЗАЛЫ СИНГУЛЯРЛЫ ҚОБАЛЖЫҒАН ЕСЕПТІҢ ШЕШІМІНІҢ БАҒАЛАУЛАРЫ. II

Гельдер кеңістігінде шекаралық шарттағы бас мүшелеріндегі екі кіші параметрлері бар параболалық теңдеулер үшін сызықтандырылған екіфазалы шеттік есеп зерттелінеді. Кіші параметрлерден тәуелсіз тұрақтысы бар есептің шешімінің бағалауы алынды, шекаралық шарттағы кіші параметрдегі шешімнің уақыт бойынша туындысының бағалауы тағайындалды.

Бижанова Г.И. ОЦЕНКИ РЕШЕНИЯ ДВУХФАЗНОЙ СИНГУЛЯРНО ВОЗМУЩЕННОЙ ЗАДАЧИ ДЛЯ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ. II

Изучается двухфазная краевая задача для параболических уравнений с двумя малыми параметрами при старших членах в граничном условии в пространстве Гельдера. Получена оценка решения задачи с константой, не зависящей от малых параметров, установлена оценка производной по времени решения при малом параметре в граничном условии.

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ОБ ОДНОЙ ЗАДАЧЕ МАТЕМАТИЧЕСКОГО МОДЕЛИРОВАНИЯ НЕСТАЦИОНАРНОГО ДИФФУЗИОННОГО ПРОЦЕССА НА ОСНОВЕ ОБРАТНОЙ ЗАДАЧИ С ФИНАЛЬНЫМ ПЕРЕОПРЕДЕЛЕНИЕМ

В работе рассматривается одно семейство задач, моделирующих процесс экстрагирования из твердых полидисперсных пористых материалов по заданным начальному и конечному (желаемому) состояниям. При их математической формулировке возникает обратная задача для уравнения диффузии, в которой вместе с решением уравнения требуется найти и неизвестную правую часть, зависящую только от пространственной переменной. Спецификой рассматриваемого семейства задач является то, что система собственных функций оператора кратного дифференцирования, подчиненного краевым условиям исходной задачи, не обладает свойством базисности. Доказано существование и единственность классического решения задачи.
Ключевые слова: *обратная задача, уравнение диффузии, начальное состояние, финальное состояние, неусиленно регулярные краевые условия, краевые условия Самарского-Ионкина, биортогональный ряд Фурье, базис Рисса, процессы экстракции.*

Keywords: *inverse problem, diffusion equation, initial state, final state, not strongly regular boundary conditions, Samarskiy-Ionkin boundary conditions, biorthogonal Fourier series, Riesz basis, extraction process.*
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Введение

На сегодняшний день общепризнано, что основным инструментом анализа тепло-массообменных процессов является математическое моделирование. Перенос тепла, массы вещества, нейтронов и других субстанций описывается уравнениями в частных производных. Эти уравнения совместно с дополнительными условиями локализации явления или процесса на пространственно - временных границах области (краевыми условиями) отображают в аналитической форме изучаемый процесс и являются математическими моделями поставленных задач. Реализация модели (интегрирование соответствующих уравнений) позволяет получить картину распределения потенциалов переноса. Полученные модели и их решения дают возможность наиболее просто установить влияние как отдельных параметров, так и их комплексов (критериев) на ход процесса и на этой основе разработать инженерные методы расчета и оптимизации процессов.

В настоящее время все большее значение приобретают разработки, связанные с созданием высокоэффективных технологий, с интенсификацией действующих производств при одновременном решении задач по повышению качественных характеристик производимых материалов. Например, одним из таких направлений, позволяющих интенсифицировать производственные процессы, снизить энергозатраты, является создание экономичных диффузионных (тепло-массообменных) аппаратов и определение режимов их работы. Такие аппараты применяются в авиационной и космической технике, энергетике химической, нефтеперерабатывающей, пищевой промышлености, в холодильной и криогенной технике, в системах отопления и горячего водоснабжения, кондиционирования, в различных тепловых двигателях. С ростом энергетических мощностей и объема производства все более увеличиваются масса и габариты применяемых теплообменных аппаратов. На их производство расходуется огромное количество легированных и цветных металлов. Уменьшение массы и габаритов теплообменных аппаратов является актуальной проблемой, а наиболее перспективным путем решения этой проблемы является интенсификация теплообмена. Наиболее перспективными материалами, позволяющими интенсифицировать теплоотдачу в десятки раз, являются пористые материалы. Кроме того, такие пористые материалы, как пенометаллы, относящиеся к сравнительно новому классу конструкционных материалов, сочетают в себе высокую жесткость, малый удельный вес и низкую теплопроводность. Это позволяет применять их как утеплители и звукоизоляторы (здесь используется их свойство гасить волны), в качестве заполнителей в тонкостенных балках для опорных индустриальных конструкций как ударозащищающие конструкции.

Для построения новых технологий, связанных с диффузионными процессами, оказывается необходимым решение задачи о восстановлении параметров, характеризующих свойства материала или процесса, необходимых для получения финального состояния за конечный промежуток времени. Такие математические задачи называются обратными задачами теории начально-краевых задач для уравнений в частных производных.

В работе рассматривается одно семейство задач, моделирующих процесс экстрагирования из твердых полидисперсных пористых материалов по заданным начальному и конечному (желаемому) состоянию. При их математической формулировке возникает обратная задача для уравнения диффузии, в которой вместе с решением уравнения требуется найти и неизвестную правую часть, зависящую только от пространственной переменной. Спецификой рассматриваемого семейства задач является то, что система собственных функций оператора кратного дифференцирования, подчиненного краевым условиям исходной задачи, не обладает свойством базисности.

Математическому моделированию нестационарных диффузионных процессов на основе обратных задач на сегодняшний день посвящено большое количество научных работ и математиков и технологов. Вопросы разрешимости различных обратных задач для параболических уравнений изучались во многих работах (см., например, [1–8]). В отличие от предыдущих работ, нами исследуется обратная задача для уравнения теплопроводности с краевыми условиями по пространственной переменной, при которых соответствующая спектральная задача для обыкновенного дифференциального оператора имеет систему собственных функций, не образующую базис. Наиболее близкими к тематике настоящей статьи являются наши работы [9–11].

1 Постановка задачи

В области $\Omega = \{(x,t): \ 0 < x < 1, \ 0 < t < T\}$ рассмотрим задачу о нахождении правой части $f(x)$ уравнения диффузии

$$u_t(x,t) - u_{xx}(x,t) = f(x) \tag{1}$$

и его решения $u(x,t)$, удовлетворяющего начальному и конечному условиям

$$u(x,0) = \varphi(x), \quad u(x,T) = \psi(x), \quad 0 \le x \le 1, \tag{2}$$

и краевым условиям

$$u_x(0,t) = u_x(1,t) + \alpha u(1,t), \quad u(0,t) = 0, \quad 0 \le t \le T. \tag{3}$$

Параметр α — любое положительное число, а $\varphi(x)$ и $\psi(x)$ — заданные функции. При $\alpha = 0$ краевые условия (3) хорошо известны и носят в литературе название условий Самарского-Ионкина.

В исследуемой нами модели уравнение (1) означает баланс массы в микропорах (в безразмерных переменных длина микропоры взята равной единице), а краевые условия (3) соответствуют переходу экстрагируемого целевого компонента на следующую стадию. Второе из граничных условий (3) моделирует взаимосвязь концентраций целевого компонента на границах макро - и микропор частиц. Начальные и конечные условия (2) моделируют начальное состояние (количество целевой компоненты в начале процесса) и финальное (желаемое) состояние - финальную концентрацию целевой компоненты. В качестве управления выбирается внешнее воздействие, которое не зависит от времени и может изменяться в пространственных координатах. В реальных задачах такое воздействие описывает влияние процессов в макропоре на процессы в микропорах.

В работе предыдущих авторов обратная задача (с простыми краевыми условиями) решается численно методом Бубнова-Галеркина. Однако удачное применение этого метода зависит от выбора базисных решений, которые определяются граничными условиями. При граничных условиях (3), а также входных параметрах возникает необходимость выбора нового базиса. А для задач с неусиленно регулярными краевыми условиями такое возможно не всегда.

Применение метода Фурье для решения задачи (1)–(3) приводит к спектральной задаче для оператора l, заданного дифференциальным выражением и краевыми условиями

$$\begin{aligned} l(y) \equiv -y''(x) = \lambda y(x), \quad 0 < x < 1, \\ y'(0) = y'(1) + \alpha y(1), \quad y(0) = 0. \end{aligned} \tag{4}$$

Краевые условия в (4) являются регулярными, но не усиленно регулярными [12]. Система корневых функций оператора l является полной системой, но не образует даже обычного базиса в $L_2(0,1)$ [13]. Однако, как показано в [14], на основе этих собственных функций может быть построен базис, позволяющий применить метод разделения переменных для решения начально-краевой задачи с краевым условием (3).

В [7–8] рассмотрены три частных случая обратной задачи (1)–(2), когда краевые условия являются неусиленно регулярными — случай периодических краевых условий и случай условий типа Самарского-Ионкина (краевые условия (3) при $\alpha = 0$). Однако методика исследования этих задач не может быть автоматически перенесена на задачи с краевыми условиями (3) при $\alpha \neq 0$. Это связано с существенным использованием в [7–8] базисности системы собственных и присоединенных функций соответствующей спектральной задачи для оператора кратного дифференцирования. В настоящей работе предлагается использование методики работы [14] для решения обратной задачи (1)–(3). Отметим, что в наших работах [9–11] была использована несколько иная методика и получены близкие по содержанию результаты.

2 Построение базиса из собственных функций задачи (4)

В этом пункте приведем необходимые нам для дальнейшей работы результаты из [14]. Спектральная задача (4) имеет две серии собственных значений

$$\lambda_k^{(1)} = (2\pi k)^2, \; k = 1, 2, \dots, \quad \lambda_k^{(2)} = (2\beta_k)^2, \; k = 0, 1, 2, \dots .$$

Здесь β_k — корни уравнения $\operatorname{tg}\beta = \alpha/2\beta, \; \beta > 0$, они удовлетворяют неравенствам $\pi k < \beta_k < \pi k + \pi/2, \; k = 0, 1, 2, \dots$, и для разности $\delta_k = \beta_k - \pi k$ при достаточно больших k выполняются двусторонние оценки

$$\frac{\alpha}{2\pi k}\left(1 - \frac{1}{2\pi k}\right) < \delta_k < \frac{\alpha}{2\pi k}\left(1 + \frac{1}{2\pi k}\right). \tag{5}$$

Собственные функции задачи (4) имеют вид

$$y_k^{(1)}(x) = \sin 2\pi kx,\ k = 1, 2, \dots, \quad y_k^{(2)}(x) = \sin 2\beta_k x,\ k = 0, 1, 2, \dots.$$

Эта система является почти нормированной, но не образует даже обычного базиса в $L_2(0,1)$. Построенная из нее вспомогательная система

$$y_0(x) = y_0^{(2)}(x)(2\beta_0)^{-1},\ y_{2k}(x) = y_k^{(1)}(x),$$

$$y_{2k-1}(x) = \left(y_k^{(2)}(x) - y_k^{(1)}(x)\right)(2\delta_k)^{-1},\ k = 1, 2, \dots,$$

образует базис Рисса в $L_2(0,1)$, а биортогональной к ней является система

$$v_0(x) = 2\beta_0 v_0^{(2)}(x),\ v_{2k}(x) = v_k^{(2)}(x) + v_k^{(1)}(x),$$

$$v_{2k-1}(x) = 2\delta_k v_k^{(2)}(x),\ k = 1, 2, \dots,$$

построенная из собственных функций сопряженной к (4) задачи

$$v_k^{(1)}(x) = C_k^{(1)}\cos(2\pi kx + \gamma_k),\ k = 1, 2, \dots,$$

$$v_k^{(2)}(x) = C_k^{(2)}\cos(\beta_k(1-2x)),\ k = 0, 1, 2, \dots.$$

Константы $C_k^{(j)}$ выбираются из соотношения биортогональности $\left(y_k^{(j)}, v_k^{(j)}\right) = 1,\ j = 1, 2.$

Если $y(x) \in C^2[0,1]$ и удовлетворяет краевым условиям (4), то ее ряд Фурье по системе $\{y_k(x)\}$ сходится абсолютно.

Нетрудно вычислить, что

$$\begin{gathered} y_0''(x) = -\lambda_0^{(2)} y_0(x), \quad y_{2k}''(x) = -\lambda_k^{(1)} y_{2k}(x), \\ y_{2k-1}''(x) = -\lambda_k^{(2)} y_{2k-1}(x) - \frac{\lambda_k^{(2)} - \lambda_k^{(1)}}{2\delta_k} y_{2k}(x). \end{gathered} \tag{6}$$

3 Построение формального решения задачи (1)–(3)

Как следует из раздела 2, любое решение $u(x,t), f(x)$ задачи (1)–(3) представимо в виде биортогональных рядов:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)y_k(x),\quad f(x)=\sum_{k=0}^{\infty}f_k y_k(x), \tag{7}$$

где $u_k(t)=(u(x,t),v_k(x))$, $f_k=(f(x),v_k(x))$.

Подставляя (7) в уравнение (1), в начальные и конечные условия (2), с учетом (6), для нахождения неизвестных функций $u_k(t)$ и коэффициентов f_k получаем следующие задачи:

$$u_0'(t)+\lambda_0^{(2)}u_0(t)=f_0,\quad u_0(0)=\varphi_0,\ u_0(T)=\psi_0; \tag{8}$$

$$u_{2k-1}'(t)+\lambda_k^{(2)}u_{2k-1}(t)=f_{2k-1},\ u_{2k-1}(0)=\varphi_{2k-1},\ u_{2k-1}(T)=\psi_{2k-1}; \tag{9}$$

$$u_{2k}'(t)+\lambda_k^{(1)}u_{2k}(t)=-\frac{\lambda_k^{(2)}-\lambda_k^{(1)}}{2\delta_k}u_{2k-1}(t)+f_{2k},\ u_{2k}(0)=\varphi_{2k},\ u_{2k}(T)=\psi_{2k}, \tag{10}$$

где φ_k,ψ_k — коэффициенты Фурье разложения функций $\varphi(x)$ и $\psi(x)$ в биортогональный ряд по системе $\{y_k(x)\}$: $\varphi_k=(\varphi(x),v_k(x))$, $\psi_k=(\psi(x),v_k(x))$.

Решение задач (8) и (9) существует, единственно и может быть выписано в явном виде:

$$\begin{aligned}
u_0(t)&=e^{-\lambda_0^{(2)}t}\varphi_0+\frac{1-e^{-\lambda_0^{(2)}t}}{1-e^{-\lambda_0^{(2)}T}}\left(\psi_0-e^{-\lambda_0^{(2)}T}\varphi_0\right),\\
f_0&=\frac{\lambda_0^{(2)}}{1-e^{-\lambda_0^{(2)}T}}\left(\psi_0-e^{-\lambda_0^{(2)}T}\varphi_0\right);\\
u_{2k-1}(t)&=e^{-\lambda_k^{(2)}t}\varphi_{2k-1}+\frac{1-e^{-\lambda_k^{(2)}t}}{1-e^{-\lambda_k^{(2)}T}}\left(\psi_{2k-1}-e^{-\lambda_k^{(2)}T}\varphi_{2k-1}\right),\\
f_{2k-1}&=\frac{\lambda_k^{(2)}}{1-e^{-\lambda_k^{(2)}T}}\left(\psi_{2k-1}-e^{-\lambda_k^{(2)}T}\varphi_{2k-1}\right),\ k=1,2,\ldots.
\end{aligned} \tag{11}$$

Подставляя найденные функции (11) в правую часть уравнений (10), получим, что задача (10) также имеет единственное решение, и оно записывается в виде

$$u_{2k}(t) = e^{-\lambda_k^{(1)}t}\varphi_{2k} + \frac{1-e^{-\lambda_k^{(1)}t}}{\lambda_k^{(1)}} f_{2k} + \frac{e^{-\lambda_k^{(2)}t}-e^{-\lambda_k^{(1)}t}}{2\delta_k}\varphi_{2k-1} +$$

$$+ \frac{\psi_{2k-1}-e^{-\lambda_k^{(2)}T}\varphi_{2k-1}}{1-e^{-\lambda_k^{(2)}T}} \left\{ \frac{\lambda_k^{(2)}-\lambda_k^{(1)}}{2\delta_k}\frac{1-e^{-\lambda_k^{(1)}t}}{\lambda_k^{(1)}} - \frac{e^{-\lambda_k^{(2)}t}-e^{-\lambda_k^{(1)}t}}{2\delta_k} \right\},$$

$$f_{2k} = \frac{\lambda_k^{(1)}}{1-e^{-\lambda_k^{(1)}T}} \left\{ \left(\psi_{2k-1} - e^{-\lambda_k^{(2)}T}\varphi_{2k-1}\right) - \frac{e^{-\lambda_k^{(2)}T}-e^{-\lambda_k^{(1)}T}}{2\delta_k}\varphi_{2k-1} \right\} +$$

$$+ \frac{\psi_{2k-1}-e^{-\lambda_k^{(2)}T}\varphi_{2k-1}}{1-e^{-\lambda_k^{(2)}T}} \left\{ \frac{\lambda_k^{(2)}-\lambda_k^{(1)}}{2\delta_k} + \frac{\lambda_k^{(1)}}{1-e^{-\lambda_k^{(1)}T}}\frac{e^{-\lambda_k^{(2)}T}-e^{-\lambda_k^{(1)}T}}{2\delta_k} \right\}. \qquad (12)$$

Подставляя функции (11) и (12) в ряды (7), получаем формальное решение задачи.

4 Основная теорема

Для завершения исследования необходимо (аналогично методу Фурье) обосновать гладкость полученного формального решения и сходимость всех полученных рядов. Сформулируем основной результат работы.

Теорема. *Если $\varphi(x)$, $\psi(x) \in C^4[0,1]$ и функции $\varphi(x)$, $\psi(x)$, $\varphi''(x)$, $\psi''(x)$ удовлетворяют краевым условиям (4), то существует единственное классическое решение $u(x,t) \in C^{2,1}_{x\,t}\left(\overline{\Omega}\right)$, $f(x) \in C[0,1]$ задачи (1)–(3).*

Доказательство. Так как $\varphi''(x)$, $\psi''(x) \in C^2[0,1]$ и удовлетворяют краевым условиям (4), то (см. п. 2) они разлагаются в абсолютно сходящиеся ряды Фурье по системе $\{y_k(x)\}$. Отсюда и из (6) имеем абсолютную и равномерную сходимость рядов

$$\varphi''(x) = -\lambda_0^{(2)}\varphi_0 y_0(x) - \sum_{k=1}^{\infty}\left\{\lambda_k^{(1)}\varphi_{2k}y_{2k}(x) + \lambda_k^{(2)}\varphi_{2k-1}y_{2k-1}(x)\right\} - \\ -\sum_{k=1}^{\infty}\frac{\lambda_k^{(2)}-\lambda_k^{(1)}}{2\delta_k}\varphi_{2k-1}y_{2k}(x), \tag{13}$$

$$\psi''(x) = -\lambda_0^{(2)}\psi_0 y_0(x) - \sum_{k=1}^{\infty}\left\{\lambda_k^{(1)}\psi_{2k}y_{2k}(x) + \lambda_k^{(2)}\psi_{2k-1}y_{2k-1}(x)\right\} - \\ -\sum_{k=1}^{\infty}\frac{\lambda_k^{(2)}-\lambda_k^{(1)}}{2\delta_k}\psi_{2k-1}y_{2k}(x). \tag{14}$$

Легко видеть, что $\left(\lambda_k^{(2)}-\lambda_k^{(1)}\right)/2\delta_k = 4\pi k + 2\delta_k$. Поэтому из (11), (12) с учетом (5) несложно получить равномерные по k оценки

$$|u_{2k-1}(t)| \le C\left(|\varphi_{2k-1}| + |\psi_{2k-1}|\right),$$

$$|u_{2k}(t)| \le C\left(|\varphi_{2k}| + |\varphi_{2k-1}| + |\psi_{2k}| + |\psi_{2k-1}|\right);$$

$$|u'_{2k-1}(t)| \le C\left(|\varphi_{2k-1}| + |\psi_{2k-1}|\right)|\lambda_k^{(1)}|,$$

$$|u'_{2k}(t)| \le C\left(|\varphi_{2k}| + |\varphi_{2k-1}| + |\psi_{2k}| + |\psi_{2k-1}|\right)|\lambda_k^{(2)}|,$$

$$|f_{2k-1}| \le C\left(|\varphi_{2k-1}| + |\psi_{2k-1}|\right)|\lambda_k^{(1)}|,$$

$$|f_{2k}| \le C\left(|\varphi_{2k}| + |\varphi_{2k-1}| + |\psi_{2k}| + |\psi_{2k-1}|\right)|\lambda_k^{(2)}|.$$

Отсюда, из абсолютной сходимости рядов (13) и (14) следует сходимость рядов (7) и принадлежность решения задачи (1)–(3) классам $u(x,t) \in C_{x\,t}^{2,1}\left(\overline{\Omega}\right)$, $f(x) \in C[0,1]$.

Так как система $\left\{y_k(x)\right\}$ образует базис Рисса пространства $L_2(0,1)$, то любое решение задачи (1)–(3) из данного класса представимо рядами (7). Из однозначности построения решений (11), (12) задач (8)–(10) следует единственность решения задачи (1)–(3). Теорема доказана полностью. □

5 Заключение

Таким образом, в работе нами показана корректность обратной задачи процесса диффузии, рассмотренного в отдельно взятой микропоре. В качестве управления выбрано внешнее воздействие, которое не зависит от времени и может изменяться в пространственных координатах. В реальных задачах такое воздействие описывает влияние процессов в макропоре на процессы в микропорах. Результат настоящей работы дает возможность построения математической модели процесса диффузии целевой компоненты из полидисперсных пористых материалов, учитывающей совместное и взаимное влияние процессов в макро- и микропорах.

Литература

1 Аниконов Ю.Е., Белов Ю.Я. Об однозначной разрешимости одной обратной задачи для параболического уравнения // Докл. АН СССР. — 1989. — Т. 306, №6. — С. 1289–1293.

2 Аниконов Ю.Е., Бубнов Б.А. Существование и единственность решения обратной задачи для параболического уравнения // Докл. АН СССР. — 1988. — Т. 298, №4. — С. 777–779.

3 Бубнов Б.А. Существование и единственность решения обратных задач для параболических и эллиптических уравнений // Сб. "Неклассические уравнения математической физики". — Новосибирск: Изд-во Ин-та математики. — 1986. — С. 25–29.

4 Прилепко А.И., Костин А.Б. Об обратных задачах для параболических уравнений с финальным и интегральным наблюдением // Мат. сб. — 1992. — Т. 183, № 4. — С. 49–68.

5 Кожанов А.И. Нелинейные нагруженные уравнения и обратные задачи // Журн. вычисл. матем. и матем. физ. — 2004. — Т. 44, №4. — С. 722–744.

6 Монахов В.Н. Обратные задачи для релаксационных моделей гидродинамики // Вестн. Новосиб. гос. ун-та. Сер. Математика, механика, информатика. — 2003. — Т. 3, вып. 3. — С. 72–80.

7 Калиев И.А., Первушина М.М. Обратные задачи для уравнения теплопроводности // Сб. "Совр. проблемы физики и математики. Т. 1". —

Уфа: Гилемю — 2004. — С. 50–55.

8 Калиев И.А., Сабитова М.М. Задачи определения температуры и плотности источников тепла по начальной и конечной температурам // Сибирский журнал индустр. математики. — 2009. — Т.12, № 1. — С. 89–97.

9 Оразов И., Садыбеков М.А. Задачи определения температуры и плотности источников тепла по начальной и конечной температурам при неусиленно регулярных краевых условиях // Узбекский математический журнал. — 2012, № 1. — С. 107–114.

10 Оразов И., Садыбеков М.А. Об одной нелокальной задаче определения температуры и плотности источников тепла // Известия вузов. Математика. — 2012, № 2. — С. 70–75.

11 Оразов И., Садыбеков М.А. Об одном классе задач определения температуры и плотности источников тепла по начальной и конечной температурам // Сиб. мат. журнал. — 2012 — Т. 53, № 1. — С. 180–186.

12 Наймарк М.А. Линейные дифференциальные операторы. — М.: Наука, 1969. — 528 с.

13 Lang P., Locker J. Spectral Theory of Two-Point Differential Operators Determined by -D^2 // J. Math. Anal. And Appl. — 1990. — V. 146, № 1. —P. 148–191.

14 Мокин А.Ю. Об одном семействе начально-краевых задач для уравнения теплопроводности // Дифференц. уравнения. — 2009. — Т. 45, № 1. — С. 123–137.

Literatura

1 Anikonov Yu.E., Belov Yu.Ya. Ob odnoznachnoy razreshimosti odnoy obratnoy zadachi dlya parabolicheskogo uravneniya // Dokl. AN SSSR. — 1989. — T. 306, № 6. — S. 1289–1293.

2 Anikonov Yu.E., Bubnov B.A. Sushestvovanie i edinstvennost' resheniya obratnoy zadachi dlya parabolicheskogo uravneniya // Dokl. AN SSSR. — 1988. — T. 298, № 4. — S. 777–779.

3 Bubnov B.A. Sushhestvovanie i edinstvennost' resheniya obratnyh zadach dlya parabolicheskih i ellipticheskih uravneniy // Sb. "Neklassicheskie uravneniya matematicheskoy fiziki". — Novosibirsk: Izd-vo In-ta matematiki. — 1986. — S. 25–29.

4 Prilepko A.I., Kostin A.B. Ob obratnyh zadachah dlya parabolicheskih uravneniy s final'nym i integral'nym nablyudeniem // Matem. sb. — 1992. — T. 183, № 4. — S. 49–68.

5 Kozhanov A.I. Nelineynye nagruzhennye uravneniya i obratnye zadachi // Zhurn. vychisl. matem. i matem. fiz. — 2004. — T. 44, № 4. — S. 722–744.

6 Monakhov V.N. Obratnye zadachi dlya relaksatcionnyh modeley gidrodinamiki // Vestn. Novosib. gos. un-ta. Ser. Matematika, mehanika, informatika. — 2003. — T. 3, vyp. 3. — S. 72–80.

7 Kaliev I.A., Pervushina M.M. Obratnye zadachi dlya uravneniya teploprovodnosti // Sb. "Sovr. problemy fiziki i matematiki. T. 1". — Ufa: Gilemyu — 2004. — S. 50–55.

8 Kaliev I.A., Sabitova M.M. Zadachi opredeleniya temperatury i plotnosti istochnikov tepla po nachal'noy i konechnoy temperaturam // Sibirskiy zhurnal industr. matematiki. — 2009. — T.12, № 1. — S. 89–97.

9 Orazov I., Sadybekov M.A. Zadachi opredeleniya temperatury i plotnosti istochnikov tepla po nachal'noy i konechnoy temperaturam pri neusilenno regulyarnyh kraevyh usloviyah // Uzbekskiy matematicheskiy zhurnal. — 2012, № 1. — S. 107–114.

10 Orazov I., Sadybekov M.A. Ob odnoy nelokal'noy zadache opredeleniya temperatury i plotnosti istochnikov tepla // Izvestiya vuzov. Matematika. — 2012, № 2. — S. 70–75.

11 Orazov I., Sadybekov M.A. Ob odnom klasse zadach opredeleniya temperatury i plotnosti istochnikov tepla po nachal'noy i konechnoy temperaturam // Sib. mat. zhurnal. — 2012 — T. 53, № 1. — S. 180–186.

12 Naimark M.A. Lineynye differencial'nye operatory. — M.: Nauka, 1969. — 528 s.

13 Lang P., Locker J. Spectral Theory of Two-Point Differential Operators Determined by -D^2 // J. Math. Anal. And Appl. — 1990. — V. 146, № 1. —P. 148–191.

14 Mokin A.Yu. Ob odnom semeistve nachal'no-kraevyh zadach dlya uravneniya teploprovodnosti // Differentc. uravneniya. — 2009. — T. 45, № 1. — S. 123–137.

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Оразов И., Садыбеков М.А. АҚЫРҒЫ ҚАЙТА АНЫҚТАЛЫМДЫ КЕРІ ЕСЕП НЕГІЗІНДЕГІ СТАЦИОНАРЛЫҚ ЕМЕС ДИФФУЗИЯЛЫҚ ПРОЦЕССТІ МАТЕМАТИКАЛЫҚ МОДЕЛДЕУДІҢ БІР ЕСЕБІ ТУРАЛЫ

Бұл жұмыста қатты полидисперсиялы кеуек материалдарды бастапқы және соңғы (қалаулы) күйлерінің берілген шамасы бойынша бөліп алу процесін моделдейтін есептердің бір топтамасы қарастырылады. Бұл есептің математикалық тұжырымдамасы диффузиялық теңдеудің кері есебіне әкеледі, мұнда есептің тек шешімін табумен ғана шектелмей, қосымша оның кеңістік айнымалыдан ғана тәуелді болатын белгісіз оң жақ бөлігін табу талап етіледі. Қарастырылып отырған есептер топтамасының ерекшелігі бастапқы есептің шекаралық шартына тәуелді болатын еселі дифференциалдау операторының меншікті функциялар жүйесі базистік қасиетке ие болмайтындығында. Есептің классикалық шешімінің бар және жалғыз екендігі дәлелденді.

Orazov I., Sadybekov M.A. ON A MATHEMATICAL MODELING PROBLEM OF NONSTATIONARY DIFFUSION PROCESS BASED ON THE INVERSE PROBLEM WITH FINAL OVERDETERMINATION

In the paper there is considered a set of the problems modelling the extraction process from solid polydisperse porous materials under the given values of the initial and final (desired) states. The mathematical formulation of these problems leads to the inverse problem for the diffusion equation, in which it is required to find both together a solution of the equation and its right-hand side depending only on a spatial variable. A specific feature of the considered set of the problems is that the system of eigenfunctions of the multiple differentiation operator subjected to the boundary conditions of the original problem, does not have the basis property. It is proved the existence and uniqueness of the classical solution of the problem.

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CANONICAL MINI CONSTRUCTION OF FINITELY AXIOMATIZABLE THEORIES AS A WEAK RELEASE OF THE UNIVERSAL CONSTRUCTION

In this work, based on an available version of the canonical construction of finitely axiomatizable theories, we deduce some weaken release of the same construction which is said to be the canonical-mini construction. The obtained construction has a standard formulation of the universal construction of finitely axiomatizable theories controlling not large (nevertheless, non-trivial) semantic layer of model-theoretic properties. Thus, the canonical-mini construction can be said to be the universal-under-canonical construction. Supporting a sublayer of the infinitary semantic layer, the canonical-mini construction can perform the role of a weak release of the universal construction; moreover, it favorably differs from the latter by a much simpler and understandable proof.

Keywords: *first-order logic, Tarski-Lindenbaum algebra, Stone space, natural binary tree, compact binary tree, universal construction of finitely axiomatizable theories.*

Introduction

The universal construction of finitely axiomatizable theories, [1], Ch.4, plays a key role in investigations concerning expressive power of first-order

Keywords: *first-order logic, Tarski-Lindenbaum algebra, Stone space, natural binary tree, compact binary tree, universal construction of finitely axiomatizable theories*
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predicate logic. Starting from an input computably axiomatizable theory T this construction yields a finitely axiomatizable theory F together with a computable isomorphism $\mu : \mathcal{L}(T) \to \mathcal{L}(F)$ of their Tarski-Lindenbaum algebras passing from complete extensions of theory T to corresponding complete extensions of theory F all model-theoretic properties within the infinitary semantic layer MQL. Fundamental nature as well as maximality of the infinitary layer MQL is established in the work [2]. Although there are weaker versions of the universal construction with a reduced or even omitted rigidity mechanism (thus controlling smaller layers of model-theoretic properties), nevertheless, all they represent large complexity in studying almost indistinguishable from that of any stronger version of the universal construction.

In the work, we present an even weaker version of the universal construction that is a simple enough consequence of the canonical construction $\mathbb{F}_{\mathbb{C}}(m, s)$ described in [1], Ch.3. It is said to be the *canonical-mini* construction, alternatively, *universal-under-canonical* construction. In Section 8, we present a routine deduction of the canonical-mini construction from the canonical construction. The other parts of the paper represent auxiliary notions; they are included for the sake of completeness of the text. Principal advantage of the canonical-mini construction is that it is essentially simpler in studying in comparison with that of any normal version of the universal construction. Although the new construction supports not large semantic layer of model-theoretic properties, nevertheless, this construction represents a non-trivial statement and can find useful applications.

Preliminaries. We consider theories in first-order predicate logic *with equality* and use general concepts of model theory, algorithm theory, and constructive models found in Hodres [3], Rogers [4], and Goncharov and Ershov [5]. Generally, *incomplete* theories are considered. In the work, the signatures are considered only, which admit a Gödel numbering of the formulas. Such a signature is called *enumerable*.

A finite signature is called *rich*, if it contains at least one n-ary predicate or function symbol for $n \geqslant 2$, or two unary function symbols. A nulary predicate symbol is often said to be a propositional variable. The following notations are used: $FL(\sigma)$ is the set of all formulas of signature σ, $FL_k(\sigma)$ is the set of all formulas of signature σ with free variables $x_0, \ldots, x_{k-1}$, $SL(\sigma)$ is the set of all sentences (i.e., closed formulas) of signature σ. Let σ be a signature, and Σ be a subset of $SL(\sigma)$. Denote by $[\Sigma]^{\sigma}$ a theory of signature σ generated by Σ as a set of its axioms. There is another variant of the definition. Let $\Sigma \subseteq SL(\sigma)$ be a set of sentences. By $[\Sigma]^{\star}$, we denote a theory of a signature $\sigma' \subseteq \sigma$ generated by the set Σ as a set of its axioms, where σ' contains only those symbols from σ that occur in formulas of the set Σ.

We use Cantor's function $c(x, y) = (x^2 + 2xy + y^2 + 3x + y)/2$ presenting a one-one mapping from the set of pairs $\mathbb{N}\times\mathbb{N}$ onto $\mathbb{N}$, cf. [4] Sec.5.3. We use more common notation $\langle x_1, \ldots, x_m\rangle$ for a m-ary computable function presenting a one-one mapping from the set $\mathbb{N}^m$ onto $\mathbb{N}$; for instance, in the case $k = 4$, it is possible to define this function as follows: $\langle x_1, x_2, x_3, x_4\rangle = c(x_1, c(x_2, c(x_3, x_4)))$. By $\varphi_n(x)$, we denote nth partial computable function in Kleene's numbering, while $\varphi_n^X(t)$ denotes nth partial function in computation with an oracle X. Often we use binary universal computable functions defined by the rules $\varphi_n^{(2)}(x, y) = \varphi_n(c(x, y))$ and $\varphi_n^{A,(2)}(x, y) = \varphi_n^A(c(x, y))$, denoted $\varphi_n(x, y)$ and $\varphi_n^A(x, y)$ for short. By W_n, we denote nth computably enumerable set in Post's numbering, while W_n^A denotes nth computably enumerable set in computation with an oracle A, cf. [4] Ch.5, Ch.9. These universal numberings are defined by the following rules $W_n = \mathrm{Dom}(\varphi_n(x))$ and $W_n^A = \mathrm{Dom}(\varphi_n^A(x))$ for all $n \in \mathbb{N}$ and $A \subseteq \mathbb{N}$.

The set of all finite tuples α of the form $\alpha = \langle\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\rangle$, $\alpha_i \in \{0, 1\}$, $k \geqslant 0$, is denoted by $2^{<\omega}$, while 2^{ω} is the set of all infinite tuples of the form $\alpha = \langle\alpha_i; i < \omega\rangle$, $\alpha_i \in \{0, 1\}$. We use symbol $\preccurlyeq$ for a partial order on the set $2^{<\omega} \cup 2^{\omega}$ defined by the rule: $\varepsilon \preccurlyeq \varepsilon' \Leftrightarrow \varepsilon$ is an initial segment of ε'. If Φ is a formula and $\alpha \in \{0, 1\}$, Φ^{α} means Φ for $\alpha = 1$, and $\neg\Phi$ for $\alpha = 0$. For a theory, c.a. means *computably axiomatizable*, while f.a. means *finitely axiomatizable*.

We denote by σ^{∞} a fixed maximum large enumerable signature. Namely, signature σ^{∞} contains countably many constant symbols, symbols of propositional variables, and predicate and function symbols of each arity $n \geqslant 1$. It is supposed that each considered signature σ is a proper part of the «universal» signature σ^{∞}. We use a fixed Gödel numbering Φ_k, $k \in \mathbb{N}$, for the set of sentences of a fixed signature σ, and Φ_k^{∞}, $k \in \mathbb{N}$, for the set of sentences of the infinite signature σ^{∞}. An entry $\mathrm{Nom}\,\Psi$ denotes the number of a sentence Ψ in an accepted Gödel numbering of the set of sentences.

Based on the Post numbering of the family of all computably enumerable sets W_n, $n \in \mathbb{N}$, we construct an effective numbering for the class of all computably axiomatizable theories. There are two versions of indices. The first one represents c.e. indices of c.a. theories of a given enumerable or finite signature σ. If a theory T of signature σ is defined by the set of axioms $\{\Phi_i \mid i \in W_m\}$, the number m is called a *computably enumerable index* or simply *c.e. index* of the theory T. The second version represents indices of c.a. theories of arbitrary enumerable signatures, which are subsets of the «universal» signature σ^∞. Given $m \in \mathbb{N}$. Consider the set of axioms $\Sigma = \{\Phi_i^\infty \mid i \in W_m\}$ and construct the theory $T = [\Sigma]^\star$. The number m is called a *weak computably enumerable index* or simply *weak c.e. index* of the theory T. As for a finitely axiomatizable theory F, it is defined by a finite system A of axioms, and therefore, by a single formula Φ which is a conjunction of the formulas from A. A Gödel number n for this formula Φ is normally considered as a *Gödel number* or *strong index* of the theory F.

Lemma 0.1. *There is an effective transformation from c.e. indices of computably axiomatizable theories of a fixed enumerable signature σ to their weak c.e. indices.*

Proof. Immediately. □

Normally, so-called *static* method of operating with signatures of theories is applied. According to this method, first, we fix a signature; then we define a Gödel numbering of the set of formulas; further, we describe a theory by enumerating its axioms; and finally, turn to studying its properties. This method is often applicable whenever we use normal c.e. indices of theories.

In contrary, while using weak indices of theories, we are forced to change the order of consideration applying a so-called *dynamic* method of operating with signatures. Having a weak c.e. index of a theory T, first, we enumerate formulas provable in T obtaining its signature $\sigma \subseteq \sigma^\infty$; then, we have to define a Gödel numbering of the set of formulas of signature σ as well as its subsets $SL(\sigma)$, $FL_k(\sigma)$, $k < \omega$, etc., as needed; finally, we turn to studying properties of the theory under consideration.

The following statement represents a technical basis for the dynamic method.

Lemma 0.2. *Fix a Gödel numbering $\varphi_t^\infty(\overline{x}_t)$, $t \in \mathbb{N}$, for the set of all formulas of the maximum large signature σ^∞. There are total computable functions $f(n, i)$*

and $h(n,k,i)$, such that, if n is a weak c.e. index of a theory T of signature σ, we have:

(a) $\varphi^{\infty}_{f(n,i)}(\overline{x}_{f(n,i)})$, $i \in \mathbb{N}$, *represents a Gödel numbering of the set of all formulas of signature* σ,

(b) $\varphi^{\infty}_{h(n,0,i)}(\overline{x}_{h(n,0,i)})$, $i \in \mathbb{N}$, *represents a Gödel numbering of the set* $SL(\sigma)$ *of all sentences of signature* σ,

(c) $\varphi^{\infty}_{h(n,k,i)}(\overline{x}_{h(n,k,i)})$, $i \in \mathbb{N}$, *represents a Gödel numbering of the set* $FL_k(\sigma)$ *of all formulas of signature* σ *with* k *free variables, for any fixed* k, $1 \leqslant k < \omega$.

Proof. Immediately. □

Lemma 0.1 shows that the dynamic method of operating with signatures can also be applied to the normal version of indices for c.e. theories of a fixed enumerable or finite signature σ.

A statement concerning particular models of a complete theory.

Claim 0.3. *Given a complete theory* T *of an enumerable signature whose models are infinite. The following assertions are satisfied*:

(a) [6]: T *has a prime model* $\Leftrightarrow$ *principal types form a dense subset in the set of all types in* $\mathcal{L}_k(T)$ *for all* k *satisfying* $1 \leqslant k < \omega$,

(b) [7, 8]: *in the case when* T *has a prime model* $\mathfrak{N}$, *the model* $\mathfrak{N}$ *is strongly constructivizable* $\Leftrightarrow$ T *is decidable and the set of all principal types of* T *is computable*,

(c) [9]: *in the case when* T *has a strongly constructivizable prime model* $\mathfrak{N}$, *the model* $\mathfrak{N}$ *has algorithmic dimension* 1 *relative to strong constructivizations* $\Leftrightarrow$ *the set of all atomic formulas of* T *is computably enumerable* $\Leftrightarrow$ *the set of all atomic formulas of* T *is computable*,

(d) [6]: T *has a countable saturated model* $\Leftrightarrow$ *the set of all types in* $\mathcal{L}_k(T)$ *is at most countable for all* k *satisfying* $1 \leqslant k < \omega$,

(e) [10]: *in the case when* T *has a countable saturated model* $\mathfrak{M}$, *the model* $\mathfrak{M}$ *is strongly constructivizable* $\Leftrightarrow$ T *is decidable and the set of all types of* T *is computable.*

We apply these statements further in Section 8 in considering preservation of the layer of model-theoretic properties involved in the canonical-mini construction.

Give some definitions concerning numerated Boolean algebras.

A Boolean algebra $\mathcal{B}$ together with a numeration $\nu : \mathbb{N} \xrightarrow{onto} |\mathcal{B}|$, denoted by

$(\mathcal{B}, \nu)$, is said to be a *numerated Boolean algebra* if its signature operations are uniformly presentable by total computable functions on the ν-numbers; i.e., if there are total computable functions $u(x, y)$, $v(x, y)$, and $w(x)$ that represent operations in the Boolean algebra as follows for all $i, j \in \mathbb{N}$:

$$\begin{aligned} &\nu(i) \cup \nu(j) = \nu(u(i, j)), \\ &\nu(i) \cap \nu(j) = \nu(v(i, j)), \\ &-\nu(i) = \nu(w(i)). \end{aligned} \tag{0.1}$$

A numerated Boolean algebra $(\mathcal{B}, \nu)$ is called a *c.e. Boolean algebra* if the equality relation in $\mathcal{B}$ is computably enumerable in this numeration ν; i.e., there is a relation $E(x, y)$ in Σ^0_1, such that, for all $i, j \in \mathbb{N}$

$$\nu(i) = \nu(j) \ \Leftrightarrow \ E(i, j). \tag{0.2}$$

Two numerated Boolean algebras $(\mathcal{B}_1, \nu_1)$ and $(\mathcal{B}_2, \nu_2)$ are called *equivalent*, or *computably isomorphic*, denoted $(\mathcal{B}_1, \nu_1) \cong (\mathcal{B}_2, \nu_2)$, if there is an isomorphism μ between $\mathcal{B}_1$ and $\mathcal{B}_2$ and two total computable functions $f(x)$ and $g(x)$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{N} & \underset{g}{\overset{f}{\rightleftarrows}} & \mathbb{N} \\ {\scriptstyle \nu_1}\downarrow & & \downarrow{\scriptstyle \nu_2} \\ \mathcal{B}_1 & \xrightarrow{\ \mu\ } & \mathcal{B}_2 \end{array} \tag{0.3}$$

Introduce indices for numerated Boolean algebras.

Definition 0.A. *Given a numerated Boolean algebra $(\mathcal{B}, \nu)$. An integer m is said to be a relativized characteristic index of the algebra $(\mathcal{B}, \nu)$ in computation with an oracle A if, by decoding the number m as follows: $m = \langle m_1, m_2, m_3, m_4 \rangle$, we obtain four integers satisfying the following properties*:

(a) *functions $u(x, y) = \varphi_{m_1}(x, y)$, $v(x, y) = \varphi_{m_2}(x, y)$, $w(x) = \varphi_{m_3}(x)$ realize all requirements posed in* (0.1),

(b) *function $\varphi^A_{m_4}(x, y)$ is a total function having the domain $\{0, 1\}$; moreover, the relation $E(x, y) \Leftrightarrow \varphi^A_{m_4}(x, y) = 1$ realizes the requirement posed in* (0.2).

1 Table conditions and the parametric Stone space

A *propositional variable* is a variable X whose value can be either *true* or *false*. For brevity, the Boolean values «true» and «false» may be written as 1 and 0. The term "propositional variable" is used as a synonym for "nulary predicate".

Consider the following sequence of propositional variables

$$\sigma^\circ = \{X_0, X_1, X_2, \dots, X_k, \dots\,;\ k \in \mathbb{N}\}. \tag{1.1}$$

It is said to be a *special propositional signature*. By $FRM(\sigma^\circ)$, we denote the set of all formulas of signature σ° constructed with the only connective $|$ called the *Sheffer stroke*. This connective is defined by the rule $X|Y \leftrightarrow \neg(X \wedge Y)$. It is a known fact that the operation $|$ represents a complete system for propositional logic. As for the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$, we consider them as suitable Boolean expressions via the basic connective $|$.

We use Cantor's function $c(x, y)$ presenting a one-one mapping from the set $\mathbb{N}\times\mathbb{N}$ onto $\mathbb{N}$. Based on this primitive recursive function, we define a mapping 'nom' from the set of formulas $FRM(\sigma^\circ)$ into the set of natural numbers by the following rule:

$$\begin{aligned} &\mathrm{nom}(X_k) = 2\,k, \quad k \in \mathbb{N}, \\ &\mathrm{nom}(\Psi'|\Psi'') = 2\,c\big(\mathrm{nom}(\Psi'), \mathrm{nom}(\Psi'')\big) + 1, \quad \Psi', \Psi'' \in FRM(\sigma^\circ). \end{aligned} \tag{1.2}$$

It is possible to check that the function 'nom' is a bijective mapping from $FRM(\sigma^\circ)$ onto $\mathbb{N}$. For $k \in \mathbb{N}$, we denote $\mathfrak{P}_k = \mathrm{nom}^{-1}(k)$. As a result, we have obtained a sequence of propositional formulas

$$\mathfrak{P}_k,\ k \in \mathbb{N}. \tag{1.3}$$

This sequence represents a Gödel numbering of the set of all formulas of the propositional signature σ°.

In detail, the sequence of formulas (1.3) looks as follows:

$$\mathfrak{P}_k(X_0, X_1, \dots, X_{\mathfrak{a}}), \quad \mathfrak{a} = \mathfrak{a}(k),\ k \in \mathbb{N}, \tag{1.4}$$

where $\mathfrak{a}(k)$ is a primitive recursive function such that all variables of the formula $\mathfrak{P}_k$ are contained in the list $X_0, X_1, \dots, X_{\mathfrak{a}(k)}$; although, some of them may be absent in the formula $\mathfrak{P}_k$.

Often, some expressions of a true-false type can be substituted in formula $\mathfrak{P}_k$ instead of its variables X_i, $i \leqslant \mathfrak{a}(k)$. Particularly, we can consider the formula $\mathfrak{P}_k$ as a *truth-table condition* (or briefly *tt-condition*); for this, we should replace each of its variables X_k, $k \leqslant \mathfrak{a}(k)$, by elementary truth-table conditions of the form

$$\mathcal{X}_k = \text{«}\textit{the set includes element } k\text{»}; \tag{1.5}$$

thereby, $\mathfrak{P}_k$ will turn out to be applied to an arbitrary set $A \subseteq \mathbb{N}$. The claim that *tt*-condition $\mathfrak{P}_k$ is satisfied on A is written as $A \vDash \mathfrak{P}_k$. Alternatively, for a set $A \subseteq \mathbb{N}$, we can count that an entry $A \models \mathfrak{P}_k$ means a truth-value of the term $\mathfrak{P}_k(\chi_A(0), ..., \chi_A(\mathfrak{a}(k)))$, where $\chi_A(x)$ is a characteristic function of the set $A \subseteq \mathbb{N}$. Thereby, we count that the same sequence (1.3) of propositional formulas represents a *Gödel numbering* of the set of all *tt-conditions.*

Let $m \in \mathbb{N}$ be an integer parameter. We introduce the following notation for a subset of the power-set $\mathcal{P}(\mathbb{N})$:

$$\Omega(m) = \{A \subseteq \mathbb{N} \mid (\forall k \in W_m)[A \vDash \mathfrak{P}_k]\}. \tag{1.6}$$

It is called the *parametric Stone space*, while the number m is called its *index.* The set (1.6) is used as a technical tool for presenting Stone spaces of computably axiomatizable and finitely axiomatizable theories.

Give a characterization of the parametric Stone space:

Lemma 1.1. *The following conditions are equivalent with each other for all $m \in \mathbb{N}$ and $A \subseteq \mathbb{N}$:*

(a) $A \in \Omega(m)$,

(b) $(\forall k \in W_m)[A \vDash \mathfrak{P}_k]$.

Proof. It is a direct consequence of definition (1.6). □

2 Tarski-Lindenbaum algebras of theories

Let T be a theory of signature σ. On the set of sentences $SL(\sigma)$, an equivalence relation $\sim_T$ is defined by the rule

$$\Phi \sim_T \Psi \;\Leftrightarrow\; T \vdash \Phi \leftrightarrow \Psi,$$

denoted by $\sim$ for brevity. The logical connectives $\vee$, $\wedge$, and $\neg$ generate Boolean operations $\cup$, $\cap$, and, respectively, $-$ on the quotient set ${}^{SL(\sigma)}/_{\sim}$ by the rule

$$\begin{aligned}
&[\Phi]_\sim \cup [\Psi]_\sim =_{dfn} [\Phi \vee \Psi]_\sim, \quad [\Phi]_\sim \cap [\Psi]_\sim =_{dfn} [\Phi \wedge \Psi]_\sim,\\
&-[\Phi]_\sim =_{dfn} [\neg\Phi]_\sim,\\
&\mathbf{0} =_{dfn} [(\forall x)(x \neq x)]_\sim, \quad \mathbf{1} =_{dfn} [(\forall x)(x = x)]_\sim.
\end{aligned}$$

It can easily be checked that, these operations are well-defined on the $\sim$-classes; i.e., these operations are defined independently of the choice of representatives.

As a result, we obtain an algebra of the form

$$\mathcal{L}(T) = \left({}^{SL(\sigma)}/_{\sim}\, ;\ \cup, \cap, -, \mathbf{0}, \mathbf{1}\right).$$

In fact, this is a Boolean algebra. It is called the *Tarski-Lindenbaum algebra* of the theory T, denoted by $\mathcal{L}(T)$.

The set of ultrafilters of $\mathcal{L}(T)$ is said to be *Stone space* of theory T, denoted by $\mathrm{St}(\mathcal{L}(T))$, or $\mathrm{St}(T)$ for short. By definition, its elements are ultrafilters of the Boolean algebra $\mathcal{L}(T)$ presenting complete extensions of the theory T. Thus, we have

$$\mathrm{St}(T) = \{T' \mid T' \textit{ is a complete extension of } T\}.$$

Study generating sets of the Tarski-Lindenbaum algebras of theories.

Let T be a theory of signature σ. A set $G \subseteq SL(\sigma)$ is said to be a *generating set* for the Tarski-Lindenbaum algebra $\mathcal{L}(T)$, if any sentence $\Phi \in SL(\sigma)$ is equivalent in T to a Boolean expression under sentences of the set G.

Give a criterion for a set of sentences to be generating in a theory.

Lemma 2.1. *Let T be a theory of signature σ. A set $G \subseteq SL(\sigma)$ is a generating set for the Tarski-Lindenbaum algebra $\mathcal{L}(T)$ if and only if for any subset $G' \subseteq G$, the theory determined by the following set of sentences*

$$T[G'] = T \cup G' \cup \{\neg\Psi \mid \Psi \in G \setminus G'\}$$

is either complete or inconsistent.

Proof. The necessity is obvious.

Now, we prove the sufficiency. Assume, that $T[G']$ is either complete or inconsistent for any set $G' \subset G$, and let Φ be a sentence of signature σ. Let us enumerate the set G and present it in the form $\{\Psi_i \mid i \in \mathbb{N}\}$. Consider the set $\mathfrak{D}$ of all finite tuples of the form

$$\langle \alpha_0, \alpha_1, \ldots, \alpha_{s-1} \rangle, \quad \alpha_i \in \{0, 1\},$$

such that neither Φ nor its negation is provable in the theory $T \cup \{\Psi_k^{\alpha_k} \mid k < s\}$; here, ordinary notations $\Psi^0 = \neg\Psi$, $\Psi^1 = \Psi$ are applied.

Relative to the natural lexicographic order, the set $\mathfrak{D}$ is a 2-branching tree. Therefore, it cannot be infinite, since otherwise a tuple of length ω could be constructed whose finite initial segments belong to $\mathfrak{D}$. However, this tuple would determine a set $G' \subseteq G$ such that neither Φ nor its negation is provable in the theory $T[G']$, that contradicts the assumptions. Thus, the set $\mathfrak{D}$ must be finite. Thereby, it is possible to choose a natural k exceeding length of any tuple in $\mathfrak{D}$.

Denote by Ω a disjunction of all those elementary conjunctions of the form

$$C(\alpha) = \Psi_0^{\alpha_0} \wedge \Psi_1^{\alpha_1} \wedge \ldots \wedge \Psi_{k-1}^{\alpha_{k-1}}, \quad \alpha = \langle \alpha_0, \ldots, \alpha_{k-1} \rangle, \quad \alpha_i \in \{0, 1\},$$

for which sentence Φ is provable in the theory $T \cup \{C(\alpha)\}$. It is obvious, that the sentence Ω has the form of a Boolean expression constructed from the sentences Ψ_i, $i \in \mathbb{N}$. It is possible to check that both sentences Φ and Ω are either simultaneously true or false in each complete extension of T. Thereby, the sentence Φ is equivalent in T to the pointed out sentence Ω.

Thereby, an arbitrary sentence Φ is equivalent in T to a Boolean expression of sentences from G. By definition, this means that the G is indeed a generating set for the Tarski-Lindenbaum algebra $\mathcal{L}(T)$.

Lemma 2.1 is proven. □

We are going to describe a technical method of constructing isomorphisms between the Tarski-Lindenbaum algebras of computably axiomatizable theories.

Lemma 2.2. *Let T_1 and T_2 be computably axiomatizable theories of signatures σ_1 and, respectively, σ_2, while $\gamma_i : \mathbb{N} \xrightarrow{onto} SL(\sigma_i)$, $i = 1, 2$, be Gödel numberings of their sets of sentences. Let also*

$$\{\Phi_0, \Phi_1, \ldots, \Phi_k, \ldots \;;\; k \in \mathbb{N}\} \subseteq SL(\sigma_1), \tag{2.1}$$

$$\{\Psi_0, \Psi_1, \ldots, \Psi_k, \ldots \;;\; k \in \mathbb{N}\} \subseteq SL(\sigma_2), \tag{2.2}$$

be computably enumerable sequences of sentences in these theories that represent generating sets in the Tarski-Lindenbaum algebras $\mathcal{L}(T_1)$ and $\mathcal{L}(T_2)$.

The following assertions are equivalent:

(a) $(\forall \mathfrak{P}_i \in FRM(\sigma^\circ))\big[T_1 \vdash \mathfrak{P}_i(\Phi_0, \Phi_1, \ldots, \Phi_{\mathfrak{a}(i)}) \;\Leftrightarrow\; T_2 \vdash \mathfrak{P}_i(\Psi_0, \Psi_1, \ldots, \Psi_{\mathfrak{a}(i)})\big]$,

(b) *partial mapping μ' defined by the rule $\mu'(\Phi_i) = \Psi_i$, $i \in \mathbb{N}$, can be expanded up to a computable isomorphism μ between the numerated Tarski-Lindenbaum algebras $(\mathcal{L}(T_1), \gamma_1)$ and $(\mathcal{L}(T_2), \gamma_2)$ of these theories.*

Proof. Implication (b)⇒(a) is obvious. For (a)⇒(b), we apply a known algebraic method. The map μ' pointed out in (b) establishes a one-to-one correspondence between the generating sequences (2.1) and (2.2). The partial mapping μ' can be expanded up to an isomorphism between the algebras whenever the set of dependencies relative to generating elements in the first algebra $\mathcal{L}(T_1)$ exactly coincides with that relative to corresponding generating elements in the second algebra $\mathcal{L}(T_2)$. Obviously, the pointed out algebraic condition is exactly equivalent to the demand given in Part (a). Thereby, an isomorphism $\mu : \mathcal{L}(T_1) \to \mathcal{L}(T_2)$ expanding μ' indeed exists. Having an initial correspondence $\Phi_i \mapsto \Psi_i$, $i \in \mathbb{N}$, we can conclude that sentence $\mathfrak{P}_k(\Phi_0, \ldots, \Phi_{\mathfrak{a}(k)})$ of theory T_1 must correspond to sentence $\mathfrak{P}_k(\Phi_0 \ldots, \Phi_{\mathfrak{a}(k)})$ of theory T_2 for all $k \in \mathbb{N}$; additionally, the closure operations under the equivalence relations $\sim_1$ in T_1 and $\sim_2$ in T_2 may be applied. Since the sequences (2.1) and (2.2) are generating in corresponding Tarski-Lindenbaum algebras, we obtain finally the following presentation for μ, for all sentences $\Phi \in SL(\sigma_1)$ and $\Psi \in SL(\sigma_2)$:

$$\mu([\Phi]_{\sim_1}) = [\Psi]_{\sim_2} \Leftrightarrow (\exists k)\big[\Phi \sim_1 \mathfrak{P}_k(\Phi_0, \ldots, \Phi_{\mathfrak{a}(k)}) \;\wedge\; \mathfrak{P}_k(\Psi_0, \ldots, \Psi_{\mathfrak{a}(k)}) \sim_2 \Psi\big]. \tag{2.3}$$

From this, based on the fact that both theories T_1 and T_2 are computably axiomatizable and both sequences (2.1) and (2.2) are effective, it is possible to show that the isomorphism μ is, in fact, computable.

Lemma 2.2 is proven. □

3 Compact binary trees

The concept of a compact binary tree corresponds to that introduced in [1], Ch.2.

A *full compact binary tree* is a partially ordered set $\mathcal{O} = \langle \mathbb{N}, \preccurlyeq \rangle$ of the form shown in Fig. 3.1. Particularly, we have the following relations between the elements: $1 \preccurlyeq 8$, $6 \preccurlyeq 6$, $\neg(6 \preccurlyeq 9)$, $\neg(9 \preccurlyeq 6)$, etc. Notice that, the basic relation $\preccurlyeq$ is computable. The other natural relations and functions for trees described later are also computable.

There are two following natural operations on elements of a tree:

$$L(n) = \textit{the left successor of element } n,$$
$$R(n) = \textit{the right successor of element } n.$$

A *tree* is a set $\mathcal{D} \subseteq \mathbb{N}$ for which the following conditions are satisfied:

$$n \preccurlyeq m \wedge m \in \mathcal{D} \;\Rightarrow\; n \in \mathcal{D}, \textit{ for all } n, m \in \mathbb{N},$$
$$L(n) \in \mathcal{D} \;\Leftrightarrow\; R(n) \in \mathcal{D}, \textit{ for all } n \in \mathbb{N}.$$

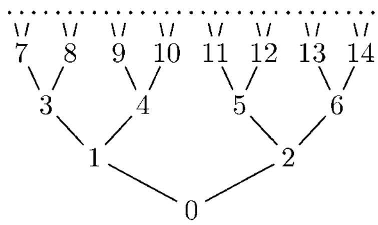

Fig. 3.1 Full binary tree

Three examples of trees are presented in Fig. 3.2, where (a) and (b) are finite trees while the tree (c) is infinite.

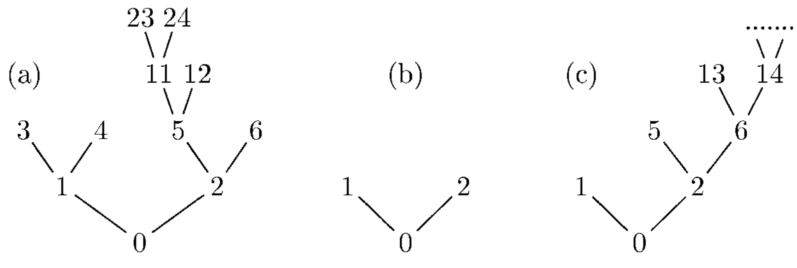

Fig. 3.2 Examples of trees

An element $n \in \mathcal{D}$ such that $L(n) \notin \mathcal{D}$ is called a *dead end* of the tree $\mathcal{D}$. The set of all dead-end elements of a tree $\mathcal{D}$ is denoted by $\mathrm{Dend}(\mathcal{D})$. A tree is called *atomic* if, above each of its elements, there is at least one dead-end element. All trees presented in Fig. 3.2 are atomic, while the full tree is not.

A *chain* is a set $\pi \subseteq \mathbb{N}$ for which two following conditions are satisfied:

$$n \preccurlyeq m \wedge m \in \pi \;\Rightarrow\; n \in \pi, \textit{ for all } n, m \in \mathbb{N},$$
$$n, m \in \pi \;\Rightarrow\; n \preccurlyeq m \vee m \preccurlyeq n, \textit{ for all } n, m \in \mathbb{N}.$$

A chain may be either finite or infinite. A finite chain whose maximal element is a is denoted by $\pi[a]$. It is uniquely determined by the maximal element a.

Consider two simple properties of trees and chains.

LEMMA 3.1. *Let $\mathcal{D}$ be a tree. The following assertions are equivalent*:
(a) $\mathcal{D}$ *is computable*,
(b) $\mathrm{Dend}(\mathcal{D})$ *is computable.*

Proof. If $\mathcal{D}$ is computable, $\mathrm{Dend}(\mathcal{D})$ is also computable by definition of this set. Conversely, suppose, that $\mathrm{Dend}(\mathcal{D})$ is computable. The following relation

$$x \in \mathcal{D} \;\Leftrightarrow\; x \in \mathrm{Dend}(\mathcal{D}) \vee \pi[x] \cap \mathrm{Dend}(\mathcal{D}) = \varnothing$$

shows that $\mathcal{D}$ is a computable set. □

A chain π is called a *maximal chain in a tree* $\mathcal{D}$, if $\pi \subseteq \mathcal{D}$, and no chain π' exists such that $\pi \subseteq \pi' \subseteq \mathcal{D}$ and $\pi \neq \pi'$. The set of all maximal chains of a tree $\mathcal{D}$ is denoted by $\Pi(\mathcal{D})$, while the set of all finite maximal chains of this tree is denoted by $\Pi^{fin}(\mathcal{D})$.

It is obvious that for any tree $\mathcal{D}$, the following claims are satisfied:

$$\begin{aligned} &\text{(a)} \quad \Pi(\mathcal{D}) \textit{ is infinite } \Leftrightarrow \mathcal{D} \textit{ is infinite}, \\ &\text{(b)} \quad |\Pi(\mathcal{D})| \leqslant 2^{\omega}, \\ &\text{(c)} \quad |\Pi^{fin}(\mathcal{D})| \leqslant \omega. \end{aligned} \tag{3.1}$$

Let $A \subseteq \mathbb{N}$. Denote by $\mathrm{ClTree}[A]$ the *closure* of A *up to a tree*, i.e., $\mathrm{ClTree}[A]$ means a minimal tree $\mathcal{D}'$ satisfying $A \subseteq \mathcal{D}'$. In the case of a singleton element, we use a brief entry $\mathcal{D}[a]$ instead of complete $\mathrm{ClTree}[\{a\}]$.

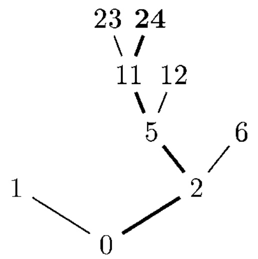

Fig. 3.3 Closure operation of a singleton set upto a tree

The following immediate formula takes place

$$\mathcal{D}[a] \;=\; \pi[a] \,\cup\, \mathrm{Nb}\big(\pi[a] \smallsetminus \{0\}\big), \tag{3.2}$$

where a function Nb(x) pointing out a 'neighboring' element is defined by the rule:

$$y = \mathrm{Nb}(x) \Leftrightarrow (y \neq x) \wedge (\exists z)\big(\,\{x, y\} = \{R(z), L(z)\}\,\big).$$

Notice that, Nb(0) is not defined. Fig. 3.3 presents a demonstration of the formula (3.2) for $a = 24$.

Characterize the closure operation of an arbitrary set $A \subseteq \mathbb{N}$ up to a tree.

Lemma 3.2. *The following assertions are held:*

(c) *for any set* $A \subseteq \mathbb{N}$*, a closure of* A *up to a tree is defined,*

(d) $\mathrm{ClTree}[A] = \bigcup_{a \in A} \mathcal{D}[a]$, *for all* $A \subseteq \mathbb{N}$,

(e) $\mathrm{ClTree}\big[\mathrm{ClTree}[A]\big] = \mathrm{ClTree}[A]$, *for all* $A \subseteq \mathbb{N}$,

(f) *a set* $A \subseteq \mathbb{N}$ *is a tree if and only if* $A = \mathrm{ClTree}[A]$.

Proof. Immediately, by using the definition of a tree and the description of the closure $\mathcal{D}[a]$ up to a tree of a singleton $\{a\}$. □

We introduce some standard numbering for computably enumerable trees.

Let W_n, $n \in \mathbb{N}$, be the standard Post numbering of the family of all computably enumerable sets, while W_n^A, $n \in \mathbb{N}$, be the standard numbering of the sets which are computably enumerable with an oracle $A \subseteq \mathbb{N}$. Introduce the following notations:

$$\mathcal{D}_n =_{dfn} \mathrm{ClTree}[W_n], \quad n \in \mathbb{N}, \tag{3.3}$$

$$\mathcal{D}_n^A =_{dfn} \mathrm{ClTree}[W_n^A], \quad n \in \mathbb{N}, \; A \subseteq \mathbb{N}. \tag{3.4}$$

By Lemma 3.2 (e,f), these sets are trees, since they are constructed by the closure operation up to a tree. They play the role of a universal computable numbering of the family of all computably enumerable trees and, respectively, the family of all trees computably enumerable with an oracle A. It follows from Lemma 3.2 (f) that all possible computably enumerable trees are presented in the sequence (3.3), and all possible computably enumerable with an oracle A trees are presented in the sequence (3.4).

By standard methods of algorithm theory, one can prove that there are total computable functions $f(x)$ and $g(x)$ reducing numberings (3.3) and (3.4) to standard numberings of computably enumerable sets and, respectively, sets computably enumerable with an oracle, by the following rules:

$$\mathcal{D}_n = W_{f(n)}, \ \textit{for all } n \in \mathbb{N}, \tag{3.5}$$

$$\mathcal{D}^A_n = W^A_{g(n)}, \ \textit{for all } n \in \mathbb{N}, \ A \subseteq \mathbb{N}. \tag{3.6}$$

In the case with an oracle, the reducing function $g(x)$ does not depend on the contents of an oracle A.

Now, we study the class of superatomic trees.

Consider the set $\Pi(\mathcal{D})$ of all maximal chains in an arbitrary tree $\mathcal{D}$. Let G be a subset of $\Pi(\mathcal{D})$. A chain $\pi \in G$ is called *isolated in* G, if there is an element $t \in \pi$ such that π is the only chain from G passing through t. By G', we denote the set of all chains $\pi \in G$ which are not isolated in G. By induction on ordinals, we define subsets $\Pi_\alpha(\mathcal{D}) \subseteq \Pi(\mathcal{D})$ as follows:

$$\begin{aligned} &\Pi_0(\mathcal{D}) = \Pi(\mathcal{D}),\\ &\Pi_{\alpha+1}(\mathcal{D}) = (\Pi_\alpha(\mathcal{D}))',\\ &\Pi_\gamma(\mathcal{D}) = \bigcap\{\Pi_\beta(\mathcal{D}) \mid \beta < \gamma\}, \ \textit{if } \gamma \textit{ is a limit ordinal.} \end{aligned}$$

The least α such that $\Pi_{\alpha+1}(\mathcal{D}) = \Pi_\alpha(\mathcal{D})$, is said to be *rank of the tree* $\mathcal{D}$, denoted by $\mathrm{Rank}(\mathcal{D})$. *Rank of a chain* $\pi \in \Pi(\mathcal{D})$, denoted by $\mathrm{rank}(\pi)$, is an ordinal α such that $\pi \in \Pi_\alpha(\mathcal{D}) \smallsetminus \Pi_{\alpha+1}(\mathcal{D})$. Clearly, the rank function in general is partially defined on the set $\Pi(\mathcal{D})$. A compact tree $\mathcal{D}$ is called a *tree with total rank function* or a *superatomic tree*, if the rank function is totally defined on the set $\Pi(\mathcal{D})$. In other words, a tree $\mathcal{D}$ is superatomic whenever $\Pi_\alpha(\mathcal{D}) = \varnothing$ for some α.

The following relations are immediate consequences of the definitions:

$$\text{(a)}\quad (\forall\alpha < \mathrm{Rank}(\mathcal{D}))\,(\exists\pi \in \Pi(\mathcal{D}))\,\big[\,\mathrm{rank}(\pi) = \alpha\,\big], \tag{3.7}$$
$$\text{(b)}\quad \mathrm{Rank}(\mathcal{D}) = \sup\{\mathrm{rank}(\pi) | \pi \in \Pi^{\mathrm{rank}}(\mathcal{D})\},$$

where $\Pi^{\mathrm{rank}}(\mathcal{D})$ is the set of all chains in $\Pi(\mathcal{D})$ for which the rank function is defined.

Specify some properties of the rank function on chains.

LEMMA 3.3. *For any compact tree $\mathcal{D}$, the set of chains $\pi \in \Pi(\mathcal{D})$ having rank is at most countable.*

Proof. Let $\pi \in \Pi_\alpha(\mathcal{D}) \setminus \Pi_{\alpha+1}(\mathcal{D})$. By $I(\pi)$, we denote the set of elements isolating chain π in $\Pi_\alpha(\mathcal{D})$. One can easily check that, if chains π_1 and π_2 have ranks, then $I(\pi_1) \cap I(\pi_2) = \varnothing$ is satisfied whenever $\pi_1 \neq \pi_2$. Since the set $\mathbb{N}$ is countable, any set consisting of its nonempty pairwise disjoint subsets must be at most countable. □

LEMMA 3.4. *Rank of any compact tree is a countable ordinal.*

Proof. The statement is a consequence of Lemma 3.3 together with the properties (3.7)(a) and (3.7)(b). □

LEMMA 3.5. *Let π_i, $i \in \mathbb{N}$ be an infinite sequence of maximal chains in a compact tree $\mathcal{D}$. Then, one can find an infinite chain π^* in $\mathcal{D}$ such that there are infinitely many chains in the given sequence passing through each of its elements $t \in \pi^*$.*

Proof. First, note an obvious fact that all chains π_i, $i \in \mathbb{N}$, pass through the root 0 of the tree. Therefore, infinitely many of them should pass either through $L(0)$ or through $R(0)$. Suppose, that it is the case for $L(0)$. Among the chains passing through $L(0)$, an infinite number of them should pass either through $L(L(0))$ or through $R(L(0))$. Continuing this process for ω steps, we will construct a chain π^* satisfying the required property. □

LEMMA 3.6. *Let $\mathcal{D}$ be a superatomic tree. Then,* $\mathrm{Rank}(\mathcal{D})$ *is a non-limiting ordinal.*

Proof. Assume, that $\mathcal{D}$ is a superatomic tree. By virtue of properties (3.7), it is enough to show that, if γ is a limiting countable ordinal, such that, for all

$\beta < \gamma$, there are chains of rank β in $\Pi(\mathcal{D})$, the family must contain a chain of rank not less than γ. Indeed, choose a sequence of chains π_k, $k \in \mathbb{N}$, in $\Pi(\mathcal{D})$ such that $\sup\{\mathrm{rank}(\pi_k) | k \in \mathbb{N}\} = \gamma$. By virtue of Lemma 3.5, we can construct a chain $\pi^* \in \pi(\mathcal{D})$ such that there are infinitely many chains of the sequence π_k passing through each of its elements. Since the tree $\mathcal{D}$ is superatomic, rank of the chain π^* must be defined. Moreover, we have $\mathrm{rank}(\pi^*) \geqslant \gamma$ by the definition. From this, we obtain $\mathrm{Rank}(\mathcal{D}) \geqslant \gamma + 1$, that is exactly what is required. □

Let us characterize superatomic trees.

THEOREM 3.7. *Let $\mathcal{D}$ be a tree. The following conditions are equivalent to each other:*

(a) *tree $\mathcal{D}$ is superatomic,*

(b) *the set $\Pi(\mathcal{D})$ is no more than countable,*

(c) *the power of the set $\Pi(\mathcal{D})$ is less than 2^ω,*

(d) *each countable set $G \subseteq \Pi(\mathcal{D})$ has an isolated chain,*

(e) *each set $G \subseteq \Pi(\mathcal{D})$ has an isolated chain.*

Proof is done by the following scheme: (a)⇒(b)⇒(c)⇒(d)⇒(e) ⇒(a).

The implication (a)⇒(b) is a consequence of Lemma 3.3.

The implication (b)⇒(c) is obvious.

For the proof of implication (c)⇒(d), assume the opposite: let there is a countable set $G \subseteq \Pi(\mathcal{D})$ not having isolated chains. Then, the set of the form $V = \bigcup\{\pi | \pi \in G\}$ is a subset of $\mathcal{D}$ and satisfies the following conditions

$$x \preccurlyeq y \wedge y \in V \Rightarrow x \in V,$$
$$(\forall x \in V)(\exists yz \in V)[x \preccurlyeq y \wedge x \preccurlyeq z \wedge \neg(y \preccurlyeq z) \wedge \neg(z \preccurlyeq y)].$$

Therefore, among subsets of V, there are 2^ω various infinite chains, and all of them belong to $\Pi(\mathcal{D})$, that is impossible since this contradicts Part (c).

The implication (d)⇒(e) is true, since from any nonempty subset $G \subseteq \Pi(\mathcal{D})$ not having isolated chains, it is possible to choose a countable subset $G_0 \subseteq G$ in which there are no isolated chains.

The implication (e)⇒(a) is a consequence of the definitions. □

LEMMA 3.8. *If $\mathcal{D}$ is a superatomic tree, then $\mathcal{D}$ is an atomic tree.*

Proof. Suppose that $\mathcal{D}$ is not atomic. Then, there is an element k without dead-end elements over it; thus, there are 2^ω chains in $\Pi(\mathcal{D})$ passing through k. By Theorem 3.7, this contradicts to the assumption that $\mathcal{D}$ is superatomic. □

Consider some natural operations on the class of trees.

Define a total computable function $\mathrm{EmbTree}(n,x)$ by the following scheme, for all $n,x\in\mathbb{N}$:

$$\begin{cases}\mathrm{EmbTree}(n,0)=n,\\ \mathrm{EmbTree}(n,L(x))=L(\mathrm{EmbTree}(n,x)),\\ \mathrm{EmbTree}(n,R(x))=R(\mathrm{EmbTree}(n,x)).\end{cases}$$

It is possible to check that, for a fixed parameter $n\in\mathbb{N}$, the unary function $\lambda x\,\mathrm{EmbTree}(n,x)$ is an isomorphic mapping from the full tree $\mathcal{D}=\mathbb{N}$ into the region $\{x|n\preccurlyeq x\}$ of the full tree.

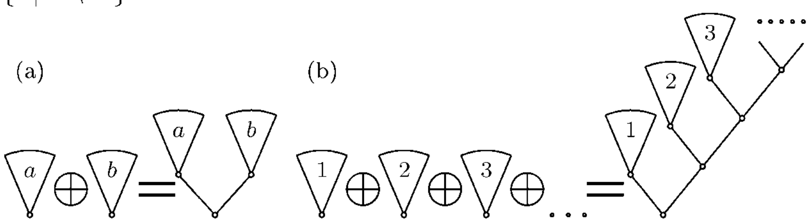

Fig. 3.4. Scheme of the sum operations on compact binary trees

The *direct sum* of *two trees* $\mathcal{D}_1$ and $\mathcal{D}_2$ is a set defined by the following rule

$$\mathcal{D}_1\oplus\mathcal{D}_2=\{0\}\cup\mathrm{EmbTree}(1,\mathcal{D}_1)\cup\mathrm{EmbTree}(2,\mathcal{D}_2). \tag{3.8}$$

It is possible to check that $\mathcal{D}_1\oplus\mathcal{D}_2$ is indeed a compact tree. The tree $\mathcal{D}_1\oplus\mathcal{D}_2$ can be obtained by attaching isomorphic copies of the source trees $\mathcal{D}_1$ and $\mathcal{D}_2$ to the two dead-end elements of the three-element tree presented in Fig. 3.2 (b).

The *direct sum* of *a sequence of trees* $\mathcal{D}_k$, $k\in\mathbb{N}$, is a set of the following form:

$$\begin{aligned}\textstyle\bigoplus_{k\in\mathbb{N}}\mathcal{D}_k\ &=\{0\}\cup\{2,6,\ldots,t_k+1,\ldots\}\cup\mathrm{EmbTree}(1,\mathcal{D}_0)\cup\\ &\cup\mathrm{EmbTree}(5,\mathcal{D}_1)\cup\ldots\cup\mathrm{EmbTree}(t_k,\mathcal{D}_k)\cup\ldots,\end{aligned} \tag{3.9}$$

where $t_k = 2^{k+2} - 3$. This tree is obtained by attaching isomorphic copies of the trees $\mathcal{D}_k$ to subsequent dead-end elements of the infinite tree shown in Fig. 3.2 (c).

Schematic rules for both types of the operation are shown in Fig. 3.4(a,b).

Lemma 3.9. *The following assertions take place*:

(a) $\mathcal{D}_1 \oplus \mathcal{D}_2$ *is a computably enumerable tree if and only if both* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are computably enumerable trees.*

(b) $\bigoplus_{k\in\mathbb{N}} \mathcal{D}_k$ *is a computably enumerable tree if and only if* $\mathcal{D}_k$, $k \in \mathbb{N}$, *is a computable sequence of c.e. trees.*

Proof. Immediately. □

Now, we consider some structural properties of the operations on the trees.

Lemma 3.10. *The following assertions hold*:

(a) $\mathcal{D}_1 \oplus \mathcal{D}_2$ *is an atomic tree if and only if both* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are atomic trees.*

(b) $\bigoplus_{k\in\mathbb{N}} \mathcal{D}_k$ *is an atomic tree if and only if all trees in the sequence* $\mathcal{D}_k, k \in \mathbb{N}$, *are atomic.*

Proof. By immediate check of the definition. □

Lemma 3.11. *The following assertions hold*:

(a) *tree* $\mathcal{D}_1 \oplus \mathcal{D}_2$ *is superatomic if and only if both trees* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are superatomic.*

(b) *tree* $\bigoplus_{k\in\mathbb{N}} \mathcal{D}_k$ *is superatomic if and only if all trees in the sequence* $\mathcal{D}_k$, $k \in \mathbb{N}$, *are superatomic.*

Proof. By immediate check with the help of characterization of superatomic trees given in Theorem 3.7. □

4 Natural binary trees and indices for them

A set $D \subseteq 2^{<\omega}$ is said to be a *natural binary tree* if the following conditions are satisfied:

$$\text{(a)}\quad \varnothing \in D, \tag{4.1}$$
$$\text{(b)}\quad \varepsilon \in D \;\Leftrightarrow\; \big(\varepsilon 0 \in D\big) \vee \big(\varepsilon 1 \in D\big),\ \textit{for all}\ \varepsilon \in 2^{<\omega}.$$

It is possible to check that an arbitrary natural tree D is closed downwards, and is semi-closed upwards; i.e., the following relations are satisfied:

$$\text{(a)}\quad \varepsilon \preccurlyeq \varepsilon' \wedge \varepsilon' \in D \Rightarrow \varepsilon \in D, \tag{4.2}$$
$$\text{(b)}\quad \varepsilon \in D,\ \mathrm{length}(\varepsilon) < k < \omega \ \Rightarrow\ (\exists \varepsilon' \in D)\big(\varepsilon \preccurlyeq \varepsilon' \wedge \mathrm{length}(\varepsilon') = k\big).$$

Now, we consider natural trees jointly with the compact trees.

Hereafter, we will use a specified symbol D for natural binary trees and another symbol $\mathcal{D}$ for compact binary trees. A natural tree D may be said to be a *tuple tree* because it is formed by a tuple logical structure. As for a *compact tree*, it presents the structure of branchings in a more compact form; this explains the choice of the term. Mention that both types of trees we are considering are supposed to be *binary*. Given a structure $\mathfrak{S}$ of some kind that can produce a tree. We denote by $\mathrm{tree}(\mathfrak{S})$ a natural tree, while by $\mathrm{Tree}(\mathfrak{S})$, a compact tree determined by the structure $\mathfrak{S}$. For instance, $\mathrm{tree}(\mathcal{B}, \nu)$ means a natural tree determined by a numerated Boolean algebra $(\mathcal{B}, \nu)$, while $\mathrm{Tree}(\mathcal{B}, \nu)$ means a compact tree determined by the algebra $(\mathcal{B}, \nu)$.

Let $D \subseteq 2^{<\omega}$ be a natural tree. A natural number n is said to be a *characteristic* index of D, if nth Kleene's computable function $\varphi_n(x)$ is a characteristic function for the set $\mathrm{Nom}(D)$ of Gödel numbers of the tree; i.e., the function $\varphi_n(x)$ is total and satisfies the following relation for all $\varepsilon \in 2^{<\omega}$:

$$\varphi_n(\mathrm{Nom}(\varepsilon)) = \begin{cases} 1, & \textit{if } \varepsilon \in D, \\ 0, & \textit{if } \varepsilon \notin D. \end{cases}$$

We now turn to a relativized version. A number n is said to be a *relativized characteristic* index of a natural tree D in computation *with an oracle* $A \subseteq \mathbb{N}$, if nth computable function $\varphi_n^A(x)$ with an oracle is total; moreover, this function is characteristic for the set $\mathrm{Nom}(D)$.

Let us introduce the following notations for $n \in \mathbb{N}$ and $A \subseteq \mathbb{N}$:

D_n *is the natural tree with a characteristic index* n, (4.3)

D_n^A *is the natural tree with a characteristic index* n *under an oracle* A. (4.4)

Notice that, D_n may be undefined for some n; the same concerns D_s^A. As for indices for compact trees, they are defined for all n and A by rules (3.3) and (3.4).

5 Chains and the rank function in a natural tree

We are going to rewrite for natural binary trees all main concepts available for compact binary trees.

Definition of a natural binary tree is given in (4.1). Let $D \subseteq 2^{<\omega}$ be a natural tree. An element $\varepsilon \in D$ is called *atomic* in the tree D if it is indivisible upwards within D; i.e., the following is satisfied:

$$(\forall \delta, \zeta \in 2^{<\omega})[\varepsilon \preccurlyeq \delta \ \wedge \ \varepsilon \preccurlyeq \zeta \ \wedge \ \delta, \zeta \in D \Rightarrow \delta \preccurlyeq \zeta \vee \zeta \preccurlyeq \delta].$$

By $\mathrm{At}(D)$, we denote the set of all atomic elements in a natural tree D. A natural tree is said to be *atomic* if above each of its elements, there is at least one atomic element.

A set of finite sequences $\varepsilon \subseteq 2^{<\omega}$ is said to be a (*natural*) *chain*, if there is an infinite sequence $\vec{\varepsilon} \in 2^{\omega}$ such that ε consists of all finite initial segments of $\vec{\varepsilon}$; i.e., the following equality is satisfied

$$\varepsilon = \{\delta \in 2^{<\omega} \mid \delta \preccurlyeq \vec{\varepsilon}\}. \tag{5.1}$$

Obviously, the infinite sequence $\vec{\varepsilon}$ in (5.1) is uniquely determined by the chain ε, and conversely, the chain ε is uniquely determined by the sequence $\vec{\varepsilon}$. In the case when (5.1) is satisfied, $\vec{\varepsilon}$ is said to be the *limit* sequence of the chain ε, denoted by $\lim \varepsilon$; as for the inverse passage $\vec{\varepsilon} \mapsto \varepsilon$, we use notation $\varepsilon = \mathrm{chain}(\vec{\varepsilon})$ in this case. A chain ε in a natural tree D is called *isolated* if there is $\delta \in \varepsilon$ such that δ is an atomic element in the tree D. By definition, any chain in a natural tree D is its maximal chain.

The set of all chains of a natural tree D is denoted by $\Pi(D)$, while the set of all isolated maximal chains of this tree is denoted by $\Pi^{iso}(D)$. Let G be a subset of $\Pi(D)$. A chain $\tau \in G$ is called *isolated in* G if there is an element $\delta \in \tau$ such that τ is the only chain from G passing through δ. By G', we denote the set of all chains $\tau \in G$ which are not isolated in G. By induction on ordinals, define subsets $\Pi_\alpha(D) \subseteq \Pi(D)$ as follows:

$$\begin{aligned}
&\Pi_0(D) = \Pi(D),\\
&\Pi_{\alpha+1}(D) = (\Pi_\alpha(D)),\\
&\Pi_\gamma(D) = \bigcap\{\Pi_\beta(D) \mid \beta < \gamma\}, \ \textit{if } \gamma \textit{ is a limit ordinal.}
\end{aligned}$$

The least α such that $\Pi_{\alpha+1}(D) = \Pi_\alpha(D)$, denoted by $\mathrm{Rank}(D)$, is called *rank of the natural tree D*. *Rank of a chain* $\tau \in \Pi(D)$ is an ordinal α, denoted $\mathrm{rank}(\tau)$, such that $\tau \in \Pi_\alpha(D) \setminus \Pi_{\alpha+1}(D)$. Clearly, the rank function in general is partially defined on the set $\Pi(D)$. A natural tree D is called a *superatomic tree*, if the rank function is totally defined on $\Pi(D)$. In other words, a natural tree D is superatomic if $\Pi_\alpha(D) = \varnothing$ for some α.

6 Transformations between Boolean algebras and trees

We describe main types of transformations needed for our main theorem.

(*a*) *Transformation $ct2L_k$ from a complete theory to the Tarski-Lindenbaum algebra with k free variables.*

We advance concept of the Tarski-Lindenbaum algebra given in Section 2.

Let T be a complete theory of signature σ and k be an integer; T and k represent a pair of input parameters of the transformation. Fix a Gödel numbering $\gamma : \mathbb{N} \xrightarrow{onto} FL_k(\sigma)$. On the set of formulas $FL_k(\sigma)$, we define an equivalence relation $\sim_T$ by the rule

$$\varphi(\overline{x}) \sim_T \psi(\overline{x}) \iff T \vdash (\forall \overline{x})[\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})],$$

denoted by $\sim$ for short. The logical connectives $\vee$, $\wedge$, and $\neg$ generate Boolean operations $\cup$, $\cap$, and $-$ on the quotient set ${}^{FL_k(\sigma)}/_{\sim}$. As a result, we obtain an algebra of the form

$$\mathcal{L}_k(T) = \big({}^{FL_k(\sigma)}/_{\sim}\,;\ \cup, \cap, -, \mathbf{0}, \mathbf{1}\big), \tag{6.1}$$

that, in fact, is a Boolean algebra. It is called the *Tarski-Lindenbaum algebra* of theory T under formulas with k free variables $x_0, \ldots, x_{k-1}$, denoted by $\mathcal{L}_k(T)$. The set of ultrafilters of the Boolean algebra $\mathcal{L}_k(T)$ is called *Stone space of dimension k* of theory T, denoted by $\mathrm{St}(\mathcal{L}_k(T))$, or $\mathrm{St}_k(T)$ for short. By definition, its elements are ultrafilters of the Boolean algebra $\mathcal{L}_k(T)$; in model theory, these ultrafilters are said to be *complete types* of T with k free variables $x_0, \ldots, x_{k-1}$. Thus the whole transformation looks as follows:

$$ct2L_k\ :\ (T, k)\ \mapsto\ \mathcal{L}_k(T) = \big({}^{FL_k(\sigma)}/_{\sim}\,;\ \cup, \cap, -, \mathbf{0}, \mathbf{1}\big). \tag{6.2}$$

Formulate a main statement describing the transformation:

Lemma 6.1. *Transformation* $ct2L_k$ *is uniformly effective. More precisely: there is a partial computable function* $e(x,z)$*, such that, for any* $n \in \mathbb{N}$*, if* n *is a relative characteristic index of a complete theory* T *of signature* σ*,* $e(n,k)$ *is defined; moreover, the number* $m = e(n,k)$ *is a relative characteristic index of the algebra* (6.1) *considered as a numerated algebra with a Gödel numbering* γ *of the set* $FL_k(\sigma)$*.*

Namely, by decoding the number m*, we obtain four numbers*

$$e(n,k) \ = m = \langle m_1, m_2, m_3, m_4 \rangle \tag{6.3}$$

satisfying the following properties:

(a) *functions* $\varphi_{m_1}(x,y)$*,* $\varphi_{m_2}(x,y)$*, and* $\varphi_{m_3}(x)$ *are total; moreover, they present signature operations* $\cup$*,* $\cap$*, and, respectively,* $-$ *in numeration* γ *of the algebra* (6.1)*; i.e, the following relations are satisfied for all* $i,j \in \mathbb{N}$:

$$\begin{aligned} \gamma(i) \cup \gamma(j) \ &\sim \ \gamma(\varphi_{m_1}(i,j)), \\ \gamma(i) \cap \gamma(j) \ &\sim \ \gamma(\varphi_{m_2}(i,j)), \\ -\gamma(i) \ &\sim \ \gamma(\varphi_{m_3}(i)); \end{aligned} \tag{6.4}$$

(b) *binary function* $\varphi^A_{m_4}(x,y)$ *with an oracle* A *is total and has the domain* $\{0,1\}$*; moreover, this function presents the equality relation in numeration* γ *of the algebra* (6.1)*; i.e., the following relation is satisfied for all* $i,j \in \mathbb{N}$:

$$\gamma(i) = \gamma(j) \ \Leftrightarrow \ \varphi^A_{m_4}(i,j) = 1. \tag{6.5}$$

Proof. Immediately, from the procedure of construction of the Tarski-Lindenbaum algebra that is effective in a characteristic index of the input theory T in computation with an oracle. □

(*b*) *Transformation* $b2n$ *from a Boolean algebra to a natural binary tree.*

First, we consider this transformation in a more common form.

Let $(\mathcal{B}, \nu)$ be a numerated Boolean algebra, and $G = \langle g_i \mid i < \omega \rangle$ be a computably enumerable in numeration ν sequence of elements of $\mathcal{B}$ presenting a generating set for this algebra. We normally count that $g_i = \nu(i)$, for all $i \in \mathbb{N}$; although, another form of this sequence G is also admitted, when needed.

Consider the following transformation (in fact, it is effective in relative indices):

$$(\mathcal{B},\nu) \;\mapsto\; D = \mathrm{tree}(\mathcal{B},\nu) = \{\varepsilon \in 2^{<\omega} \mid \varepsilon = \langle \varepsilon_0, ..., \varepsilon_{k-1}\rangle,\; g_0^{\varepsilon_0} \cap ... \cap g_{k-1}^{\varepsilon_{k-1}} \neq \mathbf{0}\}. \tag{6.6}$$

It can be easily checked that the set D is a natural tree; moreover, this tree represents the algebra $\mathcal{B}$ itself. Actually, this is a well-known construction used in applications in many works.

Now, we turn to a particular form of this transformation.

Consider the Tarski-Lindenbaum algebra $\mathcal{L}_k(T)$, (6.1), of a complete theory T of signature σ under formulas with k free variables $x_0, \dots, x_{k-1}$; by construction, it is a Boolean algebra. Fix a computably enumerable sequence of formulas $G = \langle \varphi_i(\overline{x}) : i \in \mathbb{N}\rangle$, $\overline{x} = (x_0, \dots, x_{k-1})$ presenting a generating set for the algebra $\mathcal{L}_k(T)$. Normally, unless otherwise specified, we count that $\varphi_i(\overline{x})$, $i \in \mathbb{N}$, is a Gödel numbering of the set $FL_k(\sigma)$.

Consider the following transformation

$$\begin{aligned}&(\mathcal{L}_k(T), \gamma) \;\mapsto \\ &D = \mathrm{tree}(\mathcal{L}_k(T), \gamma) = \\ &\{\varepsilon \in 2^{<\omega} \mid \varepsilon = \langle \varepsilon_0, \dots, \varepsilon_{k-1}\rangle,\; T \vdash (\exists \overline{x})\big(\varphi_0^{\varepsilon_0}(\overline{x}) \wedge \dots \wedge \varphi_{k-1}^{\varepsilon_{k-1}}(\overline{x})\big)\}.\end{aligned} \tag{6.7}$$

We can easily check that D is a natural tree; in a known sense, it represents the Tarski-Lindenbaum algebra $\mathcal{L}_k(T)$. Obviously, this form of the operation is a particular case of the common form (6.6).

Study main properties of the transformation.

LEMMA 6.2. *Transformation $b2n$ is effective under the relative indices in computation with an oracle A. More precisely: there is a partial computable function $e(x)$ such that, for any natural n, if n is a relative characteristic index of a numerated Boolean algebra $(\mathcal{L}_k(T), \gamma)$, $e(n)$ is defined; moreover, $e(n)$ is a relative characteristic index of the natural tree $D = tree(\mathcal{L}_k(T), \gamma)$; i.e., $D = D^A_{e(n)}$, cf. (4.4).*

Proof. Immediately, from the description of the transformation of a numerated Boolean algebra to a natural tree that is effective in corresponding indices. □

(c) *Transformation $n2c$ from a natural tree to a compact tree.*

We are going to describe a natural transformation from the class of natural trees into the class of compact trees. Its idea is that only branchings are important, while any lengthy segment without branchings may be reduced. Part (a) in Fig. 6.1 represents a natural tree $D \subseteq 2^{<\omega}$ marking it with bold lines in the set $2^{<\omega}$, while Part (b) demonstrates the binary tree $\mathcal{D} = \mathrm{Tree}(D)$ obtained by reducing any lengthy segment of D without branchings.

Turn to a formal specification.

Let $D \subseteq 2^{<\omega}$ be a natural tree. We are in a position to describe an operation of tree transformation

$$n2c: \; D \mapsto \mathcal{D} = \mathrm{Tree}(D), \tag{6.8}$$

It is called the *compactification operation.* Define a mapping $\lambda : D \to \mathbb{N}$ inductively as follows (this function may be partial):

$$\lambda(\varnothing) = 0; \tag{6.9}$$

$$\lambda(\varepsilon 0) = \begin{cases} \lambda(\varepsilon), & \textit{if} \;\; \varepsilon 0 \in D \;\wedge\; \varepsilon 1 \notin D, \\ L(\lambda(\varepsilon)), & \textit{if} \;\; \varepsilon 0, \varepsilon 1 \in D\,; \end{cases}$$

$$\lambda(\varepsilon 1) = \begin{cases} \lambda(\varepsilon), & \textit{if} \;\; \varepsilon 0 \notin D \;\wedge\; \varepsilon 1 \in D, \\ R(\lambda(\varepsilon)), & \textit{if} \;\; \varepsilon 0, \varepsilon 1 \in D\,. \end{cases}$$

After that, we put $\mathcal{D} = \mathrm{Tree}(D) =_{dfn} \lambda(D)$. It is possible to check that the set $\mathrm{Tree}(D)$ is indeed a compact binary tree.

We now pass to effectiveness of the operation.

Lemma 6.3. *Transformation $n2c$ is effective under the relative indices in computation with an oracle A. More precisely: there are partial computable functions $e(x)$ and $h(x)$, such that, for any natural n, if n is a relative characteristic index of a natural tree D, both $e(n)$ and $h(n)$ are defined; moreover, the numbers $m = e(n)$ and $m_1 = h(n)$ satisfy the following properties*:

(a) *m is a relative c.e. index of the target tree $\mathcal{D}$; i.e.,* $\mathcal{D} = \mathrm{Tree}(D) = \mathcal{D}^A_m$,

(b) *m_1 is a relative index of the function $\lambda(x)$, cf.* (6.9), *i.e.,* $\lambda(x) = \varphi^A_{m_1}(x)$.

Proof. Immediately, from definition of the operation $D \mapsto \mathrm{Tree}(D)$ that is effective under corresponding indices in computation with an oracle. Mention that the values of $e(n)$ and $h(x)$ do not depend on A because these functions are built based on a relative version of s-m-n-Theorem. □

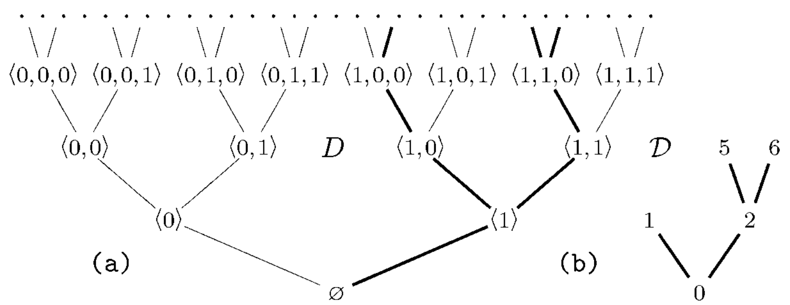

Fig. 6.1. Transformation scheme of a natural tree in compact tree

Describe a natural correspondence between the natural and compact trees.

Let D be an arbitrary natural tree and $\mathcal{D}$ be a compact tree, such that $\mathcal{D}$ is related to D by the compactification operation $n2c$. This transformation defines a correspondence (6.9) between the trees

$$\lambda : D \xrightarrow{onto} \mathcal{D}, \tag{6.10}$$

such that $\varnothing$ is mapped by λ in 0, each separate non-branching path of the natural tree D (linking two subsequent branchings) is mapped by λ to an element $a \in \mathcal{D} \smallsetminus \mathrm{Dend}(\mathcal{D})$ of the compact tree $\mathcal{D}$, and each non-branching path of the tree D (from a branching element to infinity) is mapped by λ in an element $a \in \mathcal{D} \cap \mathrm{Dend}(\mathcal{D})$.

Moreover, the following general properties are held:

Lemma 6.4. *The mapping λ pointed out in* (6.9) *bijectively maps the maximal chains of D in the maximal chains of $\mathcal{D}$; moreover, isolated chains of D are mapped in finite chains of $\mathcal{D}$. Particularly, the following relations are held:*

(a) *π is finite $\Leftrightarrow$ τ is isolated, whenever $\pi = \lambda(\tau)$ for $\tau \in \Pi(D)$,*

(b) *$\mathcal{D}$ is atomic if and only if D is atomic,*

(c) $\mathrm{rank}(\pi) = \mathrm{rank}(\tau)$, *whenever $\pi = \lambda(\tau)$ for $\tau \in \Pi(D)$,*

(d) $\mathrm{Rank}(\mathcal{D}) = \mathrm{Rank}(D)$,

(e) *$\mathcal{D}$ is superatomic if and only if D is superatomic.*

Proof. Immediately, from the definition of the transformation operations $n2c$ given in (6.8).

Lemma 6.4 is proven. □

To unify the terminology, in the case of natural trees, we often use an alternative term a *finite chain* instead of an *isolated chain.*

Transformations (6.6) and (6.7) provide a natural method of presentation of Boolean algebras. Main disadvantage of natural trees is that only characteristic indices are available for them, not c.e. indices. On the other hand, main advantage of compact trees is that both c.e. indices and c.e. indices over an oracle are available for compact trees; however, compact trees have a more complicated relation with Boolean algebras.

7 A statement presenting the canonical construction

In this section, we present a main statement of the canonical construction. Here, $\Omega(m)$ means *parametric Stone space* with an index m, it is defined in (1.6). Further, $\mathcal{D}_s^A$ means a *c.e. tree with an index s* in computation with an oracle A, see (3.4).

Theorem 7.1. [Canonical construction, basic version] *Effectively in a pair of natural numbers m, s and a Gödel number g for a finite rich signature σ, one can construct a finitely axiomatizable model-complete theory $F = \mathbb{F}\mathbb{C}(m,s,\sigma)$ of signature σ together with a computable sequence of sentences θ_n, $n \in \mathbb{N}$, of signature σ, such that the family of extensions of F defined for each $A \subseteq \mathbb{N}$ by*

$$F[A] = F \cup \{\theta_i | i \in A\} \cup \{\neg\theta_j | j \in \mathbb{N} \setminus A\}, \tag{7.1}$$

satisfies the following properties (presenting the canonical semantic layer $M1\mathbb{L}$ in Part 3, and the canonical basic semantic layer $M1L$ in Part 3(a-f)):

1. *For any $A \subseteq \mathbb{N}$, the theory $F[A]$ is either complete or contradictory.*
2. *The theory $F[A]$, $A \subseteq \mathbb{N}$, is consistent if and only if $A \in \Omega(m)$.*
3. *For an arbitrary $A \in \Omega(m)$, the following assertions hold:*

(a) *theory $F[A]$ has a prime model if and only if the tree $\mathcal{D}_s^A$ is atomic,*

(b) *a prime model of theory $F[A]$, if it exists, is strongly constructivizable if and only if the set A is computable and the family of chains $\Pi^{fin}(\mathcal{D}_s^A)$ is computable,*

(c) *a prime model of theory $F[A]$, if it exists and is strongly constructivizable, has the algorithmic dimension 1 if and only if the tree $\mathcal{D}_s^A$ is computable,*

(d) *theory $F[A]$ has a countable saturated model if and only if the thee $\mathcal{D}_s^A$ is superatomic,*

(e) *a countable saturated model of theory $F[A]$, if it exists, is strongly* *constructivizable if and only if the set A is computable and the family of chains $\Pi(\mathcal{D}_s^A)$ is computable,*

(f) *theory $F[A]$ is ω-stable if and only if the tree $\mathcal{D}_s^A$ is superatomic.*

An entry $\mathbb{F}\mathbb{C}(m,s)$ is used instead of $\mathbb{F}\mathbb{C}(m,s,\sigma)$ when σ is clear from context.

Proof: See detailed proof in [1] Ch. 3. □

In addition, we mention three following statements

$$\{F[A] \mid A \in \Omega(m)\} \text{ is the set of all complete extensions of } F, \tag{7.2}$$

$$F[A] \text{ is decidable } \Leftrightarrow A \text{ is computable, for all } A \in \Omega(m), \tag{7.3}$$

$$F[A] \vdash \mathfrak{P}_i(\theta_0, \ldots, \theta_{\mathfrak{a}(i)}) \Leftrightarrow A \models \mathfrak{P}_i, \text{ for all } A \in \Omega(m); \tag{7.4}$$

they are simple corollaries of Part 1 and Part 2 of Theorem 7.1.

Part 1 together with Part 2 ensures that $\{F[A] \mid A \in \Omega(m)\}$ is a set of complete extensions of theory F. By construction, $F[A] \neq F[A']$ for all $A, A' \in \Omega(m)$ such that $A \neq A'$. Let F' be an arbitrary complete extension of F. Obviously, $F[A_0] \subseteq F'$ is satisfied for some $A_0 \subseteq \mathbb{N}$. Since theory F' is consistent and complete, $F[A_0]$ must be consistent. By Part 1 we conclude that $A_0 \in \Omega(m)$. Thus $F[A_0]$ and F' are complete theories of the same signature s.t. $F[A_0] \subseteq F'$. This ensures their coincidence $F[A_0] = F'$. Thereby, property (7.2) indeed holds.

Prove (7.3). Suppose that $A \in \Omega(m)$ is computable. Then $F[A]$ is complete and computably axiomatizable. Therefore, this theory is decidable by Janiczack Theorem. In the opposite case, if the set $A \in \Omega(m)$ is not computable, the reduction $i \in A \Leftrightarrow T[A] \vdash \theta_i$ shows that theory $F[A]$ cannot be decidable.

Now, we turn to proof of (7.4). By construction, we have

$$T[A] \vdash \theta_i^{\alpha_i}, \text{ with } \alpha_i = \begin{cases} 1, & \text{if } i \in A, \\ 0, & \text{if } i \notin A, \end{cases} \tag{7.5}$$

where $\varphi^1 = \varphi$, and $\varphi^0 = \neg\varphi$. Lemma 1.12 in [11], Ch.4, Sec.3 states that, for all $\alpha_0, \ldots, \alpha_{\mathfrak{a}(i)} \in \{0,1\}$

$$X_0^{\alpha_0}, \ldots, X_{\mathfrak{a}(k)}^{\alpha_{\mathfrak{a}(k)}} \vdash \big(\mathfrak{P}_k(X_0, \ldots, X_{\mathfrak{a}(k)})\big)^{\alpha} \tag{7.6}$$

with $\alpha =$ the value of $\mathfrak{P}_k(\alpha_0, \ldots, \alpha_{\mathfrak{a}(k)})$. By substituting θ_i instead of X_i in (7.6) and combining with (7.5), we obtain for all $A \in \Omega(m)$ and $\alpha_0, \ldots, \alpha_{\mathfrak{a}(k)} \in$

$\{0,1\}$

$$F[A] \vdash \mathfrak{P}_k(\theta_0, \ldots, \theta_{\mathfrak{a}(k)})\,)^{\alpha}, \text{ with } \alpha = \text{ the value of "} A \models \mathfrak{P}_k\text{"}.$$

Thereby, the desired statement (7.4) is indeed satisfied.

Notice that, the first integer parameter m in Theorem 7.1 is an index for the *space* component controlling Stone space of the target theory, while the second parameter s is an index controlling model-theoretic properties of complete extensions of the theory (explanation of the notations: m represents the *main* index, while s represents a *secondary* index).

8 Main statement of the canonical-mini construction

We formulate the main statement of this paper.

Theorem 8.1. [Canonical-mini construction] *Let T be an arbitrary computably axiomatizable theory and σ be a finite rich signature. Effectively in a weak c.e. index t_0 of T, we can construct a finitely axiomatizable theory $F = \mathbb{F}\mathbb{C}^{\circ}(T,\sigma)$ of signature σ together with a computable isomorphism between their Tarski-Lindenbaum algebras $\mu : \mathcal{L}(T) \to \mathcal{L}(F)$ preserving the following layer of model-theoretic properties (called the canonical-mini semantic layer, denoted by $M1\mathbb{L}^{\circ}$):*

(a) *existence of a prime model, its strong constructivizability, and the value of its algorithmic dimension (relative to strong constructivizations);*

(b) *existence of a countable saturated model, and its strong constructivizability.*

Proof. Our aim is to derive the claim of Theorem 8.1 from the main statement of the canonical construction $\mathbb{F}\mathbb{C}(m,s)$, cf. Theorem 7.1. First, we are going to choose, effectively from t_0, a pair of integer parameters (m,s); then we will show that the theory $T = \mathbb{F}\mathbb{C}(m,s,\sigma)$ satisfies the desired properties.

We assume that T is a computably axiomatizable theory defined by a weak c.e. index t_0. Index t_0 allows us to organize an effective enumeration of the set of sentences provable in T; particularly, we can build an effective enumeration of signature $\sigma_0 \subseteq \sigma^{\infty}$ of theory T. By using the dynamic method for signatures, cf. Lemma 0.2, fix a Gödel numbering

$$\Phi_i, \quad i \in \mathbb{N}, \tag{8.1}$$

of the set of sentences $SL(\sigma_0)$. Obviously, the sequence (8.1) represents a generating set for the Tarski-Lindenbaum algebra $\mathcal{L}(T)$.

Choice of the parameter m. Since T is computably axiomatizable, the set of sentences $\{\mathfrak{P}_i \mid T \vdash \mathfrak{P}_i(\Phi_0, \ldots, \Phi_{\mathfrak{a}(i)})\}$ is computably enumerable. Therefore, we can find, effectively from t_0, an integer $m \in \mathbb{N}$ such that

$$\Sigma_0 \ =_{dfn} \ \{\mathfrak{P}_i \mid T \vdash \mathfrak{P}_i(\Phi_0, \ldots, \Phi_{\mathfrak{a}(i)})\} \ = \ \{\mathfrak{P}_i \mid i \in W_m\}. \tag{8.2}$$

Thereby, the first parameter m is chosen.

Notice that, relation (8.2) ensures the following equality:

$$T = \Sigma_0. \tag{8.3}$$

For an arbitrary set $A \subseteq \mathbb{N}$, we denote:

$$T[A] = T \ + \ \Sigma_1[A], \quad \textit{where } \Sigma_1[A] = \{\Phi_i \mid i \in A\} \cup \{\neg\Phi_j \mid j \in \mathbb{N} \smallsetminus A\}. \tag{8.4}$$

Obviously, the following property is satisfied:

$$(\forall \textit{ complete extension } T' \textit{ of } T)(\exists A \subseteq \mathbb{N})\big(T[A] \subseteq T'\big). \tag{8.5}$$

Inductively by the length of a formula, it is possible to prove that

$$\begin{aligned} &\Sigma_1[A] \vdash \big(\mathfrak{P}_i(\Phi_0, \ldots, \Phi_{\mathfrak{a}(i)})\big)^{\alpha}, \quad \textit{with} \\ &\alpha \ = \ \textit{the value of } \text{“}A \models \mathfrak{P}_i(\mathcal{X}_0, \ldots, \mathcal{X}_{\mathfrak{a}(i)})\text{”}, \ \alpha \in \{0, 1\}, \end{aligned} \tag{8.6}$$

where $\Psi^1 = \Psi$ and $\Psi^0 = \neg\Psi$, cf. Lemma 1.12 in [11] Ch.1, Sec.4. By adding all formulas of the form (8.6) with $\alpha = 1$ (provable from $\Sigma_1[A]$), we obtain a new presentation for the theory

$$\begin{aligned} &T[A] = T + \Sigma_2[A], \ \textit{where} \\ &\Sigma_2[A] \ = \{\mathfrak{P}_i(\Phi_0, \ldots, \Phi_{\mathfrak{a}(i)}) \mid A \models \mathfrak{P}_i(\mathcal{X}_0, \ldots, \mathcal{X}_{\mathfrak{a}(i)})\}. \end{aligned} \tag{8.7}$$

Moreover, we obviously have

$$\Sigma_1[A] \subseteq \Sigma_2[A]. \tag{8.8}$$

The following dependencies also take place:

$$\Sigma_2[A] \cup \bar{\Sigma}_2[A] = FL(\sigma_0), \quad \Sigma_2[A] \cap \bar{\Sigma}_2[A] = \varnothing, \ \textit{where} \qquad (8.9)$$
$$\bar{\Sigma}_2[A] \ = \{\mathfrak{P}_j(\Phi_0, \ldots, \Phi_{\mathfrak{a}(j)}) \mid A \not\models \mathfrak{P}_j(\mathcal{X}_0, \ldots, \mathcal{X}_{\mathfrak{a}(j)})\}.$$

By construction, we have either $\mathfrak{P}_i \in \Sigma_2[A]$ or $\neg\mathfrak{P}_i \in \Sigma_2[A]$ for each $i \in \mathbb{N}$. Thereby, presentation (8.7) ensures that

$$T[A] \textit{ is a complete theory whenever it is consistent.} \qquad (8.10)$$

Additionally, relation (8.9) ensures validity of the following property:

$$(\forall A \in \Omega(m))\Big[T[A] \vdash \mathfrak{P}_i(\Phi_0, \ldots, \Phi_{\mathfrak{a}(i)}) \Leftrightarrow A \models \mathfrak{P}_i(\mathcal{X}_0, \ldots, \mathcal{X}_{\mathfrak{a}(i)})\Big]. \qquad (8.11)$$

Let us prove that the following relation holds for the sets introduced above:

$$A \in \Omega(m) \ \Leftrightarrow \ \Sigma_0 \subseteq \Sigma_2[A]. \qquad (8.12)$$

First, we assume that $A \in \Omega(m)$. Consider an arbitrary $\Phi \in \Sigma_0$. By (8.2), we have $\Phi = \mathfrak{P}_{k_0}(\Phi_0, \ldots, \Phi_{\mathfrak{a}(k_0)})$ for some $k_0 \in W_m$. Since $A \in \Omega(m)$ by assumption, by virtue of definition (1.6), we conclude that $(\forall k \in W_m) A \models \mathfrak{P}_k$. Taking into account the fact that $k_0 \in W_m$, we obtain $A \models \mathfrak{P}_{k_0}(\mathcal{X}_0, \ldots, \mathcal{X}_{\mathfrak{a}(k_0)})$; thus, by definition (8.7), we have $\mathfrak{P}_{k_0}(\Phi_0, \ldots, \Phi_{\mathfrak{a}(k_0)}) \in \Sigma_2[A]$ that gives the desired inclusion $\Phi \in \Sigma_2[A]$.

Now, assume that $\Sigma_0 \subseteq \Sigma_2[A]$. For all i satisfying $T \vdash \mathfrak{P}_i(\Phi_0, \ldots, \Phi_{\mathfrak{a}(i)})$, by (8.3), we obtain $\mathfrak{P}_i(\Phi_0, \ldots, \Phi_{\mathfrak{a}(i)}) \in \Sigma_0$; by assumption, we have $\mathfrak{P}_i(\Phi_0, \ldots, \Phi_{\mathfrak{a}(i)}) \in \Sigma_2[A]$; therefore, $A \models \mathfrak{P}_i(\mathcal{X}_0, \ldots, \mathcal{X}_{\mathfrak{a}(i)})$ by (8.7). Applying again (8.2) we conclude that $(\forall i)\big[i \in W(m) \Rightarrow A \models \mathfrak{P}_i(\mathcal{X}_0, \ldots, \mathcal{X}_{\mathfrak{a}(i)})\big]$ that gives the desired inclusion $A \in \Omega(m)$ by definition (1.6).

Thus, (8.12) is indeed satisfied.

Now, we are going to show that

$$\texttt{(a)} \ \ A \in \Omega(m) \ \Rightarrow \ T[A] \textit{ is consistent and complete,} \qquad (8.13)$$
$$\texttt{(b)} \ \ A \notin \Omega(m) \ \Rightarrow \ T[A] \textit{ is contradictory.}$$

We consider two following cases.

Case 1: $A \in \Omega(m)$. From (8.7), we have $T[A] = T + \Sigma_2[A]$. Consider a finite set $\Delta = \{\Psi_0, \dots, \Psi_{t-1}\} \subseteq \Sigma_2[A]$. We are going to show that $T + \Delta$ is consistent. Let Ψ be conjunction $\Psi_0 \wedge \dots \wedge \Psi_{t-1}$. From $\Psi_i \in \Sigma_2[A]$, $i < t$, by rule (8.7), we obtain $\Psi \in \Sigma_2[A]$. By (8.9), we have $\neg\Psi \in \bar{\Sigma}_2[A]$, thus $\neg\Psi \notin \Sigma_2[A]$. By (8.3) and (8.12), we have $T = \Sigma_0 \subseteq \Sigma_2[A]$; thus, $\neg\Psi \notin T$. From this, we conclude that $T + \Psi$ is consistent; thereby, $T + \Delta$ is consistent as well. Applying Maltsev's Compactness Theorem, we obtain that theory $T[A]$ is consistent. By virtue of (8.10), this theory is complete.

Case 2: $A \notin \Omega(m)$. In this case, by (8.3) and (8.12), we obtain $T \nsubseteq \Sigma_2[A]$. Let Ψ be a sentence in $T \setminus \Sigma_2[A]$. By (8.9), we have $\Psi \in \bar{\Sigma}_2[A]$; thus, $\neg\Psi \in \Sigma_2[A]$ by virtue of (8.9). As a result, we obtain $\Psi \in T \subseteq T[A]$ and $\neg\Psi \in \Sigma_2[A] \subseteq T[A]$. This shows that the theory $T[A]$ is contradictory.

Thereby, both implications (8.13)(a) and (8.13)(b) are indeed satisfied.

Applying (8.5) and (8.13), we obtain the following principal property:

$$\{T[A] \mid A \in \Omega(m)\} \textit{ is the set of all complete extensions of } T. \tag{8.14}$$

This property shows that the set $\Omega(m)$ plays the role of a parametric presentation for Stone space of the theory T under consideration.

We also mention the following relation taking place for all $A \in \Omega(m)$:

$$T[A] \textit{ is decidable } \Leftrightarrow\ A \textit{ is computable.} \tag{8.15}$$

Indeed, if A is computable, the theory (8.4) is computably axiomatizable and complete. By Janiczack Theorem, $T[A]$ is decidable. Conversely, the relation

$$i \in A\ \Leftrightarrow\ T[A] \vdash \Phi_i, \textit{ for all } i \in \mathbb{N},$$

shows that A is computable whenever $T[A]$ is decidable, for all $A \in \Omega(m)$. Thereby, (8.15) is established.

The following more common property takes place:

there is an integer s_0 s. t. function $\varphi^A_{s_0}(t)$ is characteristic for $\mathrm{Nom}(T[A])$ *for all $A \in \Omega(m)$; moreover, s_0 can be found effectively from t_0 (by definition, s_0 is a relative characteristic index of $T[A]$).* (8.16)

Prove this statement. We use notations found in [4] Sec.9.2. For finite sets $D_u, D_v \subseteq \mathbb{N}$ of natural numbers, we introduce a notation for the following primitive propositional formula:

$$\mathfrak{P}_{u,v} = \bigwedge_{i\in D_u} X_i \ \wedge \ \bigwedge_{j\in D_v} \neg X_j.$$

Let A be an arbitrary subset of $\mathbb{N}$ and Ψ be a sentence of signature of theory T. By virtue of (8.14), theory $T[A]$ is complete for all A in $\Omega(m)$. This means that either Ψ or $\neg\Psi$ are provable in theory $T[A]$. Based on presentation of axioms for $T[A]$ in the form (8.4), we obtain that there is a pair of finite sets D_u and D_v with $D_u \cap D_v = \varnothing$ and a Boolean value $\alpha \in \{0,1\}$ satisfying the following relation

$$T \vdash \big(\mathfrak{P}_{u,v}(\Phi_0,\dots,\Phi_{\mathfrak{a}(u,v)}) \to \Psi^\alpha\big), \tag{8.17}$$

where $\Psi^0 = \neg\Psi$ and $\Psi^1 = \Psi$.

By construction, theory T is computably axiomatizable. Therefore, the set R of all sequences $\langle u, v, \mathrm{Nom}\,\Psi, \alpha\rangle$ satisfying condition (8.17) is computably enumerable. Find an integer s_0 such that

$$W_{s_0} = R = \{\langle u, v, \mathrm{Nom}\,\Psi, \alpha\rangle \mid u,\ v,\ \Psi,\ \textit{and } \alpha \textit{ satisfy } (8.17)\}. \tag{8.18}$$

By definitions in [4] Sec.1.8, the number s_0 is exactly a Gödel number of the Turing machine $\mathcal{M}$ enumerating the set R. Obviously, such a machine is built effectively from an index of theory T. Thus, we can state that

$$\textit{a value of the index } s_0 \textit{ is found effectively from } t_0. \tag{8.19}$$

From (8.17) and (8.18) we obtain the following presentation

$$T[A] \vdash \Psi^\alpha \ \Leftrightarrow\ (\exists u, v)\big[\,\langle u, v, \mathrm{Nom}\,\Psi, \alpha\rangle \in R \ \wedge\ D_u \subseteq A \ \wedge\ D_v \subseteq \mathbb{N}\smallsetminus A\,\big]. \tag{8.20}$$

Consider the passage to a normalized set $W_{s_0} \mapsto W_{\rho(s_0)}$, where both the term 'normalized' and the function $\rho(x)$ is defined in [4] Sec.9.2. By definitions (8.17) and (8.18), the set $W_{s_0} = R$ is regular relative to the cases with consistent theory $T[A]$, i.e., the value of α depending on Ψ in the left-hand side expression in (8.17) is uniquely determined whenever $A \in \Omega(m)$; as for the cases $A \notin \Omega(m)$, the value of α in (8.17) with given Ψ does not matter for our purposes. By construction, the normalization procedure may not change 'correct' cases involved in relation (8.17) with $A \in \Omega(m)$. Thus, we obtain finally the following new form of the relation which is a simple reformulation of (8.20), for all $\Psi \in SL(\sigma)$, $A \in \Omega(m)$, and $\alpha \in \{0,1\}$:

$$T[A] \vdash \Psi^{\alpha} \;\Leftrightarrow\; (\exists u, v)\big[\,\langle u, v, \mathrm{Nom}\,\Psi, \alpha\rangle \in W_{\rho(s_0)} \;\wedge\; D_u \subseteq A \;\wedge\; D_v \subseteq \mathbb{N}\smallsetminus A\big]. \tag{8.21}$$

On the other hand, we have the following standard presentation for computability with an oracle, cf. [4] Sec.9.2:

$$\varphi^{A}_{s_0}(t) = \alpha \;\Leftrightarrow\; (\exists u, v)\big[\,\langle u, v, t, \alpha\rangle \in W_{\rho(s_0)} \;\wedge\; D_u \subseteq A \;\wedge\; D_v \subseteq \mathbb{N}\smallsetminus A\big]. \tag{8.22}$$

Combining relations (8.21) and (8.22) together, we obtain the following summary statement:

$$T[A] \vdash \Psi^{\alpha} \;\Leftrightarrow\; \varphi^{A}_{s_0}(\mathrm{Nom}\,\Psi) = \alpha, \tag{8.23}$$

for all $\Psi \in SL(\sigma)$, $A \in \Omega(m)$, $\alpha \in \{0,1\}$, that is exactly what is required for (8.16); moreover, an additional statement (8.19) is also established showing effectiveness of the presentation (8.16).

Choice of the parameter s. Main aim of the parameter s is to control model theoretic-properties within the layer $M1L^{\circ}$. To choose s, we are going to manipulate with the Tarski-Lindenbaum algebras $\mathcal{L}_k(T[A])$ of the theory $T[A]$ under formulas with k free variables $x_0, x_1, \ldots, x_{k-1}$ for $1 \leqslant k < \omega$.

We are in a position to organize a computation with a pair of input parameters A and k, where A plays the role of an oracle, while k is a positive integer. A scheme below specifies both parameters and objects in this computation:

An input parameter $A \subseteq \mathbb{N}$, *a working parameter* $k \in \mathbb{N}\smallsetminus\{0\}$; (8.24)

Scheme of transformation :

$A \;\mapsto$	by rule (8.4)
$T[A] \;\mapsto$	by rule (6.2)
$\mathcal{L}_k(T[A])$, $1 \leqslant k < \omega \;\mapsto$	by rule (6.7)
$D^{(k)}[A] = \mathrm{tree}\big(\mathcal{L}_k(T[A])\big)$, $1 \leqslant k < \omega \;\mapsto$	by rule (6.8)
$\mathcal{D}^{(k)}[A] = \mathrm{Tree}\big(D^{(k)}[A]\big)$, $1 \leqslant k < \omega \;\mapsto$	by rule (3.9)

Result (*final gathering*) : $\mathcal{D}[A] = \bigoplus_{1\leqslant k<\omega}\mathcal{D}^{(k)}[A]$.

Objects involved in the computation form a chain of transformations from a given theory $T[A]$ to its Tarski-Lindenbaum algebras $\mathcal{L}_k(T[A])$, then, to normal trees, further, to compact trees, and finally, to the direct sum of the obtained sequence of compact binary trees. The right-hand side of the scheme points out rules providing the transformations at each stage. Notice that, in the chain of transformations as a whole, the set $A \subseteq \mathbb{N}$ plays the role of a free parameter, while k plays the role of a temporary bounded parameter; k disappears at the final stage having performed the role of an index in the sum operation of a sequence of compact binary trees.

Hereafter, we denote by $\mathcal{M}_p^A$ a Turing machine having Gödel number p in computation with an oracle $A \subseteq \mathbb{N}$, cf. [4] Sec.9.2, while $\mathcal{M}_p^A(\ldots)$ means a result of computation on this machine with the pointed out input parameters.

By statement (8.16), theory $T[A]$ is presented by a computable function $\varphi_{s_0}^A(t)$ defined by an index $s_0 \in \mathbb{N}$ in computation with an oracle A. By definition, cf. [4] Sec.9.2, this means that machine $\mathcal{M}_{s_0}^A(x)$ computes characteristic function of the theory $T[A]$ whenever $A \in \Omega(m)$. Using $\mathcal{M}_{s_0}^A(x)$ as a subroutine, we can build a new Turing machine $\mathcal{M}_{s_1}^A$ such that $\mathcal{M}_{s_1}^A(k, x)$ computes characteristic function of the natural tree $D^{(k)}[A]$ for all $A \in \Omega(m)$. While programming for $\mathcal{M}_{s_1}^A$, we have to use internal details of the transformation procedure from the Tarski-Lindenbaum algebras to natural trees described in (6.7). In turn, the machine $\mathcal{M}_{s_1}^A$ can be extended to a Turing machine $\mathcal{M}_{s_2}^A$ such that $\mathcal{M}_{s_2}^A(k, x)$ enumerates compact binary tree $\mathcal{D}^{(k)}[A]$; i.e., we have $\mathcal{D}^{(k)}[A] = \{t \mid \mathcal{M}_{s_2}^A(k, t)\downarrow\}$ for all $A \in \Omega(m)$. While programming for $\mathcal{M}_{s_2}^A$, we have to use internal details of the transformation procedure from natural trees to compact trees described in (6.8). Finally, the machine $\mathcal{M}_{s_2}^A$ can be extended to a Turing machine $\mathcal{M}_s^A$ (having a Gödel number s) such that $\mathcal{M}_s^A(x)$ enumerates compact binary tree $\mathcal{D}[A]$ that was obtained at the final stage of the chain (8.24); i.e., the following equality takes place for all $A \in \Omega(m)$:

$$\mathcal{D}[A] \;=\; \{t \mid \mathcal{M}_s^A(t)\downarrow\}.$$

On the other hand, we have $\{t \mid \mathcal{M}_s^A(t)\downarrow\} = W_s^A$ in accordance with the definition given in [4] Sec.9.3. Thus we obtain the following equalities for all $A \in \Omega(m)$:

$$\mathcal{D}[A] \;=\; \textstyle\bigoplus_{1 \leqslant k < \omega} \mathcal{D}^{(k)}[A] \;=\; \{t \mid \mathcal{M}_s^A(t)\downarrow\} \;=\; W_s^A. \tag{8.25}$$

Since set W_s^A in (8.25) coincides with a tree $\mathcal{D}[A]$, closure of W_s^A up to a compact binary tree must coincide with the set W_s^A itself; thus, from definition (3.4), we obtain that $\mathcal{D}_s^A = W_s^A$. From this, together with (8.25), we obtain finally

$$\mathcal{D}_s^A \ = \ \bigoplus\nolimits_{1\leqslant k<\omega}\mathcal{D}^{(k)}[A], \ \ \textit{for all } A \in \Omega(m). \tag{8.26}$$

Argumentation above shows that s is found effectively from a week c.e. index t_0 of the source theory T.

Thereby, the second parameter s is also chosen.

To this end, we have found a pair of integer parameters (m, s). Based on these parameters, we construct a finitely axiomatizable theory $F = \mathbb{F}\mathbb{C}(m, s, \sigma)$. Now, our aim is to define a computable isomorphism $\mu : \mathcal{L}(T) \to \mathcal{L}(F)$. Main idea is based on the fact that Stone spaces (7.2) and (8.14) of theories F and T are defined via the same parametric space $\Omega(m)$.

In accordance with Theorem 7.1, there is a computable sequence of sentences θ_i, $i \in \mathbb{N}$, defined in theory $\mathbb{F}\mathbb{C}(m, s)$ that satisfies all conditions posed in the formulation of the canonical construction. Using generating sequence (8.1) for $\mathcal{L}(T)$, first, we define a partial mapping

$$\mu' : \Phi_i \mapsto \theta_i, \ \ i \in \mathbb{N}. \tag{8.27}$$

The following chain of equivalences takes place for any $\mathfrak{P}_i \in \mathit{FRM}(\sigma^\circ)$, $i \in \mathbb{N}$:

$$\begin{array}{ll}
T \vdash \mathfrak{P}_i(\Phi_0, \ldots, \Phi_{\mathfrak{a}(i)}) \ \Leftrightarrow & \text{immediately} \\
\big(\forall T' \in \mathrm{St}(T)\ \big)\, T' \vdash \mathfrak{P}_i(\Phi_0, \ldots, \Phi_{\mathfrak{a}(i)}) \ \Leftrightarrow & \text{by (8.14)} \\
(\forall A \in \Omega(m))\ T[A] \vdash \mathfrak{P}_i(\Phi_0, \ldots, \Phi_{\mathfrak{a}(i)}) \ \Leftrightarrow & \text{by (8.11)} \\
(\forall A \in \Omega(m))\ A \models \mathfrak{P}_i \ \Leftrightarrow & \text{by (7.4)} \\
(\forall A \in \Omega(m))\ \mathbb{F}\mathbb{C}(m, s)[A] \vdash \mathfrak{P}_i(\theta_0, \ldots, \theta_{\mathfrak{a}(i)}) \ \Leftrightarrow & \text{by (7.2)} \\
\big(\forall F' \in \mathrm{St}(\mathbb{F}\mathbb{C}(m, s))\big)\ F' \vdash \mathfrak{P}_i(\theta_0, \ldots, \theta_{\mathfrak{a}(i)}) \ \Leftrightarrow & \text{immediately} \\
\mathbb{F}\mathbb{C}(m, s) \vdash \mathfrak{P}_i(\theta_0, \ldots, \theta_{\mathfrak{a}(i)}). &
\end{array}$$

Thereby, Lemma 2.2 is applicable ensuring that the mapping (8.27) can be extended to a computable isomorphism between the Tarski-Lindenbaum algebras

$$\mu : \mathcal{L}(T) \to \mathcal{L}(F). \tag{8.28}$$

Moreover, μ maps any complete extension $T[A]$, $A \in \Omega(m)$, of theory T in corresponding complete extension $\mathbb{F}\mathbb{C}(m,s)[A]$ of the target theory $F = \mathbb{F}\mathbb{C}(m,s)$. Let us show that μ preserves all model-theoretic properties within the semantic layer $M1L^{\circ}$.

In the chains of transformations below, initially, we pass from a complete theory to its Tarski-Lindenbaum algebra under formulas with k free variables by rule (6.2) for all k; then we pass to a natural binary tree by rule (6.7); after that, we apply the compactification operation passing to a compact binary tree by rule (6.8). Finally, we gather all these compact trees together and analyze the family of chains of the summary tree confirming preservation of the model-theoretic property under consideration. Appropriate parts of Claim 0.3 are used presenting known criteria of existence of prime and countable saturated models as well as characterization of their algorithmic complexity. Both structural and algorithmic properties of the transformations $ct2L_k$, $b2n$, and $n2c$ are also used, which are presented in Lemma 6.1, Lemma 6.2, and Lemma 6.3, as well as in Lemma 6.4.

First, we consider the property of existence of a prime model. Based on the Vaught criteria presented in Claim 0.3 (a), we have the following chain of equivalences:

$$\begin{aligned}
&T[A] \textit{ has a prime model } \Leftrightarrow && (8.29)\\
&(\forall k \in \mathbb{N}\setminus\{0\})\big(\textit{Tarski-Lindenbaum algebra } \mathcal{L}_k(T[A]) \textit{ is atomic}\big) \Leftrightarrow\\
&(\forall k \in \mathbb{N}\setminus\{0\})\big(D^{(k)}[A] \textit{ is atomic}\big) \Leftrightarrow\\
&(\forall k \in \mathbb{N}\setminus\{0\})\big(\mathcal{D}^{(k)}[A] \textit{ is atomic}\big) \Leftrightarrow\\
&\mathcal{D}^{A}_{s} \textit{ is atomic } \Leftrightarrow\\
&\mathbb{F}\mathbb{C}(m,s)[A] \textit{ has a prime model.}
\end{aligned}$$

Now, we turn to the property of strong constructivizability of a prime model. Assume that theory $T[A]$ has a prime model $\mathfrak{N}$. The case when A is not computable is trivial because, in this case, by (7.3) and (8.15), both theories $T[A]$ and $\mathbb{F}\mathbb{C}(m,s)[A]$ are undecidable; thereby, each of them does not have a s.c. prime model. Now, we consider the opposite case when A is computable. Since $T[A]$ is supposed to have a prime model, all parts in the chain (8.29) must be satisfied. Based on these assumptions together with Claim 0.3 (b), we have the following chain of equivalences:

$$\begin{aligned}
&T[A] \textit{ has a strongly constructivizable prime model } \Leftrightarrow\\
&\mathfrak{N} \textit{ is strongly constructivisable } \Leftrightarrow\\
&\textit{the family of principal types realized in } \mathfrak{N} \textit{ is computable } \Leftrightarrow\\
&\textit{the sequence } \Pi^{iso}(D^{(k)}[A]),\ k \in \mathbb{N} \setminus \{0\},\ \textit{is strongly computable; i.e.,}\\
&\textit{there is a sequence of numerations } \nu_k : \mathbb{N} \to D^{(k)}[A] \textit{ such that the set}\\
&\{\langle k, n, t\rangle \mid t \in \nu_k(n)\} \textit{ is computable } \Leftrightarrow\\
&\textit{the sequence } \Pi^{fin}(\mathcal{D}^{(k)}[A]),\ k \in \mathbb{N} \setminus \{0\},\ \textit{is computable; i.e.,}\\
&\textit{there is a sequence of numerations } \nu_k : \mathbb{N} \to \mathcal{D}^{(k)}[A] \textit{ such that the set}\\
&\{\langle k, n, t\rangle \mid t \in \nu_k(n)\} \textit{ is computably enumerable } \Leftrightarrow\\
&\Pi^{fin}(\mathcal{D}_s^A) \textit{ is computable } \Leftrightarrow\\
&\mathbb{F}\mathbb{C}(m, s)[A] \textit{ has a strongly constructivizable prime model.}
\end{aligned} \tag{8.30}$$

Further, we consider the property of algorithmic dimension of a prime model. Assume that a prime model $\mathfrak{N}$ of theory $T[A]$ exists and is strongly constructivizable. In this case, the theory $T[A]$ is decidable; thus, by (8.15), the set A must be computable; moreover, all parts in the chains of equivalences (8.29) and (8.30) mast be satisfied. Based on these assumptions together with Claim 0.3(c), we have the following chain of equivalences:

$$\begin{aligned}
&T[A] \textit{ has a strongly constructivizable prime model of the dimension } 1 \Leftrightarrow\\
&\mathfrak{N} \textit{ has the algorithmic dimension } 1 \Leftrightarrow\\
&\textit{the family of atomic formulas of } \mathrm{Th}(\mathfrak{N}) \textit{ is computable } \Leftrightarrow\\
&\textit{the family of atomic nodes in } D^{(k)}[A],\ k \in \mathbb{N} \setminus \{0\},\ \textit{is computable } \Leftrightarrow\\
&\textit{the family of dead-ends in } \mathcal{D}^{(k)}[A],\ k \in \mathbb{N} \setminus \{0\},\ \textit{is computable } \Leftrightarrow\\
&\textit{the family of dead-ends in } \mathcal{D}_s^A \textit{ is computable } \Leftrightarrow\\
&\mathbb{F}\mathbb{C}(m, s)[A] \textit{ has a strongly constructivizable prime model of the dimension } 1.
\end{aligned}$$

Now, we turn to the property of existence of a countable saturated model.

Based on Claim 0.3(d), we have the following chain of equivalences:

$$T[A] \textit{ has a countable saturated model } \Leftrightarrow \tag{8.31}$$
$$(\forall k \in \mathbb{N} \setminus \{0\})\big(\textit{Tarski-Lindenbaum algebra } \mathcal{L}_k(T[A]) \textit{ is superatomic}\big) \Leftrightarrow$$
$$(\forall k \in \mathbb{N} \setminus \{0\})\big(D^{(k)}[A] \textit{ is superatomic}\big) \Leftrightarrow$$
$$(\forall k \in \mathbb{N} \setminus \{0\})\big(\mathcal{D}^{(k)}[A] \textit{ is superatomic}\big) \Leftrightarrow$$
$$\mathcal{D}_s^A \textit{ is superatomic } \Leftrightarrow$$
$$\mathbb{F}\mathbb{C}(m,s)[A] \textit{ has a countable saturated model.}$$

Finally, we turn to the property of strong constructivizability of a countable saturated model. Assume that theory $T[A]$ has a countable saturated model $\mathfrak{M}$. The case when A is not computable is trivial because, in this case, by (7.3) and (8.15), both theories $T[A]$ and $\mathbb{F}\mathbb{C}(m,s)[A]$ are undecidable; thereby, each of them does not have a s.c. countable saturated model. Now, we consider the opposite case when A is computable. Since $T[A]$ is supposed to have a countable saturated model, all parts in the chain (8.31) must be satisfied. Based on these assumptions together with Claim 0.3 (e), we have the following chain of equivalences:

$$T[A] \textit{ has a strongly constructivizable countable saturated model } \Leftrightarrow$$
$$\mathfrak{M} \textit{ is strongly constructivisable } \Leftrightarrow$$
$$\textit{the family of all types realized in } \mathfrak{M} \textit{ is computable } \Leftrightarrow$$
$$\textit{the sequence } \Pi(D^{(k)}[A]),\ k \in \mathbb{N} \setminus \{0\}, \textit{ is strongly computable; i.e.,}$$
$$\textit{there is a sequence of numerations } \nu_k : \mathbb{N} \to D^{(k)}[A] \textit{ such that the set}$$
$$\{\langle k,n,t\rangle \mid t \in \nu_k(n)\} \textit{ is computable } \Leftrightarrow$$
$$\textit{the sequence } \Pi(\mathcal{D}^{(k)}[A]),\ k \in \mathbb{N} \setminus \{0\}, \textit{ is computable; i.e.,}$$
$$\textit{there is a sequence of numerations } \nu_k : \mathbb{N} \to \mathcal{D}^{(k)}[A] \textit{ such that the set}$$
$$\{\langle k,n,t\rangle \mid t \in \nu_k(n)\} \textit{ is computably enumerable } \Leftrightarrow$$
$$\Pi(\mathcal{D}_s^A) \textit{ is computable } \Leftrightarrow$$
$$\mathbb{F}\mathbb{C}(m,s)[A] \textit{ has a strongly constructivizable countable saturated model.}$$

This shows that, the obtained isomorphism μ indeed preserves all model-theoretic properties within the layer $M1L^{\circ}$.

Proof of Theorem 8.1 is complete. □

Conclusion

In this work, we describe a construction called canonical-mini that actually represents a mini-version of the universal construction. It is defined by a standard formulation for the universal construction supporting not so large layer of model-theoretic properties. Moreover, the canonical-mini construction turns out to be much simpler for understanding in comparison with any normal version of the universal construction.

High complexity of the universal construction represents a certain psychological barrier while studying results obtained on the base of this construction. The fact of availability of a mini-version of the universal construction that is more accessible for studying reduces this barrier. As for the results concerned with particular collections of model-theoretic properties, for such a result, we need to apply a suitable version of the universal construction that supports this set of properties. In the other case of results of a common character, a subject of the statement is the layer of properties controlled by an available version of the universal construction. In this case, without spending big efforts, we can first consider this result based on the mini-version of the universal construction. Subsequently, by attracting some stronger (however, more complicated) version of the universal construction, we will be able to extend this result up to the infinitary semantic layer MQL or maybe its large enough part.

A few open questions.

Question 9.1. Is there a routine consequence of the canonical construction presenting a weak release of the universal construction of finitely axiomatizable theories that can control the property "theory is ω-stable"?

Question 9.2. Is there a release of the universal construction of finitely axiomatizable theories controlling some model-theoretic property $\mathfrak{p}$ that is not included in the infinitary semantic layer MQL?

References

1 Peretyat'kin M.G. Finitely axiomatizable theories. Plenum. — New York, 1997. — 297 p.

2 Peretyat'kin M.G. Introduction in first-order combinatorics providing a conceptual framework for computation in predicate logic // Computation tools 2013, The Fourth International Conference on Computational Logics, Algebras, Programming, Tools, and Benchmarking, IARIA. — 2013. — P. 31–36.

3 Hodges W. A shorter model theory. — Cambridge University Press, Cambridge, 1997. — 310 p.

4 Rogers H.J. Theory of Recursive Functions and Effective Computability. — New York: McGraw-Hill Book Co., 1967. — 482 p.

5 Goncharov S.S., Ershov Yu.L. Constructive models. Plenum. — New York, 1999. — 306 p.

6 Vaught R.L. Denumerable models of complete theories, Infinitistic Methods. — London: Pregamon, , 1961. — P.303–321.

7 Goncharov S.S., Nurtazin A.T. Constructive models of complete decidable theories // Algebra and Logic. — 1973. — V. 12, No. 2. — P. 67–77.

8 Harrington L. Recursively presented prime models // J. Symbolic Logic // — 1974. — V. 39, No. 2. — P. 305–309.

9 Nurtazin A.T. Strong and weak constructivizations and computable families // Algebra and Logic. — 1974. — V. 20, No. 3. P. 177–184.

10 Morley M. Decidable models // Israel J. Math. — 1976. —V. 25, No. 3–4. — P. 233–240.

11 Mendelson E. Introduction to Mathematical Logic. — Princeton, New Jersey, Toronto, New York, London, Second edition, 1979. — 328 p.

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Перетятькин М.Г. ӘМБЕБАП ҚҰРЫЛЫМНЫҢ ӘЛСІЗ ТҮРПАТЫН ҰСЫНАТЫН АҚЫРЛЫ АКСИОМАЛАУШЫ ТЕОРИЯЛАРДЫҢ КАНОНДЫҚ МИНИ ҚҰРЫЛЫМЫ

Жұмыста, ақырлы аксиомалаушы теориялардың канондық құрылымының қолда бар нұсқасына сүйене отырып, біз канондық мини құрылым деп аталатын осы құрылымның белгілі бір әлсіретілген түрпатын шығарамыз. Шыққан құрылым теориялық-модельдік қасиеттердің салыстырмалы түрде кішігірім (дегенмен, тривиалды емес) семантикалық қабатын бақылай отырып, ақырлы аксиомалаушы теориялардың әмбебап құрылымының дағдылы тұжырымдамасын иеленген. Осыған байланысты, канондық мини құрылым әмбебапты-ішкіканондық құрылым деп те аталуы мүмкін. Канондық мини құрылым инфинитарлы семантикалық қабаттың қандайда бір ішкіқабатын сүйемелдейді және әмбебап құрылымның, түсінуге қолайлы әрі едәуір қарапайым дәлелдеумен тиімді түрде өзгешеленетін, жеңілдетілген нұсқасының рөлін атқара алады.

Перетятькин М.Г. КАНОНИЧЕСКАЯ МИНИ КОНСТРУКЦИЯ КОНЕЧНО АКСИОМАТИЗИРУЕМЫХ ТЕОРИЙ ПРЕДСТАВЛЯЮЩАЯ СЛАБУЮ ВЕРСИЮ УНИВЕРСАЛЬНОЙ КОНСТРУКЦИИ

В работе, опираясь на существующую версию канонической конструкции конечно аксиоматизируемых теорий, мы выводим некоторую ослабленную форму этой же конструкции которая называется канонической мини конструкцией. Полученная конструкция имеет стандартную формулировку универсальной конструкции конечно аксиоматизируемых теорий, контролирующих сравнительно небольшой (тем не менее, нетривиальный) семантический слой теоретико-модельных свойств. Ввиду этого, каноническая мини конструкция также может быть названа универсальной-подканонической конструкцией. Каноническая мини конструкция, поддерживающая некоторый подслой инфинитарного семантического слоя, может выполнять роль упрощенной версии универсальной конструкции, выгодно отличается от последней существенно более простым и доступным для понимания доказательством.

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RECENT ACHIEVEMENTS IN CHI-SQUARED TYPE GOODNESS-OF-FIT TESTING

During last ten years much have been done for the theory and applications of widely used modified chi-squared type goodness-of-fit tests. In this account we briefly consider the main achievements in that direction emphasizing the input made by researchers from the Institute for Mathematics and Mathematical Modeling of the Ministry of Education and Science of the Republic of Kazakhstan.
Keywords: *Modified chi-squared type tests, decomposition of tests, Wald's method, vector-valued tests, tests for multivariate normality, misusing of Pearson's sum.*

1 History and introduction

The famous chi-squared goodness-of-fit test has been discovered by Karl Pearson in 1900. If observations are grouped over r disjoined intervals Δ_i and denoting $N_i^{(n)}$ observed frequencies corresponding to a multinomial scheme and $np_i(\boldsymbol{\theta})$ expected, the Pearson's sum is written

$$\chi^2 = \sum_{i-1}^{r} \frac{(N_i^{(n)} - np_i(\boldsymbol{\theta}))^2}{np_i(\boldsymbol{\theta})} = \mathbf{V}^{(n)T}(\boldsymbol{\theta})\mathbf{V}^{(n)}(\boldsymbol{\theta}), \tag{1}$$

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where $\mathbf{V}^{(n)}(\boldsymbol{\theta})$ is a vector of standardized frequencies with components

$$v_i^{(n)}(\boldsymbol{\theta}) = (N_i^{(n)} - np_i(\boldsymbol{\theta}))/(np_i(\boldsymbol{\theta}))^{1/2},\ i = 1, \ldots, r,\ \boldsymbol{\theta} \in \boldsymbol{\Theta} \subset \mathbf{R}^s.$$

If the number of observations $n \to \infty$, the statistic (1) will follow in the limit the chi-squared probability distribution with $r-1$ degrees of freedom. We know that this remarkable result is true only for a simple null hypothesis, when a hypothetical distribution is specified uniquely (a parameter $\boldsymbol{\theta}$ is considered to be known). Up to 1934 Pearson believed that the limit distribution of the statistic (1) will be the same if unknown parameters of the null hypothesis are replaced by estimates based on a sample. In view of this it is of interest to reproduce the citation of Plackett [1], p.69, concerning the E.S. Pearson's opinion: "I knew long ago that KP (Karl Pirson) used the 'correct' degrees of freedom for (a) difference between two samples and (b) multiple contingency tables. But he could not see that χ^2 in curve fitting should be got asymptotically into the same category". Plackett explained that this crucial mistake of Pearson aroused due to Karl's assumption "that individual normality implies joint normality". Fisher [2] clearly showed that the number of degrees of freedom of the Pearson's test must be reduced by the number s of parameters estimated by a sample. To this point it must be added that Fisher's result is true if and only if parameters are estimated by the vector of frequencies (minimizing Pearson's chi-squared sum, using multinomial maximum likelihood estimates (MLEs), or by any other asymptotically equivalent procedure (see, e.g., [3], p.74)). Such estimates based on a vector of frequencies, which is not in general the vector of sufficient statistics, are not asymptotically efficient. Nowadays the Pearson's test with unknown parameters replaced by estimates $\hat{\boldsymbol{\theta}}_n$ based on the vector of frequencies is known as Pearson-Fisher (PF) test

$$X_n^2(\hat{\boldsymbol{\theta}}_n) = \sum_{i-1}^{r} \frac{(N_i^{(n)} - np_i(\hat{\boldsymbol{\theta}}_n))^2}{np_i(\hat{\boldsymbol{\theta}}_n)} = \mathbf{V}^{(n)T}(\hat{\boldsymbol{\theta}}_n)\mathbf{V}^{(n)}(\hat{\boldsymbol{\theta}}_n). \tag{2}$$

Dzhaparidze and Nikulin [4] proposed a modification of the standard Pearson's statistic (DN test) valid for any square root of n consistent estimate $\tilde{\boldsymbol{\theta}}_n$ of an unknown parameter

$$U_n^2(\tilde{\boldsymbol{\theta}}_n) = \mathbf{V}^{(n)T}(\tilde{\boldsymbol{\theta}}_n)(\mathbf{I} - \mathbf{B}_n(\mathbf{B}_n^T\mathbf{B}_n)^{-1}\mathbf{B}_n^T)\mathbf{V}^{(n)}(\tilde{\boldsymbol{\theta}}_n), \tag{3}$$

where $\mathbf{B}_n$ is an estimate of the matrix $\mathbf{B}$ with elements

$$b_{ij} = \frac{1}{\sqrt{p_i(\boldsymbol{\theta})}} \int_{\Delta_i} \frac{\partial f(x,\boldsymbol{\theta})}{\partial \theta_j} dx, \quad i = 1, \ldots, r,\ j = 1, \ldots, s,$$

where $f(x, \boldsymbol{\theta})$ is a hypothetical probability density function. This test being asymptotically equivalent to the Pearson-Fisher statistic in many cases is not powerful for equiprobable cells (see [5]) but is rather powerful if an alternative hypothesis is specified and one uses the Neyman-Pearson classes for constructing the vector of frequencies.

Even after Fisher's clarification many statisticians thought that applying Pearson's test one may use estimates (e.g., MLE) based on non-grouped (raw) data. Chernoff and Lehmann [6] showed that replacing unknown parameters in (1) by their MLEs based on non-grouped data would dramatically change the limit distribution of Pearson's sum. In this case it will follow a distribution that in general depends on unknown parameters and, hence, cannot be used for testing. In our opinion what is difficult to understand for those who apply chi-squared tests is that an estimate is a realization of a random variable with its own probability distribution and that a particular estimate can be too far from the actual unknown value of a parameter or parameters.

Thus a problem of deriving a test statistic which limiting distribution will not depend on parameters aroused. Several researchers showed that for location and scale families with proper chosen random cells the limit distribution of Pearson's sum will not depend on unknown parameters depending only on the null hypothesis. Being distribution free such tests can be used in practice, but for each specific null distribution one has to evaluate corresponding critical values. So, two ways of constructing distribution free Pearson's type tests considered are: to use proper estimates of unknown parameters (e.g., based on grouped data), or to use specially constructed grouping intervals. Another possible way is to modify the Pearson's sum such that its limit probability distribution would not depend on unknowns. Later it was shown that the limit distribution of a vector of standardized frequencies with any efficient estimator (e.g., MLE or the best asymptotically normal (BAN) estimator) instead of unknown parameter would be multivariate normal and will not depend on the fact that boundaries of cells are fixed or random. Nikulin [7] using the above results and a very general theoretical approach (nowadays known as Wald's method) solved the problem in full for any continuous and discrete probability distribution if one will use grouping intervals based on predetermined probabilities to fall into a cell. One year later Rao and Robson [8] using an heuristic approach obtained the same result for a particular case of the exponential family of distributions. Formally their result

$$Y1_n^2(\hat{\boldsymbol{\theta}}_n) = X_n^2(\hat{\boldsymbol{\theta}}_n) + \mathbf{V}^{(n)T}(\hat{\boldsymbol{\theta}}_n)\mathbf{B}_n(\mathbf{J}_n - \mathbf{J}_{gn})^{-1}\mathbf{B}_n^T\mathbf{V}^{(n)}(\hat{\boldsymbol{\theta}}_n), \tag{4}$$

where $\mathbf{J}_n$ and $\mathbf{J}_{gn} = \mathbf{B}_n^T\mathbf{B}_n$ are estimators of Fisher information matrices $\mathbf{J}$ for non-grouped and $\mathbf{J}_g$ for grouped data correspondingly, is identically equal to that of Nikulin [7]. The statistic (4) can also be presented as

$$Y1_n^2(\hat{\boldsymbol{\theta}}_n) = \mathbf{V}^{(n)T}(\hat{\boldsymbol{\theta}}_n)(\mathbf{I} - \mathbf{B}_n\mathbf{J}_n^{-1}\mathbf{B}_n^T)^{-1}\mathbf{V}^{(n)}(\hat{\boldsymbol{\theta}}_n). \tag{5}$$

The statistic (4) or (5), suggested first by Nikulin for testing the normality, will be referred to in the sequel as Nikulin-Rao-Robson (NRR) test. Nikulin assumed that only efficient estimates of unknown parameters (e.g., MLEs based on non-grouped data or BAN estimates) are used for testing.

Hsuan and Robson [9] showed that a modified statistic would be quite different in case of using moment type estimators (MMEs) of unknown parameters. They succeeded in deriving the limit covariance matrix for standardized frequencies $v_i(\bar{\theta}_n)$, where $\bar{\theta}_n$ is the MME of θ and proving the theorem that a corresponding Wald's quadratic form will follow in the limit the chi-squared distribution. Hsuan and Robson provided the test statistic explicitly for the exponential family of distributions when MMEs coincide with MLEs, thus confirming already known result of Nikulin. Hsuan and Robson were unable to derive the general modified test based on MMEs $\bar{\theta}_n$ explicitly. This was done later by Mirvaliev [10]. Taking into account the input of Hsuan and Robson and Mirvaliev, we suggest calling this test as a Hsuan-Robson-Mirvaliev (HRM) statistic

$$Y2_n^2(\bar{\boldsymbol{\theta}}_n) = X_n^2(\bar{\boldsymbol{\theta}}_n) + R_n^2(\bar{\boldsymbol{\theta}}_n) - Q_n^2(\bar{\boldsymbol{\theta}}_n). \tag{6}$$

Moore [11] based on Wald's approach suggested a general recipe for constructing modified chi-squared tests for any square root of n consistent estimator, which actually is a slight generalization of Nikulin's idea, since it includes also the case of fixed grouping cells, which is not important because nobody knows a'priory how to partite the sample space onto fixed cells if probability distribution to be tested is unknown. Moore has not specified those tests for a particular $\sqrt{n}$ - consistent estimator, but has noted that a resulting Wald's quadratic form does not depend on the way of limit covariance matrix inverting.

The important input to the theory of modified chi-squared goodness-of-fit tests has been done by McCulloch [12] and Mirvaliev [10] who considered two types of a decomposition of tests. The first is a decomposition of a test on a sum of the DN statistic and asymptotically independent on the DN test additional quadratic form. Denoting $W_n^2(\boldsymbol{\theta}) = \mathbf{V}^{(n)T}(\boldsymbol{\theta})\mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{V}^{(n)}(\boldsymbol{\theta})$ and $P_n^2(\boldsymbol{\theta}) = \mathbf{V}^{(n)T}(\boldsymbol{\theta})\mathbf{B}(\mathbf{J}-\mathbf{J}_g)^{-1}\mathbf{B}^T\mathbf{V}^{(n)}(\boldsymbol{\theta})$ the decomposition of the NRR statistic (4) in case of MLEs will be

$$Y1_n^2(\hat{\boldsymbol{\theta}}_n) = U_n^2(\hat{\boldsymbol{\theta}}_n) + (W_n^2(\hat{\boldsymbol{\theta}}) + P_n^2(\hat{\boldsymbol{\theta}}_n)), \tag{7}$$

where $U_n^2(\hat{\boldsymbol{\theta}}_n)$ is independent on $(W_n^2(\hat{\boldsymbol{\theta}}) + P_n^2(\hat{\boldsymbol{\theta}}_n))$, and on $W_n^2(\hat{\boldsymbol{\theta}})$. The decomposition of the HRM statistic (6) is

$$Y2_n^2(\bar{\boldsymbol{\theta}}_n) = U_n^2(\bar{\boldsymbol{\theta}}_n) + (W_n^2(\bar{\boldsymbol{\theta}}) + R_n^2(\bar{\boldsymbol{\theta}}_n) - Q_n^2(\bar{\boldsymbol{\theta}}_n)), \tag{8}$$

where $U_n^2(\bar{\boldsymbol{\theta}}_n)$ is independent on $(W_n^2(\bar{\boldsymbol{\theta}}) + R_n^2(\bar{\boldsymbol{\theta}}_n) - Q_n^2(\bar{\boldsymbol{\theta}}_n))$, but is asymptotically correlated with $W_n^2(\bar{\boldsymbol{\theta}})$. The second way decomposes a modified test on a sum of classical Pearson's test and a correcting term, which makes it chi-squared distributed in the limit, and independent on unknown parameters (see (4) and (6)). This representation for NRR statistic was first used by Nikulin. The case of MMEs was first investigated by Mirvaliev. The decomposition of a modified chi-squared test on a sum of the DN statistic and an additional term is of importance because the DN test based on non-grouped data is asymptotically equivalent to the Pearson-Fisher's (PF) statistic for grouped data. Hence, that additional term takes into account the Fisher's information lost due to grouping. Later it was shown (Voinov et al [5]) that the DN part like the PF test is in many cases insensitive to an alternative hypothesis in case of equiprobable cells (fixed or random) and would be sensitive to it for, e.g., non-equiprobable Neyman-Pearson classes. For equiprobable cells this suggests using the difference between a modified statistic and the DN part that will be the most powerful in case of equiprobable cells (McCulloch [12], Voinov et al [5]).

Ronald Fisher noted that "in some cases it is possible to separate the contributions to χ^2 made by the individual degrees of freedom, and so to test the separate components of a discrepancy". Cochran [13] wrote "that the usual χ^2 tests are often insensitive, and do not indicate significant results when the null hypothesis is actually false" and suggested to "use a single degree of freedom, or a group of degrees of freedom, from the total χ^2", to obtain more powerful and appropriate test. The problem of an implementation of the idea of Fisher and Cochran was that decompositions of Pearson's sum and modified test statistics were not known at that time. Voinov [14] obtained explicitly a decomposition of Pearson-Fisher's and Dzhaparidze-Nikulin's statistics. A parametric decomposition of the NRR and HRM statistics were obtained by Voinov et al [15] explicitely.

Voinov and Pya [16] introduced new vector-valued goodness-of-fit tests (based, e.g., on above discussed components of statistics or on any combination of parametric and non-parametric tests) that in some cases provide a gain in power for specified alternatives.

2 Equivalence of the NRR and HRM tests

Consider the exponential family of distributions with density

$$f(x;\boldsymbol{\theta}) = h(x)\exp\left\{\sum_{j=1}^{s}\theta_j x^j + V(\boldsymbol{\theta})\right\},\ x \in \mathcal{X} \subseteq R^1, \tag{9}$$

where $\mathcal{X}$ is open in R^1, $\mathcal{X} = \{\mathrm{x} : \mathrm{f}(\mathrm{x};\boldsymbol{\theta}) > 0\}$, and $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subset R^s$. The family in (9) contains, e.g., such distributions as Poisson, exponential, normal, and many others. Assume that the support $\mathcal{X}$ does not depend on $\boldsymbol{\theta}$, the $s \times s$ matrix with elements

$$H_{ij} = -\frac{\partial^2 V(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j},\ i,j = 1,\ldots,s,$$

is positive definite on $\boldsymbol{\Theta}$, and the population moments $m_j(\boldsymbol{\theta}) = \mathbf{E}_{\boldsymbol{\theta}}(X^j)$, $j = 1,\ldots,s$, exist.

Differentiating the equality $\int_{x\in\mathcal{X}} \mathrm{f}(\mathrm{x};\boldsymbol{\theta})\mathrm{dx} = 1$ with respect to θ_j we get

$$\int\limits_{x\in\mathcal{X}} \frac{\partial f(x;\boldsymbol{\theta})}{\partial\theta_j}dx = m_j(\boldsymbol{\theta}) + \frac{\partial V(\boldsymbol{\theta})}{\partial\theta_j} = 0.$$

It follows that moments of the distribution in (9) are $m_j(\boldsymbol{\theta}) = -\frac{\partial V(\boldsymbol{\theta})}{\partial \theta_j}$, $j = 1, \ldots, s$.

Consider a HRM statistic

$$Y2_n^2(\bar{\boldsymbol{\theta}}_n) = X_n^2(\bar{\boldsymbol{\theta}}_n) + R_n^2(\bar{\boldsymbol{\theta}}_n) - Q_n^2(\bar{\boldsymbol{\theta}}_n), \tag{10}$$

where

$$R_n^2(\bar{\boldsymbol{\theta}}_n) = \mathbf{V}^{(n)T}(\bar{\boldsymbol{\theta}}_n)\mathbf{C}(\mathbf{V} - \mathbf{C}^T\mathbf{C})^{-1}\mathbf{C}^T\mathbf{V}^{(n)}(\bar{\boldsymbol{\theta}}_n), \tag{11}$$

$$Q_n^2(\bar{\boldsymbol{\theta}}_n) = \mathbf{V}^{(n)T}(\bar{\boldsymbol{\theta}}_n)\mathbf{A}(\mathbf{C} - \mathbf{B}\mathbf{K}^{-1}\mathbf{V})\mathbf{L}^{-1}(\mathbf{C} - \mathbf{B}\mathbf{K}^{-1}\mathbf{V})^T\mathbf{A}\mathbf{V}^{(n)}(\bar{\boldsymbol{\theta}}_n), \tag{12}$$

$\mathbf{C}$ is $r \times s$ matrix with elements

$$C_{jk}(\boldsymbol{\theta}) = \frac{1}{\sqrt{p_j(\boldsymbol{\theta})}}\left(\int\limits_{\Delta_j} x^k f(x;\boldsymbol{\theta})dx - p_j(\boldsymbol{\theta})m_k(\boldsymbol{\theta})\right), \tag{13}$$

$$j = 1, \ldots, r,\ k = 1, \ldots, s,$$

$\mathbf{V}$ is the $s \times s$ matrix with elements $V_{ij} = m_{ij}(\boldsymbol{\theta}) - m_i(\boldsymbol{\theta})m_j(\boldsymbol{\theta}),\ i, j = 1, \ldots, s$, matrix $\mathbf{A}$ being $\mathbf{A} = \mathbf{I} - \mathbf{q}\mathbf{q}^T + \mathbf{C}(\mathbf{V} - \mathbf{C}^T\mathbf{C})^{-1}\mathbf{C}^T$, $\mathbf{K}$ is a $s \times s$ matrix with elements $K_{ij} = \int x^i \partial f(x;\boldsymbol{\theta})/\partial\theta_j dx,\ i, j = 1, \ldots, s$, and $\mathbf{L}$ is $\mathbf{V} + (\mathbf{C} - \mathbf{B}\mathbf{K}^{-1}\mathbf{V})^T\mathbf{A}(\mathbf{C} - \mathbf{B}\mathbf{K}^{-1}\mathbf{V})$.

In 2004 Voinov and Pya [17] proved the following

Theorem 1. *Assume conditions stated above hold. Then, for the exponential family of distributions in* (9) *the NRR* $Y1_n^2(\hat{\boldsymbol{\theta}}_n)$ *and HRM* $Y2_n^2(\bar{\boldsymbol{\theta}}_n)$ *statistics are identical.*

Proof. The complete proof of the identity $Y1_n^2(\hat{\boldsymbol{\theta}}_n) \equiv Y2_n^2(\bar{\boldsymbol{\theta}}_n)$ is given, e.g., in [18], p. 102.

Since for the exponential family of distributions in (9) MMEs can typically be constructed easier than MLEs, then due to the identity $Y1_n^2(\hat{\boldsymbol{\theta}}_n) \equiv Y2_n^2(\bar{\boldsymbol{\theta}}_n)$ one may use the statistic in (4) or in (5) that computationally is much simpler than to use the statistic in (10) with too many matrices manipulations.

3 Components of chi-squared tests

Decompositions of different chi-squared type tests that are important for constructing the most power tests are based on the specific linear transformation described, e.g. in [18], p.14.

Let $\mathbf{Z} = (Z_1, \ldots, Z_r)^T$ be a random vector such that $\mathbf{EZ} = \mathbf{0}$, $\mathbf{E}(\mathbf{Z}\mathbf{Z}^T) = \mathbf{D} = (d_{ij})$, the rank $R(\mathbf{D})$ of $\mathbf{D}$ being $k \le r$. Denote

$$\mathbf{Z}_{(i)} = (Z_1, \ldots, Z_i)^T,\ \mathbf{D}_i = \mathbf{E}(\mathbf{Z}_{(i)}\mathbf{Z}_{(i)}^T),$$

$$\mathbf{d}_{i(j)} = \mathbf{Cov}(Z_i, \mathbf{Z}_{(j)}),\ \mathbf{d}_{(i)j} = \mathbf{Cov}(\mathbf{Z}_{(i)}, Z_j),\ i, j = 1, \ldots, k.$$

Consider the linearly transformed vector $\boldsymbol{\delta}_{(t)} = (\delta_1, \ldots, \delta_t)^T$ with components

$$\delta_i = \frac{1}{\sqrt{|\mathbf{D}_{i-1}||\mathbf{D}_i|}} \left[\begin{array}{cc} \mathbf{D}_{i-1} & \mathbf{Z}_{(i-1)} \\ \mathbf{d}_{i(i-1)}^T & Z_i \end{array} \right],\ i = 1, \ldots, t,\ t = 1, \ldots, R(\mathbf{D}). \quad (14)$$

The components of $\boldsymbol{\delta}_{(t)}$ are all normalized and uncorrelated, i.e. $\mathbf{E}\boldsymbol{\delta}_{(t)} = \mathbf{0}$, $\mathbf{E}\left\{\boldsymbol{\delta}_{(t)}\boldsymbol{\delta}_{(t)}{}^T\right\} = \mathbf{I}_t$, where $\mathbf{I}_t$ is the $t \times t$ identity matrix, and $\mathbf{E}\delta_i\delta_j = 0,\ i \neq j$.

Theorem 2. *The following decomposition of the quadratic form* $\mathbf{Z}_{(t)}^T \mathbf{D}_t^{-1} \mathbf{Z}_{(t)}$ *holds:*

$$\mathbf{Z}_{(t)}^T \mathbf{D}_t^{-1} \mathbf{Z}_{(t)} = \delta_1^2 + \cdots + \delta_t^2,\ t = 1, \ldots, R(\mathbf{D}). \quad (15)$$

Corollary 1. *Let a* $r \times r$ *matrix* $\mathbf{D}$ *be non-negative definite of rank* k. *Then*

$$\mathbf{R}\mathbf{D}\mathbf{R}^T = \mathbf{I}_k,\ \mathbf{R}^T\mathbf{R} = \left(\begin{array}{cc} \mathbf{D}_k^{-1} & 0 \\ \mathbf{0}^T & 0 \end{array} \right) = \mathbf{D}^-,$$

where $\mathbf{R} = (\mathbf{R}_k \vdots \mathbf{0})$, *and* $\mathbf{R}_k$ *is a lower triangular matrix with elements*

$$r_{ii} = (|\mathbf{D}_{i-1}|/|\mathbf{D}_i|)^{1/2},\ i = 1, \ldots, k,$$

$$r_{ij} = -r_{ii}\mathbf{d}_{i(i-1)}^T(\mathbf{D}_{i-1}^{-1})_j,\ j = 1, \ldots, i-1. \quad (16)$$

In (16) $(\mathbf{D}_i)_j$ denotes the j-th column of the leading sub-matrix of order $i \times i$ of the matrix $\mathbf{D}$.

The detailed proof of these important results is given, e.g., in the book [18], pp. 15–16.

The explicit decompositions of PF (2), DN (3), NRR (5), and HRM (8) tests are all based on the Theorem 2. Decompositions of PF and DN statistics obtained in [14] can be formulated as follows.

Theorem 3. *Under the usual regularity conditions, the following decomposition of the DN statistic holds:*

$$U_n^2(\tilde{\boldsymbol{\theta}}_n) = \delta_1^2(\tilde{\boldsymbol{\theta}}_n) + \cdots + \delta_{r-s-1}^2(\tilde{\boldsymbol{\theta}}_n), \tag{17}$$

where components $\delta_i(\tilde{\boldsymbol{\theta}}_n)$ are independently distributed as χ_1^2 in the limit.

The explicit formulas for $\delta_i(\tilde{\boldsymbol{\theta}}_n)$ are provided, e.g., in [18], p.23.

The PF and DN statistics are asymptotically equivalent and possess the same decompositions. The only difference is that the DN test uses any $\sqrt{n}$-consistent estimators based on the raw data, and the PF test uses estimators (e.g., MLEs $\hat{\boldsymbol{\theta}}_n$) based on grouped data. That is why the following decomposition of the PF test holds:

$$X_n^2(\hat{\boldsymbol{\theta}}_n) = \delta_1^2(\hat{\boldsymbol{\theta}}_n) + \cdots + \delta_{r-s-1}^2(\hat{\boldsymbol{\theta}}_n).$$

Theorem 1 permits to decompose the NRR and the HRM quadratic forms as

$$Y1_n^2(\hat{\boldsymbol{\theta}}_n) = \delta_1^2(\hat{\boldsymbol{\theta}}_n) + \cdots + \delta_{r-1}^2(\hat{\boldsymbol{\theta}}_n) \tag{18}$$

and

$$Y2_n^2(\bar{\boldsymbol{\theta}}_n) = \delta_1^2(\bar{\boldsymbol{\theta}}_n) + \cdots + \delta_{r-1}^2(\bar{\boldsymbol{\theta}}_n) \tag{19}$$

respectively. Explicit formulas for $\delta_i^2(\hat{\boldsymbol{\theta}}_n)$ and $\delta_i^2(\bar{\boldsymbol{\theta}}_n)$ are given, e.g., in [18], pp. 38, and 100.

Decompositions (18) and (19) show that statistics $Y1_n^2(\hat{\boldsymbol{\theta}}_n)$ and $Y2_n^2(\bar{\boldsymbol{\theta}}_n)$ possess in the limit the chi-squared distribution with $r-1$ degrees of freedom. These facts can be considered as alternative proofs of the well-known results obtained previously by other means.

Another important feature of decompositions of test statistics is that they permit to realize the idea of Cochran [13] (see Section 1). Some numerical examples are given in [14].

4 Vector-valued tests

Vector-valued goodness-of-fit tests were introduced by Voinov and Pya [16]. It was shown that such tests may possess noticeably higher power than that of components of a vector implemented independently.

Consider the following artificial example of testing a simple null hypothesis versus a simple alternative:

$$H_0 : P(X \leq x) = F(x),\ H_a : P(X \leq x) = G(x).$$

Let Y_{1n}^2 and Y_{2n}^2 be two independent statistics such that in the limit

$$P(Y_{1n}^2 \leq y|H_0) = P(Y_{2n}^2 \leq y|H_0) = P(\chi_4^2 \leq y),$$

$$P(Y_{1n}^2 \leq y|H_a) = P(Y_{2n}^2 \leq y|H_a) = P(\chi_4^2(3.5) \leq y),$$

where χ_4^2 is the central chi-square random variable with 4 degrees of freedom, and $\chi_4^2(3.5)$ is the non-central chi-square random variable with 4 degrees of freedom and non-centrality parameter of 3.5.

Consider a two-dimensional vector-valued test $\mathbf{V}_n = (Y_{1n}^2, Y_{2n}^2)^T$ with the rejection region S_1 of the intersection-type, i.e. $S_1 = (Y_{1n}^2 > y_1) \cap (Y_{2n}^2 > y_1)$. Since Y_{1n}^2 and Y_{2n}^2 are independent and identically distributed, the probability of falling into S_1 under H_0 will be

$$P\left\{(Y_{1n}^2 > y_1|H_0) \cap (Y_{2n}^2 > y_1|H_0)\right\} =$$

$$= P(Y_{1n}^2 > y_1|H_0)P(Y_{2n}^2 > y_1|H_0) = \alpha_1^2 = \alpha,$$

where α_1 is the level of significance of any component $Y_{in}^2,\ i = 1, 2$, and α is the size (or type I error) of the vector-valued test $\mathbf{V}_n$. Assume we wish to use $\alpha = 0.05$, then $\alpha_1 = 0.2236$, in which case the critical value y_1 of χ_4^2 random variable will be $y_1 = 5.69$. In this case, the power of the vector-valued test $\mathbf{V}_n$ is determined as

$$P(\mathbf{V}_n \in S_1|H_a) = P(Y_{1n}^2 > y_1|H_a)P(Y_{2n}^2 > y_1|H_a) = 0.343.$$

At the same time, power of each component of $\mathbf{V}_n$ for the same level of significance $\alpha = 0.05$ is

$$P(Y_{1n}^2 > y_2|H_a) = P(Y_{2n}^2 > y_2|H_a) = 0.282,$$

where $y_2 = 9.49$. Thus, we see that the power of $\mathbf{V}_n$ is 1.216 times more than that of the components Y_{in}^2, $i = 1, 2$, implemented independently.

Consider the vector-valued test $\mathbf{V}_n = (Y_{1n}^2, Y_{2n}^2)^T$ with the rejection region S_2 of the union-type, i.e. $S_2 = (Y_{1n}^2 > y_3) \cup (Y_{2n}^2 > y_3)$. Suppose we set again $\alpha = 0.05$. Then

$$P(\mathbf{V}_n \in S_2|H_0) = P\left\{(Y_{1n}^2 > y_3|H_0) \cup (Y_{2n}^2 > y_3|H_0)\right\}$$

$$= P(Y_{1n}^2 > y_3|H_0) + P(Y_{2n}^2 > y_3|H_0) - P(Y_{1n}^2 > y_3|H_0)P(Y_{2n}^2 > y_3|H_0)$$

$$= 2\alpha_2 - \alpha_2^2 = 0.05,$$

which means $P(Y_{in}^2 > y_3|H_0) = \alpha_2 = 0.02532$, and so $y_3 = 11.1132$. Since $P(Y_{in}^2 > 11.1132|H_a) = 0.19525$, the power of $\mathbf{V}_n$ for the rejection region of union-type is

$$P(\mathbf{V}_n \in S_2|H_a) = P\left\{(Y_{1n}^2 > y_3|H_a) \cup (Y_{2n}^2 > y_3|H_a)\right\}$$

$$= P(Y_{1n}^2 > y_3|H_a) + P(Y_{2n}^2 > y_3|H_a) - P(Y_{1n}^2 > y_3|H_a)P(Y_{2n}^2 > y_3|H_a) = 0.352,$$

which is 1.25 times more than the power of Y_{in}^2, $i = 1, 2$, implemented individually.

This simple artificial example shows that the use of a two-dimensional vector-valued test may result in an increase in power as compared to the power of individual components of the vector-valued statistic. This effect becomes even stronger for three-dimensional vector $\mathbf{V}_n = (Y_{1n}^2, Y_{2n}^2, Y_{3n}^2)^T$. For example, the power of $\mathbf{V}_n$ for the rejection region of union-type is 1.415 times more than the power of Y_{in}^2, $i = 1, 2, 3$, implemented individually. Voinov and Pya [16] showed also that different parametric and non-parametric statistics can be combined in a vector.

Simulation studies showed that, when combining either correlated or uncorrelated nonparametric or parametric tests with approximately the same power, vector-valued tests may gain power when compared with the power of components. Examples show that the power of vector-valued goodness-of-fit tests depends on the structure of the rejection region, correlation between the components of the test, and dimensionality of the vector. Unfortunately, by the date there is no theory of vector-valued tests, and, hence, there are no recommendations on optimal structure of tests.

5 Wald's method corrected

The general Wald's method for constructing modified chi-squared goodness-of-fit tests has been elaborated by Moore (Moore [19], Moore and Spruill [20], and Moore [11]). In 2013 Voinov [21] discovered a serious mistake in that theory showing that Wald's type statistics will follow the central chi-squared distribution if and only if the limit covariance matrix of standardized frequencies will not depend on unknown parameters. In particular, it was shown that the goodness-of-fit statistic developed by Moore and Stubblebine [22] does not follow the chi-squared limit distribution, and, hence, cannot be used for testing multivariate normality.

The Moore's theory can be briefly summarized as follows. Let $X_1, \ldots, X_n$ be independent identically distributed random variables, and we intend to test the composite null hypothesis H_0 that the distribution function of X_i belongs to a parametric family of $F(x, \boldsymbol{\theta})$ continuous distribution functions, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_s)^T \in \boldsymbol{\Theta} \subset R^s$. Chi-squared tests of fit for H_0 are based on frequencies N_j^n, the number of observed values of $X_1, \ldots, X_n$ that fall into r intervals $\mathbf{I}_j$ such that $\mathbf{I}_i \cap \mathbf{I}_j = \emptyset$ for $i \neq j$ and $\mathbf{I}_1 \cup \cdots \cup \mathbf{I}_r = R^1$, $i, j = 1, \ldots, r$. Since expected probabilities $p_i(\boldsymbol{\theta}) = \int_{\mathbf{I}_i} dF(x, \boldsymbol{\theta})$ to fall into an i-th interval depend on unknown parameter $\boldsymbol{\theta}$, it can be estimated from data by any $\sqrt{n}$-consistent estimator $\boldsymbol{\theta}_n = \boldsymbol{\theta}_n(X_1, \ldots, X_n)$. Denote $\mathbf{V}^{(n)}(\boldsymbol{\theta})$ the r-vector of standardized frequencies with components $V_i^{(n)} = [N_i^{(n)} - np_i(\boldsymbol{\theta}_n)]/[np_i(\boldsymbol{\theta}_n)]^{1/2}$, $i = 1, \ldots, r$. Moore and Spruill [20] showed that the vector $\mathbf{V}^{(n)}(\boldsymbol{\theta}_n)$ asymptotically follows a multivariate normal distribution $N_r(\mathbf{0}, \boldsymbol{\Sigma})$, where $\mathbf{0}$ is a zero r-vector and $\boldsymbol{\Sigma}$ is a positive definite covariance matrix. The covariance matrix $\boldsymbol{\Sigma}$ essentially depends on the way of the parameter $\boldsymbol{\theta}$ estimation. Wald's [23] idea was as follows. Let $\mathbf{X}$ possess the $N_r(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution and is nonsingular of rank r, then the quadratic form $(\mathbf{X} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{X} - \boldsymbol{\mu})$ will be distributed in the limit as chi-squared with r degrees of freedom, χ_r^2 . Nikulin [7] was first who used this idea, standardized frequencies, and MLE $\hat{\boldsymbol{\theta}}_n$ of $\boldsymbol{\theta}$ for testing univariate normality when the matrix $\boldsymbol{\Sigma}$ is singular of rank $r - 1$. At this point it is of importance to note that in this case the limit covariance matrix of standardized frequencies is a certain matrix that does not depend on parameters. Moore [11] generalized Wald's approach and developed the method of the construction of modified chi-squared goodness-of-fit tests valid for any $\sqrt{n}$-consistent estimator $\boldsymbol{\theta}_n$.

The Moore's [11], p.134, theory is based on the

THEOREM 4. *Assume that the limit distribution of* $\mathbf{V}^{(n)}(\boldsymbol{\theta}_n)$ *follows* $N_r(\mathbf{0},\boldsymbol{\Sigma})$ *with rank of* $\boldsymbol{\Sigma}$ *being* $r-1$*, then the Wald's method statistic* $Y_n(\boldsymbol{\theta}_n) = \mathbf{V}^{(n)T}(\boldsymbol{\theta}_n)\boldsymbol{\Sigma}_n^{-}\mathbf{V}^{(n)}(\boldsymbol{\theta}_n)$ *is invariant under the choice of matrix generalized inverse* $\boldsymbol{\Sigma}_n^{-}$*. If* H_0 *is true, the* $Y_n(\boldsymbol{\theta}_n)$ *will follow* χ^2_{r-1} *distribution in the limit.*

In the proof of this theorem Moore [11], p.132, referred to his lemmas 1(b) and 2(a). Consider the Lemma 1(b) of Moore [11] in more detail. The Khatri's Lemma 9 (Khatri [24], and Rao [25]) is formulated as below.

LEMMA 1. *Let* $\mathbf{y} : n \times 1$ *be normal with mean zero and the covariance matrix* $\sigma^2\mathbf{B}$ *and let the rank of* $\mathbf{B}$ *be* r*. Then, a necessary and sufficient condition for* $\mathbf{y}^T\mathbf{G}\mathbf{y}/\sigma^2$ *to be distributed as* χ^2_r *is* $\mathbf{G} = \mathbf{B}^{-}$*, where* $\mathbf{G}$ *is a symmetric matrix.*

It is of great importance to note that in this Lemma it is assumed that the matrix $\mathbf{B}$ is certain, do not depending on unknowns. To apply this Lemma for $\mathbf{y} = \mathbf{V}^{(n)}(\boldsymbol{\theta}_n)$ we have to set $\sigma^2 = 1$. From this it follows that the limit covariance matrix of the vector of standardized frequencies should not depend on $\boldsymbol{\theta}$. It follows also that in the Theorem 4 of Moore [11] we have to add a necessary additional condition that the matrix $\boldsymbol{\Sigma}$ does not depend on $\boldsymbol{\theta}$ and formulate it as

THEOREM 5. *Assume that the limit distribution of* $\mathbf{V}^{(n)}(\boldsymbol{\theta}_n)$ *follows* $N_r(\mathbf{0},\boldsymbol{\Sigma})$ *with rank of* $\boldsymbol{\Sigma}$ *being* $r-1$*, that* **the matrix $\boldsymbol{\Sigma}$ does not depend on $\boldsymbol{\theta}$***, then the Wald's method statistic* $Y_n(\boldsymbol{\theta}_n) = \mathbf{V}^{(n)T}(\boldsymbol{\theta}_n)\boldsymbol{\Sigma}^{-}\mathbf{V}^{(n)}(\boldsymbol{\theta}_n)$ *is invariant under choice of* $\boldsymbol{\Sigma}^{-}$*. If* H_0 *is true, the* $Y_n(\boldsymbol{\theta}_n)$ *will follow* χ^2_{r-1} *in the limit.*

Proof. If one takes into account the necessary bold faced additional condition, then the Theorem 5 immediately follows from Lemmas 1 and 2(a) of Moore [11], p. 132.

Several examples of the application of the Theorem 5 for testing the univariate null hypotheses, when the additional condition of Theorem 5 is hold automatically, have been given in Voinov [21]. The serious problem aroused in an attempt to apply the NRR Wald's type test for multivariate normality developed by Moore and Stubblebine [22]. An inspection showed that, for example, in two-dimensional case the limit covariance matrix $\mathbf{\Sigma} = \mathbf{I} - \mathbf{q}\mathbf{q}^T - \mathbf{B}\mathbf{J}^{-1}\mathbf{B}^T$ of the standardized frequencies $\mathbf{V}_n(\hat{\boldsymbol{\theta}}_n)$ for equiprobable cells can be written down as $\mathbf{\Sigma} = \mathbf{I} - \mathbf{q}\mathbf{q}^T - \tilde{\mathbf{B}}\mathbf{Q}\tilde{\mathbf{B}}^T$, where $\mathbf{q}$ is the r-vector with components $1/\sqrt{r}$, and

$$\tilde{\mathbf{B}}\mathbf{Q}\tilde{\mathbf{B}}^T = \frac{r[4\sigma_{11}^2\sigma_{22}^2 - 3\sigma_{11}\sigma_{12}^2\sigma_{22} + \sigma_{12}^4]}{(\sigma_{11}\sigma_{22} - \sigma_{12}^2)^2} \begin{pmatrix} d_1^2 & d_1 d_2 \cdots & d_1 d_r \\ d_2 d_1 & d_2^2 \cdots & d_2 d_r \\ \cdots & \cdots & \cdots \\ d_r d_1 & d_r d_2 & d_r^2 \end{pmatrix},$$

where $d_i,\ i = 1, \ldots, r$, are constants that depend only on ends of grouping cells. From the last equation we see that the limit covariance matrix $\mathbf{\Sigma} = \mathbf{I} - \mathbf{q}\mathbf{q}^T - \mathbf{B}\mathbf{J}^{-1}\mathbf{B}^T$ of the standardized frequencies $\mathbf{V}_n(\hat{\boldsymbol{\theta}}_n)$ for equiprobable cells depends on unknown parameters of $\mathbf{\Sigma}$, and, hence, due to the Theorem 5, the limit distribution of the NRR statistic $Y_n(\hat{\boldsymbol{\theta}}_n) = \mathbf{V}^{(n)T}(\hat{\boldsymbol{\theta}}_n)\mathbf{\Sigma}_n^- \mathbf{V}^{(n)}(\hat{\boldsymbol{\theta}}_n)$, where $\mathbf{\Sigma}_n^-$ is the estimator of $\mathbf{\Sigma}^-$, cannot follow in the limit the chi-squared probability distribution χ^2_{r-1}, will depend on unknown parameters, and, hence, cannot be used for testing in principle. A simulation study in [21] fully confirmed this. From the above it became clear why Moore and Stubblebine [22], p.723, wrote that it is computationally complicated to use NRR test for MVN (multivariate normality) in practice.

6 New tests for the two-dimensional circular and general multivariate normality

McCulloch [12] showed that a modified chi-squared test can be represented as a sum of DN test U_n^2 and asymptotically independent on it S_n^2 statistic. He noted also that S_n^2 test for equiprobable grouping cells will possess higher power than the initial modified test. Using a simulation study, Voinov et al [5] showed that this effect works for many univariate tests. It was naturally to expect that this rule will also work for multivariate tests.

Following Moore and Stubblebine [22] consider testing for bivariate circular normality. The hypothesized probability density function is

$$f(x,y|\boldsymbol{\theta}) = (2\pi\sigma^2)^{-1} \exp\left\{-\frac{1}{2\sigma^2}\left[(x-\mu_1)^2+(x-\mu_2)^2\right]\right\}, \tag{20}$$

where $\boldsymbol{\theta} = (\mu_1, \mu_2, \sigma^2)^T$. Using a random sample $(X_1, Y_1), ..., (X_n, Y_n)$, the MLEs of μ_1, μ_2, and σ^2 can be obtained as $\bar{X} = \frac{1}{n}\sum_{j-1}^{n} X_j$, $\bar{Y} = \frac{1}{n}\sum_{j-1}^{n} Y_j$, and

$$S^2 = \frac{1}{2n}\left\{\sum_{j=1}^{n}(X_j-\bar{X})^2 + \sum_{j=1}^{n}(Y_j-\bar{Y})^2\right\}.$$

If $c_i = -2\log(1-i/r),\ i = 1,\dots,r-1,\ c_r = +\infty$, then the probability to fall into a cell will be $\hat{p}_{in} = 1/r$.

Denoting

$$\mathbf{V}_n = \left(\frac{(N_{1n}-n/r)}{\sqrt{n/r}},\dots,\frac{(N_{rn}-n/r)}{\sqrt{n/r}}\right)^T = (\tilde{N}_1,\dots,\tilde{N}_r)^T,$$

where $N_{jn},\ j = 1,\dots,r$, is the number of distances $\left[(X_i-\bar{X})^2 + (Y_i - \bar{Y})^2\right]/S^2$ that fall into the interval $[c_{j-1}, c_j), j = 1,\dots,r$, the NRR statistic is easily derived as (see, e.g., [18], p.42)

$$Y_n^2 = \sum_{i=1}^{r}\tilde{N}_i^2 + \frac{r}{4 - r\sum\limits_{i=1}^{r}\nu_i^2}\left(\sum_{i=1}^{r}\tilde{N}_i\nu_i\right)^2, \tag{21}$$

where $\nu_i = 2\left[\left(1-\frac{i}{r}\right)\log\left(1-\frac{i}{r}\right) - \left(1-\frac{i-1}{r}\right)\log\left(1-\frac{i-1}{r}\right)\right],\ i = 1,\dots,r$. In this case the DN statistic is

$$U_n^2 = \sum_{i=1}^{r}\tilde{N}_i^2 - \left(\sum_{i=1}^{r}\tilde{N}_i\nu_i\right)^2 / \sum_{i=1}^{r}\nu_i^2. \tag{22}$$

Subtracting U_n^2 from Y_n^2 we get the new test for the two-dimensional circular normality as

$$S_n^2 = \frac{4\left(\sum\limits_{i=1}^{r}\tilde{N}_i\nu_i\right)^2}{\left(4 - r\sum\limits_{i=1}^{r}\nu_i^2\right)\sum\limits_{i=1}^{r}\nu_i^2}. \tag{23}$$

Our simulation study shows that for the null hypothesis in (20), and the alternative to be the two-dimensional standard logistic distribution with independent components, the power of S_n^2 test in (23) almost does not depend on r, and is noticeably higher than the power of Y_n^2 in (21). At the same time we noticed that the DN test in (22) possesses almost no power.

Consider now a construction of a Wald's type chi-squared goodness-of-fit tests for the composite null hypothesis that a set $\mathbf{X}_1, \ldots, \mathbf{X}_n$ of n independent identically distributed (iid) p-dimensional random vectors does not contradict the following joint probability density function

$$f(\mathbf{x}|\boldsymbol{\theta}) = (2\pi)^{-p/2}|\boldsymbol{\Sigma}|^{-1/2} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right], \tag{24}$$

where $\boldsymbol{\mu}$ is a p-vector of means and $\boldsymbol{\Sigma}$ is a positive definite $p \times p$ covariance matrix. Let a hypothesized vector of unknown parameters be

$$\boldsymbol{\theta} = (\mu_1, \ldots, \mu_p, \sigma_{11}, \sigma_{12}, \sigma_{22}, \ldots, \sigma_{1j}, \sigma_{2j}, \ldots, \sigma_{jj}, \ldots, \sigma_{pp})^T.$$

The MLE $\hat{\boldsymbol{\theta}}_n$ of $\boldsymbol{\theta}$ is the vector $(\bar{\mathbf{X}}, \mathbf{S})^T$, where $\bar{\mathbf{X}}$ is the vector of sample means, and $\mathbf{S}$ is the covariance matrix. Given $\hat{\boldsymbol{\theta}}_n$, and constants $0 \le c_0 < \cdots < c_r = \infty$, where c_i are i/r points of $\chi^2(p)$ distribution, the r equiprobable $(p_{in}(\hat{\boldsymbol{\theta}}_n) = 1/r)$ grouping cells were defined by Moore and Stubblebine [22] as

$$E_{in}(\hat{\boldsymbol{\theta}}_n) = \left\{\mathbf{X} \in R^p : c_{i-1} \le (\mathbf{X}-\bar{\mathbf{X}})^T\mathbf{S}^{-1}(\mathbf{X}-\bar{\mathbf{X}}) < c_i\right\},\ i = 1, \ldots, r.$$

It was also shown in [22], p.720, that the limit covariance matrix of the vector $\mathbf{V}_n(\hat{\boldsymbol{\theta}}_n)$ of standardized frequencies with components $V_i = (N_{in} - n/r)/\sqrt{n/r}$, where observed frequency N_{in} is the number of random vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$ falling into $E_{in}(\hat{\boldsymbol{\theta}}_n),\ i = 1, \ldots, r$, is $\boldsymbol{\Sigma}_1 = \mathbf{I} - \mathbf{q}\mathbf{q}^T - \mathbf{B}\mathbf{J}^{-1}\mathbf{B}^T$. In this expression $\mathbf{B}$ is the $r \times m$ matrix with elements $B_{ij} = (p_i(\boldsymbol{\theta}))^{-1/2}\partial p_i(\boldsymbol{\theta})/\partial\theta_j,\ i = 1, \ldots, r,\ j = 1, \ldots, m$, $\mathbf{q}$ is r-vector with entries $1/\sqrt{r}$, $m = p + p(p+1)/2$, and $\mathbf{J} = \begin{pmatrix} \boldsymbol{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^{-1} \end{pmatrix}$ is the Fisher information matrix for one observation. Explicit expressions for the matrix $\mathbf{Q}$ are given in [26]. Using $\boldsymbol{\Sigma}_1$ and the Theorem 4 of Section 5 Moore and Stubblebine [22], p.722, derived the NRR statistic as

$$Y_n^2(\hat{\boldsymbol{\theta}}_n) = \mathbf{V}_n^T(\hat{\boldsymbol{\theta}}_n)\mathbf{V}_n(\hat{\boldsymbol{\theta}}_n) + \mathbf{V}_n^T(\hat{\boldsymbol{\theta}}_n)\mathbf{B}_n(\mathbf{J}_n - \mathbf{B}_n^T\mathbf{B}_n)^{-1}\mathbf{B}_n^T\mathbf{V}_n(\hat{\boldsymbol{\theta}}_n), \tag{25}$$

where $\mathbf{B}_n$ and $\mathbf{J}_n$ are MLEs of $\mathbf{B}$ and $\mathbf{J}$. Based on incorrect Theorem 4 with the necessary condition omitted (see Section 5) Moore and Stubblebine [22], p.722, erroneously decided that the NRR test in (25) will be invariant and distributed in the limit as χ^2_{r-1}. In Section 5 we have shown that in two-dimensional case the statistic in (25) will be chi-squared distributed only for a diagonal matrix $\mathbf{\Sigma}$ of the null hypothesis. Our intensive simulation study showed that for any dimensionality p the statistic in (25) will be chi-squared distributed in the limit if and only if $\mathbf{\Sigma}$ is a diagonal matrix.

We succeeded in deriving explicit expressions for NRR Y_n^2, DN U_n^2, and S_n^2 if $\mathbf{\Sigma}$ is a diagonal matrix of any dimensionality. They are:

$$Y_n^2 = \sum V_i^2 + \frac{2pr(\sum V_i d_i)^2}{1 - 2pr\sum d_i^2}, \tag{26}$$

$$U_n^2 = \sum V_i^2 - \frac{1}{\sum d_i^2}(\sum V_i d_i)^2, \tag{27}$$

and

$$S_n^2 = \frac{(\sum V_i d_i)^2}{(1 - 2pr\sum d_i^2)\sum d_i^2}, \tag{28}$$

if $d_i = (c_{i-1}^{p/2} e^{-c_{i-1}/2} - c_i^{p/2} e^{-c_i/2}) b_p/2$, where $b_p = [p(p-2)\dots(4)(2)]^{-1}$ if p is even, and $b_p = (2/\pi)^{1/2}[p(p-2)\dots(5)(3)]^{-1}$ if p is odd (see [22], p.720).

The above results for testing the MVN null hypothesis with a diagonal covariance matrix $\mathbf{\Sigma}$ suggest the following procedure: 1) produce the Karhunen-Loéve transformation of a sample data that will diagonalize a sample covariance matrix, and 2) use the statistics Y_n^2, U_n^2, and S_n^2 as defined in formulas (26), (27), and (28). Let $\mathbf{\Phi} = [\mathbf{e}_1\ \mathbf{e}_2 \cdots \mathbf{e}_p]$ be a matrix whose columns $\mathbf{e}_1, \mathbf{e}_2, \cdots \mathbf{e}_p$ are orthogonal normalized eigen-vectors of $\mathbf{S}$, then the Karhunen-Loéve transformation will be $\mathbf{Y}_i = \mathbf{\Phi}^T \mathbf{X}_i,\ i = 1, \dots, n$. From this it follows that tests in (26), (27), and (28) with frequencies $N_{in},\ i = 1, \dots, n$, defined by the number of observed vectors $\mathbf{Y}_1, \dots, \mathbf{Y}_n$ that will fall into intervals $\tilde{E}_{in}(\hat{\boldsymbol{\theta}}_n) = \{\mathbf{Y} \in R^p : c_{i-1} \le (\mathbf{Y} - \bar{\mathbf{Y}})^T \mathbf{S}_y^{-1} (\mathbf{Y} - \bar{\mathbf{Y}}) < c_i\},\ i = 1, \dots, r$, where $\mathbf{S}_y$ is the sample covariance matrix of $\mathbf{Y}_1, \dots, \mathbf{Y}_n$, can be used. So are new tests for MVN suggested.

A detailed derivation of the explicit expressions (26), (27), and (28) for the new tests is given in [26]. Since the limit covariance matrix of $\mathbf{V}_n(\hat{\boldsymbol{\theta}}_n)$ does not depend on unknown parameters of the hypothetical null MVN distribution, then from the Theorem 5 of Section 5 it immediately follows that statistics (26), (27), and (28) are invariant and, hence, can be used for hypotheses testing.

From the theory of quadratic forms it is known that if $\mathbf{Y} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ is a positive definite matrix, then the two quadratic forms $\mathbf{Y}^T\mathbf{AY}$ and $\mathbf{Y}^T\mathbf{BY}$ are distributed independently if $\mathbf{A\Sigma B} = \mathbf{0}$. Since asymptotically $\mathbf{V}_n(\hat{\boldsymbol{\theta}}_n) \sim N(\mathbf{0}, \mathbf{I} - \mathbf{BJ}^{-1}\mathbf{B}^T)$, and

$$\{\mathbf{I} - \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-}\mathbf{B}^T\}\{\mathbf{I} - \mathbf{BJ}^{-1}\mathbf{B}^T\}\{\mathbf{B}[(\mathbf{J} - \mathbf{B}^T\mathbf{B})^{-1} + (\mathbf{B}^T\mathbf{B})^{-}]\mathbf{B}^T\} = \mathbf{0},$$

then independence of U_n^2 and S_n^2 follows, and, hence one may use S_n^2 on its own right. The detailed derivation of this result is given in [26].

In [26] we have shown also that under any alternative $\lim\limits_{n\to\infty} P(Y_n^2 > \chi^2_{\alpha,r-1}) = 1$, where $\chi^2_{\alpha,r-1}$ is a critical point of the chi-square distribution with $r-1$ df (degrees of freedom). In other words, the probability to fall into rejection region under any alternative tends to one, if the sample size increases unboundedly. From this it follows consistency of Y_n^2. Consistency of U_n^2 and S_n^2 is proved analogously.

An intensive simulation study conducted in [26] showed that (at least with respect to Pearson Type II alternative distribution, Student t with 10 df, Khinchine, and several mixtures of normal multivariate distributions) the power of S_n^2, like in the univariate case, is significantly higher than that of U_n^2 and is noticeably higher than that of Y_n^2. It was shown also that the power of S_n^2 is comparable with the power of the well-known tests of Henze and Zirkler [27], Royston [28], Székely and Rizzo [29], and Doornik and Hansen [30].

At the same time it has to be noted that the application of S_n^2 test is much simpler than implementations of tests introduced in [27-30]. This simplicity is evident, because we have a simple explicit expression for S_n^2 statistic that follows in the limit the well known chi-squared distribution with one degree of freedom, and so there is no need to conduct simulations to define critical values of tests.

It has to be added also that the test S_n^2 is much more stable as compared to tests in [27–30], because the variance of it, which is 2, is always smaller than that of other tests.

7 21-st century's misusing of the classical Pearson and power-divergence tests

Moore and Stubblebine [22] considered a possibility to use the classical Pearson's test for testing the multivariate normality (MVN). Using the fundamental result of Chernoff and Lehmann [6] they show that under the null hypothesis of MVN the Pearson's statistic with parameters estimated by MLEs based on raw data for equiprobable fixed grouping cells will follow the weighted sum of two independent chi-squared random variables with $r-2$ and 1 degree of freedom, where r stands for the number of equiprobable fixed grouping cells. This result has not been taken into consideration by Cardoso De Oliveira and Ferreira [31], and Batsidis, Martin, Pardo, Zografos [32], who consider tests for MVN based on unbiased estimates that are asymptotically equivalent to MLEs. The authors of those papers erroneously decided that their tests approximately follow the chi-squared distribution with $r-1$ degrees of freedom in asymptotic.

Consider a construction of a chi-squared type goodness-of-fit test for the composite null hypothesis that a set $\mathbf{X}_1,\ldots,\mathbf{X}_n$ of n iid (independent identically distributed) p-dimensional random vectors does not contradict the joint probability density function (24). The unbiased estimator $\hat{\boldsymbol{\theta}}_n$ of $\boldsymbol{\theta}$ is the vector $(\bar{\mathbf{X}},\mathbf{S})^T$, where $\bar{\mathbf{X}}=\sum_{j-1}^{n}X_j/n$, and $\mathbf{S}=\sum_{j-1}^{n}(\mathbf{X}_j-\bar{\mathbf{X}})(\mathbf{X}_j-\bar{\mathbf{X}})^T/(n-1)$.

Cardoso De Oliveira and Ferreira [31] considered the squared radii

$$r_j^2=(\mathbf{X}_j-\bar{\mathbf{X}})^T\mathbf{S}^{-1}(\mathbf{X}_j-\bar{\mathbf{X}}),\ j=1,\ldots,n.$$

It is known that if $\mathbf{X}_j$ follows (24), then statistics

$$b(\mathbf{X}_j)=\frac{n}{(n-1)^2}r_j^2,\ j=1,\ldots,n, \tag{29}$$

follow the beta-distribution with parameters $p/2$ and $(n-p-1)/2$ (Gnanadesikan and Kettering [33]). Define grouping cells $E_i(\hat{\boldsymbol{\theta}}_n)=\{\mathbf{X}\in R^p: c_{i-1}\le b(\mathbf{X}_i)<c_i\}$, $i=1,\ldots,r$, and expected cell probability $p_i(\hat{\boldsymbol{\theta}}_n)=\int_{E_i(\hat{\boldsymbol{\theta}}_n)}f(\mathbf{x}|\hat{\boldsymbol{\theta}}_n)dx$. Let ends of equiprobable grouping cells be $0=c_0<c_1<\cdots<c_r=1$, where c_i, $i=1,\ldots,r-1$, is the i/r point of the beta-distribution with above parameters. If N_i denotes the number of $b(\mathbf{X}_j)$, $j=1,\ldots,n$, falling

in $E_i(\hat{\boldsymbol{\theta}}_n)$, and, since the expected frequency equals to $np_i(\hat{\boldsymbol{\theta}}_n) = n/r$, then the statistic of Cardoso De Oliveira and Ferreira [31] Z, that actually is the classical Pearson's sum, will be

$$Z = \sum_{i=1}^{r} r(N_i - n/r)^2/n. \tag{30}$$

For brevity, in the sequel we shall denote this test as the CarFer's statistic. Note that N_i, $i = 1, \ldots, n$, depend on unbiased estimators $\bar{\mathbf{X}}$ and $\mathbf{S}$ of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, and that those estimators are based on the non-grouped (raw) data.

Cardoso De Oliveira and Ferreira [31] have not taken into account this fact as well as the Moore and Stubblebine [22] results about limiting null distribution of (30) who have proved that this distribution is not chi-squared but is a weighted sum of chi-squared random variables. Strictly speaking, Moore and Stubblebine [22] obtained their result for MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. But, since in this case MLEs and unbiased estimators are asymptotically equivalent, the Moore and Stubblebine [22] theory can also be applied for the statistic (30). One needs only to derive correct weights for the sum of chi-squared random variables.

Under the regularity conditions of Moore and Spruill [20], p.602, which hold in our case, the r-vector $\mathbf{V}_n(\hat{\boldsymbol{\theta}}_n)$ of standardized frequencies with components $V_i(\hat{\boldsymbol{\theta}}_n) = (N_i - n/r)/\sqrt{n/r}$, $i = 1, \ldots, r$, follow asymptotically the r-dimensional multivariate normal distribution $N_r(\mathbf{0}, \boldsymbol{\Sigma}_v)$ with $\mathbf{0}$-vector of means and the covariance matrix

$$\boldsymbol{\Sigma}_v = \mathbf{I} - \mathbf{q}\mathbf{q}^T - \mathbf{B}\mathbf{J}^{-1}\mathbf{B}^T. \tag{31}$$

Moore and Stubblebine [22], p.720, justified this result for MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Since unbiased estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are asymptotically equivalent to their MLEs, the formula in (31) is valid in our case as well. Thus the asymptotic null distribution of Z is determined by the eigen-values of the matrix (31). The rank of $\mathbf{B}$ and also rank of $\mathbf{B}\mathbf{J}^{-1}\mathbf{B}^T$ is 1. From the above it follows that eigen-values of (31) are $r-2$ 1's, one 0 and $0 < \Lambda < 1$ which is nonzero eigen-value of $\mathbf{B}\mathbf{J}^{-1}\mathbf{B}^T$. The explicit expression for Λ is given in [34] as

$$\Lambda = 1 - 2rp \sum_{i=1}^{r} d_i^2,$$

where

$$d_i = \left[c_{i-1}^{p/2} \exp\left(-\frac{(n-1)^2}{2n} c_{i-1}\right) - c_i^{p/2} \exp\left(-\frac{(n-1)^2}{2n} c_i\right)\right] \frac{(n-1)^p b_p}{2n^{p/2}},$$

$b_p = [p(p-2)\dots(4)(2)]^{-1}$ if p is even, and $b_p = (2/\pi)^{1/2}[p(p-2)\dots(5)(3)]^{-1}$ if p is odd.

THEOREM 6. *The limit null probability distribution function of the CarFer's statistic Z is*

$$F(z) = \frac{1}{\sqrt{\pi}\Gamma((r-2)/2)} \gamma\left(\frac{1}{2}, \frac{z}{2\Lambda}\right) \gamma\left(\frac{r-2}{2}, \frac{z}{2}\right), \; r > 2, \tag{32}$$

where $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$, $s > 0$, is the incomplete gamma-function.

Proof. Since eigen-values of $\mathbf{\Sigma}_v = \mathbf{I} - \mathbf{q}\mathbf{q}^T - \mathbf{B}\mathbf{J}^{-1}\mathbf{B}^T$ are $r-2$ 1's, one 0 and $0 < \Lambda < 1$, the limit null distribution of the CarFer's quadratic form will be the same as that of $X + \Lambda Y$, where the probability density function of X is $f(x) = \chi^2_{r-2}(x) = x^{(r-4)/2}e^{-x/2}/\left[2^{(r-2)/2}\Gamma((r-2)/2)\right]$, $x > 0$, and the probability density function of Y is $f(y) = \chi^2_1(y) = e^{-y/2}/\sqrt{2\pi y}$, $y > 0$. Due to the independence of X and Y the limit null probability distribution function of Z will be

$$F(z) = P(Z \le z) = \int\limits_{x+\Lambda y \le z}\int f(x)f(y)dxdy = \int_0^{z/\Lambda} f(y)dy \int_0^z f(x)dx$$

$$= \frac{1}{\sqrt{\pi}\Gamma((r-2)/2)} \gamma\left(\frac{1}{2}, \frac{z}{2\Lambda}\right) \gamma\left(\frac{r-2}{2}, \frac{z}{2}\right), \; r > 2.$$

The corresponding probability density function is

$$f(z) = \frac{\chi^2_1(z/\Lambda)}{\Lambda\Gamma((r-2)/2)} \gamma\left(\frac{r-2}{2}, \frac{z}{2}\right) + \frac{\chi^2_{r-2}(z)}{\sqrt{\pi}} \gamma\left(\frac{1}{2}, \frac{z}{2\Lambda}\right), \; r > 2. \tag{33}$$

The intensive simulation study performed in [34] clearly shows that the limit distribution of the CarFer statistic Z actually follows (33), and not to χ^2_{r-1} as it was announced in [31]. In our opinion, if the simple null hypothesis were the beta-distribution with known parameters $p/2$ and $(n-p-1)/2$, then indeed the statistic Z would follow in the limit the chi-squared probability distribution with $r-1$ df. But the authors of [31] intended to test the hypothesis (24) with unknown (hence, to be estimated by raw data) parameters of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Because of this, due to the theory of Chernoff and Lehmann [6], and Moore and Stubblebine [22] the CarFer statistic Z cannot in principle follow the chi-squared probability distribution with $r-1$ df. Actually, the simulation study in [34] shows that the limit distribution of Z in many practical situations can be well approximated by the chi-squared probability distribution with $r-2$ df, and not by χ^2_{r-1}.

Moreover, that simulation study showed that if the CarFer statistic Z will be used correctly, e.g., with critical values defined by (33), then the power of Z will still be much smaller than that of the well-known tests of Henze and Zirkler [27], Royston [28], Székely and Rizzo [29], and Doornik and Hansen [30]. From all these it follows that the CarFer statistic Z cannot be recommended for applications.

Based on results of [31] Batsidis et al. [32] suggest to use the family of power divergence statistics

$$Z(\lambda)=\begin{cases}\frac{2}{\lambda(\lambda+1)}\sum\limits_{i=1}^{r}O_i\left(\left(\frac{O_i}{E_i}\right)^{\lambda}-1\right),\ -\infty<\lambda<\infty,\ \lambda\neq-1,0,\\ 2\sum\limits_{i=1}^{r}E_i\log\frac{E_i}{O_i},\ \lambda=-1,\\ 2\sum\limits_{i=1}^{r}O_i\log\frac{O_i}{E_i},\ \lambda=0,\end{cases}\tag{34}$$

where O_i and E_i are the observed and expected frequencies for the i-th equiprobable interval $i=1,\dots,r$. Particular values of a parameter λ correspond to: CarFer chi-squared test ($\lambda=1$) considered in detail before, likelihood ratio test ($\lambda=0$), Freeman-Tukey statistic ($\lambda=-0.5$), modified chi-squared test ($\lambda=-2$), and Cressie-Read statistic ($\lambda=2/3$). The authors of Batsidis et al. [32], p. 2256 announced that under the same procedure as in Section 7 the limiting null distribution of $Z(\lambda)$ will follow the chi-squared distribution with $r-1$ degrees of freedom. We have already shown that this is not true if $\lambda=1$. The same conclusion can be made for other values of λ. To avoid theoretical work for deriving the limit null distributions of $Z(\lambda)$ for different λ, we investigated simulated critical values of $Z(\lambda)$ having compared them with those for the chi-squared distribution with $r-2$ and $r-1$ degrees of freedom. This study showed that the histograms of simulated values of $Z(\lambda)$ are well approximated by the chi-squared distribution with $r-2$ df (not $r-1$ df as it was announced in Batsidis et al. [32], p. 2256). It was shown also that the power of such power-divergence tests (at least with respect to some alternatives close to MVN) is much lower than that for other well-known by the date tests for MVN. Once again, we have to conclude that power-divergence tests of Batsidis et al [32] cannot be considered as "necessary" and, hence, are not recommended for applications.

References

1 Plackett R.L. Karl Pearson and chi-squared test // International Statistical Review. — 1983. — V. 51. — P. 59–72.

2 Fisher R.A. The condition under which χ^2 measures the discrepancy between observation and hypothesis // Journal of the Royal Statistical Society. — 1924. — V. 87. — P. 442–450.

3 Greenwood P.E., Nikulin M.S. A Guide to Chi-squared Testing. — New York: John Wiley and Sons, 1996. — 304 p.

4 Dzhaparidze K.O., Nikulin M.S. On a modification of the standard statistic of Pearson // Theory of Probability and its Applications. — 1974. — V. 19. — P. 851–853.

5 Voinov V., Pya N., Alloyarova R. A comparative study of some modified chi-squared tests // Communications in Statistics-Simulation and Computation. — 2009. — V. 38. — P. 355–367.

6 Chernoff H., Lehmann E.L. The use of maximum likelihood estimates in tests for goodness-of-fit // The Annals of Mathematical Statistics. — 1954. — V. 25. — P. 579–589.

7 Nikulin M. Chi-square test for continuous distributions // Theory of Probability and its Applications. — 1973. — V. 18. — P. 638–639.

8 Rao K.C., Robson D.S. A chi-squared statistic for goodness-of-fit tests within the exponential family // Communications in Statistics. — 1974. — V. 3. — P. 1139–1153.

9 Hsuan T.A., Robson D.S. The goodness-of-fit tests with moment type estimators // Communications in Statistics — Theory and Methods. — 1976.

— V. 5. — P. 1509–1519.

10 Mirvaliev M. An investigation of generalized chi-squared type statistics // Doctoral Dissertation. Academy of Sciences of the Republic of Uzbekistan, Tashkent. — 2001. — 294 p.

11 Moore D.S. Generalized inverses, Wald's method and the construction of chi-squared tests of fit // Journal of the American Statistical Association. — 1977. — V. 72. — P. 131–137.

12 McCulloch C.E. Relationships among some chi-squared goodness-of-fit statistics // Communications in Statistics – Theory and Methods. — 1985. — V. 14. — P. 593–603.

13 Cochran W.G. Some methods for strengthening the common χ^2 tests // Biometrics. — 1954. — V. 10. — P. 417–451.

14 Voinov V. A Decomposition of Pearson-Fisher and Dzhaparidze-Nikulin Statistics and Some Ideas for a More Powerful Test Construction // Communications in Statistics – Theory and Methods. — 2010. — V. 39. — P. 667–677.

15 Voinov V., Nikulin M., Pya N. Independent χ_1^2 distributed in the limit components of some chi-squared tests // Skiadas, C.H. (Ed.), Recent Advances in Stochastic Modelling and Data Analysis. World Scientific Publishers, New Jersey. — 2007. — P. 243–250.

16 Voinov V., Pya N. A note on vector-valued goodness-of-fit tests // Communications in Statistics – Theory and Methods. — 2010. — V. 39. — P. 452–459.

17 Voinov V., Pya N. On the power of modified chi-squared goodness-of-fit tests for the family of logistic and normal distributions // Proceedings of the 7-th Iranian Statistical Conference. Allameh Tabatabaie University. — 2004. — P. 385–403.

18 Voinov, V., Nikulin, M., Balakrishnan, N. Chi-squared Goodness-of-Fit Tests with Applications. Amsterdam, etc.: Academic Press, Elsevier. — 2013. — 229 p.

19 Moore D.S. A chi-square statistic with random cell boundaries // The Annals of Mathematical Statistics. — 1971. — V. 42. — P. 147–156.

20 Moore D.S., Spruill M.C. Unified large-sample theory of general chi-squared statistics for tests of fit // The Annals of Statistics. — 1975. — V. 3. — P. 599–616.

21 Voinov V. A note on Wald's type chi-squared tests of fit // Communications in Statistics – Theory and Methods. — 2013.
http://www.tandfonline.com/doi/full/ 10.1080/03610926.2013.768668.

22 Moore D.S., Stubblebine J.B. Chi-square tests for multivariate normality with application to common stock prices // Communications in Statistics – Theory and Methods. — 1981. — V. **A** (10). — P. 713–738.

23 Wald A. Tests of statistical hypotheses concerning several parameters when the number of observations is large // Transactions of the American Mathematical Society. — 1943. — V. 54. — P. 426–482.

24 Khatri C.G. Further contributions of Wishartness and independence of second degree polynomials in normal vectors // Journal of the Indian Statistical Association. — 1963. — V. 2. — P. 60–70.

25 Rao C.R. Generalized inverse for matrices and its applications in mathematical statistics // Festschrift Volume for J. Neyman. Research Papers in Statistics edited by F. N. David. — 1966. — Wiley: New York. — P. 263–299.

26 Voinov V., Pya N., Makarov R., Voinov Y. New invariant and consistent chi-squared type goodness-of-fit tests for multivariate normality and a related comparative simulation study // Communications in Statistics — Theory and Methods. — 2014. (To appear).

27 Henze N., Zirkler B. A class of invariant consistent tests for multivariate normality // Communications in Statistics – Theory and Methods. — 1990. — V. 19. — P. 3595–3617.

28 Royston P. Approximating the Shapiro-Wilk W-test for non-normality // Statistics and Computing. — 1992. — V. 2. — P. 117–119.

29 Székely G.J., Rizzo M.L. A new test for multivariate normality // Journal of Multivariate Analysis. — 2005. — V. 93. — P. 58–80.

30 Doornik J.A., Hansen H. Practitioners' corner: An omnibus test for univariate and multivariate normality // Oxford Bulletin of Economics and Statistics. — 2008. — V. 70 supplement. — P. 927–939.

31 Cardoso De Oliveira I.R., Ferreira D.F. Multivariate extension of chi-squared univariate normality test // Journal of Statistical Computation and Simulation. — 2010. — V. 80. — P. 513–526.

32 Batsidis A., Martin N., Pardo L., Zografos K. A necessary power divergence type family tests of multivariate normality // Communications in Statistics – Simulation and Computation. — 2013. — V. 42. — P. 2253–2271.

33 Gnanadesikan R., Kettenring J.R. Robust estimates, residuals, and outlier detection with multiresponse data // Biometrics. — 1972. — V. 28. — P. 81–124.

34 Voinov, V., Makarov, R. On 21-st Century's misusing of the classical Pearson's sum and power divergence statistics // Proceedings of SMTDA 2014 International Conference, June 10-14, 2014, Lisbon, Portugal. — 2014. — To be published.

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Воинов В.Г. ХИ-КВАДРАТ ТЕКТЕС КРИТЕРИЙЛЕР КӨМЕГIМЕН КЕЛIСIМ ТЕСТIЛЕУДЕГI ҚАЗIРГI ЖЕТIСТIКТЕР

Соңғы он жылда кең түрде пайдаланатын хи-квадрат тектес келiсiмнiң түрлендiрiлген критерийлерiнiң теориясы мен қолданыстары үшiн көп нәрсе жасалды. Бұл шолуда бiз осы бағыттағы басты жетiстiктердi, Қазақстан Республикасы Бiлiм және ғылым министрлiгi Математика және математикалық моделдеу институтының ғалымдарының үлесiн атап көрсете отырып, қысқаша қарастырамыз.

Воинов В.Г. СОВРЕМЕННЫЕ ДОСТИЖЕНИЯ В ТЕСТИРОВАНИИ СОГЛАСИЯ С ПОМОЩЬЮ КРИТЕРИЕВ ТИПА ХИ-КВАДРАТ

В последние десять лет много было сделано для теории и приложений широко используемых модифицированных критериев согласия типа хи-квадрат. В этом обзоре мы кратко рассмотрим основные достижения в этом направлении, подчеркивая вклад ученых Института математики и математического моделирования Министерства образования и науки Республики Казахстан.

Правила "Математического журнала" для авторов статей

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1 Мынбаев К.Т., Отелбаев М.О. Весовые функциональные пространства и спектр дифференциальных операторов. — М.: Наука, 1988. — 288 с. (для монографий)

2 Женсыкбаев А.А. Моносплайны минимальной нормы и наилучшие квадратурные формулы // Успехи матем. наук. — 1981. — Т. 36, вып. (или №) 4. — С. 107–159.

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